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RISK MODELS IN INSURANCE AND EPIDEMICS: A BRIDGE THROUGH RANDOMIZED POLYNOMIALS

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The purpose of this work is to construct a bridge between two classical topics in applied probability: the finite-time ruin probability in insurance and the final outcome distribution in epidemics. The two risk problems are reformulated in terms of the joint right-tail and left-tail distributions of order statistics for a sample of uniforms. This allows us to show that the hidden algebraic structures are of polynomial type, namely Appell in insurance and Abel–Gontcharoff in epidemics. These polynomials are defined with random parameters, which makes their mathematical study interesting in itself.

1. INTRODUCTION

Risk is present in most daily living activities. This is true in fields as diverse as economics and health or medicine. The present paper is concerned with two specific risks in these areas: the risk of ruin in insurance and the risk of infection in epidemiology.

Both types of risk have been widely investigated in applied probability. For the ruin probability, the reader is referred, for example, to the books by Rolski et al. [42], Kaas et al. [24], Asmussen and Albrecher [2]. For the final outcome of epidemics, we refer, for example, to the books by Bailey [3], Daley and Gani [14], Andersson and Britton [1]. To the best of our knowledge, these two risks have been studied independently so far.

The purpose of this work is to construct a bridge between these insurance and epidemic risks. For that, the two risk problems are reformulated in terms of the order statistics for a finite sample of independent $(0,1)$-uniform random variables. A key observation is that the joint right-tail and left-tail distributions of such order statistics correspond, up to some constants, to two families of polynomials, similar but distinct, the Appell and Abel–Gontcharoff (A–G) polynomials. In our framework, however, the parameters involved in the polynomials are not fixed but random, so that the computations become more heavy in practice. Nevertheless, we will establish two nice properties satisfied by such polynomials and that allow us to greatly simplify the general results derived.

Remarkable families of polynomials are known to be a useful mathematical tool for the study of problems in probability and statistics. For the Appell polynomials, we mention, for example, Giraitis and Surgailis [20], Picard and Lefèvre [38,39], Schoutens and Teugels [44], Ignatov and Kaishev [22], Salminen [43], Das and Kratz [15], Lefèvre and Picard [31]. For the A–G polynomials, see for example, Daniels [12,13], Lefèvre and Picard [28], Picard and Lefèvre [38], Ball and O'Neill [5], Clancy [9,10], Ball, Sirl, and Trapman [6]. It is also worth indicating that these two families of polynomials are related to the theory of umbral calculus (see e.g., Di Bucchianico [17]).

We organize the paper as follows. Section 2 gives a short review of the families of Appell and A–G polynomials. In Section 3, these polynomials are interpreted as the right-tail and left-tail distributions of order statistics. We then derive two important results for the cases where the parameters of the polynomials correspond to the partial sums of exchangeable random variables or to the partial products of i.i.d. random variables. Section 4 deals with the finite-time ruin probability in insurance. We point out that, for several different risk models, the underlying algebraic structure is of Appell polynomial type. Section 5 is concerned with the final outcome of S–I–R epidemic models. This time, we show that the underlying algebraic structure is of A–G polynomial type. The nice operational properties satisfied by the polynomials allow a simple and systematic study of the two risk problems.

2. APPELL AND A–G POLYNOMIALS

The family of Appell polynomials is well-known in mathematics (see e.g., Kaz'min [25]). It covers several classical classes of polynomials, in particular the Hermite, Bernoulli, and Euler polynomials. A similar but different family of polynomials is given by the A–G polynomials, introduced by Gontcharoff [21] for the interpolation of entire functions. They are much less standard in the literature. We begin with some reminders on these two families of polynomials (in the univariate case).

2.1. Appell Polynomials

Following Picard and Lefèvre [38,39], let us consider any given real sequence $U = \{u_i,\, i \geq 1\}$, nondecreasing for instance. To U is attached a unique family of Appell polynomials $\{A_n(x|U),\, n \geq 0\}$, of degree n in x, defined as follows.

Definition 2.1: *The $A_n(x|U)$'s satisfy the conditions*

$$A'_n(x|U) = A_{n-1}(x|U), \quad n \geq 1, \tag{2.1}$$

with

$$A_0(x|U) = 1 \text{ and } A_n(u_n|U) = 0, \quad n \geq 1. \tag{2.2}$$

Observe that $A_n(x|U)$ depends on U through the n terms $\{u_1, \ldots, u_n\}$ only. The property that the family U is not affected by differentiation is the key Appell characteristic. Clearly, the symbol U in the notation of $A_n(x|U)$ might be omitted. This is precisely what is usually done; for us, however, it will be convenient to keep U in the notation. By (2.1), $A_n(x|U)$ admits an integral expression given by

$$A_n(x|U) = \int_{y_n=u_n}^{x} dy_n \int_{y_{n-1}=u_{n-1}}^{y_n} dy_{n-1} \ldots \int_{y_1=u_1}^{y_2} dy_1, \quad n \geq 1. \tag{2.3}$$

An equivalent definition of the $A_n(x|U)$'s is through their generating function, viewed as a formal series.

