Canad. Math. Bull. Vol. **34** (3), 1991 pp. 321–328

THE HILBERT PROBLEM—A DISTRIBUTIONAL APPROACH

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ABSTRACT. A distributional solution to the Hilbert problem in dimension > 1 is given.

1. **Introduction.** Let $F(z)$ be a holomorphic function in the region $\operatorname{Im} z \neq 0$ of the n-dimensional complex space $\mathbb{C}^n$. Assume that

$$F_+(x) = \lim_{y \to 0_+} F(z) \text{ in } D'_{L^p}(\mathbb{R}^n) \tag{1.1}$$

and

$$F_-(x) = \lim_{y \to 0_-} F(z) \text{ in } D'_{L^p}(\mathbb{R}^n) \tag{1.2}$$

and

$$z = (z_1, z_2, \ldots, z_n) = (x_1 + iy_1, x_2 + iy_2, \ldots, x_n + iy_n)$$

and $y \longrightarrow 0_+$ means $y_1 \longrightarrow 0_+$, $y_2 \longrightarrow 0_+, \ldots, y_n \longrightarrow 0_+$ simultaneously, with a similar interpretation for $y \longrightarrow 0_-$. $\operatorname{Im} z \neq 0$ means $\operatorname{Im} z_i \neq 0$ for $i = 1, 2, 3, \ldots n$. We shall consider the following Hilbert Problem. Let $f \in D'_{L^p}(\mathbb{R}^n)$. Then we wish to find a function $F(z) = F(z_1, z_2, \ldots, z_n)$ holomorphic in the region $\operatorname{Im} z_i \neq 0 \; \forall i = 1, 2, 3, \ldots n$ such that

$$F_+(x) + F_-(x) = f(x), \tag{1.3}$$

where $F_+(x)$, $F_-(x)$ are as defined in (1.1) and (1.2) respectively. The convergence in (1.1), (1.2) and the equality (1.3) is interpreted in the sense of $D'_{L^p}(\mathbb{R}^n)$. We will show that in one dimension the Hilbert Problem can always be solved while in higher dimensions a number of compatibility conditions must be satisfied by $f(x)$.

2. **Preliminaries.** An infinitely differentiable complex valued function $\varphi(x)$ defined over $\mathbb{R}^n$ is said to belong to the space $D_{L^p}(\mathbb{R}^n)$ if and only if $D^\alpha \varphi(x) \in L^p(\mathbb{R}^n)$, for every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_1, \alpha_2, \ldots, \alpha_n$ being non-negative integers. The space $D_{L^p}(\mathbb{R}^n)$ is equipped with the topology generated by the separating and countable collection of semi-norms $\{\gamma_m\}_{m=0}^{\infty}$, given by

$$\gamma_m(\varphi) = \Big[\sum_{|\alpha|=m} \int_{\mathbb{R}^n} |D^\alpha \varphi(x)|^p \, dx \Big]^{1/p}, \quad [1, 13] \tag{2.1}$$

The work of the second author was supported by NSERC grant A-5298.
Received by the editors November 20, 1989; revised: July 18, 1990.
AMS subject classification: Primary: 46F12; secondary: 44A15.
Key words and phrases: Hilbert transform, Hilbert problem, distributional Hilbert transform in n dimension.

where

$$|\alpha| = \sum_{j=1}^{n} \alpha_j.$$

Hence, a sequence $\{\varphi_m\}_{m=1}^{\infty}$ in $D_{L^p}(\mathbb{R}^n)$ converges to φ in $D_{L^p}(\mathbb{R}^n)$ if and only if $\gamma_{|\alpha|}(\varphi_m - \varphi) \longrightarrow 0$ as $m \longrightarrow \infty$ for each $|\alpha| = 0, 1, 2, \ldots$. The space $D_{L^p}(\mathbb{R}^n)$ is a locally convex, sequentially complete, Hausdorff linear space [10,13]. Note that if $\varphi \in D_{L^p}(\mathbb{R}^n)$ then $D^\alpha\varphi(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$ for each $|\alpha| \in \mathbb{N}$ [10], and if $\varphi_m \longrightarrow 0$ in $D_{L^p}(\mathbb{R}^n)$ as $m \longrightarrow \infty$, then $\phi_m \longrightarrow 0$ uniformly for all $x \in \mathbb{R}^n$ along with all its derivatives [10].

