On a reaction–diffusion system modelling infectious diseases without lifetime immunity

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In this paper, we study a mathematical model for an infectious disease caused by a virus such as Cholera without lifetime immunity. Due to the different mobility for susceptible, infected human and recovered human hosts, the diffusion coefficients are assumed to be different. The resulting system is governed by a strongly coupled reaction–diffusion system with different diffusion coefficients. Global existence and uniqueness are established under certain assumptions on known data. Moreover, global asymptotic behaviour of the solution is obtained when some parameters satisfy certain conditions. These results extend the existing results in the literature. The main tool used in this paper comes from the delicate theory of elliptic and parabolic equations. Moreover, the energy method and Sobolev embedding are used in deriving *a priori* estimates. The analysis developed in this paper can be employed to study other epidemic models in biological, ecological and health sciences.

1 Introduction

In an ecological environment, various infectious diseases in human or animals occur frequently (see [26]). The current global COVID-19 pandemic is one such an example. Other recent examples include epidemics caused by HIV, Cholera and Zika viruses. Scientists often use the well-known Susceptibility–Infection–Recovery (SIR) model ([13]) to describe how a virus spreads and evolves in future. The SIR model and its various extensions have been studied by many scientists (see [1, 15, 17, 21, 24, 27, 31], e.g.). A good review for the model written by Hethcote in 2000 can be seen in [11]. The most of these studies focus on understanding the complicated dynamics of interaction between different hosts and viruses. However, this SIR model and its extensions cannot include the mobility of human hosts around different geographical regions. In order to take the movement of human hosts into consideration, one must develop a new mathematical model which can reflect these factors (see [9, 20, 37]). Towards this goal, considerable progress has been made for different types of infectious diseases. In particular, many researchers have studied the mathematical model for the epidemic caused by the Cholera virus (see [2, 17, 18, 27, 28, 31, 32, 33, 34], etc.).

In this paper, we consider the mathematical model for the cholera epidemic with diffusion processes. A novel feature for the model is that there is no lifetime immunity. This leads to a coupled reaction–diffusion system with different diffusion coefficients. We begin by describing the model system recently studied in [28, 32, 34, 35].

Let Ω be a bounded domain in R^n with C^2-boundary $\partial\Omega$. Let $Q_T = \Omega \times (0, T]$ for any $T > 0$. When $T = \infty$, we denote Q_∞ by Q. Let $S(x,t)$, $I(x,t)$ and $R(x,t)$ represent, respectively, susceptible, infected and recovered human hosts. Let $B(x,t)$ be the concentration of bacteria. Then, by the population growth and the conservation laws, we see that S, I, R and B satisfy the following reaction–diffusion system in Q_T (see [9, 11, 27]):

$$S_t - \nabla[d_1(x, t)\nabla S] = b(x, t) - \beta_1 SI - \beta_2 S \cdot h(B) - dS + \sigma R, \tag{1.1}$$

$$I_t - \nabla[d_2(x, t)\nabla I] = \beta_1 SI + \beta_2 S \cdot h(B) - (d + \gamma)I, \tag{1.2}$$

$$R_t - \nabla[d_3(x, t)\nabla R] = \gamma I - (d + \sigma)R, \tag{1.3}$$

$$B_t - \nabla[d_4(x, t)\nabla B] = \xi I + gB(1 - \frac{B}{K}) - \delta B \tag{1.4}$$

subject to the following initial and boundary conditions:

$$(\nabla_\nu S, \nabla_\nu I, \nabla_\nu R, \nabla_\nu B) = 0, \qquad (x, t) \in \partial\Omega \times (0, T], \tag{1.5}$$

$$(S(x, 0), I(x, 0), R(x, 0), B(x, 0)) = (S_0(x), I_0(x), R_0(x), B_0(x)), x \in \Omega, \tag{1.6}$$

where ν represents the outward unit normal on $\partial\Omega$, $h(B)$ is a differential function with

$$h(0) = 0, 0 \le h(B) \le 1.$$

A typical example used in [34] for $h(B)$ is

$$h(B) = \frac{B}{B + K}.$$

For reference, we list various parameters in the model as in [9, 27, 32, 33]:

$$d = \text{ the natural death rate}$$
$$\gamma = \text{ the recovery of infectious individuals}$$
$$b = \text{ the influx of susceptible host}$$
$$\sigma = \text{ the rate at which recovered individuals lose immunity}$$
$$\delta = \text{ the natural death rate of bacteria}$$
$$\xi = \text{ the shedding rate of bacteria by infectious human hosts,}$$
$$d_i = \text{ the diffusion coefficients, } i = 1, 2, 3, 4,$$
$$K = \text{ the maximum capacity of the bacteria}$$

All these parameters are positive. The interested readers may find the value range of those parameters from the website of the World Health Organization.

We would like to point out that the analysis for the reaction–diffusion system (1.1)–(1.4) is often difficult since there is no comparison principle ([19, 23]). Some basic questions such as global existence and uniqueness for the system are extremely challenging since equations (2.1) and (2.2) contain some quadratic terms in the system ([5, 7, 12]). Many of the quantitative properties are still open. For example, from the physical point of view, the concentration, $S(x, t)$, $I(x, t)$, $R(x, t)$ and $B(x, t)$ must be nonnegative and bounded in Q_T. However, it appears that there has been no rigorous proof in the previous research when the space dimension is greater than 1. Answering these open questions is one of the motivations for this paper. Moreover, with different diffusion coefficients in a reaction–diffusion system, the dynamics of a solution may be very different from that of an ODE system. The most striking example is the Turing phenomenon in which the solution of an ODE system is stable, while the solution of the corresponding reaction–diffusion system is unstable when one diffusion coefficient is much larger than the other ([10]).

When the space dimension is equal to 1, the authors of [18, 24, 32, 34, 35] studied the system (1.1)–(1.6). The global well-posedness is established for the model. Moreover, some global dynamical analysis for the solution is carried out in these papers. However, when the spatial dimension n is greater than 1, the analysis becomes much more complicated. More recently, by using a dual argument, Klements–Morgan–Tang ([14]) and Morgan–Tang ([22]) obtained some interesting results for a class of reaction–diffusion system. Particularly, a global bound of the solution can be obtained if the coefficients are constants. The purpose of this paper is to study the reaction–diffusion system (1.1)–(1.6) when the space dimension is greater than 1 and the diffusion coefficients in the model for susceptible, infected and recovered human hosts are different. Furthermore, the diffusion coefficients in our model allow to depend on space and time, which are certainly more realistic than in the previous model. This fact is observed from the recent COVID-19 pandemic in which the mobility for infected patients is close to zero due to required global quarantine and travel restriction. It is also clear that for flu-like viruses, the rate of infection is much higher in the winter than in the summer.

In this paper, we establish a global existence and uniqueness result for any space dimension n. Particularly, we prove that the weak solution is actually classical as long as the initial data and coefficients are smooth. Hence, global well-posedness for the model problem (1.1)–(1.6) is established without any restriction on space dimension. Moreover, under a condition on some parameters, we are able to prove that the solution is uniformly bounded and converges to the steady-state solution (global attractor, see [25]) for any space dimension as time evolves. These results improve the previous research obtained by others where the well-posedness is proved when the space dimension is equal to 1 (see [24, 33, 34, 35]). The main idea for establishing global existence is to derive various *a priori* estimates (see [37]). Our analysis in this paper uses a lot of very delicate results for elliptic and parabolic equations ([16, 19, 36]). Particularly, we use a subtle form of Campanato–John–Nirenberg–Morrey space to obtain the regularity of weak solution ([36]). A uniform bound in Q for the solution is even more subtle, where a special form of Gagliardo–Nirenberg's inequality is used. The global asymptotic analysis is based on accurate energy estimates for the solution of the system ([8]). The method developed in this paper can also be used to deal with different epidemic models caused by some viruses such as avian influenza for birds ([30]).

The paper is organised as follows. In Section Section 2, we first recall some basic function spaces and then state the main results. In Section Section 3, we prove the first part of the main results on global solvability of the system (1.1)–(1.6) (Theorems 2.1 and 2.2). The long-time

behaviour of the solution in the second part is proved in Section Section 4 (Theorems 2.3 and 2.4). In Section Section 5, we give some concluding remarks.

2 Notations and the statement of main results

For reader's convenience, we recall some basic Sobolev spaces which are standard in dealing with elliptic and parabolic partial differential equations.