Proposition 2.2: *The $A_n(x|U)$'s are such that*

$$\sum_{n=0}^{\infty} A_n(x|U)s^n = e^{sx} \sum_{n=0}^{\infty} A_n(0|U)s^n, \tag{2.4}$$

with the condition (2.2).

The representation (2.3) is little appropriate for a numerical evaluation. It is simpler to use the Taylor expansion (2.5) with the recursion (2.6) below.

Property 2.3: *The $A_n(x|U)$'s can be expanded as*

$$A_n(x|U) = \sum_{j=0}^{n} A_{n-j}(0|U) \frac{x^j}{j!}, \quad n \geq 1, \tag{2.5}$$

where the coefficients $A_{n-j}(0|U)$ are obtained recursively by

$$A_n(0|U) = -\sum_{j=1}^{n} A_{n-j}(0|U) \frac{u_n^j}{j!}, \quad n \geq 1. \tag{2.6}$$

Proof: First, applying Taylor's formula to $A_n(x|U)$ and using (2.1) yields the expansion (2.5). Now, taking $x = u_n$ in (2.5) and using (2.2) gives the recursion (2.6) for the $A_{n-j}(0|U)$'s. ■

For example, we get

$$A_1(x|U) = -u_1 + x,$$
$$A_2(x|U) = -u_2^2/2 + u_1u_2 - u_1x + x^2/2,$$
$$A_3(x|U) = -u_3^3/6 + u_1u_3^2/2 + u_2^2u_3/2 - u_1u_2u_3 + (-u_2^2/2 + u_1u_2)x - u_1x^2/2 + x^3/6.$$

It is easily seen that, putting $a + bU = \{a + bu_i,\, i \geq 1\}$,

$$A_n(a + bx|a + bU) = b^n A_n(x|U), \quad n \geq 1. \tag{2.7}$$

Various other properties of these polynomials are known. A simple special case is when U is an affine sequence.

Property 2.4: *If $u_i = a + bi$, $i \geq 1$,*

$$A_n(x|\{a + bi,\, i \geq 1\}) = (x - a - bn)(x - a)^{n-1}/n!, \quad n \geq 1. \tag{2.8}$$

Remarks: Many works refer to a small variant for the definition of Appell polynomials. These polynomials, $\tilde{A}_n(x)$ say (U is omitted here), are constructed by substituting the condition

$$\tilde{A}'_n(x) = n\tilde{A}_{n-1}(x), \quad n \geq 1,$$

for (2.1). This relation shows that the $\tilde{A}_n$'s behave like power functions. Obviously, $\tilde{A}_n(x)/n!$, $n \geq 1$, satisfy (2.1).

The condition (2.2) provides the constants of integration. In the context of central limit theorems, a different condition is often considered, namely

$$\tilde{A}_0(x|U) = 1 \text{ and } E[\tilde{A}_n(X)] = 0, \quad n \geq 1,$$

for some random variable X with finite moments (e.g., Giraitis and Surgailis [20], Salminen [43], Ta [46]). In that case,

$$\sum_{n=0}^{\infty} \tilde{A}_n(x|U)\frac{s^n}{n!} = \frac{e^{sx}}{E(e^{sX})}.$$

These polynomials are called Wick powers or polynomials in physics.

2.2. A–G Polynomials

Following Lefèvre and Picard [28] and Picard and Lefèvre [38], let us consider again a real sequence $U = \{u_i,\, i \geq 0\}$. Note that, for convenience reasons, u_0 is now added inside U. To U is attached a unique family of A–G polynomials $\{G_n(x|U),\, n \geq 0\}$, of degree n in x, defined as follows.

