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ON THE EXCEPTIONAL SPECIALIZATIONS OF BIG HEEGNER POINTS

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Abstract We extend the p-adic Gross–Zagier formula of Bertolini *et al.* [Generalized Heegner cycles and p-adic Rankin L-series, *Duke Math. J.* **162**(6) (2013), 1033–1148] to the semistable non-crystalline setting, and combine it with our previous work [Castella, On the p-adic variation of Heegner points, Preprint, 2014, arXiv:1410.6591] to obtain a derivative formula for the specializations of Howard's big Heegner points [Howard, Variation of Heegner points in Hida families, *Invent. Math.* **167**(1) (2007), 91–128] at exceptional primes in the Hida family.

Keywords: Hida families; Heegner points; p-adic L-functions; L-invariants

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Contents

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Introduction

Fix a prime $p \geqslant 5$, an integer $N > 0$ prime to p, and let $f \in S_2(\Gamma_0(Np))$ be a newform. Throughout this paper, we shall assume that f is *split multiplicative* at p, meaning that

$$f(q) = q + \sum_{n=2}^{\infty} a_n(f) q^n \quad \text{with } a_p(f) = 1.$$

Fix embeddings $\mathbf{C} \overset{\imath_\infty}{\hookleftarrow} \overline{\mathbf{Q}} \overset{\imath_p}{\hookrightarrow} \mathbf{C}_p$, let L be a finite extension of $\mathbf{Q}_p$ containing $\imath_p\imath_\infty^{-1}(a_n(f))$ for all n, and let $\mathcal{O}_L$ be the ring of integers of L. Since the U_p-eigenvalue of f is $a_p(f) = 1$ by hypothesis, the form f is ordinary at p, and hence there is a Hida family

$$\mathbf{f} = \sum_{n=1}^{\infty} \mathbf{a}_n q^n \in \mathbb{I}[\![q]\!]$$

passing through f. Here $\mathbb{I}$ is a finite flat extension of the power series ring $\mathcal{O}_L[\![T]\!]$, which for simplicity in this introduction will be assumed to be $\mathcal{O}_L[\![T]\!]$ itself. Embed $\mathbf{Z}$ in the space $\mathcal{X}_{\mathcal{O}_L}(\mathbb{I})$ of continuous $\mathcal{O}_L$-algebra homomorphisms $\nu : \mathbb{I} \longrightarrow \overline{\mathbf{Q}}_p$ by identifying $k \in \mathbf{Z}$ with the homomorphism $\nu_k : \mathbb{I} \longrightarrow \overline{\mathbf{Q}}_p$ defined by $1 + T \mapsto (1+p)^{k-2}$. The Hida family $\mathbf{f}$ is then uniquely characterized by the property that for every $k \in \mathbf{Z}_{\geqslant 2}$ its *weight k specialization*

$$\mathbf{f}_k := \sum_{n=1}^{\infty} \nu_k(\mathbf{a}_n) q^n$$

gives the q-expansion of a p-ordinary p-stabilized newform $\mathbf{f}_k \in S_k(\Gamma_0(Np))$ with $\mathbf{f}_2 = f$.

Let K be an imaginary quadratic field equipped with an integral ideal $\mathfrak{N} \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N} \simeq \mathbf{Z}/N\mathbf{Z}$, assume that p splits in K, and write $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$ with $\mathfrak{p}$ the prime above p induced by $\imath_p$. If A is an elliptic curve with complex multiplication (CM) by $\mathcal{O}_K$, then the pair $(A, A[\mathfrak{N}\mathfrak{p}])$ defines a *Heegner point* P_A on $X_0(Np)$ defined over the Hilbert class field H of K. Taking the image of the degree zero divisor $(P_A) - (\infty)$ under the composite map

$$J_0(Np) \xrightarrow{\text{Kum}} H^1(H, \mathrm{Ta}_p(J_0(Np))) \longrightarrow H^1(H, V_f) \xrightarrow{\mathrm{Cor}_{H/K}} H^1(K, V_f) \tag{0.1}$$

yields a class $\kappa_f \in \mathrm{Sel}(K, V_f)$ in the Selmer group for the p-adic Galois representation

$$\rho_f : G_{\mathbf{Q}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{Aut}_L(V_f) \simeq \mathbf{GL}_2(L)$$

associated with f. On the other hand, by working over a p-tower of modular curves, Howard [15] constructed a so-called *big Heegner point* $\mathfrak{Z}_0 \in \mathrm{Sel}_{\mathrm{Gr}}(K, \mathbf{T}^\dagger)$ in the Selmer group for a self-dual twist of the big Galois representation

$$\rho_{\mathbf{f}} : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{I}}(\mathbf{T}) \simeq \mathbf{GL}_2(\mathbb{I})$$

associated with $\mathbf{f}$. The image of $\mathfrak{Z}_0$ under the *specialization map* $\nu_2 : \mathrm{Sel}_{\mathrm{Gr}}(K, \mathbf{T}^\dagger) \longrightarrow \mathrm{Sel}(K, V_f)$ induced by $\nu_2 : \mathbb{I} \longrightarrow \overline{\mathbf{Q}}_p$ yields a second class of 'Heegner type' in $\mathrm{Sel}(K, V_f)$; thus the question of comparing κ_f with $\nu_2(\mathfrak{Z}_0)$ naturally arises.

For $k > 2$, the question of relating the specializations $\nu_k(\mathfrak{Z}_0)$ to higher dimensional Heegner cycles was considered in [2]. In that case, one could show (see [2, (5.31)]) that

$$\mathrm{loc}_{\mathfrak{p}}(\nu_k(\mathfrak{Z}_0)) = u^{-1}\left(1 - \frac{p^{k/2-1}}{\nu_k(\mathbf{a}_p)}\right)^2 \cdot \mathrm{loc}_{\mathfrak{p}}(\kappa_{\mathbf{f}_k}), \tag{0.2}$$

where $u := |\mathcal{O}_K^\times|/2$, $\mathrm{loc}_{\mathfrak{p}} : H^1(K, V_{\mathbf{f}_k}) \longrightarrow H^1(K_{\mathfrak{p}}, V_{\mathbf{f}_k})$ is the localization map, and $\kappa_{\mathbf{f}_k}$ is a class given by the p-adic étale Abel–Jacobi images of certain Heegner cycles on a Kuga–Sato variety of dimension $k-1$. However, for the above newform f, the main result of [2] does not immediately yield a similar relation between $\nu_2(\mathfrak{Z}_0)$ and $\kappa_{\mathbf{f}_2} = \kappa_f$, since in [2] a crucial use is made of the fact that the p-adic Galois representations associated with the eigenforms under consideration are (potentially) crystalline at p, whereas V_f is well known to be semistable but non-crystalline at p. Moreover, it is easy to see that the expected relation between these two classes may not be given by the naive extension of (0.2) with $k = 2$: indeed, granted the injectivity of $\mathrm{loc}_{\mathfrak{p}}$, by the Gross–Zagier formula the class $\mathrm{loc}_{\mathfrak{p}}(\kappa_f)$ is non-zero as long as $L'(f/K, 1) \neq 0$, whilst (0.2) for $k = 2$ would imply the vanishing of $\mathrm{loc}_{\mathfrak{p}}(\nu_2(\mathfrak{Z}_0))$ in all cases, since

$$\left(1 - \frac{p^{k/2-1}}{\nu_k(\mathbf{a}_p)}\right)\bigg|_{k=2} = \left(1 - \frac{1}{a_p(f)}\right) = 0. \tag{0.3}$$

As shown in [15], the class $\mathfrak{Z}_0$ fits in the compatible system of similar classes $\mathfrak{Z}_\infty = \{\mathfrak{Z}_n\}_{n \geqslant 0}$ over the anticyclotomic $\mathbf{Z}_p$-extension of K; thus $\mathfrak{Z}_0$ might be seen as the value of $\mathfrak{Z}_\infty$ at the trivial character. As suggested by the above discussion, in this paper we will show that the class $\mathrm{loc}_{\mathfrak{p}}(\nu_2(\mathfrak{Z}_0))$ vanishes, and prove an 'exceptional zero formula' relating its derivative at the trivial character (in a precise sense to be defined) to the geometric class κ_f. To state the main result, let h be the class number of K, write $\mathfrak{p}^h = \pi_{\mathfrak{p}}\mathcal{O}_K$, and define

$$\mathscr{L}_{\mathfrak{p}}(f, K) := \mathscr{L}_p(f) - \frac{\log_p(\varpi_{\mathfrak{p}})}{\mathrm{ord}_p(\varpi_{\mathfrak{p}})}, \tag{0.4}$$

where $\mathscr{L}_p(f)$ is the $\mathscr{L}$-invariant of f (see [21, § II.14] for example), $\varpi_{\mathfrak{p}} := \pi_{\mathfrak{p}}/\overline{\pi}_{\mathfrak{p}} \in K_{\mathfrak{p}} \simeq \mathbf{Q}_p$, and $\log_p : \mathbf{Q}_p^\times \longrightarrow \mathbf{Z}_p$ is Iwasawa's branch of the p-adic logarithm.

Theorem. *Let $f \in S_2(\Gamma_0(Np))$ be a newform split multiplicative at p, and define $\mathcal{Z}_{\mathfrak{p},f,\infty} = \{\mathcal{Z}_{\mathfrak{p},f,n}\}_{n \geqslant 0}$ by $\mathcal{Z}_{\mathfrak{p},f,n} := \mathrm{loc}_{\mathfrak{p}}(\nu_2(\mathfrak{Z}_n))$. Then $\mathcal{Z}_{\mathfrak{p},f,0} = 0$ and*

$$\mathcal{Z}'_{\mathfrak{p},f,0} = \mathscr{L}_{\mathfrak{p}}(f, K) \cdot \mathrm{loc}_{\mathfrak{p}}(\kappa_f).$$

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In Lemma 3.10, we define the 'derivative' $\mathcal{Z}'_\infty$ for any compatible system of classes $\mathcal{Z}_\infty = \{\mathcal{Z}_n\}_{n\geqslant 0}$ with $\mathcal{Z}_0 = 0$. The above result, which corresponds to Theorem 3.11 in the body of the paper, may thus be seen as an exceptional zero formula relating the derivative of $\mathrm{loc}_{\mathfrak{p}}(\nu_2(\mathfrak{z}_\infty))$ at the trivial character to classical Heegner points.

Remark 1. As suggested in [20, § 8], one might view p-adic L-functions (as described in [28] and [29, Chapter 8]) as 'rank 0' Euler–Iwasawa systems. In this view, it is natural to expect higher rank Euler–Iwasawa systems to exhibit exceptional zero phenomena similar to their rank 0 counterparts. We would like to see the main result of this paper as an instance of this phenomenon in 'rank 1'.

Remark 2. It would be interesting to study the formulation of our main result in the framework afforded by Nekovář's theory of Selmer complexes [23], similarly in the manner that the exceptional zero conjecture of Mazur *et al.* [21] has recently been proved by Venerucci [31] in the rank 1 case.

Remark 3. The second term in the definition (0.4) is precisely the $\mathscr{L}$-invariant $\mathscr{L}_{\mathfrak{p}}(\chi_K)$ appearing in the exceptional zero formula of Ferrero–Greenberg [9] and Gross–Koblitz [12] for the Kubota–Leopoldt p-adic L-function associated with the quadratic Dirichlet character χ_K corresponding to K. It would be interesting to find a conceptual explanation for the rather surprising appearance of $\mathscr{L}_{\mathfrak{p}}(\chi_K)$ in our derivative formula; we expect this to be related to a comparison of p-adic periods (cf. [4]).

The proof of the above theorem is obtained by computing in two different ways the value of a certain anticyclotomic p-adic L-function $L_{\mathfrak{p}}(f)$ at the norm character $\mathbf{N}_K$. The p-adic L-function $L_{\mathfrak{p}}(f)$ is defined by the interpolation of the central critical values for the Rankin–Selberg convolution of f with the theta series attached to Hecke characters of K of infinity type $(2+j,-j)$ with $j\geqslant 0$. The character $\mathbf{N}_K$ thus lies *outside* the range of interpolation of $L_{\mathfrak{p}}(f)$, and via a suitable extension of the methods of Bertolini *et al.* [1] to our setting, in Theorem 2.11 we show that

$$L_{\mathfrak{p}}(f)(\mathbf{N}_K) = (1-a_p(f)p^{-1})\cdot\langle \log_{V_f}(\mathrm{loc}_{\mathfrak{p}}(\kappa_f)),\omega_f\rangle. \tag{0.5}$$

On the other hand, in [3] we constructed a two-variable p-adic L-function $L_{\mathfrak{p},\xi}(\mathbf{f})$ of the variables (ν,ϕ) interpolating (a shift of) the p-adic L-functions $L_{\mathfrak{p}}(\mathbf{f}_k)$ for all $k\geqslant 2$, and established the equality

$$L_{\mathfrak{p},\xi}(\mathbf{f}) = \mathcal{L}^{\omega}_{\mathscr{F}+\mathbb{T}}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{z}_\infty^{\xi^{-1}})), \tag{0.6}$$

where $\mathcal{L}^{\omega}_{\mathscr{F}+\mathbb{T}}$ is a two-variable Coleman power series map whose restriction to a certain 'line' interpolates

$$\left(1-\frac{p^{k/2-1}}{\nu_k(\mathbf{a}_p)}\right)^{-1}\left(1-\frac{\nu_k(\mathbf{a}_p)}{p^{k/2}}\right)\cdot\log_{V_{\mathbf{f}_k}}$$

for all $k>2$. A second evaluation of $L_{\mathfrak{p}}(f)(\mathbf{N}_K)$ should thus follow by specializing (0.6) at $(\nu_2,\mathbb{1})$. However, because of the vanishing equation (0.3), we may not directly specialize $\mathcal{L}^{\omega}_{\mathscr{F}+\mathbb{T}}$ at $(\nu_2, \mathbb{1})$, and we are led to utilize a different argument reminiscent of Greenberg–Stevens' [11]. In fact, from the form of the p-adic multipliers appearing in the interpolation property defining $\mathcal{L}^{\omega}_{\mathscr{F}+\mathbb{T}}$, we deduce a factorization

$$E_{\mathfrak{p}}(\mathbf{f}) \cdot L_{\mathfrak{p},\xi}(\mathbf{f}) = \widetilde{\mathcal{L}}^{\omega}_{\mathscr{F}+\mathbb{T}}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_0^{\xi^{-1}}))$$

upon restricting (0.6) to an appropriate 'line' (different from the above) passing through $(\nu_2, \mathbb{1})$, where $\widetilde{\mathcal{L}}^{\omega}_{\mathscr{F}+\mathbb{T}}$ is a modification of $\mathcal{L}^{\omega}_{\mathscr{F}+\mathbb{T}}$ and $E_{\mathfrak{p}}(\mathbf{f})$ is a p-adic analytic function vanishing at that point. The vanishing of $\mathcal{Z}_{\mathfrak{p},f,0}$ thus follows, and exploiting the 'functional equation' satisfied by $\mathfrak{Z}_{\infty}$, we arrive at the equality

$$\mathscr{L}_{\mathfrak{p}}(f, K) \cdot L_{\mathfrak{p}}(f)(\mathbf{N}_K) = (1 - a_p(f)p^{-1}) \cdot \langle \log_{V_f}(\mathcal{Z}'_{\mathfrak{p},f,0}), \omega_f \rangle \tag{0.7}$$

using a well-known formula for the $\mathscr{L}$-invariant as a logarithmic derivative of $\nu_k(\mathbf{a}_p)$ at $k = 2$. The proof of our exceptional zero formula then follows by combining (0.5) and (0.7).

1. Preliminaries

For a more complete and detailed discussion of the topics that we touch upon in this section, we refer the reader to [1, 6].

1.1. Modular curves

Keep N and $p \nmid N$ as in the Introduction, and let

$$\Gamma := \Gamma_1(N) \cap \Gamma_0(p) \subset \mathbf{SL}_2(\mathbf{Z}).$$

An *elliptic curve with* Γ*-level structure* over a $\mathbf{Z}[1/N]$-scheme S is a triple (E, t, α) consisting of

- an elliptic curve E over S;
- a section $t : S \longrightarrow E$ of the structure morphism of E/S of exact order N;
- a p-isogeny $\alpha : E \longrightarrow E'$.

The functor on $\mathbf{Z}[1/N]$-schemes assigning to S the set of isomorphism classes of elliptic curves with Γ-level structure over S is representable, and we let $Y/\mathbf{Z}[1/N]$ be the corresponding fine moduli scheme. The same moduli problem for *generalized* elliptic curves with Γ-level structure defines a smooth geometrically connected curve $X/\mathbf{Z}[1/N]$ containing Y as an open subscheme, and we refer to $Z_X := X \smallsetminus Y$ as the *cuspidal subscheme* of X. Removing the data of α from the above moduli problem, we obtain the modular curve $X_1(N)$ of level $\Gamma_1(N)$.

For our later use (see, in particular, Theorem 2.4), recall that if a is any integer coprime to N, the rule

$$\langle a \rangle (E, t, \alpha) = (E, a \cdot t, \alpha)$$

defines an action of $(\mathbf{Z}/N\mathbf{Z})^{\times}$ on X defined over $\mathbf{Z}[1/N]$, and we let $X_0(Np) = X/(\mathbf{Z}/N\mathbf{Z})^{\times}$ be the quotient of X by this action.

