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RELATIVELY AMENABLE ACTIONS OF THOMPSON'S GROUPS

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Abstract

We investigate the notion of relatively amenable topological action and show that the action of Thompson's group T on S^1 is relatively amenable with respect to Thompson's group F. We use this to conclude that F is exact if and only if T is exact. Moreover, we prove that the groupoid of germs of the action of T on S^1 is Borel amenable.

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1. Introduction

In [14], Spielberg showed that every *Kirchberg* (that is, simple, nuclear, purely infinite and separable) *algebra* which satisfies the universal coefficient theorem (UCT) admits a Hausdorff groupoid model and hence admits a Cartan subalgebra. Conversely, it was shown by Barlak and Li in [2] that any separable and nuclear C^*-algebra which has a Cartan subalgebra satisfies the UCT.

Given an étale non-Hausdorff groupoid G, there are dynamical criteria which ensure that the essential C^*-algebra of G is a Kirchberg algebra. Since, in general, $C^*_{\mathrm{ess}}(G)$ does not admit any obvious Cartan subalgebra, it seems natural to look at such groupoids as potential sources of counterexamples to the UCT problem (which asks whether every separable nuclear C^*-algebra satisfies the UCT).

Let $G(T,S^1)$ be the groupoid of germs of the action of Thompson's group T on S^1. In [7], Kalantar and the author showed that the reduced C^*-algebra of $G(T,S^1)$ is not simple, even though $G(T,S^1)$ is minimal and effective. Moreover, as observed in [7], it follows from results of Kwaśniewski and Meyer [8] that $C^*_{\mathrm{ess}}(G(T,S^1))$ is purely infinite and simple. In this paper, we show that $G(T,S^1)$ is Borel amenable. Since, as observed by Renault in [13], the results on nuclearity of groupoid C^*-algebras from the work of Anantharaman-Delaroche and Renault [1] use only Borel amenability and hold in the non-Hausdorff setting as well, we conclude that $C^*_{\mathrm{ess}}(G(T,S^1))$ is a

Kirchberg algebra. We leave open the question of whether $C^*_{\mathrm{ess}}(G(T, S^1))$ admits a Cartan subalgebra (equivalently, whether it satisfies the UCT).

Let Γ be a group acting on a locally compact Hausdorff space X and on a set K. In [11], Ozawa studied the existence of nets of continuous approximately equivariant maps $\mu_i \colon X \to \mathrm{Prob}(K)$. Clearly, the existence of such maps generalises both topological amenability (in the case $K = \Gamma$) and set-theoretical amenability (in the case that X consists of a single point). If such a property holds in the case that K is a set of left cosets Γ/Λ, we say that X is (Γ, Λ)-*amenable*. We show that this property generalises the notion of *relative co-amenability* introduced by Caprace and Monod in [6] (in the more general setting of locally compact groups).

Consider Thompson's groups $F \leq T$. We show that S^1 is (T, F)-amenable and use this fact to conclude that F is exact if and only if T is exact.

2. Relatively amenable actions

Given a set Y, we consider $\mathrm{Prob}(Y) := \{\mu \in \ell^1(Y) : \mu \geq 0, \|\mu\|_1 = 1\}$ equipped with the pointwise convergence topology.

Given a group Γ acting by homeomorphisms on a locally compact Hausdorff space X, we say that X is a *locally compact*Γ-*space*. Given $\Lambda \leq \Gamma$, we say that X is (Γ, Λ)-*amenable* if there exists a net of continuous functions $\mu_i \colon X \to \mathrm{Prob}(\Gamma/\Lambda)$ which is *approximately invariant* in the sense that

$$\lim_i \sup_{x \in K} \|s\mu_i(x) - \mu_i(sx)\|_1 = 0$$

for all $s \in \Gamma$ and $K \subset X$ compact. If $\Lambda = \{e\}$, this is the usual notion of (*topologically*) amenable action on a space X [1, Example 2.2.14(2)]. If Λ is co-amenable in Γ, then any Γ-space is (Γ, Λ)-amenable.

We will need the following result.

PROPOSITION 2.1 [4, Proposition 5.2.1]. *Let X be a compact Γ-space which is (Γ, Λ)-amenable for some $\Lambda \leq \Gamma$. If Λ is exact, then Γ is exact.*

Let us now characterise (Γ, Λ)-amenability in the case of a discrete Γ-space.