In conformity with the notation of L. Schwartz [10], we denote the dual space of $D_{L^q}(\mathbb{R}^n)$ by $D'_{L^p}(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1, q > 1$.

DEFINITION 2.1. The space $X(\mathbb{R}^n)$ is a subspace of the Schwartz testing function space $D(\mathbb{R}^n)$ consisting of all the finite linear combinations of the functions of the type $\prod_{i=1}^{n} \varphi_i(t_i)$, where $\varphi_i(t_i) \in D(\mathbb{R})$. The space $X(\mathbb{R}^n)$ is endowed with the topology induced on it by the space $D(\mathbb{R}^n)$. The space $X(\mathbb{R}^n)$ is dense in $D(\mathbb{R}^n)$ [11]. The space $D(\mathbb{R}^n)$ is dense in $D_{L^p}(\mathbb{R}^n)$ [10]. Since the topology of $X(\mathbb{R}^n)$ is the same as the topology induced on it by that of $D(\mathbb{R}^n)$ and the topology of the space $D(\mathbb{R}^n)$ is stronger than the topology induced on it by the space $D_{L^p}(\mathbb{R}^n)$, it follows that the space $X(\mathbb{R}^n)$ is dense in the space $D_{L^p}(\mathbb{R}^n)$. Hence, for an element $\varphi(x) \in D_{L^p}(\mathbb{R}^n)$, we can find a sequence $\{\varphi_v\}_{v=1}^{\infty}$ in $X(\mathbb{R}^n)$ such that

$$||D^\alpha(\varphi_v - \varphi)|| \longrightarrow 0, \text{ as } v \longrightarrow \infty,$$

for each $|\alpha| = 0, 1, 2, \ldots$.

DEFINITION 2.2. The n-dimensional Hilbert transform, $(Hf)(x)$, of $f \in L^p(\mathbb{R}^n)$ is defined by

$$(2.2) \quad \begin{aligned} (Hf)(x) &= \frac{1}{\pi^n} \lim_{|\varepsilon| \to 0} \int_{\substack{|t_i - x_i| > \varepsilon_i \\ i=1,2,\ldots,n}} \frac{f(t)}{\prod_{i=1}^{n}(t_i - x_i)}\, dt \\ &= \frac{1}{\pi^n} P \int \frac{f(t)}{\prod_{i=1}^{n}(t_i - x_i)}\, dt, \quad [2, 10] \end{aligned}$$

where

$$|\varepsilon| = \left(\sum_{i=1}^{n} \varepsilon_i^2\right)^{1/2}.$$

Note that $(Hf)(x)$ exists almost everywhere and that H is a bounded linear operator from $L^p(\mathbb{R}^n)$ into itself i.e.,

$$(2.3) \quad ||Hf||_p \leq C_p^n ||f||_p, \quad [4, 12]$$

where C_p is a constant independent of f. The first nontrivial result on multidimensional Hilbert transforms was due to C. Fefferman [3]. Recently, Singh and Pandey [11] established the following inversion formula for H:

$$(2.4) \quad H^2 f = (-1)^n f,$$

https://doi.org/10.4153/CMB-1991-052-2 Published online by Cambridge University Press

and obtained that, for $\varphi \in D_{L^p}(\mathbb{R}^n)$ $(p > 1)$,

$$D^\alpha(H\varphi) = H(D^\alpha \varphi). \tag{2.5}$$

A consequence of (2.4) and (2.5) was a very simple proof of the fact that the Hilbert transform operator H is a homeomorphism from $D_{L^p}(\mathbb{R}^n)$ onto itself [11]. For $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\int_{\mathbb{R}^n} (Hf)(x)\, g(x)\, dx = \int_{\mathbb{R}^n} f(x)(-1)^n (Hg)(x)\, dx.$$

In analogy with this fact, the operator H of the Hilbert transform on $D'_{L^p}(\mathbb{R}^n)$ was defined in [9,11] as follows:

$$\langle Hf, \varphi \rangle = \langle f, (-1)^n H\varphi \rangle, \quad \varphi \in D_{L^q}(\mathbb{R}^n), \tag{2.6}$$

where the generalized function space $D'_{L^p}(\mathbb{R}^n)$ is the dual space of $D_{L^q}(\mathbb{R}^n)$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, and $H\varphi$ is the Hilbert transform of φ given by (2.2).