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The special fiber $X_{\mathbf{F}_p} := X \times_{\mathbf{Z}[1/N]} \mathbf{F}_p$ is non-smooth. In fact, it consists of two irreducible components, denoted by C_0 and C_∞, meeting transversally at the singular points SS. Let Frob be the absolute Frobenius of an elliptic curve over $\mathbf{F}_p$, and Ver = Frob$^\vee$ be the Verschiebung. The maps

$$\gamma_V : X_1(N)_{\mathbf{F}_p} := X_1(N) \times_{\mathbf{Z}[1/N]} \mathbf{F}_p \longrightarrow X_{\mathbf{F}_p} \quad \gamma_F : X_1(N)_{\mathbf{F}_p} \longrightarrow X_{\mathbf{F}_p}$$

defined by sending a pair $(E, t)_{/\mathbf{F}_p}$ to $(E, t, \ker(\mathrm{Ver}))$ and $(E, t, \ker(\mathrm{Frob}))$, respectively, are closed immersions sending $X_1(N)_{\mathbf{F}_p}$ isomorphically onto C_0 and C_∞, and mapping the supersingular points in $X_1(N)_{\mathbf{F}_p}$ bijectively onto SS. The non-singular geometric points of C_0 (resp. C_∞) thus correspond to the moduli of triples (E, t, α) in characteristic p with $\ker(\alpha)$ étale (resp. connected).

Corresponding to the preceding description of $X_{\mathbf{F}_p}$ there is a covering of X as rigid analytic space over $\mathbf{Q}_p$. Consider the reduction map

$$\mathrm{red}_p : X(\mathbf{C}_p) \longrightarrow X_{\mathbf{F}_p}(\overline{\mathbf{F}}_p), \tag{1.1}$$

let $\mathcal{W}_0$ and $\mathcal{W}_\infty$ be the inverse images of C_0 and C_∞, respectively, and let $\mathcal{Z}_0 \subset \mathcal{W}_0$ and $\mathcal{Z}_\infty \subset \mathcal{W}_\infty$ be the inverse images of their non-singular points. In the terminology of [5], $\mathcal{W}_0$ (resp. $\mathcal{W}_\infty$) is a *basic wide open* with underlying affinoid $\mathcal{Z}_0$ (resp. $\mathcal{Z}_\infty$). If $x \in SS$, then $\mathcal{A}_x := \mathrm{red}_p^{-1}(x)$ is conformal to an open annulus in $\mathbf{C}_p$, and by definition we have

$$X(\mathbf{C}_p) = \mathcal{W}_0 \cup \mathcal{W}_\infty = \mathcal{Z}_0 \cup \mathcal{Z}_\infty \cup \mathcal{W},$$

where $\mathcal{W} = \mathcal{W}_0 \cap \mathcal{W}_\infty = \bigcup_{x \in SS} \mathcal{A}_x$ is the union of the supersingular annuli.

1.2. Modular forms and cohomology

In this section, we regard the modular curve X as a scheme over a fixed base field F. Let $\mathcal{E} \xrightarrow{\pi} X$ be the universal generalized elliptic curve with Γ-level structure, set $\tilde{Z}_X = \pi^{-1}(Z_X)$, and consider the invertible sheaf on X given by

$$\underline{\omega} := \pi_* \Omega^1_{\mathcal{E}/X}(\log \widetilde{Z}_X).$$

The space of algebraic *modular forms* (resp. *cusp forms*) of weight k and level Γ defined over F is

$$M_k(X; F) := H^0(X, \underline{\omega}_F^{\otimes k}) \quad (\text{resp. } S_k(X; F) := H^0(X, \underline{\omega}_F^{\otimes k} \otimes \mathcal{I})),$$

where $\underline{\omega}_F$ is the pullback of $\underline{\omega}$ to $X \times_{\mathbf{Q}} F$, and $\mathcal{I}$ is the ideal sheaf of $Z_X \subset X$. If there is no risk of confusion, F will be often neglected from the notation. Alternatively, on the open modular curve Y a form $f \in S_k(X; F) \subset M_k(X; F)$ is a rule on quadruples $(E, t, \alpha, \omega)_{/A}$, consisting of an A-valued point $(E, t, \alpha) \in Y(A)$ and a differential $\omega \in \Omega^1_{E/A}$ over arbitrary F-algebras A, assigning to any such quadruple a value $f(E, t, \alpha, \omega) \in A$ subject to the *weight k condition*

$$f(E, t, \alpha, \lambda\omega) = \lambda^{-k} \cdot f(E, t, \alpha, \omega) \quad \text{for all } \lambda \in A^\times,$$

depending only on the isomorphism class of the quadruple, and compatible with base change of F-algebras. The two descriptions are related by

$$f(E, t, \alpha) = f(E, t, \alpha, \omega)\omega^k,$$

for any chosen generator $\omega \in \Omega^1_{E/A}$.

There is a third way of thinking about modular forms that will be useful in the following. Consider the *relative de Rham cohomology* of $\mathcal{E}/X$:

$$\mathcal{L} := \mathbb{R}^1\pi_*(0 \longrightarrow \mathcal{O}_{\mathcal{E}} \longrightarrow \Omega^1_{\mathcal{E}/X}(\log \widetilde{Z}_X) \longrightarrow 0),$$

which fits in a short exact sequence

$$0 \longrightarrow \underline{\omega} \longrightarrow \mathcal{L} \longrightarrow \underline{\omega}^{-1} \longrightarrow 0 \tag{1.2}$$

of sheaves on X and is equipped with a non-degenerate pairing

$$\langle\,,\rangle : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{O}_X \tag{1.3}$$

coming from the Hodge filtration and the Poincaré pairing on the de Rham cohomology of the fibers. By the Kodaira–Spencer isomorphism

$$\sigma : \underline{\omega}^{\otimes 2} \cong \Omega^1_X(\log Z_X)$$

given by $\sigma(\omega \otimes \eta) = \langle \omega, \nabla\eta\rangle$, where

$$\nabla : \mathcal{L} \longrightarrow \mathcal{L} \otimes \Omega^1_X(\log Z_X)$$

is the Gauss–Manin connection, a modular form f of weight $r+2$ and level Γ defines a section ω_f of the sheaf $\underline{\omega}^{\otimes r} \otimes \Omega^1_X(\log Z_X)$ by the rule

$$\omega_f(E, t, \alpha) := f(E, t, \alpha, \omega)\omega^r \otimes \sigma(\omega^2).$$

If f is a cusp form, then the above rule defines a section ω_f of $\underline{\omega}^{\otimes r} \otimes \Omega^1_X$, thus yielding an identification

$$S_{r+2}(X) \simeq H^0(X, \underline{\omega}^{\otimes r} \otimes \Omega^1_X).$$

For each $r \geqslant 0$, let $\mathcal{L}_r := \mathrm{Sym}^r \mathcal{L}$ (with $\mathcal{L}_0 := \mathcal{O}_X$), and define the *de Rham cohomology* of X (attached to $\mathcal{L}_r$) as the hypercohomology group

$$H^1_{\mathrm{dR}}(X, \mathcal{L}_r, \nabla) := \mathbb{H}^1(\mathcal{L}_r^\bullet : \mathcal{L}_r \xrightarrow{\nabla} \mathcal{L}_r \otimes \Omega^1_X(\log Z_X)). \tag{1.4}$$

Twisting by the ideal sheaf $\mathcal{I}$ gives rise to the subcomplex $\mathcal{L}_r^\bullet \otimes \mathcal{I} \longrightarrow \mathcal{L}_r^\bullet$, and the weight $r+2$ *parabolic cohomology* of X is defined by

$$H^1_{\mathrm{par}}(X, \mathcal{L}_r, \nabla) := \mathrm{image}(\mathbb{H}^1(\mathcal{L}_r^\bullet \otimes \mathcal{I}) \longrightarrow H^1_{\mathrm{dR}}(X, \mathcal{L}_r, \nabla)). \tag{1.5}$$

The exact sequence (1.2) induces the short exact sequence

$$0 \longrightarrow H^0(X, \underline{\omega}^{\otimes r} \otimes \Omega^1_X) \longrightarrow H^1_{\mathrm{par}}(X, \mathcal{L}_r, \nabla) \longrightarrow H^1(X, \underline{\omega}^{\otimes -r}) \longrightarrow 0, \tag{1.6}$$

and hence the above assignment $f \mapsto \omega_f$ identifies $S_{r+2}(X)$ with a subspace of $H^1_{\mathrm{par}}(X, \mathcal{L}_r, \nabla)$. In addition, the pairing (1.3) induces a non-degenerate pairing

$$\langle\,,\rangle : H^1_{\mathrm{par}}(X, \mathcal{L}_r, \nabla) \times H^1_{\mathrm{par}}(X, \mathcal{L}_r, \nabla) \longrightarrow F \tag{1.7}$$

with respect to which (1.6) is self-dual.

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1.3. p-newforms

Consider the two degeneracy maps

$$\pi_1, \pi_2 : X \longrightarrow X_1(N)$$

defined by sending, under the moduli interpretation, a triple (E, t, α) to the pairs (E, t) and $(\alpha(E), \alpha(t))$, respectively. These morphisms induce maps

$$\pi_1^*, \pi_2^* : H^1_{\mathrm{par}}(X_1(N), \mathcal{L}_r, \nabla) \longrightarrow H^1_{\mathrm{par}}(X, \mathcal{L}_r, \nabla),$$

where $H^1_{\mathrm{par}}(X_1(N), \mathcal{L}_r, \nabla)$ is defined as in (1.5) using the analogous objects over $X_1(N)$.

Lemma 1.1. *The map $\pi_1^* \oplus \pi_2^*$ is injective.*

Proof. This is [6, Proposition 4.1]. □

Define the *p-old* subspace $H^1_{\mathrm{par}}(X, \mathcal{L}_r, \nabla)^{p\text{-old}}$ of $H^1_{\mathrm{par}}(X, \mathcal{L}_r, \nabla)$ to be the image of $\pi_1^* \oplus \pi_2^*$, and the *p-new* subspace $H^1_{\mathrm{par}}(X, \mathcal{L}_r, \nabla)^{p\text{-new}}$ to be the orthogonal complement of the p-old subspace under the Poincaré pairing (1.7). The space of p-new cusp forms of weight k and level Γ is defined by

$$S_{r+2}(X)^{p\text{-new}} := S_{r+2}(X) \cap H^1_{\mathrm{par}}(X, \mathcal{L}_r, \nabla)^{p\text{-new}},$$

viewing $S_{r+2}(X)$ as the subspace of $H^1_{\mathrm{par}}(X, \mathcal{L}_r, \nabla)$ in the form described above.

1.4. p-adic modular forms

Recall that the *Hasse invariant* is a modular form H over $\mathbf{F}_p$ of level 1 and weight $p-1$ with the property that an elliptic curve E over an $\mathbf{F}_p$-algebra B is *ordinary* if and only if $H(E, \omega)$ is a unit in B for some (or equivalently, any) generator $\omega \in \Omega^1_{E/B}$.

Let R be a p-adic ring, i.e., a ring that is isomorphic to its pro-p completion. A *p-adic modular form* of tame level N and weight k defined over R is a rule assigning to every triple $(E, t, \omega)_{/A}$, over an arbitrary p-adic R-algebra A, consisting of

- an elliptic curve E/A such that the reduction $E \times_A A/pA$ is ordinary;
- a section $t : \mathrm{Spec}(A) \longrightarrow E$ of the structure morphism of E/A of exact order N; and
- a differential $\omega \in \Omega^1_{E/A}$,

an element $f(E, t, \omega) \in A$ depending only on the isomorphism class of $(E, t, \omega)_{/A}$, homogeneous of degree $-k$ in the third entry, and compatible with base change of p-adic R-algebras. Let $\mathcal{M}_k(N; R)$ be the R-module of p-adic modular forms of weight k and level N defined over R; as before, if there is no risk of confusion R will be often neglected from the notation.

Similarly to that for classical modular forms, it will be convenient to think of p-adic modular forms of weight k as sections of the sheaf $\underline{\omega}^{\otimes k}$ over a certain subset of the rigid analytic space $X(\mathbf{C}_p)$. Let E_{p-1} be the normalized Eisenstein series of weight $p-1$ (recall that $p \geqslant 5$), and define the *ordinary locus* of $X_1(N)$ by

$$X_1(N)^{\mathrm{ord}} := \{x \in X_1(N)(\mathbf{C}_p) : |E_{p-1}(E_x, \omega_x)|_p \geqslant 1\},$$

where $E_x/\mathbf{C}_p$ is a generalized elliptic curve corresponding to x under the moduli interpretation, $\omega_x \in \Omega^1_{E_x/\mathbf{C}_p}$ is a regular differential on E_x, chosen so that it extends to a regular differential over $\mathcal{O}_{\mathbf{C}_p}$ if E_x has good reduction at p, or corresponds to the canonical differential on the Tate curve if x lies in the residue disc of a cusp, and $|\cdot|_p$ is the absolute value on $\mathbf{C}_p$ normalized so that $|p|_p = p^{-1}$. Since E_{p-1} reduces to the Hasse invariant H modulo p, it follows that the points $x \in X_1(N)^{\mathrm{ord}}$ correspond to pairs (E_x, t_x) with E_x having ordinary reduction modulo p. Thus the assignment $f \mapsto (x \mapsto f(E_x, t_x, \omega_x)\omega_x^k)$, for any chosen generator $\omega_x \in \Omega^1_{E_x/\mathbf{C}_p}$, defines an identification

$$\mathcal{M}_k(N) \simeq H^0(X_1(N)^{\mathrm{ord}}, \underline{\omega}^{\otimes k}).$$

Let $I := \{v \in \mathbf{Q} : 0 < v \leqslant \frac{p}{p+1}\}$, and for any $v \in I$ define

$$X_1(N)(v) := \{x \in X_1(N)(\mathbf{C}_p) : |E_{p-1}(E_x, \omega_x)|_p > p^{-v}\}.$$

The space of *overconvergent p-adic modular forms* of weight k and tame level N is given by

$$\mathcal{M}_k^{\dagger}(N) = \varinjlim_{v} H^0(X_1(N)(v), \underline{\omega}^{\otimes k}),$$

where the transition maps $H^0(X_1(N)(v), \underline{\omega}^{\otimes k}) \longrightarrow H^0(X_1(N)(v'), \underline{\omega}^{\otimes k})$, for $v' < v$ in I, are given by restriction; since these maps are injective, $\mathcal{M}_k^{\dagger}(N)$ is naturally a subspace of $\mathcal{M}_k(N)$.

By the theory of the *canonical subgroup* (see [18, Theorem 3.1]), if (E_x, t_x) corresponds to a point x in $X_1(N)(\frac{p}{p+1})$, the elliptic curve E_x admits a distinguished subgroup $\mathrm{can}(E_x) \subset E_x[p]$ of order p reducing to the kernel of Frobenius in characteristic p. The rule

$$(E_x, t_x) \mapsto (E_x, t_x, \alpha_{\mathrm{can}}),$$

where $\alpha_{\mathrm{can}} : E_x \mapsto E_x/\mathrm{can}(E_x)$ is the projection, defines rigid morphism $X_1(N)(\frac{p}{p+1}) \longrightarrow \mathcal{W}_\infty$, and hence if f is a modular form of weight k and level Γ, then the restriction $f|_{\mathcal{W}_\infty}$ gives an overconvergent p-adic modular form of weight k and tame level N.

1.5. Ordinary CM points

Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$ equipped with a cyclic ideal $\mathfrak{N} \subset \mathcal{O}_K$ such that

$$\mathcal{O}_K/\mathfrak{N} \simeq \mathbf{Z}/N\mathbf{Z}.$$

Fix an elliptic curve A defined over the Hilbert class field H of K with $\mathrm{End}_H(A) \simeq \mathcal{O}_K$ having good reduction at the primes above p, and choose a $\Gamma_1(N)$-level structure $t_A \in A[\mathfrak{N}]$ and a regular differential $\omega_A \in \Omega^1_{A/H}$. The identification $\mathrm{End}_H(A) = \mathcal{O}_K$ is normalized so that $\lambda \in \mathcal{O}_K$ acts as

$$\lambda^*\omega = \lambda\omega \quad \text{for all } \omega \in \Omega^1_{A/H}.$$

For every integer $c \geqslant 1$ prime to Np, let $\mathcal{O}_c = \mathbf{Z} + c\mathcal{O}_K$ be the order of K of conductor c, and denote by $\mathrm{Isog}_c^{\mathfrak{N}}(A)$ the set of elliptic curves A' with CM by $\mathcal{O}_c$ equipped with an isogeny $\varphi : A \longrightarrow A'$ satisfying $\ker(\varphi) \cap A[\mathfrak{N}] = \{0\}$.

The semigroup of projective rank one $\mathcal{O}_c$-modules $\mathfrak{a} \subset \mathcal{O}_c$ prime to $\mathfrak{N} \cap \mathcal{O}_c$ acts on $\mathrm{Isog}_c^{\mathfrak{N}}(A)$ by the rule

$$\mathfrak{a} * (\varphi : A \longrightarrow A') = \varphi_{\mathfrak{a}}\varphi : A \longrightarrow A' \longrightarrow A'_{\mathfrak{a}},$$

where $A'_{\mathfrak{a}} := A'/A'[\mathfrak{a}]$ and $\varphi_{\mathfrak{a}} : A' \longrightarrow A'_{\mathfrak{a}}$ is the natural projection. It is easily seen that this induces an action of $\mathrm{Pic}(\mathcal{O}_c)$ on $\mathrm{Isog}_c^{\mathfrak{N}}(A)$.

Throughout this paper, we shall assume that $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K, and let $\mathfrak{p}$ be the prime of K above p induced by our fixed embedding $\overline{\mathbf{Q}}_p \overset{\iota_p}{\hookrightarrow} \mathbf{C}_p$. Thus if A' is an elliptic curve with CM by $\mathcal{O}_c$ defined over the ring class field H_c of K of conductor c, then A' has ordinary reduction at p, and $A'[\mathfrak{p}] \subset A'[p]$ is the canonical subgroup. In the following, we will let $\alpha'_{\mathfrak{p}} = \alpha_{\mathrm{can}} : A' \longrightarrow A'/A'[\mathfrak{p}]$ denote the projection.

1.6. Generalized Heegner cycles

For any $r > 0$, let W_r be the Kuga–Sato variety over

$$X_0 := X_1(Np)$$

obtained as the canonical desingularization of the r-fold self-product of the universal generalized elliptic curve over X_0, and define

$$X_r := W_r \times A^r, \tag{1.8}$$

where A/H is the elliptic curve with CM by $\mathcal{O}_K$ fixed in the preceding section.