Let $\alpha \in (0, 1)$. We denote by $C^{\alpha, \frac{\alpha}{2}}(\bar{Q}_T)$ the Hölder space in which every function is Hölder continuous with respect to (x,t) with exponent $(\alpha, \frac{\alpha}{2})$ in $\bar{Q}_T$.

Let $p > 1$ and V be a Banach space with norm $|| \cdot ||_v$, we denote

$$L^p(0, T; V) = \{F(t) : t \in [0, T] \to V : ||F||_{L^p(0,T;V)} < \infty\},$$

equipped with norm

$$||F||_{L^p(0,T;V)} = \left(\int_0^T ||F||_v^p dt \right)^{\frac{1}{p}}.$$

When $V = L^p(\Omega)$, we simply use

$$L^p(Q_T) = L^p(0, T; L^p(\Omega)).$$

Moreover, the $L^p(Q_T)$-norm is denoted by $|| \cdot ||_p$ and $|| \cdot ||_\infty$ when $p = \infty$ for simplicity.

Sobolev spaces $W_p^k(\Omega)$ and $W_p^{k,l}(Q_T)$ are defined the same as in the classical books such as [10] and [16].

Let $V_2(Q_T) = \{u(x, t) \in C([0, T]; W_2^1(\Omega)) : ||u||_{V_2} < \infty\}$ equipped with the norm

$$||u||_{V_2} = \sup_{0 \leq t \leq T} ||u||_{L^2(\Omega)} + \sum_{i=1}^n ||u_{x_i}||_2.$$

For any $\mu \in [0, n + 2)$, the Morrey–John–Nirenberg–Campanato space $L^{2,\mu}(Q_T)$ is defined as in [6]. Particularly, when $\mu = 0$,

$$L^{2,0}(Q_T) = L^2(Q_T),$$

and when $\mu \in (n, n + 2)$, $L^{2,\mu}(Q_T)$ is equivalent to the classical Hölder space $C^{\alpha, \frac{\alpha}{2}}(\bar{Q}_T)$ with

$$\alpha = \frac{(n + 2) - \mu}{2}.$$

We first state the basic assumptions for the diffusion coefficients and known data. All other parameters in equations (1.1)–(1.4) are assumed to be positive automatically throughout this paper. We also choose $h(s) = \frac{s}{s+K}$ for simplicity.

H(2.1). Assume that $d_i(x, t) \in L^\infty(\Omega \times (0, \infty))$ for all i. There exist two positive constants d_0 and D_0 such that

$$0 < d_0 \leq d_i(x, t) \leq D_0, \quad (x, t) \in Q_T.$$

H(2.2). Assume that all initial data $S_0(x), I_0(x), R_0(x), B_0(x)$ are nonnegative on Ω. Moreover, $U_0(x) = (S_0(x), I_0(x), R_0(x), B_0(x)) \in L^2(\Omega)^4$.

H(2.3). Let $0 \leq b(x, t) \in L^\infty(Q)$. Moreover,

$$||b||_{L^\infty(Q)} \leq b_0 < \infty.$$

For brevity, we set

$$u_1(x, t) = S(x, t), u_2(x, t) = I(x, t), u_3 = R(x, t), u_4(x, t) = B(x, t), \ (x, t) \in Q.$$

We use $U(x, t) = (u_1, u_2, u_3, u_4)$ to be a vector function defined in Q. The right-hand sides of the equations (1.1), (1.2), (1.3) and (1.4) are denoted by $f_1(U), f_2(U), f_3(U), f_4(U)$, respectively. For convenience, we use $f_1(U)$ instead of $f_1(x, t, U)$ since f_1 depends on x, t and U. With the new notation, the system (1.1)–(1.6) can be written as the following reaction–diffusion system:

$$u_{1t} - \nabla[d_1(x, t)\nabla u_1] = f_1(U), \quad (x, t) \in Q_T, \tag{2.1}$$

$$u_{2t} - \nabla[d_2(x, t)\nabla u_2] = f_2(U), \quad (x, t) \in Q_T, \tag{2.2}$$

$$u_{3t} - \nabla[d_3(x, t)\nabla u_3] = f_3(U), \quad (x, t) \in Q_T, \tag{2.3}$$

$$u_{4t} - \nabla[d_4(x, t)\nabla u_4] = f_4(U), \quad (x, t) \in Q_T, \tag{2.4}$$

subject to the initial and boundary conditions:

$$U(x, 0) = U_0(x) = (S_0(x), I_0(x), R_0(x), B_0(x)), \quad x \in \Omega, \tag{2.5}$$

$$\nabla_\nu U(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T]. \tag{2.6}$$

We define a product space X equipped with the standard product norm:

$$X = L^2(0, T; W_2^1(\Omega))^4.$$

The conjugate dual space of X is denoted by X^*.

Definition 2.1. We say $U(x, t)$ is a weak solution to the problem (2.1)–(2.6) in Q_T if $U(x, t) \in X$ such that

$$\int_0^T \int_\Omega [-u_k \cdot \phi_{kt} + d_k \nabla u_k \cdot \nabla \phi_k] dx dt$$
$$= \int_\Omega u_k(x, 0)\phi_k(x, 0)dx + \int_0^T \int_\Omega f_k(U)\phi_k(x, t)dx dt,$$

for any test function $\phi_k \in X^* \bigcap L^\infty(Q_T)$ and $\phi_{kt} \in L^2(Q_T)$ with $\phi_k(x, T) = 0$ on Ω for all $k = 1, 2, 3, 4$.

Theorem 2.1. *Under the assumptions H(2.1)–H(2.3), the problem (2.1)–(2.6) possesses at least one weak solution and the weak solution is nonnegative in Q_T.*

To obtain more regularity, we need additional regularity for known data.

H(2.4). Assume that $b(x, t) \in L^{2,\mu}(Q))$ for any $\mu \in [n, n + 2)$. Let $\nabla U_0(x) \in L^{2,\mu}(\Omega)^4$ for any $\mu \in [n - 2, n)$.

Theorem 2.2. *Under the assumptions H(2.1)–H(2.4), the problem (2.1)–(2.6) has a unique weak solution $U(x, t) \in C^{\alpha, \frac{\alpha}{2}}(\bar{Q}_T)$ for any $T > 0$ with $\alpha = \frac{\mu+2-n}{2}$. Moreover, the weak solution is classical if the coefficients d_i are smooth.*

With some additional effort, we can prove that the solution is uniformly bounded in Q for any dimension n.

Theorem 2.3. *Under the assumption H(2.1)–(2.4), the problem (2.1)–(2.6) has a unique global solution in $L^{\infty}(Q) \bigcap C^{\alpha, \frac{\alpha}{2}}(\bar{Q})$. Moreover,*

$$||U||_{L^{\infty}(Q)} \leq C,$$

where C depends only on known data.

Remark 2.1. The results in Theorems 2.1–2.3 still hold with some modification of the proofs if the system (2.1)–(2.5) contains linear convection terms.

With the result of Theorem 2.3, we can establish the global asymptotic behaviour of the solution.

H(2.5). Assume

$$\lim_{t \to \infty} b(x, t) = b_0(x), \lim_{t \to \infty} d_i(x, t) = d_{i\infty}(x), \qquad x \in \Omega, i = 1, 2, 3, 4,$$

uniformly in $\bar{\Omega}$. Moreover, $d_{i\infty}(x) \in C^{\alpha}(\bar{\Omega})$ for all i.

Let $S_{\infty}(x)$ be the steady-state solution for the following elliptic equation

$$-\nabla[d_1(x)\nabla S_{\infty}] = b_0(x) - dS_{\infty}, \qquad x \in \Omega \qquad (2.7)$$

$$\nabla_{\nu} S_{\infty}(x) = 0, \qquad x \in \partial\Omega. \qquad (2.8)$$

where $m_0 = \max_{\bar{\Omega}} S_{\infty}(x)$.

Theorem 2.4. *Let the assumption H(2.1)–H(2.5) hold. If the parameters in the system (1.1)–(1.5) satisfy either*

(a) $d_1(x, t) = d_{1\infty}(x)$ *and*

$$d > 2\beta_1 m_0 + \frac{2\beta_2 m_0 \xi}{K(\delta - g)}, \delta - g > 0$$

or

(b)

$$d > \frac{3\beta_1 m_0}{2} + \frac{\beta_2 m_0 \xi}{2K(\delta - g - \frac{\xi}{2})} + \frac{1}{4} \max\{\sigma, \gamma\}, \delta - g - \frac{\xi}{2} > 0,$$

then the corresponding steady-state system of (2.1)–(2.6) has only one steady-state solution
$U_\infty(x) := (S_\infty(x), 0, 0, 0)$. *Moreover,*

$$\lim_{t \to \infty} (S(x, t), I(x, t), R(x, t), B(x, t)) = (S_\infty(x), 0, 0, 0),$$

in $L^p(Q)$-sense for any $p > 1$. That is $(S_\infty(x), 0, 0, 0)$ is a global attractor.