PROPOSITION 2.2. *Let S be a discrete Γ-space and $\Lambda \leq \Gamma$. The space S is (Γ, Λ)-amenable if and only if there exists a unital positive Γ-equivariant linear map $\varphi \colon \ell^\infty(\Gamma/\Lambda) \to \ell^\infty(S)$.*

PROOF. We identify the space of bounded linear maps $\mathcal{L}(\ell^\infty(\Gamma/\Lambda), \ell^\infty(S))$ with $\ell^\infty(S, \ell^\infty(\Gamma/\Lambda)^*)$. Under this identification, a unital positive Γ-equivariant map $\varphi \in \mathcal{L}(\ell^\infty(\Gamma/\Lambda), \ell^\infty(S))$ corresponds to a map $\psi \colon S \to \ell^\infty(\Gamma/\Lambda)^*$ such that $\psi(s)$ is a state and $\psi(gs) = g(\psi(s))$ for every $s \in S$ and $g \in \Gamma$.

Suppose that S is (Γ, Λ)-amenable and let $\mu_i \colon S \to \mathrm{Prob}(\Gamma/\Lambda) \subset \ell^\infty(\Gamma/\Lambda)^*$ be a net of approximately invariant functions. By taking a subnet, we may assume that, for each $s \in S$, $\mu_i(s)$ converges in the weak-$*$ topology to a state $\psi(s) \in \ell^\infty(\Gamma/\Lambda)^*$. Clearly, $\psi \colon S \to \ell^\infty(\Gamma/\Lambda)^*$ has the desired properties.

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Conversely, suppose that there exists a map $\psi \in \ell^\infty(S, \ell^\infty(\Gamma/\Lambda)^*)$ which is unital, positive and Γ-equivariant. Since $\ell^1(\Gamma)$ is weak-$*$ dense in $\ell^\infty(\Gamma)^*$, we can find a net $\mu_i : S \to \mathrm{Prob}(\Gamma/\Lambda) \subset \ell^\infty(\Gamma/\Lambda)^*$ such that, for each $s \in S$, $\mu_i(s) \to \psi(s)$ in the weak-$*$ topology. By Γ-equivariance of ψ, the net $g\mu_i(s) - \mu_i(gs)$ converges to zero weakly in $\ell^1(\Gamma/\Lambda)$ for each $g \in \Gamma$ and $s \in S$.

Given $\epsilon > 0$ and finite subsets $E \subset \Gamma$ and $F \subset S$, we claim that there is a function $\mu : S \to \mathrm{Prob}(\Gamma/\Lambda)$ such that $\|g\mu(x) - \mu(gx)\|_1 < \epsilon$ for each $x \in F$ and $g \in E$. From the previous paragraph, it follows that 0 is in the weak closure of the convex set

$$\bigoplus_{\substack{g\in E\\ s\in F}} \{g\mu(s) - \mu(gs) \mid \mu : S \to \mathrm{Prob}(\Gamma/\Lambda)\} \subset \bigoplus_{\substack{g\in E\\ s\in F}} \ell^1(\Gamma/\Lambda).$$

By the Hahn–Banach separation theorem, the claim follows. Thus, S is (Γ, Λ)-amenable. □

REMARK 2.3. Given a group Γ and subgroups $\Lambda_1, \Lambda_2 \le \Gamma$, Proposition 2.2 implies that Γ/Λ_2 is (Γ, Λ_1)-amenable if and only if Λ_1 is co-amenable to Λ_2 relative to Γ in the sense of [6, Section 7.C].

For completeness, we record the following permanence property. The proof follows the argument in [4, Proposition 5.2.1].

PROPOSITION 2.4. *Let X be a locally compact Γ-space and $\Lambda_1 \le \Lambda_2 \le \Gamma$ be such that X is (Γ, Λ_2)-amenable and (Λ_2, Λ_1)-amenable. Then X is (Γ, Λ_1)-amenable.*

PROOF. Fix $E \subset \Gamma$ finite, $\epsilon > 0$ and $K \subset X$ compact. Take $\eta : X \to \mathrm{Prob}(\Gamma/\Lambda_2)$ continuous such that $\sup_{x\in K} \|s\eta^x - \eta^{sx}\| < \epsilon/2$ for all $s \in E$. By arguing as in [4, Lemma 4.3.8], we may assume that there is $F \subset \Gamma/\Lambda_2$ finite such that $\mathrm{supp}\, \eta^x \subset F$ for all $x \in X$.