The variety X_r is fibered over X_0, and the fiber over a non-cuspidal point x associated with a pair (E_x, t_x) is identified with $E_x^r \times A^r$. Thus for every isogeny $\varphi : A \longrightarrow A'$ in $\mathrm{Isog}_c^{\mathfrak{N}}(A)$, we may consider the cycle

$$\Upsilon_\varphi := (\Gamma_\varphi^{\mathrm{t}})^r \subset (A' \times A)^r \subset X_r,$$

where $\Gamma_\varphi^{\mathrm{t}}$ is the transpose of the graph of φ, and following [1, § 2.3] define the *generalized Heegner cycle* associated with φ by

$$\Delta_\varphi := \epsilon_X \Upsilon_\varphi, \tag{1.9}$$

where ϵ_X is the idempotent defined in [1, (2.1.1)] (with X_0 in place of their curve $C = X_1(N)$). By [1, Proposition 2.7], the cycles Δ_φ are homologically trivial; by abuse of notation, we shall still denote by Δ_φ the classes they define in the Chow group $\mathrm{CH}^{r+1}(X_r)_0$ with rational coefficients. For $r = 0$, set

$$\Delta_\varphi := (A', t_{A'}) - (\infty),$$

where $t_{A'} \in A'[Np]$ is a $\Gamma_1(Np)$-level structure contained in $A'[\mathfrak{N}\mathfrak{p}]$, and ∞ is the cusp $(\mathrm{Tate}(q), \zeta_{Np})$.

2. A semistable non-crystalline setting

This section is aimed at proving Theorem 2.11, which extends the p-adic Gross–Zagier formula due to Bertolini *et al.* [1] in the good reduction case to the semistable non-crystalline setting.

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2.1. p-adic Abel–Jacobi maps

Let F be a finite unramified extension of $\mathbf{Q}_p$, denote by $\mathcal{O}_F$ the ring of integers of F, and let κ be the residue field. The generalized Kuga–Sato variety X_r, which was defined in (1.8) as a scheme over $\mathbf{Z}[1/Np]$, has semistable reduction at p. In other words, there exists a proper scheme $\mathcal{X}_r$ over $\mathcal{O}_F$ with generic fiber $X_r \times_{\mathbf{Z}[1/Np]} F$ and with special fiber $\mathcal{X}_r \times_{\mathcal{O}_F} \kappa$ whose only singularities are divisors with normal crossings.

By the work of Hyodo–Kato [16], attached to X_r there are log-crystalline cohomology groups $H^j_{\text{log-cris}}(\mathcal{X}_r \times_{\mathcal{O}_F} \kappa)$, which are $\mathcal{O}_F$-modules of finite rank equipped with a semilinear Frobenius automorphism Φ and a linear nilpotent monodromy operator N satisfying

$$N\Phi = p\Phi N.$$

Moreover, for each choice of a uniformizer of $\mathcal{O}_F$ there is a comparison isomorphism

$$H^j_{\text{log-cris}}(\mathcal{X}_r \times_{\mathcal{O}_F} \kappa) \otimes_{\mathcal{O}_F} F \simeq H^j_{\text{dR}}(X_r/F)$$

endowing the algebraic de Rham cohomology groups $H^j_{\text{dR}}(X_r/F)$ with the structure of filtered (Φ, N)-modules. In the following, we shall restrict our attention to the middle degree cohomology, i.e., we set $j = 2r + 1$.

Let $G_F := \text{Gal}(\overline{F}/F)$ be the absolute Galois group of F, and consider the p-adic G_F-representation given by

$$V_r := H^{2r+1}_{\text{ét}}(X_r \times_F \overline{F}, \mathbf{Q}_p).$$

Applying Fontaine's functor $\mathbf{D}_{\text{st}}$ to V_r yields another filtered (Φ, N)-module associated with X_r.

Theorem 2.1 (Tsuji). *The p-adic G_F-representation V_r is semistable, and there is a natural isomorphism*

$$\mathbf{D}_{\text{st}}(V_r) \simeq H^{2r+1}_{\text{dR}}(X_r/F)$$

compatible with all structures. In particular, the assignment $V \mapsto \mathbf{D}_{\text{st}}(V)$ induces an isomorphism $\text{Ext}_{\text{st}}(\mathbf{Q}_p, V_r) \simeq \text{Ext}_{\text{Mod}_F(\Phi,N)}(F, H^{2r+1}_{\text{dR}}(X_r/F))$.

Here, $\text{Ext}_{\text{st}}(\mathbf{Q}_p, V_r) \simeq H^1_{\text{st}}(F, V_r) := \ker(H^1(F, V_r) \longrightarrow H^1(F, V_r \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{st}}))$ is the group of extensions of the trivial representation $\mathbf{Q}_p$ by V_r in the category of semistable p-adic G_F-representations.

The idempotent ϵ_X used in the definition (1.9) of the generalized Heegner cycles Δ_φ acts as a projector on the various cohomology groups associated with the variety X_r. Let $V_r(r+1)$ be the $(r+1)$st Tate twist of V_r, and consider the étale Abel–Jacobi map

$$\text{AJ}^{\text{ét}}_F : \text{CH}^{r+1}(X_r)_0(F) \longrightarrow \text{Ext}_{\text{Rep}_{G_F}}(\mathbf{Q}_p, \epsilon_X V_r(r+1)) = H^1(F, \epsilon_X V_r(r+1))$$

constructed in [22]. By [22, Theorem 3.1(ii)], the image of $\text{AJ}^{\text{ét}}_F$ lands in $H^1_{\text{st}}(F, \epsilon_X V_r(r+1))$, and hence via the comparison isomorphism (2.1) it can be seen as taking values in the group

$$\text{Ext}_{\text{Mod}_F(\Phi,N)}(F, \epsilon_X H^{2r+1}_{\text{dR}}(X_r/F)(r+1))$$

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of extensions of F by the twist $\epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r+1)$ in the category of filtered (Φ, N)-modules over F. This group admits the following explicit description.

Lemma 2.2. *Set $H_r := H^{2r+1}_{\mathrm{dR}}(X_r/F)$ and let $n = [F : \mathbf{Q}_p]$. The assignment*

$$\{0 \longrightarrow \epsilon_X H_r(r+1) \longrightarrow E \xrightarrow{\rho} F \longrightarrow 0\} \quad \rightsquigarrow \quad \eta_E = \eta_E^{\mathrm{hol}}(1) - \eta_E^{\mathrm{frob}}(1),$$

where $\eta_E^{\mathrm{hol}} : F \longrightarrow \mathrm{Fil}^0 E$ (resp. $\eta_E^{\mathrm{frob}} : F \longrightarrow E^{\Phi^n=1, N=0}$) is a section of ρ compatible with filtrations (resp. with Frobenius and monodromy), yields an isomorphism

$$\mathrm{Ext}_{\mathrm{Mod}_F(\Phi,N)}(F, \epsilon_X H_r(r+1)) \simeq \epsilon_X H_r(r+1)/\mathrm{Fil}^0 \epsilon_X H_r(r+1).$$

Proof. See [17, Lemma 2.1], for example. □

Define the *p-adic Abel–Jacobi map*

$$\mathrm{AJ}_F : \mathrm{CH}^{r+1}(X_r)_0(F) \longrightarrow \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r+1)/\mathrm{Fil}^0 \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r+1) \quad (2.1)$$

to be the composite of $\mathrm{AJ}_F^{\text{ét}}$ with the isomorphisms of Theorem 2.1 and Lemma 2.1. Since the filtered pieces $\mathrm{Fil}^1 \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r)$ and $\mathrm{Fil}^0 \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r+1)$ are exact annihilators under the Poincaré duality

$$\epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r) \times \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r+1) \longrightarrow F,$$

the target of AJ_F may be identified with the linear dual $(\mathrm{Fil}^{r+1} \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F))^\vee$.

Recall the coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{L}_r = \mathrm{Sym}^r \mathcal{L}$ on X introduced in § 1.2, and set

$$\mathcal{L}_{r,r} := \mathcal{L}_r \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A).$$

With the trivial extension of the Gauss–Manin connection ∇ on $\mathcal{L}_r$ to $\mathcal{L}_{r,r}$, consider the complex

$$\mathcal{L}^\bullet_{r,r} : \mathcal{L}_{r,r} \xrightarrow{\nabla} \mathcal{L}_{r,r} \otimes \Omega^1_X(\log Z_X),$$

and define $H^1_{\mathrm{par}}(X, \mathcal{L}_{r,r}, \nabla)$ as in (1.5). By [1, Proposition 2.4], we then have

$$\epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F) \simeq H^1_{\mathrm{par}}(X, \mathcal{L}_{r,r}, \nabla) = H^1_{\mathrm{par}}(X, \mathcal{L}_r, \nabla) \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A/F)$$

and

$$\mathrm{Fil}^{r+1} \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F) \simeq H^0(X, \underline{\omega}^{\otimes r} \otimes \Omega^1_X) \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A/F).$$

As a result of these identifications, we shall view the p-adic Abel–Jacobi map (2.1) as a map

$$\mathrm{AJ}_F : \mathrm{CH}^{r+1}(X_r)_0(F) \longrightarrow (H^0(X, \underline{\omega}^{\otimes r} \otimes \Omega^1_X) \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A/F))^\vee. \quad (2.2)$$

Moreover, if $\Delta = \epsilon_X \Delta \in \mathrm{CH}^{r+1}(X_r)_0(F)$ is the class of a cycle in the image of the idempotent ϵ_X supported on the fiber of $X_r \longrightarrow X$ over a point $P \in X(F)$, we see that

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$\mathrm{AJ}_F(\Delta)$ may be computed using the following recipe. Consider the commutative diagram with Cartesian squares:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{\mathrm{par}}(X, \mathcal{L}_{r,r}, \nabla)(r+1) & \longrightarrow & D_\Delta & \longrightarrow & F & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \scriptstyle{\mathrm{cl}_\Delta} & & \\ 0 & \longrightarrow & H^1_{\mathrm{par}}(X, \mathcal{L}_{r,r}, \nabla)(r+1) & \longrightarrow & H^1_{\mathrm{par}}(X \smallsetminus P, \mathcal{L}_{r,r}, \nabla)(r+1) & \longrightarrow & \mathcal{L}_{r,r}(P)(r) & \longrightarrow & 0, \end{array}$$

where the rightmost vertical map is defined by sending $1 \in F$ to the cycle class $\mathrm{cl}_P(\Delta)$. Then $\mathrm{AJ}_F(\Delta)$ is given by the linear functional

$$\mathrm{AJ}_F(\Delta) = \langle -, \eta_\Delta \rangle,$$

where $\eta_\Delta = \eta^{\mathrm{hol}}_\Delta - \eta^{\mathrm{frob}}_\Delta := \eta^{\mathrm{hol}}_{D_\Delta}(1) - \eta^{\mathrm{frob}}_{D_\Delta}(1)$ is the 'tangent vector' associated, as in Lemma 2.2, with the extension D_Δ as filtered (Φ, N)-modules, and

$$\langle\, , \rangle : H^1_{\mathrm{par}}(X, \mathcal{L}_{r,r}, \nabla)(r) \times H^1_{\mathrm{par}}(X, \mathcal{L}_{r,r}, \nabla)(r+1) \longrightarrow F \tag{2.3}$$

is the Poincaré duality.

2.2. Rigid cohomology

Recall the rigid spaces $\mathcal{Z}_\infty \subset \mathcal{W}_\infty$, $\mathcal{Z}_0 \subset \mathcal{W}_0$ introduced in § 1.1. Fix a collection of points $\{P_1, \ldots, P_t\}$ of $X(F)$ contained in $\mathcal{Z}_\infty$, containing all the cusps of $\mathcal{Z}_\infty$, and such that $\mathrm{red}_p(P_i) \neq \mathrm{red}_p(P_j)$ for $i \neq j$. Let w_p be the automorphism of X defined in terms of moduli by

$$w_p(E, t, \alpha) = (\alpha(E), \alpha(t), \alpha^\vee), \tag{2.4}$$

where $\alpha^\vee$ is the isogeny dual to α, and set $P_j^* := w_p P_j$. Then the points P_j^* factor through $\mathcal{Z}_0$, and the set

$$S := \{P_1, \ldots, P_t, P_1^*, \ldots, P_t^*\}$$

contains all the cusps of X. Since the points $Q \in S$ reduce to smooth points $\bar{Q}$ in the special fiber, the spaces $D(Q) := \mathrm{red}_p^{-1}(\bar{Q})$ are conformal to the open disc in $D(0; 1)$ in $\mathbf{C}_p$. Fix isomorphisms $h_Q : D(Q) \longrightarrow D(0; 1)$ mapping the point Q to 0, and for a collection of real numbers $r_Q < 1$ consider the annuli

$$\mathcal{V}_Q := \{x \in D(Q) : r_Q < |h_Q(x)|_p < 1\}. \tag{2.5}$$

Denote by $\mathcal{L}^{\mathrm{rig}}_{r,r}$ the sheaf for the rigid analytic topology on $X(\mathbf{C}_p)$ defined by the algebraic vector bundle $\mathcal{L}_{r,r}$. If $\mathcal{V} \subset X(\mathbf{C}_p)$ is a connected wide open contained in $Y(\mathbf{C}_p)$, the Gauss–Manin connection yields a connection

$$\nabla : \mathcal{L}^{\mathrm{rig}}_{r,r}|_\mathcal{V} \longrightarrow \mathcal{L}^{\mathrm{rig}}_{r,r}|_\mathcal{V} \otimes \Omega^1_\mathcal{V},$$

and similarly to that in (1.4) we define the ith *de Rham cohomology* of $\mathcal{V}$ attached to $\mathcal{L}^{\mathrm{rig}}_{r,r}$ by

$$H^i(\mathcal{L}^\bullet_{r,r}|_\mathcal{V}) = H^i_{\mathrm{dR}}(\mathcal{V}, \mathcal{L}^{\mathrm{rig}}_{r,r}, \nabla) := \mathbb{H}^i(\mathcal{L}^{\mathrm{rig}}_{r,r}|_\mathcal{V} \xrightarrow{\nabla} \mathcal{L}^{\mathrm{rig}}_{r,r}|_\mathcal{V} \otimes \Omega^1_\mathcal{V}).$$

In particular, if $\mathcal{V}$ is a basic wide open, then

$$H^1(\mathcal{L}^\bullet_{r,r}|_{\mathcal{V}}) \simeq \frac{\mathcal{L}^{\mathrm{rig}}_{r,r}(\mathcal{V}) \otimes \Omega^1_{\mathcal{V}}}{\nabla \mathcal{L}^{\mathrm{rig}}_{r,r}(\mathcal{V})},$$

and $H^0(\mathcal{L}^\bullet_{r,r}|_{\mathcal{V}}) \simeq \mathcal{L}^{\mathrm{rig}}_{r,r}(\mathcal{V})^{\nabla=0}$ is the space of horizontal sections of $\mathcal{L}^{\mathrm{rig}}_{r,r}$ over $\mathcal{V}$. For $r = 0$, we set

$$H^1(\mathcal{L}^\bullet_{r,r}|_{\mathcal{V}}) = H^1(\mathcal{V}) := \Omega^1_{\mathcal{V}}/d\mathcal{O}_{\mathcal{V}}, \quad H^0(\mathcal{L}^\bullet_{r,r}|_{\mathcal{V}}) = H^0(\mathcal{V}) := \mathcal{O}^{d=0}_{\mathcal{V}},$$

where $d : \mathcal{O}_{\mathcal{V}} \longrightarrow \Omega^1_{\mathcal{V}}$ is the differentiation map.

In terms of the admissible cover of $X(\mathbf{C}_p)$ by basic wide opens described in § 1.1, the classes in $H^1_{\mathrm{dR}}(X(\mathbf{C}_p), \mathcal{L}^{\mathrm{rig}}_{r,r}, \nabla)$ may be represented by hypercocycles $(\omega_0, \omega_\infty; f_{\mathcal{W}})$, where ω_0 and ω_∞ are $\mathcal{L}^{\mathrm{rig}}_{r,r}$-valued differentials on $\mathcal{W}_0$ and $\mathcal{W}_\infty$, respectively, and $f_{\mathcal{W}} \in \mathcal{L}^{\mathrm{rig}}_{r,r}(\mathcal{W})$ is such that $(\omega_\infty - \omega_0)|_{\mathcal{W}} = \nabla f_{\mathcal{W}}$; and two hypercocycles represent the same class if their difference is of the form $(\nabla f_0, \nabla f_\infty; (f_\infty - f_0)|_{\mathcal{W}})$ for some $f_0 \in \mathcal{L}^{\mathrm{rig}}_{r,r}(\mathcal{W}_0)$ and $f_\infty \in \mathcal{L}^{\mathrm{rig}}_{r,r}(\mathcal{W}_\infty)$.