Remark 2.2. Roughly speaking, the solution of (1.1)–(1.5) converges to the unique steady-state solution as long as β_1 and β_2 are suitably small.

3 Global solvability and regularity

From the physical model, the concentration must be nonnegative. Here, we state this fact as a lemma and sketch a proof since we could not find precise reference in the literature. Recall that a vector function is said to be nonnegative if each component of the vector function is nonnegative.

Lemma 3.1: *Assume the conditions H(2.1)–H(2.2) hold. Then, a weak solution U(x,t) of (2.1)– (2.6) must be nonnegative in Q:*

$$U(x, t) \geq 0, (x, t) \in Q_T.$$

Proof Recall that a vector function $F(x, t, U) := (f_1(x, t, U), f_2(x, t, U), \cdots, f_m(x, t, U))$ is said to be quasi-positive if for all $k = 1, 2, \cdots, m$,

$$f_k(x, t, u_1, u_2, \cdots, u_{k-1}, 0, u_{k+1}, \cdots, u_m) \geq 0, u_i \in [0, \infty), i = 1, 2, \cdots, m, i \neq k.$$

Since $b(x, t) \geq 0$, clearly, $F(x, t, U) = (f_1(U), \cdots, f_4(U))$ is quasi-positive. Moreover, $f_k(U)$ is locally Lipschitz continuous with respect to u_k. For a function $v(x, t) \in L^2(Q_T)$, we define

$$v^+(x, t) = \frac{|v(x, t)| + v(x, t)}{2}, v^-(x, t) = \frac{|v(x, t)| - v(x, t)}{2}.$$

Now, we consider a modified system (2.1)–(2.6) in which $f_i(U)$ is replaced by $f_i(U^+)$. Then, we choose $\phi_k(x, t) = (T - t)u_k^-$ (one may need to use a truncation technique as in Lemma 1, p. 749, in [3], if necessary) to obtain, after some routine calculation,

$$\frac{d}{dt} \int_\Omega (T - t)|U^-|^2 dx \leq C \int_\Omega (T - t)|U^-|^2 dx.$$

It follows that $u_k^-(x, t) = 0$ in Q_T, which implies that a weak solution to the modified system is the same as the original system (2.1)–(2.6). □

As in the standard analysis in deriving an *a priori* estimate for solution of a partial differential equation, we may assume that the solution is smooth in Q_T. A special attention is paid to be what a constant C depends precisely on known data in the derivation. We will denote by

$C, C_1, C_2, \cdots$, etc., generic constants in the derivation. Those constants may be different from one line to the next as long as their dependence is the same.

Lemma 3.2: Under the assumptions of H(2.1)–H(2.2), there exists a constants C_1 such that

$$\sum_{i=1}^{4} ||u_i||_{V_2(Q_T)} \leq C_1,$$

where C_1 depends only on known data and the upper bound of T.

Proof From equations (2.1) and (2.3), we use the energy method to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} u_1^2 dx + d_0 \int_{\Omega} |\nabla u_1|^2 dx + d\int_{\Omega} u_1^2 dx$$

$$\leq \int_{\Omega} u_1 b\, dx + \sigma \int_{\Omega} u_1 u_3 dx$$

$$\leq \varepsilon \int_{\Omega} u_1^2 dx + C(\varepsilon) \int_{\Omega} b^2 dx + \frac{\sigma}{2}\int_{\Omega} \left[u_1^2 + u_3^2 \right] dx.$$

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} u_3^2 dx + d_0 \int_{\Omega} |\nabla u_3|^2 dx + (d+\sigma)\int_{\Omega} u_3^2 dx$$

$$\leq \gamma \int_{\Omega} u_2 u_3 dx$$

$$\leq \frac{\gamma}{2}\int_{\Omega} \left[u_2^2 + u_3^2 \right] dx.$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} \left[u_1^2 + u_3^2 \right] dx + d_0 \int_{\Omega} \left[|\nabla u_1|^2 + |\nabla u_3|^2 \right] dx$$

$$+ \left(d - \varepsilon - \frac{\sigma}{2} \right)\int_{\Omega} u_1^2 dx + \left(d + \frac{\sigma}{2} - \frac{\gamma}{2} \right)\int_{\Omega} u_3^2 dx$$

$$\leq C(\varepsilon) \int_{\Omega} b^2 dx + \frac{\gamma}{2}\int_{\Omega} u_2^2 dx.$$

To estimate u_2, we must use the special structure of the system (1.1)–(1.4).
Let

$$v(x,t) = u_1(x,t) + u_2(x,t), \quad (x,t) \in Q.$$

Then, we see that $v(x,t)$ satisfies

$$v_t - \nabla[d_2 \nabla v] + dv = \nabla[(d_1 - d_2)\nabla u_1] + b(x,t) - \gamma u_2 + \sigma u_3, \quad (x,t) \in Q_T, \quad (3.1)$$

$$v(x,0) = S_0(x) + I_0(x), x \in \Omega; \ \nabla_v v = 0, (x,t) \in \partial\Omega \times (0,\infty). \quad (3.2)$$

The energy estimate for v from equations (3.1)–(3.2) yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 dx + d_0\int_\Omega |\nabla v|^2 dx + d\int_\Omega v^2 dx$$

$$\leq 2D_0\int_\Omega [|\nabla u_1|\cdot|\nabla v|]dx + \int_\Omega vbdx + \sigma\int_\Omega u_3 vdx$$

$$\leq \frac{d_0}{2}\int_\Omega |\nabla v|^2 dx + C\int_\Omega |\nabla u_1|^2 dx + \varepsilon\int_\Omega v^2 dx + C(\varepsilon)\int_\Omega b^2 dx + \frac{\gamma}{2}\int_\Omega [u_3^2 + v^2]dx,$$

We combine the estimates for u_1, u_3 and v to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [u_1^2 + u_3^2 + v^2]dx + \frac{d_0}{2}\int_\Omega [|\nabla u_1|^2 + |\nabla u_3|^2 + |\nabla v|^2]dx$$

$$\leq C\int_\Omega [u_1^2 + u_3^2 + v^2]dx + C\int_\Omega b^2 dx,$$

where C depends only on known data.

Gronwall's inequality yields

$$\int_\Omega [u_1^2 + u_3^3 + v^2]dx + \int_0^t\int_\Omega [|\nabla u_1|^2 + |\nabla u_3|^2 + |\nabla v|^2 dx]dxdt \leq C + C\int_0^T\int_\Omega b^2 dxdt,$$

where C depends only on known data and T.

It follows that

$$||u_1||_{V_2(Q_T)} \leq C; \ ||u_3||_{V_2(Q_T)} \leq C.$$

By the definition of v, we see

$$||u_2||_{V_2(Q_T)} \leq C,$$

where C depends only on known data and the upper bound of T.

Once u_2 is bounded in $V_2(Q_T)$, by equation (2.4), we easily obtain an estimate for u_4:

$$||u_4||_{V_2(Q_T)} \leq C,$$

where C depends only on known data. □

With the estimates from the previous lemmas, we are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. There are several ways to prove the existence of a weak solution for the system (2.1)–(2.6). We use a truncation method. For any $\varepsilon > 0$, we denote by $\chi(u)$ the Heaviside function. Denote the smooth approximation of $\chi_\varepsilon(u)$ by

$$\chi_\varepsilon(u) = \text{smooth approximation of } \chi\left(u - \tfrac{1}{\varepsilon}\right).$$

$$f_{1\varepsilon}(U) = b(x,t) - \beta_1 u_1 u_2 - \beta_2 u_1 h(u_4) - du_1 + \sigma u_3(1 - \chi_\varepsilon(u_1)),$$

$$f_{2\varepsilon}(U) = \beta_1 u_1 u_2 + \beta_2 S \cdot h(u_4) - (d+\gamma)I,$$

$$f_{3\varepsilon}(U) = \gamma u_2 - (d+\sigma)u_3,$$

$$f_{4\varepsilon}(U) = \xi u_2 + gu_4\left(1 - \frac{u_4}{K}\right) - \delta u_4.$$

Now, we consider the following approximated reaction–diffusion system:

$$u_{1\varepsilon} - \nabla[d_1(x,t)\nabla u_1] = f_{1\varepsilon}(U), \qquad (x,t) \in Q_T, \tag{3.3}$$

$$u_{2t} - \nabla[d_2(x,t)\nabla u_2] = f_{2\varepsilon}(U), \qquad (x,t) \in Q_T, \tag{3.4}$$

$$u_{3t} - \nabla[d_3(x,t)\nabla u_3] = f_{3\varepsilon}(U), \qquad (x,t) \in Q_T, \tag{3.5}$$

$$u_{4t} - \nabla[d_4(x,t)\nabla u_4] = f_{4\varepsilon}(U), \qquad (x,t) \in Q_T, \tag{3.6}$$

subject to the same initial and boundary conditions (2.5)–(2.6).