Fix a cross-section $\sigma : \Gamma/\Lambda_2 \to \Gamma$. Let

$$E^* := \{\sigma(sa)^{-1} s\sigma(a) : a \in F, s \in E\} \subset \Lambda_2$$

and

$$L := \bigcup_{a\in F} \sigma(a)^{-1} K.$$

Take $\nu : X \to \mathrm{Prob}(\Lambda_2/\Lambda_1) \subset \mathrm{Prob}(\Gamma/\Lambda_1)$ continuous such that

$$\max_{s\in E^*} \sup_{y\in L} \|s\nu(y) - \nu(sy)\|_1 < \epsilon/2.$$

Let

$$\begin{aligned} \mu : X &\to \mathrm{Prob}(\Gamma/\Lambda_1) \\ x &\mapsto \sum_{a\in F} \eta^x(a)\sigma(a)\nu^{\sigma(a)^{-1}x}. \end{aligned}$$

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Given $s \in E$ and $x \in K$,

$$\begin{aligned} s\mu(x) &= \sum_{a \in F} \eta^x(a) s\sigma(a) \nu^{\sigma(a)^{-1}x} \\ &= \sum_{a \in \Gamma/\Lambda_2} \eta^x(a)\sigma(sa)\sigma(sa)^{-1} s\sigma(a)\nu^{\sigma(a)^{-1}x} \\ &\approx_{\epsilon/2} \sum_{a \in \Gamma/\Lambda_2} \eta^x(a)\sigma(sa)\nu^{\sigma(sa)^{-1}sx} \\ &\approx_{\epsilon/2} \sum_{a \in \Gamma/\Lambda_2} \eta^{sx}(sa)\sigma(sa)\nu^{\sigma(sa)^{-1}sx} \\ &= \sum_{b \in \Gamma/\Lambda_2} \eta^{sx}(b)\sigma(b)\nu^{\sigma(b)^{-1}sx} = \mu(sx). \end{aligned}$$

□

Thompson's groups. Thompson's group V consists of piecewise linear, right continuous bijections on $[0, 1)$ which have finitely many points of nondifferentiability, all being dyadic rationals, and have a derivative which is an integer power of two at each point of differentiability.

Let $\mathcal{W}$ be the set of finite words in the alphabet $\{0, 1\}$. Given $w \in \mathcal{W}$ with length $|w|$, let $C(w) := \{(x_n) \in \{0, 1\}^{\mathbb{N}} : x_{[1,|w|]} = w\}$. Also let $\psi\colon \mathcal{W} \to [0, 1]$ be the map given by $\psi(w) := \sum_{n=1}^{|w|} x_n 2^{-n}$ for $w \in \mathcal{W}$. By identifying a set of the form $C(w)$ with the half-open interval $[\psi(w), \psi(w) + 2^{-|w|})$, we can view V as the group of homeomorphisms of $\{0, 1\}^{\mathbb{N}}$ consisting of elements g for which there exist two partitions $\{C(w_1), \dots, C(w_n)\}$ and $\{C_{z_1}, \dots, C_{z_n}\}$ of $\{0, 1\}^{\mathbb{N}}$ such that $g(w_i x) = z_i x$ for every i and infinite binary sequence x.

Let $D := \{(x_n) \in \{0, 1\}^{\mathbb{N}}$: there exists $k \in \mathbb{N}$ such that $x_l = 0$ for all $l \geq k\}$. Notice that D is V-invariant. Given $w \in \mathcal{W}$, let $w0^\infty$ be the element of D obtained by extending w with infinitely many 0's.

THEOREM 2.5. *There is a sequence of continuous maps* $\mu_N\colon \{0, 1\}^{\mathbb{N}} \to \mathrm{Prob}(D)$ *such that*

$$\lim_N \sup_{x \in \{0,1\}^{\mathbb{N}}} \|s\mu_N(x) - \mu_N(sx)\|_1 = 0 \tag{2.1}$$

for every $s \in V$.

PROOF. Given $N \in \mathbb{N}$, let $\mu_N\colon \{0, 1\}^{\mathbb{N}} \to \mathrm{Prob}(D)$ be defined by

$$\mu_N(x) := \frac{1}{N} \sum_{j=1}^{N} \delta_{x_{[1,j]}0^\infty}.$$

Clearly, for each $d \in D$ and $N \in \mathbb{N}$, the map $x \mapsto \mu_N(x)(d)$ is continuous. We claim that (μ_N) satisfies (2.1).