If $\mathcal{V}$ is a wide open annulus, associated with an orientation of $\mathcal{V}$ there is a *p-adic annular residue*

$$\mathrm{res}_{\mathcal{V}} : \Omega^1_{\mathcal{V}} \longrightarrow \mathbf{C}_p \tag{2.6}$$

defined by expanding $\omega = \sum_n a_n T^n \frac{dT}{T} \in \Omega^1_{\mathcal{V}}$ with respect to a fixed uniformizing parameter T compatible with the orientation, and setting $\mathrm{res}_{\mathcal{V}}(\omega) := a_0$ (see [5, Lemma 2.1]). Combined with the natural pairing

$$\langle\, , \rangle : \mathcal{L}^{\mathrm{rig}}_{r,r}(\mathcal{V}) \times \mathcal{L}^{\mathrm{rig}}_{r,r}(\mathcal{V}) \otimes \Omega^1_{\mathcal{V}} \longrightarrow \Omega^1_{\mathcal{V}}$$

induced by the Poincaré duality (1.3) on $\mathcal{L}_r$ (extended to $\mathcal{L}_{r,r}$ in the obvious manner), we obtain a higher p-adic annular residue map

$$\mathrm{Res}_{\mathcal{V}} : \mathcal{L}^{\mathrm{rig}}_{r,r}(\mathcal{V}) \otimes \Omega^1_{\mathcal{V}} \longrightarrow \mathcal{L}^{\mathrm{rig}}_{r,r}(\mathcal{V})^\vee \tag{2.7}$$

by setting

$$\mathrm{Res}_{\mathcal{V}}(\omega)(\alpha) = \mathrm{res}_{\mathcal{V}}\langle \alpha, \omega\rangle$$

for every $\mathcal{L}^{\mathrm{rig}}_{r,r}$-valued differential ω on $\mathcal{V}$ and every section $\alpha \in \mathcal{L}^{\mathrm{rig}}_{r,r}(\mathcal{V})$. Since $\mathrm{res}_{\mathcal{V}}$ clearly descends to a map $H^1(\mathcal{V}) = \Omega^1_{\mathcal{V}}/d\mathcal{O}_{\mathcal{V}} \longrightarrow \mathbf{C}_p$, by composing $\mathrm{Res}_{\mathcal{V}}$ with the projection $\mathcal{L}^{\mathrm{rig}}_{r,r}(\mathcal{V})^\vee \longrightarrow H^0(\mathcal{L}^\bullet_{r,r}|_{\mathcal{V}})^\vee$ it is easily seen from the Leibniz rule that we obtain a well-defined map

$$\mathrm{Res}_{\mathcal{V}} : H^1(\mathcal{L}^\bullet_{r,r}|_{\mathcal{V}}) \longrightarrow H^0(\mathcal{L}^\bullet_{r,r}|_{\mathcal{V}})^\vee. \tag{2.8}$$

If $\mathcal{V}_Q \subset D(Q)$ is the annulus attached to a non-cuspidal point $Q \in S$, it will be convenient, following the discussion after [1, Corollary 3.7], to view $\mathrm{Res}_{\mathcal{V}_Q}$ as taking values on the fiber $\mathcal{L}_{r,r}(Q)$, using the sequence of identifications

$$H^0(\mathcal{L}^\bullet_{r,r}|_{\mathcal{V}_Q})^\vee = (H^0(D(Q), \mathcal{L}_{r,r})^{\nabla=0})^\vee = \mathcal{L}_{r,r}(Q)^\vee = \mathcal{L}_{r,r}(Q) \tag{2.9}$$

arising from 'analytic continuation', the choice of an 'initial condition', and the self-duality of $\mathcal{L}_{r,r}(Q)$, respectively. (See [1, Corollary 3.7] for the case of a cusp $Q \in S$.)

For a supersingular annulus $\mathcal{A}_x$, the vector space $H^0(\mathcal{L}^\bullet_{r,r}|_{\mathcal{A}_x})$ is equipped with a pairing $\langle\,,\rangle_{\mathcal{A}_x}$, arising from an identification (similar to (2.9)) with the de Rham cohomology of a supersingular elliptic curve in characteristic p corresponding to $x \in SS$. Moreover, since $H^0(\mathcal{A}_x) \simeq \mathbf{C}_p$, the residue map (2.6) yields an isomorphism $\mathrm{res}_{\mathcal{A}_x} : H^1(\mathcal{A}_x) \longrightarrow H^0(\mathcal{A}_x)$, and using a trivialization of $\mathcal{L}^{\mathrm{rig}}_{r,r}|_{\mathcal{A}_x}$ it may be extended to an isomorphism

$$\mathrm{Res}_{\mathcal{A}_x} : H^1(\mathcal{L}^\bullet_{r,r}|_{\mathcal{A}_x}) \simeq H^0(\mathcal{L}^\bullet_{r,r}|_{\mathcal{A}_x}) \tag{2.10}$$

(see [6, Proposition 7.1]). It is then easily checked that (2.8) and (2.10) correspond to each other under the identification $H^0(\mathcal{L}^\bullet_{r,r}|_{\mathcal{A}_x})^\vee = H^0(\mathcal{L}^\bullet_{r,r}|_{\mathcal{A}_x})$ defined by $\langle\,,\rangle_{\mathcal{A}_x}$.

Let S be a set of points as introduced above, and define

$$\mathcal{W}^\sharp_\infty := \mathcal{Z}_\infty \smallsetminus \bigcup_{Q \in S \cap \mathcal{Z}_\infty} D(Q) \smallsetminus \mathcal{V}_Q, \quad \mathcal{U} := \mathcal{W}^\sharp_\infty \cup \mathcal{W}^\sharp_0,$$

where $\mathcal{W}^\sharp_0 := w_p \mathcal{W}^\sharp_\infty$, and $U := X \smallsetminus S$. The restriction of an $\mathcal{L}_{r,r}$-valued differential on X, which is regular on U defines a section of $\mathcal{L}^{\mathrm{rig}}_{r,r} \otimes \Omega^1_X$ over $\mathcal{U}$. As argued in the proof of [6, Proposition 7.2], this yields an isomorphism

$$H^1_{\mathrm{dR}}(U, \mathcal{L}_{r,r}, \nabla) \simeq H^1(\mathcal{L}^\bullet_{r,r}|_{\mathcal{U}})$$

between algebraic and rigid de Rham cohomology.

Proposition 2.3. *Let the notation be as before.*

(1) *A class $\kappa \in H^1(\mathcal{L}^\bullet_{r,r}|_{\mathcal{U}})$ belongs to the image of $H^1_{\mathrm{par}}(X, \mathcal{L}_{r,r}, \nabla)$ under restriction*

$$H^1_{\mathrm{par}}(X, \mathcal{L}_{r,r}, \nabla) \longrightarrow H^1_{\mathrm{dR}}(U, \mathcal{L}_{r,r}, \nabla) \simeq H^1(\mathcal{L}^\bullet_{r,r}|_{\mathcal{U}})$$

if and only if $\mathrm{Res}_{\mathcal{V}_Q}(\kappa) = 0$ for all $Q \in S$.

(2) *Let V be such that $\{U, V\}$ is an admissible covering of X. If $\kappa_\omega, \kappa_\eta \in H^1_{\mathrm{par}}(X, \mathcal{L}_{r,r}, \nabla)$ are represented by the hypercocycles $(\omega_U, \omega_V; \omega_{U\cap V})$, $(\eta_U, \eta_V; \eta_{U\cap V})$ respectively, with respect to this covering, then the value $\langle \kappa_\omega, \kappa_\eta \rangle$ under the Poincaré duality (2.3) is given by*

$$\langle \kappa_\omega, \kappa_\eta \rangle = \sum_{Q \in S} \mathrm{res}_{\mathcal{V}_Q} \langle F_{\omega,Q}, \eta_U \rangle,$$

where $F_{\omega,Q}$ is any local primitive of ω_U on $\mathcal{V}_Q$, i.e., such that $\nabla F_{\omega,Q} = \omega_U|_{\mathcal{V}_Q}$.

Proof. The first assertion follows from the same argument as in [1, Proposition 3.8], and the second is [6, Lemma 7.1]. □

2.3. Coleman's p-adic integration

In this section, we give an explicit description of the filtered (Φ, N)-module structure on $H^1_{\mathrm{par}}(X, \mathcal{L}_{r,r}, \nabla)$, following the work of Coleman–Iovita [8]. We state the results for $\mathcal{L}_r$, leaving their trivial extension to $\mathcal{L}_{r,r} = \mathcal{L}_r \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A)$ to the reader.

As recalled in § 1.4, for every pair (E_x, t_x) corresponding to a point $x \in X_1(N)(\frac{p}{p+1})$ there is a canonical p-isogeny $\alpha_{\mathrm{can}} : E_x \mapsto E_x/\mathrm{can}(E_x)$, where $\mathrm{can}(E_x) \subset E_x[p]$ is the canonical subgroup. The map $V : X_1(N)(\frac{1}{p+1}) \longrightarrow X_1(N)(\frac{p}{p+1})$ defined in terms of moduli by

$$V(E_x, t_x) = (\alpha_{\mathrm{can}}(E_x), \alpha_{\mathrm{can}}(t_x)) \tag{2.11}$$

is then a lift of the absolute Frobenius on $X_1(N)_{\mathbf{F}_p}$. Letting $s_1 : X_1(N)(\frac{p}{p+1}) \longrightarrow \mathcal{W}_\infty$ be defined by $(E_x, t_x) \mapsto (E_x, t_x, \mathrm{can}(E_x))$, and letting $\mathcal{W}'_\infty \subset \mathcal{W}_\infty$ be the image of $X_1(N)(\frac{1}{p+1})$ under s_1, the map ϕ_∞, defined by the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{W}'_\infty & \xrightarrow{\phi_\infty} & \mathcal{W}_\infty \\ \downarrow{\scriptstyle \pi_1} & & \uparrow{\scriptstyle s_1} \\ X_1(N)(\frac{1}{p+1}) & \xrightarrow{V} & X_1(N)(\frac{p}{p+1}), \end{array}$$

is therefore a lift of the absolute Frobenius on $X_{\mathbf{F}_p}$.

As explained in [6, p. 41] (see also the more detailed discussion in [7, p. 218]), the canonical subgroup yields a horizontal morphism $\mathrm{Fr}_\infty : \phi_\infty^* \mathcal{L}_r \longrightarrow \mathcal{L}_r|_{\mathcal{W}'_\infty}$. Define the *Frobenius endomorphism* Φ_∞ on $H^1(\mathcal{L}_r^\bullet|_{\mathcal{W}_\infty})$ by the composite map

$$H^1(\mathcal{L}_r^\bullet|_{\mathcal{W}_\infty}) \simeq \frac{\mathcal{L}_r^{\mathrm{rig}} \otimes \Omega^1_{\mathcal{W}_\infty}(\mathcal{W}_\infty)}{\nabla \mathcal{L}_r^{\mathrm{rig}}(\mathcal{W}_\infty)} \xrightarrow{(\mathrm{Fr}_\infty \otimes \mathrm{id})\phi_\infty^*} \frac{\mathcal{L}_r^{\mathrm{rig}} \otimes \Omega^1_{\mathcal{W}_\infty}(\mathcal{W}'_\infty)}{\nabla \mathcal{L}_r^{\mathrm{rig}}(\mathcal{W}'_\infty)} \simeq H^1(\mathcal{L}_r^\bullet|_{\mathcal{W}_\infty}),$$

where the last isomorphism is given by restriction (see [6, Proposition 10.3]). Setting $\mathcal{W}'_0 := w_p \mathcal{W}'_\infty \subset \mathcal{W}_0 = w_p \mathcal{W}_\infty$ and $\phi_0 := w_p^{-1} \phi_\infty w_p$, where w_p is the automorphism of X given by (2.4), we similarly define a Frobenius endomorphism Φ_0 of $H^1(\mathcal{L}_r^\bullet|_{\mathcal{W}_0})$.

Theorem 2.4 (Coleman). *Let $f = q + \sum_{n=2}^\infty a_n(f) q^n \in S_{r+2}(\Gamma_0(Np))$ be a p-new eigenform of weight $r+2 \geqslant 2$, and let $\omega_f \in H^0(X, \underline{\omega}^r \otimes \Omega^1_X) \subset H^1_{\mathrm{par}}(X, \mathcal{L}_r, \nabla)$ be the associated differential. Then for each $\star \in \{\infty, 0\}$ there exists a locally analytic section $F_{f,\star}$ of $\mathcal{L}_r$ on $\mathcal{W}_\star$ such that*

(i) $\nabla F_{f,\star} = \omega_f|_{\mathcal{W}_\star}$;

(ii) $F_{f,\star} - \frac{a_p(f)}{p^{r+1}} \phi_\star^* F_{f,\star}$ *is rigid analytic on* $\mathcal{W}'_\star$.

Moreover, $F_{f,\star}$ is unique modulo $H^0(\mathcal{L}_r^\bullet|_{\mathcal{W}_\star})$.

Proof. This follows from the discussion in [6, § 11]. By [6, Lemma 11.1] we have $\Phi_\infty = pU_p$ on the image of $S_{r+2}(X)^{p\text{-new}}$ in $H^1(\mathcal{L}_r^\bullet|_{\mathcal{W}_\infty})$. Since $U_p^2 = p^r \langle p \rangle$ on the former space and we have the relations $U_p \omega_f = a_p(f) \omega_f$ and $\langle p \rangle \omega_f = \omega_f$ by hypothesis, it follows that the polynomial

$$P(T) = 1 - \frac{a_p(f)}{p^{r+1}} T$$

is such that $P(\Phi_\infty)([\omega_f|_{\mathcal{W}_\infty}]) = 0$, and hence also $P(\Phi_0)([\omega_f|_{\mathcal{W}_0}]) = 0$. The result thus follows from [6, Theorem 10.1]. □

A locally analytic section $F_{f,\star}$ as in Theorem 2.4 is called a *Coleman primitive* of f on $\mathcal{W}_\star$.

Remark 2.5. For $r > 0$, the spaces $H^0(\mathcal{L}_r^\bullet|_{\mathcal{W}_\star})$ are trivial, and so the Coleman primitives $F_{f,\star}$ are unique. On the other hand, for $r = 0$ we have $H^0(\mathcal{L}_r^\bullet|_{\mathcal{W}_\star}) \simeq \mathbf{C}_p$, and so $F_{f,\star}$ are unique modulo a global constant on $\mathcal{W}_\star$.

2.4. Frobenius and monodromy

Denote by ι the inclusion of any rigid subspace of X into X. Associated with the exact sequence of complexes of sheaves on X

$$0 \longrightarrow \mathcal{L}_r^\bullet \longrightarrow \iota^*(\mathcal{L}_r^\bullet|_{\mathcal{W}_0}) \oplus \iota^*(\mathcal{L}_r^\bullet|_{\mathcal{W}_\infty}) \xrightarrow{\rho_\infty - \rho_0} \iota^*(\mathcal{L}_r^\bullet|_{\mathcal{W}}) \longrightarrow 0,$$

there is a Mayer–Vietoris long exact sequence

$$\begin{aligned}\cdots \longrightarrow H^0_{\text{par}}(\mathcal{L}_r^\bullet|_{\mathcal{W}_0 \sqcup \mathcal{W}_\infty}) &\xrightarrow{\beta^0} H^0_{\text{par}}(\mathcal{L}_r^\bullet|_{\mathcal{W}}) \xrightarrow{\delta} H^1_{\text{par}}(X, \mathcal{L}_r^{\text{rig}}, \nabla) \longrightarrow \\ \longrightarrow H^1_{\text{par}}(\mathcal{L}_r^\bullet|_{\mathcal{W}_0 \sqcup \mathcal{W}_\infty}) &\xrightarrow{\beta^1} H^1_{\text{par}}(\mathcal{L}_r^\bullet|_{\mathcal{W}}) \longrightarrow \cdots\end{aligned}$$

in hypercohomology. By [6, § 10] and the discussion in the preceding section, each of the non-central spaces in the resulting short exact sequence

$$0 \longrightarrow \frac{H^0_{\text{par}}(\mathcal{L}_r^\bullet|_{\mathcal{W}})}{\beta^0(H^0_{\text{par}}(\mathcal{L}_r^\bullet|_{\mathcal{W}_0 \sqcup \mathcal{W}_\infty}))} \xrightarrow{\delta} H^1_{\text{par}}(X, \mathcal{L}_r, \nabla) \longrightarrow H^1_{\text{par}}(\mathcal{L}_r^\bullet|_{\mathcal{W}_0 \sqcup \mathcal{W}_\infty})^{\beta^1 = 0} \longrightarrow 0 \quad (2.12)$$

is equipped with a Frobenius endomorphism. Therefore, to define a Frobenius action on $H^1_{\text{par}}(X, \mathcal{L}_r, \nabla)$ it suffices to construct a splitting of (2.12).

As shown in [6, § A.5], this may be obtained as follows. Assume that $\kappa \in H^1_{\text{par}}(X, \mathcal{L}_r, \nabla)$ is represented by the hypercocycle $(\omega_0, \omega_\infty; f_{\mathcal{W}})$ with respect to the covering $\{\mathcal{W}_0, \mathcal{W}_\infty\}$ of X. Since $\mathcal{W} = \bigcup_{x \in SS} \mathcal{A}_x$ is the union of the supersingular annuli, we may write $f_{\mathcal{W}} = \{f_x\}_{x \in SS}$ with $f_x \in \mathcal{L}_r^{\text{rig}}(\mathcal{A}_x)$. The assignment

$$\mathcal{A}_x \longmapsto F_{\omega_\infty}|_{\mathcal{A}_x} - F_{\omega_0}|_{\mathcal{A}_x} - f_x, \quad (2.13)$$

where $F_{\omega_\star}$ is a Coleman primitive of $\omega_\star$ on $\mathcal{W}_\star$, defines a horizontal section of $\mathcal{L}_r^{\text{rig}}$ on $\mathcal{W}$, and its image modulo $\beta^0(H^0_{\text{par}}(\mathcal{L}_r^\bullet|_{\mathcal{W}_0 \sqcup \mathcal{W}_\infty}))$ is independent of the chosen $F_{\omega_\star}$ (see Remark 2.5). It is easily checked that $s\delta = \text{id}$, and hence we may define a Frobenius operator Φ on $H^1_{\text{par}}(X, \mathcal{L}_r, \nabla)$ by requiring that its action be compatible with the resulting splitting of (2.12).

On the other hand, define the monodromy operator N on $H^1_{\text{par}}(X, \mathcal{L}_r, \nabla)$ by the composite map

$$H^1_{\text{par}}(X, \mathcal{L}_r, \nabla) \longrightarrow H^1(\mathcal{L}_r^\bullet|_{\mathcal{W}}) \xrightarrow{\bigoplus_{x \in SS} \text{Res}_{\mathcal{A}_x}} H^0(\mathcal{L}_r^\bullet|_{\mathcal{W}}) \xrightarrow{\delta} H^1_{\text{par}}(X, \mathcal{L}_r, \nabla),$$

where $\text{Res}_{\mathcal{A}_x}$ are the p-adic residue maps (2.10).