For the approximate problem (3.3)–(3.6) subject to initial-boundary conditions (2.5)–(2.6), by using the standard theory for parabolic equations (see [10] or [16]), we claim that there exists a unique weak solution in $V_2(Q_T) \bigcap L^\infty(Q_T)$ for every small $\varepsilon > 0$ and

$$\sum_{i=1}^{4} ||u_{i\varepsilon}||_{L^\infty(Q_T)} \le C(\varepsilon),$$

where $C(\varepsilon)$ depends on known data and ε.

For the completeness, we sketch a proof for the claim here. For the existence of a bounded weak solution $U_\varepsilon(x,t)$, we use Leray–Schauder's fixed-point theorem. Choose a Banach space $X = L^\infty(Q_T)^4$. For every $V(x,t) \in X$ and each $\lambda \in [0,1]$, the standard theory of parabolic equations (see [16]) implies that there exists a unique weak solution $U_\varepsilon(x,t) \in V_2(Q_T)^4 \bigcap X$ for the following linear parabolic system:

$$u_{1\varepsilon} - \nabla[d_1(x,t)\nabla u_1] = \lambda f_{1\varepsilon}(V), \quad (x,t) \in Q_T,$$

$$u_{2t} - \nabla[d_2(x,t)\nabla u_2] = \lambda f_{2\varepsilon}(V), \quad (x,t) \in Q_T,$$

$$u_{3t} - \nabla[d_3(x,t)\nabla u_3] = \lambda f_{3\varepsilon}(V), \quad (x,t) \in Q_T,$$

$$u_{4t} - \nabla[d_4(x,t)\nabla u_4] = \lambda f_{4\varepsilon}(V), \quad (x,t) \in Q_T,$$

$$\partial_\nu U(x,t) = 0, (x,t) \in \partial\Omega \times (0,T], U(x,0) = \lambda U_0(x), x \in \Omega.$$

We define a mapping M_λ from X into X as follows:

$$M_\lambda : V \in X \to U_\varepsilon = M_\lambda[V] \in X.$$

Clearly, the mapping M_λ is well-defined and $M_0[V] = 0$. Moreover, DeGiorgi–Nash's theory implies that all fixed-points of M_λ satisfies

$$||U_\varepsilon||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)} \le C(\varepsilon),$$

where $\alpha \in (0,1)$ and $C(\varepsilon)$ is a constant which depends on ε.

Since the embedding mapping from $C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$ into $L^\infty(Q_T)$ is compact, we can easily show that the mapping M_λ is continuous and compact from X to $C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$. By Leray–Schauder's fixed-point theorem, the problem (3.3)–(3.6) along with the initial-boundary conditions (2.5)–(2.6) has a weak solution $U_\varepsilon(x,t) \in V_2(Q_T \bigcap L^\infty(Q_T))$. Uniqueness follows from the fact that all weak solutions are bounded and f_k is locally Lipschtz continuous with respect to u_i.

Next, we see that all *a priori* energy estimates from Lemma 3.2 hold and these bounds are independent of ε. Note that $U_{\varepsilon t} \in L^2(0,T; H^{-1}(\Omega))$. By using the weak compactness of $L^2(Q_T)$, we can extract a subsequence of ε if necessary, as $\varepsilon \to 0$, that for $k = 1, 2, 3, 4$,

$$u_{k\varepsilon}(x,t) \to u_k(x,t), \qquad \text{strongly in } L^2(Q_T) \text{ and } a.e. \text{ in } Q_T,$$

$$\nabla u_{k\varepsilon}(x,t) \to \nabla u_k(x,t), \text{ weakly in } L^2(Q_T).$$

Moreover, by using Egorov's theorem, we see, as $\varepsilon \to 0$,

$$\int \int_{Q_T} |u_{1\varepsilon}u_{2\varepsilon} - u_1 u_2| dxdt \le \int \int_{Q_T} [|(u_{1\varepsilon} - u_1)u_{2\varepsilon}| + u_1|u_{2\varepsilon} - u_2|]dxdt \to 0.$$

On the other hand, since $u_1 \in L^\infty(0,T; W^{1,2}(\Omega))$, we see

$$\left| \left\{ (x,t) \in Q_T : u_1 > \frac{1}{\varepsilon} \right\} \right|$$

is sufficiently small, provided that ε is chosen to be sufficiently small. It follows that for any test function $\phi(x,t) \in L^2(0,T; W_2^1(\Omega)) \bigcap L^\infty(Q_T)$

$$\lim_{\varepsilon \to 0} \int \int_{Q_T} u_{3\varepsilon}(1 - \chi_\varepsilon(u_{1\varepsilon})\phi dxdt = \int \int_\Omega u_3 \phi dxdt.$$

Consequently, for all k we have, as $\varepsilon \to 0$,

$$f_{k\varepsilon}(U_\varepsilon) \to f_k(U) \ a.e. \text{ in } Q_T \text{ and strongly in } L^1(Q_T).$$

For any test function $\phi_k \in X^*$ with $\phi_k \in L^\infty(Q_T)$ and $\phi_{kt} \in X^*$, $\phi_k(x,T) = 0$, we have for all $k = 1, 2, 3, 4$,

$$\int_0^T \int_\Omega [-u_{k\varepsilon} \cdot \phi_{kt} + d_k \nabla u_\varepsilon \cdot \nabla \phi_k]dxdt = \int_\Omega u_{k0}(x)\phi_k(x,0)dx + \int_\Omega f_{\varepsilon k}(U_\varepsilon)\phi_k(x,t)dxdt.$$

After taking limit as $\varepsilon \to 0$, we obtain a weak solution $U(x,t) \in X$. $\qquad\square$

To obtain more regularity, we need a result in Campanato–John–Nirenberg–Morrey space for a linear parabolic equation. Consider the following linear parabolic equation

$$u_t - \nabla(a(x,t)\nabla u) = f(x,t) + \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}, (x,t) \in Q_T, \tag{3.7}$$

$$\nabla_\nu u(x,t) = 0, (x,t) \in \partial\Omega \times (0,T], \tag{3.8}$$

$$u(x,0) = u_0(x), x \in \Omega. \tag{3.9}$$

H(3.1): Assume that $a(x, t) \in L^\infty(Q_T)$ and

$$0 < a_0 \leq a(x, t) \leq A_0 < \infty.$$

Moreover, $\nabla u_0(x) \in L^{2,\mu}(\Omega), f \in L^{2,\mu}(Q_T)$ and $f_i(x, t) \in L^{2,\mu}(Q_T)$ with $\mu \in [0, n+2)$.

Lemma 3.3. *([36]) Under the assumption of H(3.1) for any weak solution $u(x, t) \in V_2(Q_T)$ to the problem (3.7)–(3.9) and any $\mu \in [0, n)$, there exists a constant C_1 such that*

$$||\nabla u||_{L^{2,\mu}(Q_T)} \leq C_1 \left[||\nabla u_0||_{L^{2,\mu}(\Omega)} + ||f||_{L^{2,(\mu-2)^+}(Q_T)} + \sum_{i=1}^{n} ||f_i||_{L^{2,\mu}(Q_T)} \right]. \quad (3.10)$$

Moreover, there exists a constant C_2 such that

$$||u||_{L^{2,\mu+2}(Q_T)} \leq C_2, \quad (3.11)$$

where C_1 and C_2 depend only on a_0, A_0, T and Ω.