Fix $s \in V$. There exist two partitions $\{C(w_1), \dots, C(w_n)\}$ and $\{C_{z_1}, \dots, C_{z_n}\}$ of $\{0, 1\}^{\mathbb{N}}$ such that $s(w_i x) = z_i x$ for every i and infinite binary sequence x.

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Let $k(s) := \max_i\{|w_i|, |z_i| - |w_i|\}$. Fix $1 \le i \le n$ and $x \in C(w_i)$. Let $\alpha_i := |z_i| - |w_i|$. Given $k > k(s)$,

$$s(x_{[1,k]}0^\infty) = z_i x_{[|w_i|+1,k]}0^\infty = s(x)_{[1,|z_i|]}s(x)_{[|z_i|+1,k+|z_i|-|w_i|]}0^\infty = s(x)_{[1,k+\alpha_i]}0^\infty.$$

For $N > 2k(s)$,

$$\begin{aligned}\|s\mu_N(x) - \mu_N(sx)\| &= \frac{1}{N}\Big\| \sum_{j=1}^{N} \delta_{s(x_{[1,j]}0^\infty)} - \delta_{s(x)_{[1,j]}0^\infty}\Big\| \\ &\le \frac{1}{N}\Big\| \sum_{j=k(s)+1}^{N-k(s)} \delta_{s(x_{[1,j]}0^\infty)} - \sum_{l=k(s)+1+\alpha_i}^{N-k(s)+\alpha_i} \delta_{s(x)_{[1,l]}0^\infty}\Big\| + \frac{4k(s)}{N} \\ &= \frac{4k(s)}{N}.\end{aligned}$$

□

Thompson's group T is the subgroup of V consisting of elements which have at most one point of discontinuity. By identifying $[0, 1)$ with S^1, the elements of T can be seen as homeomorphisms on S^1. Thompson's group F is the subgroup of T which stabilises $1 \in S^1$.

Corollary 2.6. *The spaces* $\{0, 1\}^{\mathbb{N}}$ *and* S^1 *are* (T, F)*-amenable.*

Proof. Notice that T acts transitively on $D \subset \{0, 1\}^{\mathbb{N}}$. Since F is the stabiliser of $0^\infty \in D$, it follows immediately from Theorem 2.5 that $\{0, 1\}^{\mathbb{N}}$ is (T, F)-amenable.

Let $\varphi\colon S^1 \to \{0, 1\}^{\mathbb{N}}$ be the map which, given $\theta \in [0, 1)$, sends $e^{2\pi i\theta}$ to the binary expansion of θ. Clearly, φ is T-equivariant and Borel measurable. Since $\{0, 1\}^{\mathbb{N}}$ is (T, F)-amenable, composition with φ gives rise to a sequence $u_n\colon S^1 \to \mathrm{Prob}(T/F)$ of approximately T-equivariant pointwise Borel maps (in the sense that for each $d \in T/F$, the map $x \mapsto u_n(x)(d)$ is Borel). It follows from [4, Proposition 5.2.1] (or [11, Proposition 11]) that S^1 is (T, F)-amenable. □

The next result follows immediately from Proposition 2.1 and Corollary 2.6.

Corollary 2.7. *Thompson's group F is exact if and only if Thompson's group T is exact.*

The next result has been recorded in [9, Section 3.2] as a consequence of hyperfiniteness of the equivalence relation of T on S^1. It also follows from the fact that stabilisers of amenable actions are amenable, Proposition 2.4 and Corollary 2.6.

Corollary 2.8 [9]. *The following conditions are equivalent:*

(i) *F is amenable;*
(ii) *$T \curvearrowright \{0, 1\}^{\mathbb{N}}$ is amenable;*
(iii) *$T \curvearrowright S^1$ is amenable.*

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3. Groupoids of germs

We say that a topological groupoid G is *étale* if its unit space $G^{(0)}$ is Hausdorff and the range and source maps $r, s\colon G \to G^{(0)}$ are local homeomorphisms. If G is also second countable, then G is said to be *Borel amenable* [13, Definition 2.1] if there exists a sequence $(m_n)_{n\in\mathbb{N}}$, where each m_n is a family $(m_n^x)_{x\in G^{(0)}}$ of probability measures on $r^{-1}(x)$ such that:

(i) for all $n \in \mathbb{N}$, m_n is Borel in the sense that for all bounded Borel functions f on G, $x \mapsto \sum_{g\in r^{-1}(x)} f(g)m_n^x(g)$ is Borel;

(ii) for all $g \in G$, we have $\sum_{h\in r^{-1}(r(g))} |m_n^{s(g)}(g^{-1}h) - m_n^{r(g)}(h)| \to 0$.