Lemma 2.6. *Let $\kappa \in H^1_{\mathrm{par}}(X, \mathcal{L}_r, \nabla)$. Then we have*

(i) *for $r > 0$, $N(\kappa) = 0 \Longleftrightarrow \mathrm{Res}_{\mathcal{A}_x}(\kappa) = 0$ for all $x \in SS$;*

(ii) *for $r = 0$, $N(\kappa) = 0 \Longleftrightarrow$ there is $C \in \mathbf{C}_p$ such that $\mathrm{res}_{\mathcal{A}_x}(\kappa) = C$ for all $x \in SS$.*

Proof. This follows immediately from the exact sequence (2.12) and the determination of the spaces $H^0(\mathcal{L}_r^{\bullet}|_{\mathcal{W}_\star})$ recalled in Remark 2.5. □

By the main result of [8], the operators Φ and N on $H^1_{\mathrm{par}}(X, \mathcal{L}_r, \nabla)$ defined above agree with the corresponding structures deduced from the comparison isomorphism of Theorem 2.1.

2.5. p-adic Gross–Zagier formula

Fix a finite extension $F/\mathbf{Q}_p$ containing the image of the Hilbert class field H of K under our fixed embedding $\overline{\mathbf{Q}} \overset{\iota_p}{\hookrightarrow} \mathbf{C}_p$, and let $c \geqslant 1$ be an integer prime to Np.

Proposition 2.7. *Let $f = q + \sum_{n=2}^{\infty} a_n(f)q^n \in S_{r+2}(\Gamma_0(Np))$ be a p-new eigenform of weight $r + 2 \geqslant 2$. Let $\varphi : A \longrightarrow A'$ be an isogeny in $\mathrm{Isog}_c^{\mathfrak{N}}(A)$, let $P_{A'} \in X(F)$ be the point defined by $(A', t_{A'})$, and let Δ_φ be the generalized Heegner cycle associated with φ. Then for all $\alpha \in \mathrm{Sym}^r H^1_{\mathrm{dR}}(A/F)$, we have*

$$\mathrm{AJ}_F(\Delta_\varphi)(\omega_f \wedge \alpha) = \langle F_{f,\infty}(P_{A'}) \wedge \alpha, \mathrm{cl}_{P_{A'}}(\Delta_\varphi)\rangle,$$

where $F_{f,\infty}$ is the Coleman primitive of $\omega_f \in H^0(X, \underline{\omega}^{\otimes r} \otimes \Omega^1_X)$ on $\mathcal{W}_\infty$ (vanishing at ∞ if $r = 0$), and the pairing on the right-hand side is the natural one on $\mathcal{L}_{r,r}(P_{A'})$.

Proof. Following the recipe described at the end of § 2.1, we have

$$\mathrm{AJ}_F(\Delta_\varphi)(\omega_f \wedge \alpha) = \langle \omega_f \wedge \alpha, \eta_\Delta^{\mathrm{hol}} - \eta_\Delta^{\mathrm{frob}}\rangle, \tag{2.14}$$

where

- $\eta_\Delta^{\mathrm{hol}}$ is a cohomology class represented by a section (still denoted by $\eta_\Delta^{\mathrm{hol}}$) of $\mathcal{L}_{r,r} \otimes \Omega^1_X(\log Z_X)$ over U having residue 0 at the cusps, and with a simple pole at $P_{A'}$ with residue $\mathrm{cl}_{P_{A'}}(\Delta_\varphi)$;
- $\eta_\Delta^{\mathrm{frob}}$ is a section of $\mathcal{L}_{r,r}^{\mathrm{rig}} \otimes \Omega^1_X$ over $\mathcal{U}$ having the same residues as $\eta_\Delta^{\mathrm{hol}}$, and satisfying $N(\eta_\Delta^{\mathrm{frob}}) = 0$ and
$$\Phi\eta_\Delta^{\mathrm{frob}} = \eta_\Delta^{\mathrm{frob}} + \nabla G, \tag{2.15}$$
for some rigid section G of $\mathcal{L}_{r,r}^{\mathrm{rig}}$ on a strict neighborhood of $(\mathcal{Z}_0 \cap \mathcal{W}_0^\sharp) \cup (\mathcal{Z}_\infty \cap \mathcal{W}_\infty^\sharp)$ in $\mathcal{U}$.

By the formula for the Poincaré pairing in Proposition 2.3, equation (2.14) may be rewritten as

$$\mathrm{AJ}_F(\Delta_\varphi)(\omega_f \wedge \alpha) = \sum_{Q \in S} \mathrm{res}_{\mathcal{V}_Q}\langle F_{f,Q} \wedge \alpha, \eta_\Delta^{\mathrm{hol}} - \eta_\Delta^{\mathrm{frob}}\rangle, \tag{2.16}$$

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where $F_{f,Q} \in \mathcal{L}_r(\mathcal{V}_Q)$ is an arbitrary local primitive of ω_f on the annulus $\mathcal{V}_Q$. (Note that here we are using the fact that the connection ∇ on $\mathcal{L}_{r,r} = \mathcal{L}_r \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A/F)$ is defined from the Gauss–Manin connection on $\mathcal{L}_r$ by extending it trivially on the second factor.)

If $F_{f,\infty}$ is a Coleman primitive of ω_f on $\mathcal{W}_\infty$, then $F^{[p]}_{f,\infty} := F_{f,\infty} - \frac{a_p(f)}{p^{r+1}}\phi^*_\infty F_{f,\infty}$ is rigid analytic on $\mathcal{W}'_\infty \subset \mathcal{W}_\infty$ by Theorem 2.4, and hence

$$\sum_{Q\in S\cap\mathcal{W}_\infty} \mathrm{res}_{\mathcal{V}_Q}\langle F^{[p]}_{f,\infty} \wedge \alpha, \eta^{\mathrm{frob}}_\Delta\rangle + \sum_{x\in SS} \mathrm{res}_{\mathcal{A}_x}\langle F^{[p]}_{f,\infty} \wedge \alpha, \eta^{\mathrm{frob}}_\Delta\rangle = 0 \tag{2.17}$$

by the residue theorem (see [1, Theorem 3.8]). Since $N(\eta^{\mathrm{frob}}_\Delta) = 0$, Lemma 2.6 implies that we can write $\eta^{\mathrm{frob}}_\Delta = \nabla G_x$ for some rigid section $G_x \in \mathcal{L}^{\mathrm{rig}}_{r,r}(\mathcal{A}_x)$ on each supersingular annulus $\mathcal{A}_x$, and hence

$$d\langle F^{[p]}_{f,\infty} \wedge \alpha, G_x\rangle = \langle \nabla F^{[p]}_{f,\infty} \wedge \alpha, G_x\rangle + \langle F^{[p]}_{f,\infty} \wedge \alpha, \eta^{\mathrm{frob}}_\Delta\rangle.$$

In particular, the right-hand side in the last equality has residue 0, and hence

$$\mathrm{res}_{\mathcal{A}_x}\langle F^{[p]}_{f,\infty} \wedge \alpha, \eta^{\mathrm{frob}}_\Delta\rangle = -\mathrm{res}_{\mathcal{A}_x}\langle \nabla F^{[p]}_{f,\infty} \wedge \alpha, G_x\rangle. \tag{2.18}$$

Plugging (2.18) into (2.17), we arrive at

$$\sum_{Q\in S\cap\mathcal{W}_\infty} \mathrm{res}_{\mathcal{V}_Q}\langle F^{[p]}_{f,\infty} \wedge \alpha, \eta^{\mathrm{frob}}_\Delta\rangle - \sum_{x\in SS} \mathrm{res}_{\mathcal{A}_x}\langle \nabla F^{[p]}_{f,\infty} \wedge \alpha, \eta^{\mathrm{frob}}_\Delta\rangle = 0. \tag{2.19}$$

An entirely parallel reasoning with $\mathcal{W}_0$ in place of $\mathcal{W}_\infty$ yields a proof of the equality

$$\sum_{Q\in S\cap\mathcal{W}_0} \mathrm{res}_{\mathcal{V}_Q}\langle F^{[p]}_{f,0} \wedge \alpha, \eta^{\mathrm{frob}}_\Delta\rangle + \sum_{x\in SS} \mathrm{res}_{\mathcal{A}_x}\langle \nabla F^{[p]}_{f,\infty} \wedge \alpha, \eta^{\mathrm{frob}}_\Delta\rangle = 0, \tag{2.20}$$

where $F_{f,0}$ is a Coleman primitive of ω_f on $\mathcal{W}_0$, and where we used the fact that the supersingular annuli acquire opposite orientations with respect to $\mathcal{W}_\infty$ and $\mathcal{W}_0$. Combining (2.19) and (2.20), we get

$$0 = \sum_{Q\in S} \mathrm{res}_{\mathcal{V}_Q}\langle F^{[p]}_{f,Q} \wedge \alpha, \eta^{\mathrm{frob}}_\Delta\rangle = \left(1 - \frac{a_p(f)}{p^{r+1}}\right) \sum_{Q\in S} \mathrm{res}_{\mathcal{V}_Q}\langle F_{f,Q} \wedge \alpha, \eta^{\mathrm{frob}}_\Delta\rangle, \tag{2.21}$$

where $F^{[p]}_{f,Q}$ denotes $F^{[p]}_{f,\infty}$ or $F^{[p]}_{f,0}$ depending on whether $Q \in \mathcal{W}_\infty$ or $\mathcal{W}_0$, respectively, using (2.15) for the second equality (see the argument [1, p. 1079]).

Since $a_p(f)^2 = p^r$, this shows that there is no contribution from $\eta^{\mathrm{frob}}_\Delta$ in (2.16). On the other hand, since by the choice of $\eta^{\mathrm{hol}}_\Delta$ we easily have

$$\sum_{Q\in S} \mathrm{res}_{\mathcal{V}_Q}\langle F_{f,Q} \wedge \alpha, \eta^{\mathrm{hol}}_\Delta\rangle = \langle F_{f,\infty}(P_{A'}) \wedge \alpha, \mathrm{cl}_{P_{A'}}(\Delta_\varphi)\rangle$$

(see [1, Lemma 3.19]), the result follows. □

Let (A, t_A, ω_A) be the CM triple introduced in § 1.5, and let $\eta_A \in H^1_{\mathrm{dR}}(A/F)$ be the class determined by the conditions

$$\lambda^*\eta_A = \lambda^\rho \eta_A \quad \text{for all } \lambda \in \mathcal{O}_K, \quad \text{and} \quad \langle \omega_A, \eta_A\rangle_A = 1,$$

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where $\lambda \mapsto \lambda^\rho$ denotes the action of the non-trivial automorphism of K, and $\langle\, ,\rangle_A$ is the cup product pairing on $H^1_{\mathrm{dR}}(A/F)$. If $(A', t_{A'}, \omega_{A'})$ is the CM triple induced from (A, t_A, ω_A) by an isogeny $\varphi \in \mathrm{Isog}_c^{\mathfrak{N}}(A)$, we define $\eta_{A'} \in H^1_{\mathrm{dR}}(A'/F)$ by the analogous recipe. For the integers j with $0 \leqslant j \leqslant r$, the classes $\omega_{A'}^j \eta_{A'}^{r-j}$ defined in [1, (1.4.6)] then form a basis of $\mathrm{Sym}^r H^1_{\mathrm{dR}}(A'/F)$.

Lemma 2.8. *Let the notation be as in Proposition 2.7. Then, for each $0 \leqslant j \leqslant r$, we have*

$$\mathrm{AJ}_F(\Delta_\varphi)(\omega_f \wedge \omega_A^j \eta_A^{r-j}) = \deg(\varphi)^j \cdot \langle F_{f,\infty}(P_{A'}), \omega_{A'}^j \eta_{A'}^{r-j} \rangle_{A'},$$

where $F_{f,\infty}$ is the Coleman primitive of $\omega_f \in H^0(X, \underline{\omega}^{\otimes r} \otimes \Omega^1_X)$ on $\mathcal{W}_\infty$ (vanishing at ∞ if $r = 0$), and the pairing $\langle\, ,\rangle_{A'}$ on the right-hand side is the natural one on $\mathrm{Sym}^r H^1_{\mathrm{dR}}(A'/F)$.

Proof. This follows from Proposition 2.7 as in [1, Lemma 3.22]. □

Recall that if $f \in S_k(X)$ is a cusp form of weight k and level Γ, then $f|_{\mathcal{W}_\infty}$ defines a p-adic modular form $f_p \in \mathcal{M}_k(N)$ of weight k and tame level N. Evaluated on a CM triple $(A', t_{A'}, \omega_{A'})$ of conductor c prime to p, we then have

$$f_p(A', t_{A'}, \omega_{A'}) = f(A', t_{A'}, \alpha'_{\mathfrak{p}}, \omega_{A'}),$$

where $\alpha'_{\mathfrak{p}} : A' \longrightarrow A'/A'[\mathfrak{p}]$ is the p-isogeny defined by the canonical subgroup of A' (see § 1.5). By abuse of notation, in the following we will denote f_p also by f. The map V defined in (2.11) yields an operator $V : \mathcal{M}_k(N) \longrightarrow \mathcal{M}_k(N)$ on p-adic modular forms whose effect on q-expansions is given by $q \mapsto q^p$. Let $a_p(f)$ be the U_p-eigenvalue of f, and define the *p-depletion* of f by

$$f^{[p]} := f - a_p(f) V f.$$

Letting $d = q\frac{d}{dq} : \mathcal{M}_k(N) \longrightarrow \mathcal{M}_{k+2}(N)$ be the Atkin–Serre operator, for any integer j the limit

$$d^{-1-j} f^{[p]} := \lim_{t \to -1-j} d^t f^{[p]}$$

is a p-adic modular form of weight $k - 2 - j$ and tame level N (see [30, Theorem 5]).

Lemma 2.9. *Let the notation be as in Proposition 2.7. Then for each $0 \leqslant j \leqslant r$ there exists a locally analytic p-adic modular form G_j of weight $r - j$ and tame level N such that*

$$\langle F_{f,\infty}(P_{A'}), \omega_{A'}^j \eta_{A'}^{r-j} \rangle_{A'} = G_j(A', t_{A'}, \omega_{A'}), \tag{2.22}$$

where $F_{f,\infty}$ is the Coleman primitive of ω_f on $\mathcal{W}_\infty$ (vanishing at ∞ if $r = 0$), and

$$G_j(A', t_{A'}, \omega_{A'}) - \frac{a_p(f)}{p^{r-j+1}} G_j(\mathfrak{p} * (A', t_{A'}, \omega_{A'})) = j! d^{-1-j} f^{[p]}(A', t_{A'}, \omega_{A'}). \tag{2.23}$$

Proof. The construction of G_j as the 'jth component' of $F_{f,\infty}$ is given in [1, p. 1083], and (2.22) then follows from the definition. On the other hand, (2.23) follows from the same calculations as in [1, Lemma 3.23 and Proposition 3.24]. □

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We now relate the expression appearing in the right-hand side of Proposition 2.7 to the value of a certain p-adic L-function associated with f.

Recall that (A, t_A, ω_A) denotes the CM triple introduced in § 1.5, and fix an elliptic curve A_0/H_c with $\mathrm{End}_{H_c}(A_c) \simeq \mathcal{O}_c$. The curve A_0 is related to A by an isogeny $\varphi_0 : A \longrightarrow A_0$ in $\mathrm{Isog}_c^{\mathfrak{N}}(A)$, and we let (A_0, t_0, ω_0) be the induced triple. Since we assume that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K, we may fix an isomorphism $\mu_{p^\infty} \simeq \mathcal{A}_0[\mathfrak{p}^\infty]$ of p-divisible groups, where $\mathcal{A}_0/\mathcal{O}_{\mathbf{C}_p}$ is a good integral model of A_0. This amounts to the choice of an isomorphism $\imath : \hat{\mathcal{A}}_0 \longrightarrow \hat{\mathbb{G}}_m$ of formal groups, and we let $\Omega_p \in \mathbf{C}_p^\times$ be the p-adic period defined by the rule

$$\omega_0 = \Omega_p \cdot \omega_{\mathrm{can}},$$

where $\omega_{\mathrm{can}} := \imath^* \frac{dt}{t}$ for the standard coordinate t on $\hat{\mathbb{G}}_m$.

Finally, consider the set $\Sigma_{k,c}^+$ of algebraic Hecke characters $\chi : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbf{C}^\times$ of conductor c, infinity type $(k+j, -j)$ with $j \geqslant 0$ (with the convention in [1, p. 1089]), and such that

$$\chi|_{\mathbb{A}_{\mathbf{Q}}^\times} = \mathbf{N}^k,$$

where $\mathbf{N}$ is the norm character on $\mathbb{A}_{\mathbf{Q}}^\times$, and for every $\chi \in \Sigma_{k,c}^+$ set

$$L_{\mathfrak{p}}(f)(\chi) := \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)} \chi^{-1}(\mathfrak{a}) \mathrm{N}(\mathfrak{a})^{-j} \cdot d^j f^{[p]}(\mathfrak{a} * (A_0, t_0, \omega_{\mathrm{can}})), \tag{2.24}$$

and define

$$L_{\mathrm{alg}}(f, \chi^{-1}) := w(f, \chi)^{-1} C(f, \chi, c) \cdot \frac{L(f, \chi^{-1}, 0)}{\Omega^{2(k+2j)}},$$

where $w(f, \chi)$ and $C(f, \chi, c)$ are the constants defined in [1, (5.1.11)] and [1, Theorem 4.6], respectively, Ω is the complex period in [1, (5.1.16)], and $L(f, \chi^{-1}, 0)$ is the central critical value of the Rankin–Selberg L-function $L(f \times \theta_{\chi^{-1}}, s)$ of f and the theta series of χ^{-1}.

As explained in [1, p. 1134], the set $\Sigma_{k,c}^+$ may be endowed with a natural p-adic topology, and we let $\hat{\Sigma}_{k,c}$ denote its completion.