Proof of Theorem 2.2. For any $\mu \in (0, 2]$, we use equation (2.1) to obtain

$$||\nabla u_1||_{L^{2,\mu}(Q_T)} \leq C[||\nabla S_0||_{L^{2,\mu}(\Omega)} + ||u_3||_{L^{2,(\mu-2)^+}(Q_T)}].$$

Similarly, by equation (2.3), we obtain

$$||\nabla u_3||_{L^{2,\mu}(Q_T)} \leq C[||\nabla R_0||_{L^{2,\mu}(\Omega)} + ||u_2||_{L^{2,(\mu-2)^+}(Q_T)}].$$

From the linear system (3.1)–(3.2) for $v = u_1 + u_2$, we see

$$||\nabla v||_{L^{2,\mu}(Q_T)} \leq C[||\nabla v_0||_{L^{2,\mu}(\Omega)} + ||\nabla u_1||_{L^{2,\mu}(Q_T)} + ||u_2||_{L^{2,(\mu-2)^+}(Q_T)}] + ||u_3||_{L^{2,(\mu-2)^+}(Q_T)}],$$

where C depends only on known data.

Since $v(x, t) = u_1(x, t) + u_2(x, t)$, we see

$$\sum_{i=1}^{3} ||\nabla u_i||_{L^{2,\mu}(Q_T)}$$

$$\leq C \left[||\nabla S_0||_{L^{2,\mu}(\Omega)} + ||\nabla I_0||_{L^{2,\mu}(\Omega)} + ||\nabla R_0||_{L^{2,\mu}(\Omega)} + \sum_{i=1}^{3} ||u_i||_{L^{2,(\mu-2)^+}(Q_T)} \right]. \quad (3.12)$$

The $L^2(Q_T)$-estimate for $u_i, i = 1, 2, 3$ implies that the above estimate holds for any $\mu \in [0, 2]$. The interpolation yields that, for any $\mu \in [0, 2]$,

$$\sum_{i=1}^{3} ||u_i||_{L^{2,\mu+2}(Q_T)} \leq C, \quad (3.13)$$

where C depends only on known data.

Now, we go back the estimate (3.12) to see that the estimate (3.13) holds for any $\mu \in [2, 4]$. We can continue this iteration process as long as the right-hand side of (3.9) is bounded. After a finite number of iterations, we see that the above estimate (3.12) holds for any $\mu \in [0, n)$.

Since equation (2.3) is linear, we can use the interpolation to obtain

$$||u_3||_{L^{2,\mu+2}(Q_T)} \leq C,$$

where $\mu \in [0, n)$ and C depends only on known data.

It follows that

$$||u_3||_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q}_T)} \leq C,$$

where C depends only on known data and $\alpha = \frac{\mu+2-n}{2}$.

Once we have the estimate for u_3 in $C^{\alpha, \frac{\alpha}{2}}(\bar{Q})_T$, we obtain

$$||u_1||_{L^\infty(Q_T)} \leq C.$$

Now, we can use the standard DiGorgi–Nash's estimate to obtain

$$||u_1||_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})_T} + |u_2||_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})_T} \leq C,$$

where C depends only on known data.

From equation (2.4), we obtain

$$||u_4||_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q}_T)} \leq C,$$

where C depends only on known data.

With these estimates, the uniqueness follows immediately.

Finally, using the general theory of parabolic equations, we see that u_i is smooth in Q_T as long as $d_i(x, t)$ is smooth for all $i = 1, 2, 3, 4$. $\qquad\square$

4 Global bounds and asymptotic behaviour of solution

In this section, we first derive a global bound for U and then prove Theorems 2.3 and 2.4. Unlike the proof of Theorem 2.2, we must pay a special attention to various estimates which must be independent of T.

In order to see a clear physical meaning for all species, we may use the original variables (S, I, R, B) or (u_1, u_2, u_3, u_4) in this section.

Lemma 4.1. *Under the assumptions H(2.1)–H(2.2), there exists a constant C_1 such that*

$$\sup_{0 < t < \infty} \sum_{k=1}^{4} ||u_k||_{L^1(\Omega)} \leq C_1,$$

where C_1 depends only on know data.

Proof: We take integration over Ω for equations (2.1)–(2.3) and then add up to see

$$\frac{d}{dt} \int_\Omega (u_1 + u_2 + u_3)dx + d \int_\Omega (u_1 + u_2 + u_3)dx$$

$$= \int_\Omega b(x, t)dx \leq b_0|\Omega|.$$

Since $d > 0$, it follows that

$$\sup_{0 < t < \infty} \int_\Omega (u_1 + u_2 + u_3)dx \leq C_1,$$

where C_1 depends only on known data.

Now, we integrate of Ω for equation (2.1) again to find

$$\frac{d}{dt} \int_\Omega u_1 dx + \beta_1 \int_\Omega u_1 u_2 dx + \beta_2 \int_\Omega u_1 h(u_4)dx = \int_\Omega b(x,t)dx + \sigma \int_\Omega u_3 dx.$$

It follows that

$$\beta_1 \int_0^T \int_\Omega u_1 u_2 dx dt + \beta_2 \int_0^T \int_\Omega u_1(u_4)dx dt \leq C_2.$$

Next, we take integration over Ω for equation (2.4) to obtain

$$\frac{d}{dt} \int_\Omega u_4 dx + \delta \int_\Omega u_4 dx + g \int_\Omega u_4^2 dx$$

$$= \xi \int_\Omega u_2 dx + g \int_\Omega u_4 dx$$

$$\leq \xi \int_\Omega u_2 dx + \frac{g}{2} \int_\Omega u_4^2 dx + C.$$

Since $L^1(\Omega)$-norm of u_2 is uniformly bounded, it follows that

$$\frac{d}{dt} \int_\Omega u_4 dx + \delta \int_\Omega u_4 dx \leq C.$$

Thus, we obtain the desired $L^1(\Omega)$-estimate for u_4. $\qquad\square$

A direct consequence of Lemma 4.1 is that when the space dimension is equal to $n = 1$, from the general theory of parabolic equations (for example, See Lemma 2.6 in [5] or [4]), we immediately see that S, R, I and B are uniformly bounded in Q.

When the space dimension is greater than 1, the derivation is much more complicated. In the derivation of various energy estimates, the following Young's inequality with a small parameter $\varepsilon > 0$ will be used frequently: for any $a, b \geq 0$,

$$ab \leq \frac{\varepsilon a^p}{p} + \frac{b^q}{q\varepsilon^{q-1}},$$

where

$$\frac{1}{p} + \frac{1}{q} = 1, p, q \in (1, \infty).$$

Lemma 4.2. Under the condition H(2.1)–H(2.3), there exists a constant C such that

$$\sup_{t > 0} \int_\Omega [S^{1+r} + I^{1+2} + R^{1+r} + B^{1+r}]dx \leq C,$$

where $r \in (0, \frac{1}{n-1})$ is arbitrary and C depends only on known data.

Proof Keep in mind that we shall always derive an estimate for *S,I* and *R* as a first step. The second step for estimating *B* will be followed easily once we have an estimate for *I* by equation (1.4).

We use the standard energy method to derive the estimate. We multiply equation (2.1) by u_1^r and equation (2.3) by R^r, respectively, and then integrate ove Ω to obtain

$$\frac{1}{1+r}\frac{d}{dt}\int_\Omega S^{1+r}dx + d_0 r\int_\Omega S^{r-1}|\nabla S|^2 dx$$

$$+\beta_1\int_\Omega S^{1+r}Idx + \beta_2\int_\Omega S^{1+r}h(B)dx + d\int_\Omega S^{1+r}dx$$

$$\leq \int_\Omega S^r bdx + \sigma\int_\Omega RS^r dx$$

$$\leq \varepsilon\int_\Omega S^{1+r}dx + C(\varepsilon)\int_\Omega b^{1+r}dx + \sigma\int_\Omega\left[\frac{r}{r+1}S^{1+r}+\frac{1}{r+1}R^{1+r}\right]dx.$$

$$\frac{1}{1+r}\frac{d}{dt}\int_\Omega R^{1+r}dx + d_0 r\int_\Omega R^{r-1}|\nabla R|^2 dx + (d+\sigma)\int_\Omega R^{1+r}dx$$

$$\leq \gamma\int_\Omega IR^r dx$$

$$\leq \gamma\int_\Omega\left[\frac{r}{1+r}R^{1+r}+\frac{1}{1+r}I^{1+r}\right]dx.$$

Similarly for *I*, we have

$$\frac{1}{1+r}\frac{d}{dt}\int_\Omega I^{1+r}dx + d_0 r\int_\Omega I^{r-1}|\nabla I|^2 dx + (d+\gamma)\int_\Omega I^{1+r}dx$$

$$\leq \beta_1\int_\Omega SI^{1+r}dx + \beta_2\int_\Omega SI^r h(B)dx$$

$$:= J_1 + J_2.$$

$$J_1 \leq \beta_1\int_\Omega[\frac{\varepsilon_1}{1+r}S^{1+r}I + \frac{r}{(1+r)\varepsilon_1^{\frac{1}{r}}}I^{1+2r}]dx$$

$$J_2 \leq \beta_2\int_\Omega[S^{1+r}h(B)+I^{1+r}]dx,$$

where we have used the fact $0 \leq h(B) \leq 1$ for the second estimate.