REMARK 3.1. Let G be a second countable étale groupoid and $A \subset G^{(0)}$ a measurable subset which is invariant in the sense that $r^{-1}(A) = s^{-1}(A)$. In this case, $G_A := s^{-1}(A)$ is a subgroupoid of G. If G is Borel amenable, then clearly G_A is also Borel amenable. Conversely, if G_A and $G_{G^{(0)}\setminus A}$ are Borel amenable, then, since $G = G_A \sqcup G_{G^{(0)}\setminus A}$, also G is Borel amenable.

Let Γ be a group acting on a compact Hausdorff space X. Given $x \in X$, let $\Gamma_x^0 := \{g \in \Gamma : g \text{ fixes pointwise a neighbourhood of } x\}$ be the *open stabiliser* at x. Consider the following equivalence relation on $\Gamma \times X$: $(g, x) \sim (h, y)$ if and only if $x = y$ and $g\Gamma_x^{(0)} = h\Gamma_x^{(0)}$. As a set, the *groupoid of germs* of $\Gamma \curvearrowright X$ is $G(\Gamma, X) := (\Gamma \times X)/\sim$. The topology on $G(\Gamma, X)$ is the one generated by sets of the form $[g, U] := \{[g, x] : x \in U\}$ for $U \subset X$ open and $g \in \Gamma$. Inversion in $G(\Gamma, X)$ is given by $[g, x]^{-1} = [g^{-1}, gx]$. Two elements $[h, y], [g, x] \in G(\Gamma, X)$ are multipliable if and only if $y = gx$, in which case $[h, y][g, x] := [hg, x]$. With this structure, $G(\Gamma, X)$ is an étale groupoid.

EXAMPLE 3.2. Let $G_{[2]}$ be the Cuntz groupoid introduced in [12, Definition III.2.1]. Since Thompson's group T can be seen as a covering subgroup of the topological full group of $G_{[2]}$ [3, Example 3.3], it follows from [10, Proposition 4.10] that $G(T, \{0, 1\}^{\mathbb{N}}) \simeq G_{[2]}$. Hence, $G(T, \{0, 1\}^{\mathbb{N}})$ is Borel amenable by [12, Proposition III.2.5].

THEOREM 3.3. *The groupoid of germs of $T \curvearrowright S^1$ is Borel amenable.*

PROOF. Let $X := \{e^{2\pi i\theta} : \theta \in \mathbb{Z}[1/2]\}$ and $Y := S^1 \setminus X$. Notice that X is T-invariant. We will show that $G(T, S^1)_X$ and $G(T, S^1)_Y$ are Borel amenable. From Remark 3.1, it will follow that $G(T, S^1)$ is Borel amenable.

Let $\varphi\colon S^1 \to \{0, 1\}^{\mathbb{N}}$ be the T-equivariant map, which, given $\theta \in [0, 1)$, sends $e^{2\pi i\theta}$ to the binary expansion of θ. Notice that $\varphi|_Y\colon Y \to \varphi(Y)$ is a homeomorphism. Furthermore, the map

$$\tilde{\varphi}\colon G(T, S^1)_Y \to G(T, \{0, 1\}^{\mathbb{N}})_{\varphi(Y)}$$
$$[g, y] \mapsto [g, \varphi(y)]$$

is an isomorphism of topological groupoids. Therefore, $G(T, S^1)_Y$ is Borel amenable by Remark 3.1 and Example 3.2.

Notice that $G(T, S^1)_X$ is a countable set. It follows from [5, Theorem 4.1] that the open stabiliser T_1^0 is equal to the commutator subgroup $[F, F]$ and $F/[F, F] \simeq \mathbb{Z}^2$. Therefore, $G(T, S^1)_X$ is Borel isomorphic to the transitive discrete groupoid $X \times X \times \mathbb{Z}^2$, which, due to the amenability of the isotropy group, is Borel amenable. □

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