Let $\{\varphi_\nu\}_{\nu=1}^\infty$ be a sequence in $X(\mathbb{R}^n)$ converging to φ in $\left(D_{L^q}(\mathbb{R}^n)\right)$, that is

$$\|D^\alpha(\varphi_\nu - \varphi)\|_q \longrightarrow 0 \text{ as } \nu \longrightarrow \infty,$$

then the Hilbert transform Hf of a generalized function $f \in D'_{L^p}(\mathbb{R}^n)$ can also be defined by

$$\langle Hf, \varphi \rangle = \lim_{\nu \to \infty} \langle f, (-1)^n H\varphi_\nu \rangle = \langle f, (-1)^n H\phi \rangle. \tag{2.7}$$

Using the above definition, it is easy to see that,

$$(D^\alpha H)f = (HD^\alpha)f, \quad f \in D'_{L^p}(\mathbb{R}^n). \quad [9, 11]$$

The definition (2.7) of the Hilbert transform of the elements of $D'_{L^p}(\mathbb{R}^n)$ is equivalent to the one given in [9,11]. Using the following structure formula

$$f = \sum_{|\alpha| \le m} (-1)^\alpha D^\alpha f_\alpha, \quad [1] \tag{2.8}$$

where each $f_\alpha \in L^p(\mathbb{R}^n)$, we obtain

$$\langle Hf, \varphi \rangle = \lim_{\nu \to \infty} \sum_{|\alpha| \le m} \int_{\mathbb{R}^n} (-1)^n f_\alpha(x) D^\alpha (H\varphi_\nu)(x)\, dx, \tag{2.9}$$

for each $\varphi \in D_{L^q}(\mathbb{R}^n)$.

The Hilbert transform technique is a powerful tool in solving some singular integral equations. For further details see [2,5,8,11].

https://doi.org/10.4153/CMB-1991-052-2 Published online by Cambridge University Press

3. **The Hilbert problem.** Given a function f on the real line satisfying certain prescribed conditions, we wish to find a holomorphic function $F(z)$ in the complex plane such that

$$F_+(x) + F_-(x) = f(x), \tag{3.1}$$

where

$$F_+(x) = \lim_{y\to 0_+} F(z), \quad z = x + iy$$

and

$$F_-(x) = \lim_{y\to 0_-} F(x). \tag{3.2}$$

The mode of convergence may be suitably chosen. The solution to the problem in the classical sense is given by Lauwerier [5] and in the distributional sense is given in [7]. We attempt to solve the n-dimensional Hilbert problem for the distribution space $D'_{L^p}(\mathbb{R}^n)$.

Let $F(z)$ be a function defined on the complex plane which is holomorphic in the upper half plane $\operatorname{Im} z > 0$ and also in the lower half plane $\operatorname{Im} z < 0$ satisfying the following conditions:

(i) $F(z) = o(1)$ as $|y| \longrightarrow \infty$ uniformly for every $x \in \mathbb{R}$,

(ii) $\sup_{x\in\mathbb{R}, y\geq\delta} |F(z)| \leq A_\delta < \infty$,

(iii) $\lim_{y\to 0_+} F(z) = F_+(x)$ in $D'_{L^p}(\mathbb{R})$,

(iv) $\lim_{y\to 0_-} F(z) = F_-(x)$ in $D'_{L^p}(\mathbb{R})$.

Then we have

$$F(z) = \frac{1}{(2\pi i)}\left\langle F_+(t) - F_-(t), \frac{1}{t-z}\right\rangle, \quad \operatorname{Im} z \neq 0. \quad [7] \tag{3.3}$$

If we consider the convergence in $D'(\mathbb{R})$, then

$$F(z) = \frac{1}{(2\pi i)}\left\langle F_+(t) - F_-(t), \frac{1}{t-z}\right\rangle + P(z), \quad \operatorname{Im} z \neq 0,$$

where $P(z)$ is a polynomial in z. From now onwards, we will consider the convergence in the space $D'_{L^p}(\mathbb{R})$ only, for $p > 1$. Writing $g = F_+ - F_-$, we have

$$F(z) = \frac{1}{(2\pi i)}\left\langle g(t), \frac{t - x + iy}{(t-x)^2 + y^2}\right\rangle.$$