We choose $\varepsilon_1 = 1 + r$ and then add up the estimates for *S,I* and *R* to obtain

$$\frac{1}{1+r}\frac{d}{dt}\int_\Omega[S^{1+r}+I^{1+r}+R^{1+r}]dx + d_0 r\int_\Omega[S^{r-1}|\nabla S|^2 + I^{r-1}|\nabla I|^2 + R^{r-1}|\nabla R|^2 dx$$

$$+\frac{d}{2}\int_\Omega[S^{1+r}+I^{1+r}+R^{1+r}]dx$$

$$\leq C_1\int_\Omega I^{1+2r}dx + C_2\int_\Omega b^{1+r}dx.$$

Now, we recall a Gagliardo–Nirenberg's inequality

$$||u||_{L^p(\Omega)} \le C||\nabla u||_{L^q(\Omega)}^\theta ||u||_{L^s(\Omega)}^{1-\theta} + C||u||_{L^1},$$

where C depends only on n and Ω, p, q, s, θ satisfy $0 \le \theta < 1, 1 \le p < \infty$, and

$$\frac{1}{p} = \theta\left(\frac{1}{q} - \frac{1}{n}\right) + (1-\theta)\frac{1}{s}.$$

We choose $q = 2$ and $s = 1$, then we see that, for $p = \frac{2(1+2r)}{1+r}$ and $\theta = \frac{n(1+3r)}{(n+2)(1+2r)}$,
Note that by Lemma 4.1, for $u = I^{\frac{1+r}{2}}$ with $r \le 1$, then

$$||u||_{L^1(\Omega)} \le C.$$

Moreover,

$$|\nabla u|^2 = r^2 I^{r-1}|\nabla I|^2.$$

If we apply Gagliardo–Nirenberg's inequality to obtain

$$\int_\Omega |u|^p dx \le C||\nabla u||_{L^2(\Omega)}^{p\theta} + C$$

Note that if r satisfies

$$0 < r < \frac{1}{n-1},$$

then

$$\frac{p\theta}{2} < 1.$$

We see

$$\int_\Omega |u|^p dx \le C||\nabla u||_{L^2(\Omega)}^{p\theta} + C$$

$$\le \varepsilon \int_\Omega |\nabla u|^2 dx + C(\varepsilon),$$

where $\varepsilon > 0$ is arbitrary and $C(\varepsilon)$ depends only on ε, $||u||_{L^1(\Omega)}$ and known data.
By choosing ε sufficiently small, we obtain

$$\frac{d}{dt} \int_\Omega [S^{1+r} + I^{1+r} + R^{1+r}]dx + \int_\Omega [S^{r-1}|\nabla S|^2 + I^{r-1}|\nabla I|^2 + R^{r-1}|\nabla R|^2 dx$$

$$+ \int_\Omega [S^{1+r} + I^{1+r} + R^{1+r}]dx$$

$$\le C_3 \int_\Omega b^{1+r}dx \le C_3 b_0^{1+r}|\Omega|.$$

Consequently, for any $r \in (0, \frac{1}{n-1})$, we have

$$\sup_{0<t<\infty} \int_\Omega [S^{1+r} + I^{1+r} + R^{1+r}]dx \leq C,$$

where C depends only on known data.

The estimate for B follows easily from equation (1.4). $\qquad\square$

Proof of Theorem 2.3: Let $k \geq 1$. We follow the same calculations as in Lemma 4.2 to obtain

$$\frac{1}{1+k}\frac{d}{dt}\int_\Omega [S^{1+k} + I^{1+k} + R^{1+k}]dx + d_0 k \int_\Omega [S^{k-1}|\nabla S|^2 + I^{k-1}|\nabla I|^2 + R^{k-1}|\nabla R|^2]dx$$

$$+ \frac{d}{2}\int_\Omega [S^{1+k} + I^{1+k} + R^{1+k}]dx$$

$$\leq C_1 \int_\Omega I^{1+2k}dx + C_2 \int_\Omega b^{1+k}dx.$$

Let $u = I^{\frac{1+k}{2}}$. As in Lemma 4.2, if we have an estimate in $L^1(\Omega)$ for a function u, then we can obtain an estimate for u in $L^p(\Omega)$ with $p = \frac{2n}{2n-(n+2)\theta}$ and $\theta = \frac{n(1+3k)}{(n+2)(1+2k)}$ as long as $k < \frac{1}{n-1}$. For any $k \geq 1$, after a finite number of steps, we obtain

$$\sup_{t\geq 0} \int_\Omega [S^{1+k} + I^{1+k} + R^{1+k}]dx \leq C(k),$$

By the standard estimate for parabolic equation (see Lemma 2.6 in [5]), we obtain

$$\sup_{t\geq 0}[||S||_{L^\infty(\Omega} + ||I||_{L^\infty(\Omega)} + ||R||_{L^\infty(\Omega)}] \leq C,$$

where C depends only on known data.

Once I is uniformly bounded, we can easily derive a uniform bound for B by using the standard theory of parabolic equation ([16]). This concludes the proof of Theorem 2.3. $\qquad\square$

With the global bound for $U(x,t)$, we are ready to prove the asymptotic behaviour of the solution for the system (1.1)–(1.6).

Proof of Theorem 2.4: From Theorem 2.3, we have

$$\sup_Q |S(x,t)| + \sup_Q |I(x,t)| + \sup_Q |R(x,t)| + \sup_Q |B(x,t)| \leq C,$$

where C depends only on known data.

We consider the following steady-state system corresponding to (2.1)–(2.6):

$$-\nabla[d_{1\infty}(x)\nabla S] = b_0(x) - \beta_1 SI - \beta_2 S \cdot h(B) - dS + \sigma R, \tag{4.1}$$

$$-\nabla[d_{2\infty}(x)\nabla I] = \beta_1 SI + \beta_2 S \cdot h(B) - (d+\gamma)I, \tag{4.2}$$

$$-\nabla[d_{3\infty}(x)\nabla R] = \gamma I - (d+\sigma)R, \tag{4.3}$$

$$-\nabla[d_{4\infty}(x)\nabla B] = \xi I + gB(1 - \frac{B}{K}) - \delta B \tag{4.4}$$

subject to the following boundary conditions:

$$(\nabla_\nu S, \nabla_\nu I, \nabla_\nu R, \nabla_\nu B) = 0, \qquad x \in \partial\Omega, \tag{4.5}$$

By using the same argument as for the system (2.1)–(2.6), we can show that the problem (4.1)–(4.5) has a weak solution $U^*(x) = (S^*(x), I^*(x), R^*(x), B^*(x)) \in W_2^1(\Omega) \bigcap L^\infty(\Omega)$. Moreover, since $b_0(x) \geq 0$, the maximum principle implied that every weak solution is nonnegative on Ω. Furthermore, since the coefficients $d_{i\infty} \in C^\alpha(\bar{\Omega})$, regularity theory for elliptic equation ([29]) yields

$$||S^*||_{C^{1+\alpha}(\bar{\Omega}} + ||I^*||_{C^{1+\alpha}(\bar{\Omega}} + ||R^*||_{C^{1+\alpha}(\bar{\Omega}} + ||B^*||_{C^{1+\alpha}(\bar{\Omega}} \leq C,$$

where C depends only on known data.

We divide the rest of the proof into three steps. In the first step, we show that there is an upper bound M_0 which is independent of β_1 and β_2. In the second step, we show that the steady-state system has a unique solution $(S_\infty(x), 0, 0, 0)$ if the parameters satisfy the conditions in Theorem 2.4, where $S_\infty(x)$ is a solution of (2.7)–(2.8). In the third step, we show that the $U(x,t)$ converges to the steady-state solution $U^*(x)$ in $L^p(\Omega)$ for any $p > 1$.