Theorem 2.10. *The assignment $\chi \mapsto L_{\mathfrak{p}}(f)(\chi)$ extends to a continuous function on $\hat{\Sigma}_{k,c}$ and satisfies the following interpolation property. If $\chi \in \Sigma_{k,c}^+$ has infinity type $(k+j, -j)$, with $j \geqslant 0$, then*

$$\frac{L_{\mathfrak{p}}(f)(\chi)^2}{\Omega_p^{2(k+2j)}} = (1 - a_p(f)\chi^{-1}(\bar{\mathfrak{p}}))^2 \cdot L_{\mathrm{alg}}(f, \chi^{-1}, 0).$$

Proof. See Theorem 5.9, Proposition 5.10, and equation (5.2.4) of [1], noting that $\beta_p = 0$ here, since f has level divisible by p. □

Let $\Sigma_{k,c}^-$ be the set of algebraic Hecke characters of K of conductor c and infinity type $(k-1-j, 1+j)$, with $j \geqslant 0$. Even though $\Sigma_{k,c}^+ \cap \Sigma_{k,c}^- = \emptyset$, any character in $\Sigma_{k,c}^-$ can be

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written as a limit of characters in $\Sigma_{k,c}^{+}$ (see [1, p. 1137]). Thus for any $\chi \in \Sigma_{k,c}^{-}$, the value $L_{\mathfrak{p}}(f)(\chi)$ is defined by continuity.

The next result extends the p-adic Gross–Zagier formula of [1, Theorem 5.13] to the semistable non-crystalline setting.

Theorem 2.11. *Let* $f = q + \sum_{n=2}^{\infty} a_n(f)q^n \in S_k(\Gamma_0(Np))$ *be a* p*-new eigenform of weight* $k = r+2 \geqslant 2$*, and suppose that* $\chi \in \Sigma_{k,c}^{-}$ *has infinity type* $(r+1-j, 1+j)$*, with* $0 \leqslant j \leqslant r$*. Then*

$$\frac{L_{\mathfrak{p}}(f)(\chi)}{\Omega_p^{r-2j}} = (1 - a_p(f)\chi^{-1}(\bar{\mathfrak{p}})) \cdot \left(\frac{c^{-j}}{j!} \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)} \chi^{-1}(\mathfrak{a})\mathrm{N}(\mathfrak{a}) \cdot \mathrm{AJ}_F(\Delta_{\varphi_{\mathfrak{a}}\varphi_0})(\omega_f \wedge \omega_A^j \eta_A^{r-j}) \right).$$

Proof. The proof of [1, Proposition 5.10] shows that the expression (2.24) extends in the obvious way to a character χ as in the statement, yielding

$$\frac{L_{\mathfrak{p}}(f)(\chi)}{\Omega_p^{r-2j}} = \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)} \chi^{-1}(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{1+j} \cdot d^{-1-j} f^{[p]}(\mathfrak{a} * (A_0, t_0, \omega_0)). \tag{2.25}$$

On the other hand, by Lemma 2.9 we have

$$j! d^{-1-j} f^{[p]}(\mathfrak{a} * (A_0, t_0, \omega_0)) = G_j(\mathfrak{a} * (A_0, t_0, \omega_0)) - \frac{a_p(f)}{p^{r-j+1}} G_j(\mathfrak{p}\mathfrak{a} * (A_0, t_0, \omega_0)). \tag{2.26}$$

Substituting (2.26) into (2.25), summing over $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)$, and noting that

$$\chi(\mathfrak{p}) p^{-1-j} = \chi^{-1}(\bar{\mathfrak{p}}) p^{r+1-j},$$

we see that

$$\begin{aligned} \frac{L_{\mathfrak{p}}(f)(\chi)}{\Omega_p^{r-2j}} &= \left(1 - a_p(f)\chi^{-1}(\bar{\mathfrak{p}})\right) \\ &\quad \times \left(\frac{1}{j!} \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)} \chi^{-1}(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{1+j} \cdot G_j(\mathfrak{a} * (A_0, t_0, \omega_0)) \right). \end{aligned} \tag{2.27}$$

Finally, since the isogeny $\varphi_{\mathfrak{a}}\varphi_0 : (A, t_A, \omega_A) \longrightarrow \mathfrak{a} * (A_0, t_0, \omega_0)$ has degree $c\mathrm{N}(\mathfrak{a})$, combining Lemmas 2.8 and 2.9 we have

$$G_j(\mathfrak{a} * (A_0, t_0, \omega_0)) = c^{-j}\mathrm{N}(\mathfrak{a})^{-j} \cdot \mathrm{AJ}_F(\Delta_{\varphi_{\mathfrak{a}}\varphi_0})(\omega_f \wedge \omega_A^j \eta_A^{r-j}), \tag{2.28}$$

and substituting (2.28) into (2.27), the result follows. □

3. Main result

In this section we prove the main result of this paper, giving an 'exceptional zero formula' for the specializations of Howard's big Heegner points at exceptional primes in the Hida family.

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3.1. Heegner points in Hida families

We begin by briefly reviewing the constructions of [15], which we adapt to our situation, referring the reader to [15, § 2] for further details.

Recall that $p \nmid N$. Let $f = \sum_{n=1}^{\infty} a_n(f) q^n \in S_k(\Gamma_0(Np))$ be a newform, fix a finite extension L of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_L$ containing the Fourier coefficients of f, and let

$$\rho_f : G_{\mathbf{Q}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{Aut}_L(V_f) \simeq \mathbf{GL}_2(L)$$

be the Galois representation associated with f. Also, let $K = \mathbf{Q}(\sqrt{-D_K})$ be an imaginary quadratic field as in § 1.5. For the rest of this paper, these will be subject to the following further hypotheses.

Assumptions 3.1. (1) f is ordinary at p, i.e., $\iota_p(a_p(f))$ is a p-adic unit;

(2) $\overline{\rho}_f$ is absolutely irreducible;

(3) $\overline{\rho}_f$ is ramified at every prime q dividing (D_K, N);

(4) $p \nmid h_K := |\mathrm{Pic}(\mathcal{O}_K)|$, the class number of K.

Note that by [15, Lemma 2.15], the first assumption forces the weight of f to be $k = 2$, which will thus be assumed for the rest of this paper. As noted in [19, Remark 7.2.7], this also implies that the residual representation $\overline{\rho}_f$ is automatically *p-distinguished*.

Definition 3.2. Set $\Lambda_{\mathcal{O}_L} := \mathcal{O}_L[\![1 + p\mathbf{Z}_p]\!]$. For any $\Lambda_{\mathcal{O}_L}$-algebra A, let $\mathcal{X}^a_{\mathcal{O}_L}(A)$ be the set of continuous $\mathcal{O}_L$-algebra homomorphisms $\nu : A \longrightarrow \overline{\mathbf{Q}}_p$ such that the composition

$$1 + p\mathbf{Z}_p \longrightarrow \Lambda_{\mathcal{O}_L}^{\times} \longrightarrow A^{\times} \xrightarrow{\nu} \overline{\mathbf{Q}}_p^{\times}$$

is given by $\gamma \mapsto \gamma^{k_\nu - 2}$, for some integer $k_\nu \geqslant 2$ with $k_\nu \equiv 2 \pmod{2(p-1)}$ called the *weight* of ν.

Since f is ordinary at p, by [13, Corollary 1.3] there exists a local reduced finite integral extension $\mathbb{I}$ of $\Lambda_{\mathcal{O}_L}$, and a formal q-expansion $\mathbf{f} = \sum_{n=1}^{\infty} \mathbf{a}_n q^n \in \mathbb{I}[\![q]\!]$ uniquely characterized by the following property. For every $\nu \in \mathcal{X}^a_{\mathcal{O}_L}(\mathbb{I})$ of weight $k_\nu > 2$, there exists a newform $f_\nu \in S_{k_\nu}(\Gamma_0(N))$ such that

$$\nu(\mathbf{f}) = f_\nu(q) - \frac{p^{k_\nu - 1}}{\nu(\mathbf{a}_p)} f_\nu(q^p), \tag{3.1}$$

and there exists a unique $\nu_f \in \mathcal{X}^a_{\mathcal{O}_L}(\mathbb{I})$ of weight 2 such that $\nu_f(\mathbf{f}) = f(q)$.

By [13, Theorem 1.2], there is a free $\mathbb{I}$-module $\mathbf{T}$ of rank 2 equipped with a continuous action

$$\rho_{\mathbf{f}} : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{I}}(\mathbf{T}) \cong \mathbf{GL}_2(\mathbb{I})$$

such that for every $\nu \in \mathcal{X}^a_{\mathcal{O}_L}$, $\nu(\rho_{\mathbf{f}})$ is isomorphic to the Galois representation $\rho_{f_\nu} : G_{\mathbf{Q}} \longrightarrow \mathbf{GL}_2(\overline{\mathbf{Q}}_p)$ associated with f_ν. Moreover, by [32, Theorem 2.2.2], if $D_p \subset G_{\mathbf{Q}}$ is the decomposition group of any place $\mathfrak{P}$ of $\overline{\mathbf{Q}}$ above p, there exists an exact sequence of $\mathbb{I}[D_p]$-modules

$$0 \longrightarrow \mathscr{F}^{+}\mathbf{T} \longrightarrow \mathbf{T} \longrightarrow \mathscr{F}^{-}\mathbf{T} \longrightarrow 0 \tag{3.2}$$

with $\mathscr{F}^{\pm}\mathbf{T}$ free of rank 1 over $\mathbb{I}$, and with the D_p-action on $\mathscr{F}^{-}\mathbf{T}$ given by the unramified character sending an arithmetic Frobenius Fr_p^{-1} to $\mathbf{a}_p \in \mathbb{I}^{\times}$.

Following [15, Definition 2.1.3], define the *critical character* $\Theta : G_{\mathbf{Q}} \longrightarrow \mathbb{I}^{\times}$ by the composite

$$\Theta : G_{\mathbf{Q}} \xrightarrow{\varepsilon_{\mathrm{cyc}}} \mathbf{Z}_p^{\times} \xrightarrow{\langle\cdot\rangle} 1 + p\mathbf{Z}_p \xrightarrow{\gamma \mapsto \gamma^{1/2}} 1 + p\mathbf{Z}_p \longrightarrow \Lambda_{\mathcal{O}}^{\times} \longrightarrow \mathbb{I}^{\times}, \tag{3.3}$$

where $\varepsilon_{\mathrm{cyc}}$ is the p-adic cyclotomic character and $\langle\cdot\rangle$ denotes the projection to the 1-units in $\mathbf{Z}_p$. Let $\mathbb{I}^{\dagger}$ be the free $\mathbb{I}$-module of rank 1 where $G_{\mathbf{Q}}$ acts via Θ^{-1}, and set

$$\mathbf{T}^{\dagger} := \mathbf{T} \otimes_{\mathbb{I}} \mathbb{I}^{\dagger}$$

equipped with the diagonal Galois action. Then, if for every $\nu \in \mathcal{X}^{a}_{\mathcal{O}_L}(\mathbb{I})$ we let V_{f_ν} be a representation space for ρ_{f_ν}, then $\nu(\mathbf{T}^{\dagger}) := \mathbf{T}^{\dagger} \otimes_{\mathbb{I},\nu} \nu(\mathbb{I})$ is isomorphic to a lattice in the self-dual Tate twist $V_{f_\nu}(k_\nu/2)$ of V_{f_ν} (see [26, Theorem 1.4.3] and [24, (3.2.4)]).

Let K_∞ be the anticyclotomic $\mathbf{Z}_p$-extension of K, and for each $n \geqslant 0$, let K_n be the subfield of K_∞ with $\mathrm{Gal}(K_n/K) \simeq \mathbf{Z}/p^n\mathbf{Z}$.

Theorem 3.3 (Howard). *There is a system of 'big Heegner points'*

$$\mathfrak{Z}_\infty = \{\mathfrak{Z}_n\}_{n\geqslant 0} \in H^1_{\mathrm{Iw}}(K_\infty, \mathbf{T}^{\dagger}) := \varprojlim_n H^1(K_n, \mathbf{T}^{\dagger})$$

with the following properties.

(1) *For each n, $\mathfrak{Z}_n$ belongs to the Greenberg–Selmer group $\mathrm{Sel}_{\mathrm{Gr}}(K_n, \mathbf{T}^{\dagger})$ of [15, Definition 2.4.2]. In particular, for every prime $\mathfrak{q}$ of K above p, we have*

$$\mathrm{loc}_{\mathfrak{q}}(\mathfrak{Z}_\infty) \in \ker\left(H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{q}}, \mathbf{T}^{\dagger}) \longrightarrow H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{q}}, \mathscr{F}^{-}\mathbf{T}^{\dagger})\right)$$

for the natural map induced by (3.2).

(2) *If $\mathfrak{Z}_\infty^*$ denotes the image of $\mathfrak{Z}_\infty$ under the action of complex conjugation, then*

$$\mathfrak{Z}_\infty^* = w \cdot \mathfrak{Z}_\infty$$

for some $w \in \{\pm 1\}$.

Proof. In the following, all the references are to [15]. The construction of $\mathfrak{Z}_\infty$ is given in §§ 2.2, 3.3 and the proof of (1) is given in Proposition 2.4.5. For the proof of (2), we need to briefly recall the definition of $\mathfrak{Z}_n$. Let $H_{p^{n+1}}$ be the ring class field of K of conduction p^{n+1}, and note that it contains K_n. By Proposition 2.3.1, the 'big Heegner points' $\mathfrak{X}_{p^{n+1}} \in H^1(H_{p^{n+1}}, \mathbf{T}^{\dagger})$ satisfy $\mathrm{Cor}_{H_{p^{n+1}}/H_{p^n}}(\mathfrak{X}_{p^{n+1}}) = U_p \cdot \mathfrak{X}_{p^n}$, and hence the classes

$$\mathfrak{Z}_n := U_p^{-n} \cdot \mathrm{Cor}_{H_{p^{n+1}}/K_n}(\mathfrak{X}_{p^{n+1}}) \tag{3.4}$$

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are compatible under corestriction. Denoting by τ the image of a class under the action of complex conjugation and using Proposition 2.3.5, we find that

$$\begin{aligned}\mathrm{Cor}_{H_{p^{n+1}}/K_n}(\mathfrak{X}_{p^{n+1}})^{\tau} &= \sum_{\sigma\in\mathrm{Gal}(H_{p^{n+1}}/K_n)} \mathfrak{X}_{p^{n+1}}^{\tau\sigma} \\ &= \sum_{\sigma\in\mathrm{Gal}(H_{p^{n+1}}/K_n)} \mathfrak{X}_{p^{n+1}}^{\sigma^{-1}\tau} \\ &= w\cdot \mathrm{Cor}_{H_{p^{n+1}}/K_n}(\mathfrak{X}_{p^{n+1}}) \qquad (3.5)\end{aligned}$$

for some $w \in \{\pm 1\}$. Combining (3.4) and (3.5), the result follows. □

3.2. Two-variable p-adic L-functions

As in the preceding section, let $f \in S_2(\Gamma_0(Np))$ be a newform split multiplicative at p, and let $\mathbf{f} \in \mathbb{I}[\![q]\!]$ be the Hida family passing through f. Recall the spaces of characters $\Sigma^{\pm}_{k,c}$ and $\hat{\Sigma}_{k,c}$ introduced in § 2.5. In the following, we only consider the case $c = 1$, which will henceforth be neglected from the notation.

By [1, Proposition 5.10] (see also Theorem 2.10), for every $\nu \in \mathcal{X}^a_{\mathcal{O}_L}(\mathbb{I})$ the assignment

$$\chi \longmapsto L_{\mathfrak{p}}(f_\nu)(\chi) := \sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_K)} \chi^{-1}(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{-j}\cdot d^j f_\nu^{[p]}(\mathfrak{a} * (A_0, t_0, \omega_{\mathrm{can}}))$$

extends to a continuous function on $\hat{\Sigma}_{k_\nu}$. Using the explicit expression for these values, it is easy to show the existence of a two-variable p-adic L-function interpolating $L_{\mathfrak{p}}(f_\nu)$ for varying ν. For the precise statement, denote by $h = h_K$ the class number of K (which we assume is prime to p), and let ϕ_o be the unramified Hecke character defined on fractional ideals by the rule

$$\phi_o(\mathfrak{a}) = \alpha/\overline{\alpha}, \quad \text{where } (\alpha) = \mathfrak{a}^h. \qquad (3.6)$$

Assume that $\mathcal{O}_L$ contains the values of ϕ_o, and denote by $\langle\phi_o\rangle$ the composition of ϕ_o with the projection onto the $\mathbf{Z}_p$-free quotient of $\mathcal{O}_L^\times$, which then is valued in $1 + p\mathbf{Z}_p$, and define $\xi : K^\times\backslash\mathbb{A}_K^\times \longrightarrow \mathbb{I}^\times$ by

$$\xi : K^\times\backslash\mathbb{A}_K^\times \xrightarrow{\phi_o} \mathcal{O}_L^\times \xrightarrow{\langle\cdot\rangle} 1 + p\mathbf{Z}_p \xrightarrow{\gamma\mapsto\gamma^{1/2h}} 1 + p\mathbf{Z}_p \longrightarrow \Lambda_{\mathcal{O}_L}^\times \longrightarrow \mathbb{I}^\times. \qquad (3.7)$$

Similarly, recall the critical character $\Theta : G_{\mathbf{Q}} \longrightarrow \mathbb{I}^\times$ from (3.3), and define $\chi : K^\times\backslash\mathbb{A}_K^\times \longrightarrow \mathbb{I}^\times$ by

$$\chi(x) = \Theta(\mathrm{rec}_{\mathbf{Q}}(\mathrm{N}_{K/\mathbf{Q}}(x))),$$

where $\mathrm{rec}_{\mathbf{Q}} : \mathbb{A}_{\mathbf{Q}}^\times \longrightarrow \mathrm{Gal}(\mathbf{Q}^{\mathrm{ab}}/\mathbf{Q})$ is the *geometrically* normalized global reciprocity map. Let $\Gamma_\infty := \mathrm{Gal}(K_\infty/K)$ be the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension of K, and denote by $\mathcal{X}^a_{\mathcal{O}_L}(\Gamma_\infty)$ the set of continuous $\mathcal{O}_L$-algebra homomorphisms $\mathcal{O}_L[\![\Gamma_\infty]\!] \longrightarrow \mathbf{Q}_p^\times$ induced by a character ϕ of the form $\phi = \phi_o^{\ell_\phi/h}$ for some integer $\ell_\phi \geqslant 0$ with $\ell_\phi \equiv 0 \pmod{p-1}$. Finally, let

$$\mathbf{N}_K : K^\times\backslash\mathbb{A}_K^\times \xrightarrow{\mathrm{N}_{K/\mathbf{Q}}} \mathbf{Q}^\times\backslash\mathbb{A}_{\mathbf{Q}}^\times \xrightarrow{\mathbf{N}} \mathbf{C}^\times$$

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be the norm character of K, and for every $\nu \in \mathcal{X}^a_{\mathcal{O}_L}(\mathbb{I})$, let ξ_ν and χ_ν be the composition of ξ and χ with ν, respectively.