Then we have

$$\lim_{y\to 0_+} F(z) = F_+(x) = \frac{1}{2i}[Hg + iIg], \tag{3.4}$$

and

$$\lim_{y\to 0_-} F(z) = F_-(x) = \frac{1}{2i}[Hg - iIg], \tag{3.5}$$

where I is the identity operator. A detailed proof of the identities (3.4) and (3.5) is given in [7]. Adding (3.4) and (3.5), we obtain

$$F_+(x) + F_-(x) = \frac{1}{i} Hg = f. \tag{3.6}$$

Hence, using the inversion formula (2.4), we deduce

$$g = -iHf,$$

so the required function $F(z)$, holomorphic for $\operatorname{Im} z \neq 0$, is given by

$$F(z) = -\frac{1}{2\pi}\left\langle Hf, \frac{1}{t-z}\right\rangle, \quad \operatorname{Im} z \neq 0. \tag{3.7}$$

We now extend the problem to $D'_{L^p}(\mathbb{R}^n)$. Let $f \in D'_{L^p}(\mathbb{R}^n)$ and let $F(z_1, z_2)$ be a function holomorphic in the region $\operatorname{Im} z_1 \neq 0$, $\operatorname{Im} z_2 \neq 0$ satisfying similar conditions as in the case of one dimension, i.e.,

(1) $F(z_1, z_2) = o(1)$ as $|y_1|, |y_2| \to \infty$,

(2) $\sup_{\substack{|y_1| \geq \delta_1 > 0 \\ |y_2| \geq \delta_2 > 0}} |F(z_1, z_2)| \leq A_\delta < \infty \quad \delta = (\delta_1, \delta_2)$.

(3) (i) $\lim_{y_1 \to 0_+, y_2 \to 0_+} F(z_1, z_2) = F_{++}(x_1, x_2)$,

(ii) $\lim_{y_1 \to 0_+, y_2 \to 0_-} F(z_1, z_2) = F_{+-}(x_1, x_2)$,

(iii) $\lim_{y_1 \to 0_-, y_2 \to 0_+} F(z_1, z_2) = F_{-+}(x_1, x_2)$, and

(iv) $\lim_{y_1 \to 0_-, y_2 \to 0_-} F(z_1, z_2) = F_{--}(x_1, x_2)$,

in $D'_{L^p}(\mathbb{R}^2)$, where

$$z_j = x_j + iy_j, \quad j = 1, 2.$$

Then we have

$$F(z_1, z_2) = \left(\frac{1}{2\pi i}\right)^2 \left\langle (F_{++} - F_{+-} - F_{-+} + F_{--})(t), \frac{1}{(t_1 - z_1)(t_2 - z_2)}\right\rangle. \quad [9] \tag{3.8}$$

Writing $g = F_{++} - F_{+-} - F_{-+} + F_{--}$, we have

$$F(z_1, z_2) = \frac{1}{(2\pi i)^2}\left\langle g(t), \frac{1}{(t_1 - z_1)(t_2 - z_2)}\right\rangle.$$

It was proved in [9,11] that

$$F_{++} = \frac{1}{(2i)^2}(H_1 + iI_1)(H_2 + iI_2)g,$$

where I_1, I_2 are the identity operators i.e.,

$$I_1 g(t_1, t_2) = g(x_1, t_2),$$
$$I_2 g(t_1, t_2) = g(t_1, x_2),$$
$$H_1\big(g(t_1, t_2)\big) = \frac{1}{\pi} P \int_{\mathbb{R}} \frac{g(t_1, t_2)}{t_1 - x_1}\, dt_1,$$

and

$$H_2\big(g(t_1,t_2)\big) = \frac{1}{\pi} P \int_{\mathbb{R}} \frac{g(t_1,t_2)}{(t_2 - x_2)}\, dt_2.$$

Similarly we have

$$F_{--} = \frac{1}{(2i)^2}(H_1 - iI_1)(H_2 - iI_2)g.$$

Hence $f = F_{++} + F_{--}$ gives

$$-\frac{1}{2}[H_1H_2 - i_1I_2]g = f,$$

that is

$$(H - I)g = -2f, \tag{3.9}$$

where $H = H_1H_2$ and $I = I_1I_2$ are the 2-dimensional Hilbert transform and identity operators on $D'_{L^p}(\mathbb{R}^2)$ respectively. Using the inversion formula (2.4), we obtain