Step 1. There exists a constant M_0 which is independent β_1 and β_2 such that

$$\sup_\Omega S^*(x) \leq M_0.$$

Indeed, it is easy to see

$$d\int_\Omega [S^* + I^* + R^*]dx = \int_\Omega b_0(x)dx,$$

$$(\delta - g)\int_\Omega B^*dx + g\int_\Omega B^{*2}dx = \xi\int_\Omega I^*dx.$$

Hence, L^1-bound of (S^*, I^*, R^*, B^*) is independent of β_1 and β_2. Moreover,

$$\int_\Omega B^{*2}dx \leq \frac{\xi}{g}\int_\Omega I^*dx.$$

To derive an L^2-bound, we define $V(x) = S(x) + I(x)$. Then

$$-\nabla[d_{1\infty}(x)\nabla S] = b_0(x) - \beta_1 SI - \beta_2 S \cdot h(B) - dS + \sigma R, \tag{4.6}$$

$$-\nabla[d_{2\infty}(x)\nabla V] = \nabla[(d_{1\infty} - d_{2\infty}\nabla S] + b_0(x) - dV + \sigma R, \tag{4.7}$$

$$-\nabla[d_{3\infty}(x)\nabla R] = \gamma I - (d + \sigma)R, \tag{4.8}$$

$$-\nabla[d_{4\infty}(x)\nabla B] = \xi I + gB\left(1 - \frac{B}{K}\right) - \delta B, \tag{4.9}$$

By using the energy method, we immediately obtain an $L^2(\Omega)$-bound for S,V,R,B and the bound is independent β_1 and β_2. By using the same $L^{2,\mu}$-argument, we can deduce an $L^\infty(\Omega)$ bound for $S^*(x)$ and the bound is independent β_1 and β_2.

Step 2. We show that a steady-state solution $U^*(x)$ to (4.3)–(4.7) must equal to $(S_\infty(x), 0, 0, 0)$ if the conditions in Theorem 2.4 hold.

Let

$$\hat{U}(x) = (\hat{S}(x), \hat{I}(x), \hat{R}(x), \hat{B}) = (S^*(x) - S_\infty(x), I^*(x), R^*(x), B^*(x)), x \in \Omega.$$

It is easy to see that $(\hat{S}, \hat{I}, \hat{R}, \hat{B})$ satisfies the following elliptic system (for convenience, we use I^*, R^*, B^* instead of $(\hat{I}, \hat{R}, \hat{B})$):

$$-\nabla[d_{1\infty}(x)\nabla\hat{S}] = -\beta_1[\hat{S}I^* + S_\infty I^*] - \beta_2[\hat{S}h(B^*) + S_\infty h(B^*)] - d\hat{S} + \sigma R^*, \quad (4.10)$$

$$-\nabla[d_{2\infty}(x)\nabla I^*] = \beta_1[\hat{S}I^* + S_\infty I^*] + \beta_2[\hat{S} \cdot h(B^*) + S_\infty h(B^*)] - (d + \gamma)I^*, \quad (4.11)$$

$$-\nabla[d_{3\infty}(x)\nabla R^*] = \gamma I^* - (d + \sigma)R^*, \quad (4.12)$$

$$-\nabla[d_{4\infty}(x)\nabla B^*] = \xi I + gB^*(1 - \frac{B^*}{K}) - \delta B \quad (4.13)$$

subject to the following boundary conditions:

$$(\nabla_\nu\hat{S}, \nabla_\nu I^*, \nabla_\nu R^*, \nabla_\nu B^*) = 0, \qquad x \in \partial\Omega, \quad (4.14)$$

We first use $L^1(\Omega)$-estimate to show that the steady-state system (4.6)–(4.9) with homogeneous Neumann condition has a unique solution $U_\infty(x) = (S_\infty(x), 0, 0, 0)$. From the elliptic equation (4.6), we multiply the sign function $sgn\hat{S}$ (one may use a smooth approximation of sign function first and then take the limit) to obtain the following estimates:

$$d \int_\Omega |\hat{S}|dx + \int_\Omega [\beta_1|\hat{S}|I^* + \beta_2|\hat{S}|h(B^*)]dx$$

$$\leq \beta_1 \int_\Omega S_\infty I^* dx + \beta_2 \int_\Omega S_\infty h(B^*)dx + \sigma \int_\Omega R^* dx$$

$$\leq \beta_1 m_0 \int_\Omega I^* dx + \frac{\beta_2 m_0}{K} \int_\Omega B^* dx + \sigma \int_\Omega R^* dx, \quad (4.15)$$

where

$$m_0 = \max_{\bar{\Omega}} S_\infty(x).$$

Similarly, from equation (4.7), we have

$$(d + \gamma) \int_\Omega I^* dx$$

$$\leq \beta_1 \int_\Omega [|\hat{S}|I^* + S_\infty I^*]dx + \beta_2 \int_\Omega [|\hat{S}|h(B^*) + S_\infty h(B^*)]dx. \quad (4.16)$$

From equations (4.8)–(4.9), we find

$$(d + \sigma) \int_\Omega R^* dx \le \gamma \int_\Omega I^* dx. \qquad (4.17)$$

$$(\delta - g) \int_\Omega B^* dx \le \xi \int_\Omega I^* dx. \qquad (4.18)$$

We add up the estimates (4.15)–(4.17) to obtain

$$d \int_\Omega |\hat{S}| dx + (d - 2\beta_1 m_0 - \frac{2\beta_2 m_0 \xi}{K(\delta - g)}) \int_\Omega I^* dx + d \int_\Omega R^* dx \le 0,$$

where we have used the estimate (4.18) for the L^1-norm of B^*. If

$$d > 2\beta_1 m_0 + \frac{2\beta_2 m_0 \xi}{K(\delta - g)},$$

then we conclude

$$\hat{S}(x) = I^*(x) = R^*(x) = 0, \qquad x \in \Omega.$$

Consequently, the inequality (4.18) implied $B^*(x, t) = 0$ on Ω.

When $d_1(x, t)$ depends on t, the above conditions for the parameters are not enough to prove $U(x,t)$ converges to $U^*(x)$. We need a different set of conditions on parameters to prove the uniqueness of $U^*(x)$ via L^2-estimate. Indeed, we multiply equation (4.6) by $\hat{S}$ and integrate over Ω to obtain

$$d_0 \int_\Omega |\nabla \hat{S}|^2 dx + d \int_\Omega \hat{S}^2 dx + \beta_1 \int_\Omega \hat{S}^2 I^* dx + \beta_2 \int_\Omega \hat{S}^2 h(B^*) dx$$

$$\le \frac{\beta_1 m_0}{2} \int_\Omega [\hat{S}^2 + I^{*2}] dx + \frac{\beta_2 m_0}{2K} \int_\Omega [\hat{S}^2 + B^{*2}] dx + \sigma \int_\Omega [\varepsilon R^{*2} + \frac{1}{4\varepsilon} \hat{S}^2] dx, \qquad (4.19)$$

where we have used fact $h(B^*) \le \frac{B^*}{K}$ and Cauchy–Schwarz's inequality with a parameter $\varepsilon > 0$.

Similarly, from equation (4.7), we have

$$d_0 \int_\Omega |\nabla I^*|^2 dx + (d + \gamma) \int_\Omega I^{*2} dx \le \beta_1 M_0 \int_\Omega I^{*2} dx + \frac{\beta_2 M_0}{2K} \int_\Omega [I^{*2} + B^{*2}] dx, \qquad (4.20)$$

From equations (4.8)–(4.9), we find

$$d_0 \int_\Omega |\nabla R^*|^2 dx + (d + \sigma) \int_\Omega R^{*2} dx \le \gamma \int_\Omega [\varepsilon I^{*2} + \frac{1}{4\varepsilon} R^{*2}] dx; \qquad (4.21)$$

$$d_0 \int_\Omega |\nabla B^*|^2 dx + (\delta - g) \int_\Omega B^{*2} dx \le \frac{\xi}{2} \int_\Omega [B^{*2} + I^{*2}] dx. \qquad (4.22)$$

From the estimate (4.18), we see

$$(\delta - g - \frac{\xi}{2}) \int_\Omega B^{*2} dx \le \frac{\xi}{2} \int_\Omega I^{*2} dx.$$

If we combine the estimates (4.15)–(4.18) and choose ε sufficiently close to 1 to conclude that

$$\hat{S}(x) = I^*(x) = R^*(x) = 0, \qquad x \in \Omega$$

provided that

$$d > \frac{3\beta_1 m_0}{2} + \frac{\beta_2 m_0 \xi}{2K(\delta - g - \frac{\xi}{2})} + \frac{1}{4} max\{\sigma, \gamma\}, \ \delta - g > \frac{\xi}{2}.$$

The inequality (4.18) yields

$$B^*(x) = 0, \qquad x \in \Omega.$$

That is

$$U^*(x) = (S_\infty(x), 0, 0, 0), \qquad x \in \Omega.$$

Consequently, $M_0 = m_0$.