Theorem 3.4. *The exists a continuous function $L_{\mathfrak{p},\xi}(\mathbf{f})$ on $\mathcal{X}^a_{\mathcal{O}_L}(\mathbb{I}) \times \mathcal{X}^a_{\mathcal{O}_L}(\Gamma_\infty)$ such that for every $\nu \in \mathcal{X}^a_{\mathcal{O}_L}(\mathbb{I})$ we have*

$$L_{\mathfrak{p},\xi}(\mathbf{f})(\nu, \phi) = L_{\mathfrak{p}}(f_\nu)(\phi\xi_\nu\chi_\nu\mathbf{N}_K)$$

as functions of $\phi \in \mathcal{X}^a_{\mathcal{O}_L}(\Gamma_\infty)$.

Proof. See [3, Theorem 1.4]. (Note that if $(\nu, \phi) \in \mathcal{X}^a_{\mathcal{O}_L}(\mathbb{I}) \times \mathcal{X}^a_{\mathcal{O}_L}(\Gamma_\infty)$, then $\phi\xi_\nu\chi_\nu\mathbf{N}_K$ is an unramified Hecke character of infinity type $(k_\nu + \ell_\phi - 1, -(\ell_\phi - 1))$, thus lying in the domain of $L_{\mathfrak{p}}(f_\nu)$.) □

3.3. Ochiai's big logarithm maps

By our assumption that $p \nmid h = h_K$, the extension K_∞/K is totally ramified at every prime $\mathfrak{q}$ above p; let $K_{\infty,\mathfrak{q}}$ be the completion of K_∞ at the unique prime above $\mathfrak{q}$, and set $\Gamma_{\mathfrak{q},\infty} = \mathrm{Gal}(K_{\infty,\mathfrak{q}}/K_{\mathfrak{q}})$. Even though $\Gamma_{\mathfrak{q},\infty}$ may be identified with Γ_∞, in the following it will be convenient to maintain the distinction between them. Write $\mathfrak{q}^h = \pi_{\mathfrak{q}}\mathcal{O}_K$, and set $\varpi_{\mathfrak{q}} = \pi_{\mathfrak{q}}/\pi_{\overline{\mathfrak{q}}} \in K_{\mathfrak{q}}^\times$; in particular, note that $\varpi_{\overline{\mathfrak{p}}} = \varpi_{\mathfrak{p}}^{-1}$.

Recall the $\mathbb{I}$-adic Hecke character introduced in (3.7), and let $\xi : G_K \longrightarrow \mathbb{I}^\times$ also denote the Galois character defined by

$$\xi(\sigma) := [\langle\hat{\phi}_o(\sigma)\rangle^{1/2h}],$$

where $\hat{\phi}_o : G_K \longrightarrow \mathcal{O}_L^\times$ is the p-adic avatar of the Hecke character ϕ_o in (3.6). Finally, set

$$\mathbb{T}_{\mathfrak{p}} := \mathbf{T}^\dagger|_{G_K} \otimes \xi^{-1}, \quad \mathbb{T}_{\overline{\mathfrak{p}}} := \mathbf{T}^\dagger|_{G_K} \otimes \xi,$$

and for every $\nu \in \mathcal{X}^a_{\mathcal{O}_L}(\mathbb{I})$ denote by V_ν the specialization of $\mathbb{T}_{\mathfrak{q}}$ at ν (it will be clear from the context which prime $\mathfrak{q}$ above p is meant).

Theorem 3.5. *Let $\lambda = \mathbf{a}_p - 1$ and set $\tilde{\mathbb{I}} := \mathbb{I}[\lambda^{-1}] \otimes_{\mathbf{Z}_p} \hat{\mathbf{Z}}_p^{\mathrm{nr}}$. For each $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, there exists an $\mathbb{I}[\![\Gamma_{\mathfrak{q},\infty}]\!]$-linear map*

$$\mathcal{L}^{\omega}_{\mathscr{F}^+\mathbb{T}_{\mathfrak{q}}} : H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{q}}, \mathscr{F}^+\mathbb{T}_{\mathfrak{q}}) \longrightarrow \tilde{\mathbb{I}}[\![\Gamma_{\mathfrak{q},\infty}]\!]$$

such that for every $\mathfrak{Y}_\infty \in H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{q}}, \mathscr{F}^+\mathbb{T}_{\mathfrak{q}})$ and every $(\nu, \phi) \in \mathcal{X}^a_{\mathcal{O}_L}(\tilde{\mathbb{I}}) \times \mathcal{X}^a_{\mathcal{O}_L}(\Gamma_{\mathfrak{q},\infty})$ we have

$$\left(1 - \frac{(p\varpi_{\overline{\mathfrak{q}}}^{1/h})^{k_\nu/2-1}}{\nu(\mathbf{a}_p)\varpi_{\overline{\mathfrak{q}}}^{\ell_\phi/h}}\right)\mathcal{L}^{\omega}_{\mathscr{F}^+\mathbb{T}_{\mathfrak{q}}}(\mathfrak{Y}_\infty)(\nu, \phi) = \ell_\phi!^{-1}\left(1 - \frac{\nu(\mathbf{a}_p)\varpi_{\overline{\mathfrak{q}}}^{\ell_\phi/h}}{p(p\varpi_{\overline{\mathfrak{q}}}^{1/h})^{k_\nu/2-1}}\right)\langle\log(\nu(\mathfrak{Y}_\infty)^\phi), \breve{\omega}_\nu\rangle,$$

where $\log = \log_{\mathscr{F}^+V_\nu\otimes\phi} : H^1(K_{\mathfrak{q}}, \mathscr{F}^+V_\nu \otimes \phi) \longrightarrow D_{\mathrm{dR}}(\mathscr{F}^+V_\nu \otimes \phi)$ *is the Bloch–Kato logarithm map, and $\nu(\mathfrak{Y}_\infty)^\phi \in H^1(K_{\mathfrak{q}}, \mathscr{F}^+V_\nu \otimes \phi)$ is the ϕ-specialization of $\nu(\mathfrak{Y}_\infty)$.*

Proof. See [3, Proposition 4.3]. □

Remark 3.6. Fix a compatible system $\zeta_\infty = \{\zeta_r\}_{r\geqslant 0}$ of p-power roots of unity, and let $\zeta_\infty t^{-1}$ be the associated basis element of $D_{\mathrm{dR}}(\mathbf{Q}_p(1))$. In Theorem 3.5, ω denotes a generator of the module

$$\mathbb{D} := (\mathscr{F}^+\mathbb{T}_{\mathfrak{q}}(-1)\hat{\otimes}_{\mathbf{Z}_p}\hat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{K_{\mathfrak{q}}}},$$

which by [25, Lemma 3.3] is free of rank one over $\mathbb{I}$. (Note that $\mathscr{F}^+\mathbb{T}_{\mathfrak{q}}(-1)$ is unramified.) As explained in [25, Lemma 3.3], for each $\nu \in \mathcal{X}^a_{\mathcal{O}_L}(\mathbb{I})$ there is a specialization map

$$\nu_* : \mathbb{D} \longrightarrow \mathbb{D}_\nu \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \simeq D_{\mathrm{dR}}(\mathscr{F}^+V_\nu(-1)).$$

Then, letting ω_ν denote the image of $\nu_*(\omega)\otimes\zeta_\infty t^{-1}$ in $D_{\mathrm{dR}}(\mathscr{F}^+V_\nu(-1))\otimes D_{\mathrm{dR}}(\mathbf{Q}_p(1)) \simeq D_{\mathrm{dR}}(\mathscr{F}^+V_\nu)$, the class $\breve{\omega}_\nu \in D_{\mathrm{dR}}(\mathscr{F}^-V_\nu^*(1))$ in the above interpolation formulas is defined by requiring that

$$\langle \omega_\nu, \breve{\omega}_\nu\rangle = 1$$

under the de Rham pairing $\langle\,,\rangle : D_{\mathrm{dR}}(\mathscr{F}^+V_\nu)\times D_{\mathrm{dR}}(\mathscr{F}^-V_\nu^*(1)) \longrightarrow F_\nu$.

The big logarithm map $\mathcal{L}^\omega_{\mathscr{F}^+\mathbb{T}_{\mathfrak{q}}}$ of Theorem 3.5 may not be specialized at any pair $(\nu, \mathbb{1})$ with $\nu \in \mathcal{X}^a_{\mathcal{O}_L}(\mathbb{I})$ such that $\nu(\lambda) = 0$, i.e., $\nu(\mathbf{a}_p) = 1$. Since such arithmetic primes are in fact the main concern in this paper, the following construction of an 'improved' big logarithm will be useful.

Proposition 3.7. *There exists an* $\mathbb{I}$*-linear map*

$$\widetilde{\mathcal{L}}^\omega_{\mathscr{F}^+\mathbb{T}_{\mathfrak{p}}} : H^1(K_{\mathfrak{p}}, \mathscr{F}^+\mathbb{T}_{\mathfrak{p}}) \longrightarrow \mathbb{I}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$$

such that for every $\mathfrak{Y}_0 \in H^1(K_{\mathfrak{p}}, \mathscr{F}^+\mathbb{T}_{\mathfrak{p}})$ *and every* $\nu \in \mathcal{X}^a_{\mathcal{O}_L}(\mathbb{I})$, *we have*

$$\nu\left(\widetilde{\mathcal{L}}^\omega_{\mathscr{F}^+\mathbb{T}_{\mathfrak{p}}}(\mathfrak{Y}_0)\right) = \left(1 - \frac{\nu(\mathbf{a}_p)p^{-1}}{(p\varpi_{\bar{\mathfrak{p}}}^{1/h})^{k_\nu/2-1}}\right)\langle \log_{\mathscr{F}^+V_\nu}(\nu(\mathfrak{Y}_0)), \breve{\omega}_\nu\rangle.$$

Proof. This can be shown by adapting the methods of Ochiai [25, § 5]. Indeed, let

$$\mathcal{L}_{\mathscr{F}^+\mathbb{T}_{\mathfrak{p}}} : H^1(K_{\mathfrak{p}}, \mathscr{F}^+\mathbb{T}_{\mathfrak{p}})\otimes\mathbf{Q}_p \longrightarrow \mathbb{D}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$$

be the inverse of the map $\exp_{\mathbb{T}}$ constructed in [31, Proposition 3.8] (see Remark 3.6 for the definition of $\mathbb{D}$), and define

$$\mathcal{L}^\omega_{\mathscr{F}^+\mathbb{T}_{\mathfrak{p}}} : H^1(K_{\mathfrak{p}}, \mathscr{F}^+\mathbb{T}_{\mathfrak{p}}) \longrightarrow \mathbb{I}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$$

by the relation $\mathcal{L}_{\mathscr{F}^+\mathbb{T}_{\mathfrak{p}}}(-) = \mathcal{L}^\omega_{\mathscr{F}^+\mathbb{T}_{\mathfrak{p}}}(-)\cdot\omega$. Setting

$$\widetilde{\mathcal{L}}^\omega_{\mathscr{F}^+\mathbb{T}_{\mathfrak{p}}} = \left(1 - \frac{\mathbf{a}_p p^{-1}}{\Theta^{-1}\xi(\mathrm{Fr}_{\mathfrak{p}})}\right)\mathcal{L}^\omega_{\mathscr{F}^+\mathbb{T}_{\mathfrak{p}}} : H^1(K_{\mathfrak{p}}, \mathscr{F}^+\mathbb{T}_{\mathfrak{p}}) \longrightarrow \mathbb{I}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p,$$

the result follows. □

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Corollary 3.8. *For any* $\mathfrak{Y}_\infty = \{\mathfrak{Y}_n\}_{n\geqslant 0} \in H^1_{\rm Iw}(K_{\infty,\mathfrak{p}}, \mathscr{F}^+\mathbb{T}_\mathfrak{p})$*, we have the factorization in* $\tilde{\mathbb{I}}$*:*

$$\left(1 - \frac{\Theta^{-1}\xi\,(\mathrm{Fr}_\mathfrak{p})}{\mathbf{a}_p}\right)\cdot \varepsilon\left(\mathcal{L}^\omega_{\mathscr{F}^+\mathbb{T}_\mathfrak{p}}(\mathfrak{Y}_\infty)\right) = \widetilde{\mathcal{L}}^\omega_{\mathscr{F}^+\mathbb{T}_\mathfrak{p}}(\mathfrak{Y}_0),$$

where $\varepsilon : \tilde{\mathbb{I}}[\![\Gamma_\infty]\!] \longrightarrow \tilde{\mathbb{I}}$ *is the augmentation map.*

Proof. Comparing the interpolation formulas in Theorem 3.5 and Proposition 3.7, we see that

$$\left(1 - \frac{\Theta_\nu^{-1}\xi_\nu(\mathrm{Fr}_\mathfrak{p})}{\nu(\mathbf{a}_p)}\right)\mathcal{L}^\omega_{\mathscr{F}^+\mathbb{T}_\mathfrak{p}}(\mathfrak{Y}_\infty)(\nu, \mathbb{1}) = \nu\left(\widetilde{\mathcal{L}}^\omega_{\mathscr{F}^+\mathbb{T}_\mathfrak{p}}(\mathfrak{Y}_0)\right)$$

for every $\nu \in \mathcal{X}^a_{\mathcal{O}_L}(\tilde{\mathbb{I}})$; since these primes are dense in $\tilde{\mathbb{I}}$, the corollary follows. □

The proof of our main result will rely crucially on the relation found in [3, § 4] between the p-adic L-function $L_{\mathfrak{p},\xi}(\mathbf{f})$ of Theorem 3.4 and Howard's system of big Heegner points $\mathfrak{Z}_\infty$. We conclude this section by briefly recalling that relation.

By [15, Lemma 2.4.4], for every prime $\mathfrak{q}$ of K above p the natural map

$$H^1_{\rm Iw}(K_{\infty,\mathfrak{q}}, \mathscr{F}^+\mathbf{T}^\dagger) \longrightarrow H^1_{\rm Iw}(K_{\infty,\mathfrak{q}}, \mathbf{T}^\dagger)$$

induced by (3.2) is injective. In light of Theorem 3.3, in the following we will thus view $\mathrm{loc}_\mathfrak{q}(\mathfrak{Z}_\infty)$ as residing inside $H^1_{\rm Iw}(K_{\infty,\mathfrak{q}}, \mathscr{F}^+\mathbf{T}^\dagger)$.

Theorem 3.9. *There is a generator* $\omega = \omega_\mathbf{f}$ *of the module* $\mathbb{D}$ *such that*

$$\mathcal{L}^\omega_{\mathscr{F}^+\mathbb{T}_\mathfrak{p}}(\mathrm{loc}_\mathfrak{p}(\mathfrak{Z}_\infty^{\xi^{-1}})) = L_{\mathfrak{p},\xi}(\mathbf{f})$$

as functions on $\mathcal{X}^a_{\mathcal{O}_L}(\tilde{\mathbb{I}}) \times \mathcal{X}^a_{\mathcal{O}_L}(\Gamma_\infty)$.

Proof. The construction of the basis element $\omega = \omega_\mathbf{f}$ of $\mathbb{D}$ is deduced in [19, Proposition 10.1.2] from Ohta's work [27], and it has the property that $\langle \omega_\nu, \omega_{\mathbf{f}_\nu}\rangle = 1$, for all $\nu \in \mathcal{X}^a_{\mathcal{O}_L}(\mathbb{I})$, where $\omega_{\mathbf{f}_\nu}$ is the class in $\mathrm{Fil}^1 D_{\rm dR}(V_\nu^*) \simeq D_{\rm dR}(\mathscr{F}^- V_\nu^*(1))$ associated with the p-stabilized newform (3.1); in particular,

$$\breve{\omega}_{\nu_f} = \omega_f$$

in the notation of Remark 3.6. The result is then the content of [3, Theorem 4.4]. □

3.4. Exceptional zero formula

Let $f = \sum_{n=1}^\infty a_n(f)q^n \in S_2(\Gamma_0(Np))$ be an ordinary newform as in § 3.1, and assume in addition that f is *split multiplicative* at p, meaning that $a_p(f) = 1$. Recall the CM triple $(A, t_A, \alpha_\mathfrak{p}) \in X(H)$ introduced in § 1.5, which maps to the point $P_A = (A, A[\mathfrak{N}\mathfrak{p}]) \in X_0(Np)$ under the forgetful map $X \longrightarrow X_0(Np)$. Let ∞ be any cusp of $X_0(Np)$ rational over $\mathbf{Q}$, and let $\kappa_f \in H^1(K, V_f)$ be the image of $(P_A) - (\infty)$ under the composite map

$$J_0(Np)(H) \xrightarrow{\mathrm{Kum}} H^1(H, \mathrm{Ta}_p(J_0(Np)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p) \longrightarrow H^1(H, V_f) \xrightarrow{\mathrm{Cor}_{H/K}} H^1(K, V_f). \quad (3.8)$$

If $\mathbf{f} \in \mathbb{I}[[q]]$ is the Hida family passing through f, and $\nu_f \in \mathcal{X}^a_{\mathcal{O}_L}(\mathbb{I})$ is the arithmetic prime of $\mathbb{I}$ such that $\nu_f(\mathbf{f}) = f$, it would be natural to expect a relation between the class κ_f and the specialization at ν_f of Howard's big Heegner point $\mathfrak{Z}_0$. As done in [14, § 3], one can trace through the construction of $\mathfrak{Z}_0$ to deduce a relation between the *generic* (in the sense of [14, Definition 2]) weight 2 specializations of $\mathfrak{Z}_0$ and the Kummer images of certain CM points. However, the arithmetic prime ν_f is not generic in that sense, and in fact one does not expect a similar direct relation between $\nu_f(\mathfrak{Z}_0)$ and κ_f (see the discussion in [14, p. 813]).