$$(I - H)g = -2Hf. \tag{3.10}$$

Adding (3.9) and (3.10), we deduce that

$$f + Hf = 0. \tag{3.11}$$

Hence, if f does not satisfy (3.11), the solution of the aforesaid Hilbert problem does not exist. In [11], it was shown that there do exist functions satisfying (3.11). So let f satisfy (3.11) and let $g_1, g_2, \ldots, g_m$ in $D_{L^p}(\mathbb{R}^n)$ be such that they satisfy

$$y - Hy = 0. \tag{3.12}$$

Then we have that

$$g = \sum_{j=1}^{m} c_j g_j + f, \tag{3.13}$$

where c_j $(j = 1, \ldots, m)$ are constants, satisfies (3.9). Substituting $F_{++} - F_{+-} - F_{-+} + F_{--}$ for g in (3.13), a class of solutions to the Hilbert problem is obtained.

Let us now consider the solution to the Hilbert problem in the next higher dimension. Let $F(z_1, z_2, z_3)$, where $z_j = x_j + iy_j$ $(j = 1, 2, 3)$ be a function of z_1, z_2, z_3 which is analytic in the region

$$\{(z_1, z_2, z_3) : \operatorname{Im} z_1 \neq 0, \quad \operatorname{Im} z_2 \neq 0, \quad \operatorname{Im} z_3 \neq 0\}$$

of C^3 and satisifes the following conditions:

(i) $|F(z_1, z_2, z_3)| = o(1)$ as $|y_1|, |y_2|, |y_3| \to \infty$, the asymptotic order being valid uniformly $\forall x_1, x_2, x_3 \in \mathbb{R}^n$

(ii) $\lim_{\substack{y_1 \to 0_\pm \\ y_2 \to 0_\pm \\ y_3 \to 0_\pm}} F(z_1, z_2, z_3) = F_{\pm\pm\pm}$ in $D'_{L^p}(\mathbb{R}^n)$

(iii) $\sup_{\substack{|y_1|\geq\delta_1>0\\|y_2|\geq\delta_2>0\\|y_3|\geq\delta_3>0}} |F(z_1,z_2,z_3)| = A_\delta < \infty$, where $\delta = (\delta_1,\delta_2,\delta_3)$.

Now in view of the results proved in [9,11] there exists $g \in D'_{L^p}(\mathbb{R}^n)$ such that

$$F(z_1,z_2,z_3) = \frac{1}{(2\pi i)^3}\left\langle g(t), \frac{1}{\prod_{i=1}^3 (t_1 - z_i)}\right\rangle.$$

Therefore using results in [9,11] we obtain

$$F_{+++} = \frac{1}{(2i)^3}(H_1 + iI_1)(H_2 + iI_2)(H_3 + iI_3)g$$

and

$$F_{---} = \frac{1}{(2i)^3}(H_1 + iI_1)(H_2 + iI_2)(H_3 + iI_3)g,$$

so that

$$f = F_{+++} + F_{---} = \frac{2}{(2i)^3}(H_1H_2H_3 - H_1 - H_2 - H_3)g,$$

that is

$$-4if = (H - H_1 - H_2 - H_3)g. \tag{3.14}$$

Applying the operation $(H + H_1 + H_2 + H_3)$ to both sides of (3.14) we deduce

$$\begin{aligned}
&-4i(H + H_1 + H_2 + H_3)f \\
&\quad = [H^2 - (H_1 + H_2 + H_3)^2]g \\
&\quad = [-1 - (H_1^2 + H_2^2 + H_3^2 + 2H_1H_2 + 2H_1 + H_3 + 2H_2H_3)]g \\
&\quad = [-1 + 3 - 2(H_1H_2 + H_2H_3 + H_1H_3)]g \\
&\quad = [2 + 2H(H_1 + H_2 + H_3)]g.
\end{aligned} \tag{3.15}$$

Applying the operator $2H$ to both sides of (3.14) and adding the result to (3.15), we obtain

$$-4iHf - 4iH(H + H_1 + H_2 + H_3)f = 2H^2g + 2g = 0$$

or

$$f + (H + H_1 + H_2 + H_3)f = 0 \tag{3.16}$$

If the given f satisfies (3.16) then and only then a solution to the Hilbert problem exists. If f satisfies (3.16) then the solution to the Hilbert problem can be obtained by solving for g from (3.14) and substituting in the expression for $F(z_1,z_2,z_3)$. As we go to higher and higher dimensions the problem becomes more and more difficult. We leave this as an open problem.

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