Definition 2.5: *The $G_n(x|U)$'s satisfy the conditions*

$$G'_n(x|U) = G_{n-1}(x|\mathcal{E}U), \quad n \geq 1, \tag{2.9}$$

where $\mathcal{E}U \equiv \{u_{i+1},\, i \geq 0\}$, together with

$$G_0(x|U) = 1 \text{ and } G_n(u_0|U) = 0, \quad n \geq 1. \tag{2.10}$$

Note that $G_n(x|U)$ depends on U through the n terms $\{u_0, \ldots, u_{n-1}\}$ only. Since the family U is shifted after differentiation, the presence of U in the notation becomes here compulsory. By (2.9), $G_n(x|U)$ can also be expressed under integral form, given by

$$G_n(x|U) = \int_{y_0=u_0}^{x} dy_0 \int_{y_1=u_1}^{y_0} dy_1 \ldots \int_{y_{n-1}=u_{n-1}}^{y_{n-2}} dy_{n-1}, \quad n \geq 1. \tag{2.11}$$

Clearly, for each n, $G_n(x|U)$ and $A_n(x|U)$ are linked by the simple relation

$$G_n(x|u_0, \ldots, u_{n-1}) = A_n(x|u_{n-1}, \ldots, u_0), \tag{2.12}$$

but the two families of polynomials (i.e., considered for all n) are different.

An A–G family can also be characterized as follows (the proof is easy).

Property 2.6: *The $G_n(x|U)$'s are such that*

$$G_n^{(j)}(u_j|U) = \delta_{n,j}, \quad j, n \geq 0. \tag{2.13}$$

An important property of an A–G family is that it may be used as a basis for Abelian-type expansions.

PROPERTY 2.7: *Any polynomial* $R(x)$ *of degree* n *can be expanded as*

$$R(x) = \sum_{j=0}^{n} R^{(j)}(u_j) G_j(x|U). \tag{2.14}$$

PROOF: The $G_n(x|U)$ being linearly independent, we can write

$$R(x) = \sum_{k=0}^{n} a_k G_k(x|U),$$

for some coefficients a_k. Using (2.13), we then get

$$R^{(j)}(u_j) = \sum_{k=0}^{n} a_k G_k^{(j)}(u_j|U) = a_j, \quad 0 \le j \le n,$$

and formula (2.14) follows. ■

Choosing $R(x) = x^n/n!$ in (2.14) leads to the recursion below for the evaluation of the A–G polynomials:

PROPERTY 2.8:

$$G_n(x|U) = \frac{x^n}{n!} - \sum_{j=0}^{n-1} \frac{u_j^{n-j}}{(n-j)!} G_j(x|U), \quad n \ge 1. \tag{2.15}$$

For example, we get

$$G_1(x|U) = -u_0 + x,$$
$$G_2(x|U) = -u_0^2/2 + u_0 u_1 - u_1 x + x^2/2,$$
$$G_3(x|U) = -u_0^3/6 + u_0^2 u_2/2 + u_0 u_1^2/2 - u_0 u_1 u_2 - (u_1^2/2 - u_1 u_2)x - u_2 x^2/2 + x^3/6.$$

By (2.12), the identity (2.7) holds too for $G_n(x|U)$. In the affine case, the A–G polynomials reduce to the classical Abel polynomials.

PROPERTY 2.9: *If* $u_i = a + bi$, $i \ge 0$,

$$G_n(x|\{a + bi,\, i \ge 0\}) = (x - a)(x - a - bn)^{n-1}/n!, \quad n \ge 1.$$

3. JOINT ORDER STATISTICS

3.1. Tail Distributions

Let $(U_{(1:n)}, \ldots, U_{(n:n)})$ be the order statistics for a sample of n (≥ 1) independent $(0,1)$-uniform random variables. From the integral representations (2.3) for A_n and (2.11) for G_n, we directly get the following probabilistic interpretation of these polynomials (see also Denuit, Lefèvre, and Picard [16]).

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Proposition 3.1: *For* $0 \le u_1 \le \cdots \le u_n \le x \le 1$,

$$P[U_{(1:n)} \ge u_1, \ldots, U_{(n:n)} \ge u_n \ \textit{and}\ U_{(n:n)} \le x] = n!\, A_n(x|u_1, \ldots, u_n). \tag{3.1}$$

For $0 \le x \le u_1 \le \cdots \le u_n \le 1$,

$$P[U_{(1:n)} \le u_1, \ldots, U_{(n:n)} \le u_n \ \textit{and}\ U_{(1:n)} \ge x] = n!(-1)^n\, G_n(x|u_1, \ldots, u_n). \tag{3.2}$$

In other words, up to some constants, A_n corresponds to the joint survival function of $(U_{(1:n)}, \ldots, U_{(n:n)})$ and G_n to their joint distribution function.