In Theorem 3.11 we will show that in fact the localization at $\mathfrak{p}$ of $\nu_f(\mathfrak{Z}_0)$ vanishes, but that nonetheless it can be related to κ_f upon taking a certain 'derivative' in the following sense, where we let $\log_p : \mathbf{Q}_p^\times \longrightarrow \mathbf{Z}_p$ be Iwasawa's branch of the p-adic logarithm, i.e., such that $\log_p(p) = 0$.

Lemma 3.10. *Let T be a free $\mathcal{O}_L$-module of finite rank equipped with a linear action of $G_{\mathbf{Q}_p}$, let $k_\infty/\mathbf{Q}_p$ be a $\mathbf{Z}_p$-extension, and let $\gamma \in \mathrm{Gal}(k_\infty/\mathbf{Q}_p)$ be a topological generator. Assume that $T^{G_{k_\infty}} = \{0\}$, and let $\mathcal{Z}_\infty = \{\mathcal{Z}_n\}_{n\geqslant 0} \in H^1_{\mathrm{Iw}}(k_\infty, T)$ be such that $\mathcal{Z}_0 = 0$. Then there exists a unique $\mathcal{Z}'_{\gamma,\infty} = \{\mathcal{Z}'_{\gamma,n}\}_{n\geqslant 0} \in H^1_{\mathrm{Iw}}(k_\infty, T)$ such that*

$$\mathcal{Z}_\infty = (\gamma - 1)\cdot \mathcal{Z}'_{\gamma,\infty}.$$

Moreover, if $\eta : \mathrm{Gal}(k_\infty/\mathbf{Q}_p) \simeq \mathbf{Z}_p$ is any group isomorphism, then

$$\mathcal{Z}'_0 := \frac{\mathcal{Z}'_{\gamma,0}}{\log_p(\eta(\gamma))} \in H^1(\mathbf{Q}_p, T[1/p])$$

is independent of the choice of γ.

Proof. Consider the module $T_\infty := T\hat{\otimes}_{\mathcal{O}_L}\mathcal{O}_L[[\mathrm{Gal}(k_\infty/\mathbf{Q}_p)]]$ equipped with the diagonal Galois action, where $G_{\mathbf{Q}_p}$ acts on the second factor via the projection $G_{\mathbf{Q}_p} \longrightarrow \mathrm{Gal}(k_\infty/\mathbf{Q}_p)$. By Shapiro's lemma, we then have

$$H^1(\mathbf{Q}_p, T_\infty) \simeq H^1_{\mathrm{Iw}}(k_\infty, T),$$

and the assumption that $T^{G_{k_\infty}} = \{0\}$ implies that $H^1(\mathbf{Q}_p, T_\infty)$ is torsion-free. Therefore, the exact sequence of $\mathcal{O}_L[[\mathrm{Gal}(k_\infty/\mathbf{Q}_p)]]$-modules

$$0 \longrightarrow T_\infty \xrightarrow{\gamma-1} T_\infty \longrightarrow T \longrightarrow 0$$

induces the cohomology exact sequence

$$0 \longrightarrow H^1(\mathbf{Q}_p, T_\infty) \xrightarrow{\gamma-1} H^1(\mathbf{Q}_p, T_\infty) \longrightarrow H^1(\mathbf{Q}_p, T),$$

giving the proof of the first claim, and the second follows from an immediate calculation. □

Recall the uniformizer $\varpi_{\mathfrak{p}} = \pi_{\mathfrak{p}}/\pi_{\overline{\mathfrak{p}}} \in K_{\mathfrak{p}}^\times \simeq \mathbf{Q}_p^\times$ introduced in § 3.3, and define

$$\mathscr{L}_{\mathfrak{p}}(f, K) := \mathscr{L}_p(f) - \mathscr{L}_{\mathfrak{p}}(\chi_K), \tag{3.9}$$

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where $\mathscr{L}_p(f)$ is the $\mathscr{L}$-invariant of f (as defined in [21, § II.14], for example), and

$$\mathscr{L}_{\mathfrak{p}}(\chi_K) := \frac{\log_p(\varpi_{\mathfrak{p}})}{\mathrm{ord}_p(\varpi_{\mathfrak{p}})} = -\frac{2\log_p(\pi_{\overline{\mathfrak{p}}})}{h}$$

is the $\mathscr{L}$-invariant of the quadratic character χ_K associated with K (see [10, § 1], for example), with ord_p the p-adic valuation on $\mathbf{Q}_p$ with the normalization $\mathrm{ord}_p(p) = 1$.

The following derivative formula is the main result of this paper.

Theorem 3.11. *Let $f \in S_2(\Gamma_0(Np))$ be a newform split multiplicative at p, let $\mathbf{f} \in \mathbb{I}[\![q]\!]$ be the Hida family passing through f, let $\mathfrak{Z}_\infty \in H^1_{\mathrm{Iw}}(K_\infty, \mathbf{T}^\dagger)$ be Howard's system of big Heegner points, and define $\mathcal{Z}_{\mathfrak{p},f,\infty} := \{\mathcal{Z}_{\mathfrak{p},f,n}\}_{n\geqslant 0} \in H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}}, \mathscr{F}^+V_f)$ by*

$$\mathcal{Z}_{\mathfrak{p},f,n} := \mathrm{loc}_{\mathfrak{p}}(\nu_f(\mathfrak{Z}_n)),$$

where $\nu_f \in \mathcal{X}^a_{\mathcal{O}_L}(\mathbb{I})$ is such that $f = \nu_f(\mathbf{f})$. Then $\mathcal{Z}_{\mathfrak{p},f,0} = 0$ and

$$\mathcal{Z}'_{\mathfrak{p},f,0} = \mathscr{L}_{\mathfrak{p}}(f, K) \cdot \mathrm{loc}_{\mathfrak{p}}(\kappa_f), \tag{3.10}$$

where $\mathscr{L}_{\mathfrak{p}}(f, K)$ is the $\mathscr{L}$-invariant (0.4), and $\kappa_f \in H^1(K, V_f)$ is the image of the degree zero divisor $(A, A[\mathfrak{N}\mathfrak{p}]) - (\infty)$ under the Kummer map (3.8).

Proof. By Proposition 3.7, Corollary 3.8, and Theorems 3.9 and 3.4, respectively, we see that

$$\begin{aligned}\left(1 - a_p(f)p^{-1}\right)\cdot\langle\log(\mathcal{Z}_{\mathfrak{p},f,0}), \omega_f\rangle &= \lim_{\nu\to\nu_f} \nu\left(\widetilde{\mathcal{L}}^{\omega}_{\mathscr{F}^+\mathbb{T}}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_0^{\xi^{-1}}))\right)\\ &= \lim_{\nu\to\nu_f}\left(1 - \frac{\Theta_\nu^{-1}\xi_\nu(\mathrm{Fr}_{\mathfrak{q}})}{\nu(\mathbf{a}_p)}\right)\mathcal{L}^{\omega}_{\mathscr{F}^+\mathbb{T}}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\infty^{\xi^{-1}}))(\nu, \mathbb{1})\\ &= \lim_{\nu\to\nu_f}\left(1 - \frac{\Theta_\nu^{-1}\xi_\nu(\mathrm{Fr}_{\mathfrak{q}})}{\nu(\mathbf{a}_p)}\right)L_{\mathfrak{p},\xi}(\mathbf{f})(\nu, \mathbb{1})\\ &= \left(1 - a_p(f)^{-1}\right)\cdot L_{\mathfrak{p}}(f, \mathbf{N}_K).\end{aligned}$$

Since $a_p(f) = 1$ by hypothesis, this shows that $\langle\log(\mathcal{Z}_{\mathfrak{p},f,0}), \omega_f\rangle = 0$, and the vanishing of $\mathcal{Z}_{\mathfrak{p},f,0}$ follows. Now we turn to the proof of the derivative formula (3.10).

Denote by $L_{\mathfrak{p},\xi}(\mathbf{f})^\iota$ the image of $L_{\mathfrak{p},\xi}(\mathbf{f})$ under the involution of $\tilde{\mathbb{I}}[\![\Gamma_\infty]\!]$ induced by complex conjugation, so that $L_{\mathfrak{p},\xi}(\mathbf{f})^\iota(\chi) = L_{\mathfrak{p},\xi}(\mathbf{f})(\chi^{-1})$ for every character χ of Γ_∞. One immediately checks the commutativity of the diagram

$$\begin{array}{ccccc} H^1_{\mathrm{Iw}}(K_\infty, \mathscr{F}^+\mathbb{T}_{\mathfrak{p}}) & \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} & H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}}, \mathscr{F}^+\mathbb{T}_{\mathfrak{p}}) & \xrightarrow{\mathcal{L}^{\omega}_{\mathscr{F}^+\mathbb{T}_{\mathfrak{p}}}} & \tilde{\mathbb{I}}[\![\Gamma_{\mathfrak{p},\infty}]\!] \\ \downarrow * & & \downarrow * & & \downarrow \iota \\ H^1_{\mathrm{Iw}}(K_\infty, \mathscr{F}^+\mathbb{T}_{\overline{\mathfrak{p}}}) & \xrightarrow{\mathrm{loc}_{\overline{\mathfrak{p}}}} & H^1_{\mathrm{Iw}}(K_{\infty,\overline{\mathfrak{p}}}, \mathscr{F}^+\mathbb{T}_{\overline{\mathfrak{p}}}) & \xrightarrow{\mathcal{L}^{\omega}_{\mathscr{F}^+\mathbb{T}_{\overline{\mathfrak{p}}}}} & \tilde{\mathbb{I}}[\![\Gamma_{\overline{\mathfrak{p}},\infty}]\!], \end{array}$$

where the left and middle vertical arrows denote the action of complex conjugation.

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From the discussion in [11, § 2.6], we may find a disc $U \subset \mathbf{Z}/2(p-1)\mathbf{Z} \times \mathbf{Z}_p$ contained in the residue class of 2, and a unique morphism of $\Lambda_{\mathcal{O}_L}$-modules

$$\mathscr{M} = \mathscr{M}_f : \mathbb{I} \longrightarrow \mathscr{A}_U$$

such that $\mathscr{M}(r)|_{k=2} = \nu_f(r)$ for every $r \in \mathbb{I}$, where $\mathscr{A}_U \subset L[\![k-2]\!]$ denotes the subring of power series convergent for $k \in U$ endowed with the $\Lambda_{\mathcal{O}_L}$-algebra structure induced by the character $1 + p\mathbf{Z}_p \longrightarrow \mathscr{A}_U^\times$, which sends $\gamma \in 1 + p\mathbf{Z}_p$ to the power series $\gamma^{k-2} := \exp((k-2)\log_p(\gamma))$.

For every $k \in U \cap \mathbf{Z}_{\geqslant 2}$ the composition of $\mathscr{M}$ with the evaluation map at k defines an element $\nu_k \in \mathcal{X}^a_{\mathcal{O}_L}(\mathbb{I})$. Set $\mathbf{a}_p(k) := \mathscr{M}(\mathbf{a}_p)$, and for $(k, \phi_o^{t/h}) \in (U \cap \mathbf{Z}_{\geqslant 2}) \times \mathcal{X}^a_{\mathcal{O}_L}(\Gamma_\infty)$, define the functions

$$\mathcal{L}_{\mathfrak{p}}(k,t) := \left(1 - \frac{(p\varpi_{\overline{\mathfrak{p}}}^{1/h})^{k/2-1}}{\mathbf{a}_p(k)\varpi_{\overline{\mathfrak{p}}}^{t/h}}\right) L_{\mathfrak{p},\xi}(\mathbf{f})(\nu_k, \phi_o^{t/h}),$$

$$\mathcal{L}_{\overline{\mathfrak{p}}}(k,t) := \left(1 - \frac{(p\varpi_{\mathfrak{p}}^{1/h})^{k/2-1}}{\mathbf{a}_p(k)\varpi_{\mathfrak{p}}^{t/h}}\right) L_{\mathfrak{p},\xi}(\mathbf{f})(\nu_k, \phi_o^{-t/h}).$$

By the combination of Theorems 3.5 and 3.9, we then have

$$\mathcal{L}_{\mathfrak{p}}(k,t) = \frac{1}{t!}\left(1 - \frac{\mathbf{a}_p(k)\varpi_{\overline{\mathfrak{p}}}^{t/h}}{p(p\varpi_{\overline{\mathfrak{p}}}^{1/h})^{k/2-1}}\right)\langle \log(\mathrm{loc}_{\mathfrak{p}}(\nu_k(\mathfrak{Z}_\infty)^{\phi_o^{(1-k/2+t)/h}})), \breve{\omega}_{\nu_k}\rangle,$$

and by the above diagram we also have

$$\mathcal{L}_{\overline{\mathfrak{p}}}(k,t) = \frac{1}{t!}\left(1 - \frac{\mathbf{a}_p(k)\varpi_{\mathfrak{p}}^{t/h}}{p(p\varpi_{\mathfrak{p}}^{1/h})^{k/2-1}}\right)\langle \log(\mathrm{loc}_{\overline{\mathfrak{p}}}(\nu_k(\mathfrak{Z}_\infty^*)^{\phi_o^{(k/2-1-t)/h}})), \breve{\omega}_{\nu_k}\rangle.$$

By the 'functional equation' satisfied by $\mathfrak{Z}_\infty$ (see Theorem 3.3), it follows that the function

$$\mathcal{L}_p(k,t) := \mathcal{L}_{\mathfrak{p}}(k,t) - w\mathcal{L}_{\overline{\mathfrak{p}}}(k, k-2-t)$$

vanishes identically along the 'line' $t = k/2-1$. By [15, Proposition 2.3.6], the sign w is the *opposite* of the sign in the functional equation for the p-adic L-function $L_p(f,s)$ associated with f in [21]. Thus, if $w = 1$, then $\mathrm{ord}_{s=1} L_p(f,s) > 2$, and by [31, Lemma 6.1] the right-hand side of (3.10) vanishes; since the vanishing of the left-hand side follows easily from the construction of $\mathcal{Z}'_{\gamma,\infty}$ in Lemma 3.10, we conclude that (3.10) reduces to the identity '$0 = 0$' when $w = 1$. As a consequence, in the following we shall assume that $w = -1$.

Using the formula for the $\mathscr{L}$-invariant of f as the logarithmic derivative of $\mathbf{a}_p(k)$ at $k = 2$ (see [11, Theorem 3.18], for example) and noting that $(p\varpi_{\overline{\mathfrak{p}}}^{1/h})^{k/2-1} = \pi_{\overline{\mathfrak{p}}}^{(k-2)/h}$ by definition, we find

$$\frac{\partial}{\partial k}\mathcal{L}_p(k,t)\big|_{(2,0)} = \left[\frac{d}{dk}\mathbf{a}_p(k)\big|_{k=2} - \frac{\log_p(\pi_{\overline{\mathfrak{p}}})}{h} - w\left(\frac{d}{dk}\mathbf{a}_p(k)\big|_{k=2} - \frac{\log_p(\pi_{\overline{\mathfrak{p}}})}{h}\right)\right] L_{\mathfrak{p}}(f)(\mathbf{N}_K)$$
$$= -\left[\frac{(1-w)}{2}\left(\mathscr{L}_p(f) - \mathscr{L}_{\mathfrak{p}}(\chi_K)\right)\right] L_{\mathfrak{p}}(f)(\mathbf{N}_K)$$
$$= -\mathscr{L}_{\mathfrak{p}}(f,K)\cdot L_{\mathfrak{p}}(f)(\mathbf{N}_K). \tag{3.11}$$

Using the aforementioned vanishing of $\mathcal{L}_p(k, k/2-1)$ for the first equality, we also find

$$\frac{\partial}{\partial k}\mathcal{L}_p(k,t)\big|_{(2,0)} = -\frac{1}{2}\frac{\partial}{\partial t}\mathcal{L}_p(k,t)\big|_{(2,0)} = -\frac{(1-w)}{2}\left(1 - a_p(f)p^{-1}\right)\langle \log(\mathcal{Z}'_{\mathfrak{p},f,0}), \omega_f\rangle$$
$$= -(1-p^{-1})\cdot\langle \log(\mathcal{Z}'_{\mathfrak{p},f,0}), \omega_f\rangle, \tag{3.12}$$

and comparing (3.11) and (3.12), we arrive at the equality

$$(1-p^{-1})\cdot\langle \log(\mathcal{Z}'_{\mathfrak{p},f,0}), \omega_f\rangle = \mathscr{L}_{\mathfrak{p}}(f,K)\cdot L_{\mathfrak{p}}(f)(\mathbf{N}_K). \tag{3.13}$$

On the other hand, letting $\varphi_0 : A \longrightarrow A$ be the identity isogeny, by Theorem 2.11 we have

$$L_{\mathfrak{p}}(f)(\mathbf{N}_K) = (1 - a_p(f)p^{-1}) \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O}_K)} \langle \mathrm{AJ}_F(\Delta_{\varphi_{\mathfrak{a}}\varphi_0}), \omega_f\rangle$$
$$= (1-p^{-1})\cdot\langle \log(\mathrm{loc}_{\mathfrak{p}}(\kappa_f)), \omega_f\rangle,$$

which combined with (3.13) concludes the proof of Theorem 3.11. □

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