Step 3: In this step, we show that $U(x,t)$ converges to the steady-state solution $U^*(x) = (S_\infty(x), 0, 0, 0)$ as $t \to \infty$.

For convenience, we set

$$W(x, t) := (w_1, w_2, w_3, w_4) = U(x, t) - U^*(x) = (S - S_\infty, I, R, B), (x, t) \in Q.$$

A direct calculation shows that $W(x,t)$ satisfies the following system:

$$w_{1t} - \nabla[d_1(x, t)\nabla w_1] = m_1(x, t) + f_1(x, t, U) - f_1(x, U^*), (x, t) \in Q, \qquad (4.23)$$

$$w_{2t} - \nabla[d_2(x, t)\nabla w_2] = m_2(x, t) + f_2(U) - f_2(U^*), \quad (x, t) \in Q, \qquad (4.24)$$

$$w_{3t} - \nabla[d_3(x, t)\nabla w_3] = m_3(x, t) + f_3(U) - f_3(U^*), \quad (x, t) \in Q, \qquad (4.25)$$

$$w_{4t} - \nabla[d_4(x, t)\nabla w_4] = m_4(x, t) + f_4(U) - f_4(U^*), \quad (x, t) \in Q, \qquad (4.26)$$

subject to the boundary conditions:

$$\nabla_\nu(w_1, w_2, w_3, w_4) = 0, \qquad (x, t) \in \partial\Omega \times (0, \infty), \qquad (4.27)$$

where

$$m_i(x, t) = \nabla[(d_i(x, t) - d_{i\infty}(x))\nabla u_i^*], i = 1, \cdots, 4.$$

When $d_i(x, t)$ is independent of t, we see $m_i(x, t) = 0$ in Q. For this case, we can use the same method as in Step 2 to show that $W(x,t)$ converges to $U^*(x)$ in $L^1(\Omega)$ if the conditions in Step 2 hold.

Now, we consider the general case when d_i depends on t. Note that $d_{i\infty}(x)$ are uniformly bounded in $C^\alpha(\bar{\Omega})$, we know that ∇u_i^* is uniformly bounded in Ω. Hence, for each i,

$$| \int_\Omega m_i(x)w_i dx| \le \varepsilon_1 \int_\Omega |\nabla w_i|^2 dx + C(\varepsilon_1) \int_\Omega (d_i - d_{i\infty})^2 dx,$$

where ε_1 is a small parameter.

For any $t_0 > 0$. Let $Q(t_0) = \Omega \times [t_0, \infty)$ and

$$m_0^* = \sup_{Q(t_0)} s^*(x, t).$$

By using the same argument as in Step 1, we see m_0^* is independent of β_1 and β_2.

Observe $h(0) = 0$, hence

$$f_1(x, t, U) - f_1(x, U^*)$$

$$= b(x,t) - b_0(x) - \beta_1[w_1 w_2 + S_\infty w_2] - \beta_2[w_1 h(B) + S_\infty(h(B))] - dw_1 + \sigma w_3.$$

Moreover, from the definition of $h(B)$,

$$|h(B)| \leq \frac{|w_4|}{K}.$$

It follows that

$$|w_1 S_\infty w_2| \leq \frac{m_0}{2}[w_1^2 + w_2^2];$$

$$|w_1 S_\infty h(w_4)| \leq \frac{m_0}{2K}(w_1^2 + w_4^2).$$

Now, we use the standard energy estimate to obtain

$$\frac{d}{dt}\frac{1}{2}\int_\Omega w_1^2 dx + (d_0 - \varepsilon_1)\int_\Omega |\nabla w_1|^2 dx + (d - 2\varepsilon_1)\int_\Omega w_1^2 dx$$

$$\leq \frac{\beta_1 m_0}{2}\int_\Omega [w_1^2 + w_2^2]dx + \frac{m_0 \beta_2}{2K}\int_\Omega (w_1^2 + w_4^2)]dx$$

$$+ C_1(\varepsilon_1)\int_\Omega [(b - b_0)^2 + (d_1 - d_{1\infty})^2]dx + \sigma\int_\Omega [\varepsilon_2 w_3^2 + \frac{1}{4\varepsilon_2}w_1^2]dx,$$

where ε_1 and ε_2 are arbitrarily constants.

Similarly, since $f_2(U^*) = f_3(U^*) = f_4(U^*) = 0$, we have

$$\frac{d}{dt}\frac{1}{2}\int_\Omega w_2^2 dx + (d_0 - \varepsilon_1)\int_\Omega |\nabla w_2|^2 dx + (d + \gamma - \varepsilon_1)\int_\Omega w_2^2 dx$$

$$\leq \beta_1 m_0^* \int_\Omega w_2^2 dx + \frac{\beta_2 m_0^*}{2K}\int_\Omega [(w_2^2 + w_4^2)]dx + C_2(\varepsilon_1)\int_\Omega [(d_2 - d_{2\infty})^2]dx;$$

$$\frac{d}{dt}\frac{1}{2}\int_\Omega w_3^2 dx + (d_0 - \varepsilon_1)\int_\Omega |\nabla w_3|^2 dx + (d + \sigma - \varepsilon_1)\int_\Omega w_1^2 dx$$

$$\leq \gamma \int_\Omega [\varepsilon_2 w_2^2 + \frac{1}{4\varepsilon_2}w_3^2]dx + C_3(\varepsilon_1)\int_\Omega (d_3 - d_{3\infty})^2 dx;$$

$$\frac{d}{dt}\frac{1}{2}\int_\Omega w_4^2 dx + (d_0 - \varepsilon_1)\int_\Omega |\nabla w_4|^2 dx + (\delta - g)\int_\Omega w_4^2 dx$$

$$\leq \frac{\xi}{2}\int_\Omega [w_2^2 + w_4^2]dx + C_4(\varepsilon_1)\int_\Omega (d_4 - d_{4\infty})^2 dx.$$

Suppose the parameters in the system (1.1)–(1.4) satisfy the following conditions:

$$d > \frac{1}{4}\max\{\sigma, \gamma\} + \frac{3\beta_1 M_0}{2} + \frac{\beta_2 M_0 \xi}{2K(\delta - g - \frac{\xi}{2})}, \tag{4.28}$$

$$\delta - g > \frac{\xi}{2}, \tag{4.29}$$

where $M_0 = \max\{m_0, m_0^*\}$.

We choose ε_1 sufficiently small and ε_2 sufficiently close to 1, then we combine the above estimates to see that there exists a constant $\alpha_1 > 0$ such that

$$Y'(t) + \alpha_1 Y(t) \leq CG(t),$$

where

$$Y(t) = \sum_{i=1}^{4} \int_{\Omega} w_i^2 dx, \, G(t) = \int_{\Omega} [(b - b_0)^2 + \sum_{i=1}^{4} (d_i - d_{i\infty})^2] dx.$$

Since $G(t)$ converges to 0 as $\to \infty$, we conclude that

$$\lim_{t \to \infty} Y(t) = 0,$$

which implies, for any $p > 1$,

$$\lim_{t \to \infty} ||U - U^*||_{L^p(\Omega)} = 0$$

since $U(x,t)$ and $U^*(x)$ are uniformly bounded.

Finally, we can choose m_0^* sufficiently close to m_0 as long as t_0 is sufficiently large. Therefore, the convergence of $U(x,t)$ to $U^*(x)$ holds if the conditions for parameters in Theorem 2.4 hold.

The proof of Theorem 2.4 is completed. $\qquad\square$

5 Conclusion

In this paper, we studied a mathematical model for the Cholera epidemic without lifetime immunity. The model equations are governed by a coupled reaction–diffusion system with different diffusion coefficients for each species. We established the global well-posedness for the coupled reaction–diffusion system under a certain condition on known data. Moreover, the long-time behaviour of the solution is obtained for any space dimension n. Particularly, we prove that there is a global attractor for the system under appropriate conditions on known data. These results justify the mathematical model and provide scientists a deeper understanding of the dynamics of interaction between bacteria and susceptible, infected and recovered human hosts. The mathematical model with the help of real data analysis provides a scientific foundation for policymakers to make better decisions for the general public in health and medical sciences. The main tools used in this paper come from some delicate theories for elliptic and parabolic equations. The method developed in this paper can be used to study other models such as the avian influenza for birds.

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