Remark: A forthcoming paper will be devoted to bivariate Appell and A–G polynomials. To be able to extend (3.1) and (3.2), it is convenient to provide an equivalent interpretation of these formulas in terms of a particular random walk. Consider the path with unit steps going from point 0 to point n, that is, $\{(i-1, i),\, 1 \le i \le n\}$. To each step $(i-1, i)$ we associate a level u_i that is nondecreasing in i. Let us now follow this path by means of the ordered statistics $(U_{(1:n)}, \ldots, U_{(n:n)})$. A step $(i-1, i)$ is taken by $U_{(i:n)}$ if $U_{(i:n)} \ge u_i$; so, u_i represents a lower bound to be exceeded. Then, the joint survival function of $(U_{(1:n)}, \ldots, U_{(n:n)})$ can be viewed as the probability that there is a path that allows us to reach point n. By (3.1), we thus have

$$P[\text{there is a path from 0 to } n \ \text{ and }\ U_{(n:n)} \le x] = n!\, A_n(x|u_1, \ldots, u_n). \tag{3.3}$$

A similar interpretation holds for (3.2) when each u_i corresponds to a level not to be exceeded by $U_{(i:n)}$.

3.2. Randomized Boundary

A randomization of the boundary $\{u_i\}$ above leads us to consider Appell and A–G polynomials with random parameters. Such polynomials can be easily evaluated in the two special cases discussed here.

The first case is when the successive parameters are partial sums of exchangeable random variables. Let $S_j = X_1 + \cdots + X_j$, $j \ge 1$, where the X_j's are exchangeable random variables.

Proposition 3.2: *For any integer* $l \ge n \ge 1$,

$$E\left[A_n(x|S_1, \ldots, S_n)\,|\, S_l\right] = \frac{x^{n-1}}{(n-1)!}\left(\frac{x}{n} - \frac{S_l}{l}\right) \ \text{a.s.} \tag{3.4}$$

Proof: Let us first establish the identity (3.4) when $x = 0$, that is,

$$E[A_n(0|S_1, \ldots, S_n)\,|\, S_l] = -\delta_{n,1}\, S_l/l, \quad l \ge n \ge 1. \tag{3.5}$$

When $n = 1$, as $A_1(x) = x - S_1$ and the X_i's are exchangeable,

$$E[A_1(0|S_1)\,|\, S_l] = -E(S_1|S_l) = -S_l/l, \tag{3.6}$$

as desired. When $n \ge 2$, we have by (2.6) that

$$A_n(0|S_1, \ldots, S_n) = -\sum_{j=1}^{n} A_{n-j}(0|S_1, \ldots, S_{n-j})\frac{S_n^j}{j!}.$$

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Using $E(.|S_l) = E[E(.|S_n, S_l)|S_l] = E[E(.|S_n)|S_l]$ (as $l \geq n$), we then write that

$$E[A_n(0|S_1,\ldots,S_n)\,|\,S_l] = -\sum_{j=1}^{n} E\left\{\frac{S_n^j}{j!}\,E[A_{n-j}(0|S_1,\ldots,S_{n-j})|S_n,S_l]|S_l\right\}$$
$$= -E\left(\frac{S_n^n}{n!}|S_l\right) - E\left\{\frac{S_n^{n-1}}{(n-1)!}\,E[A_1(0|S_1)\,|\,S_n]|S_l\right\}$$
$$-\sum_{j=1}^{n-2} E\left\{\frac{S_n^j}{j!}\,E[A_{n-j}(0|S_1,\ldots,S_{n-j})|S_n]|S_l\right\}. \qquad \textbf{(3.7)}$$

By (3.6), we know that $E[A_1(0|S_1)\,|\,S_n] = -S_n/n$, so that the right-hand side (r.h.s.) of (3.7) reduces to the term $-\sum_{j=1}^{n-2} E\{\ldots\}$. Applying induction, we thus deduce that this term is equal to 0, hence (3.5) holds too for $n \geq 2$. Now, for $x \neq 0$, Taylor's expansion (2.5) of $A_n(x|S_1,\ldots,S_n)$ together with formula (3.5) yields the announced result (3.4). ■

The special case where $l = n$ is proved in Lemma 2.1 of Lefèvre and Picard [31] through a different method. It is easily seen to be a generalization of formula (2.8).

The second case is when the successive parameters are partial products of i.i.d. random variables. This time, we will see that the expectation of the randomized polynomials reduces to similar polynomials with adapted fixed parameters. More precisely, let $\Pi_j = X_1 \ldots X_j$, $j \geq 1$, where the X_j's are i.i.d. random variables (distributed as X), and put $\Pi_0 = 1$. Let Y be a random variable independent of the X_j's. All the variables are assumed to have finite moments.

PROPOSITION 3.3: *For any reals α, β, x,*

$$E\left[Y^\alpha (\Pi_n)^\beta G_n(x|Y\Pi_i,\, i \geq 0)\right]$$
$$= \sum_{j=0}^{n} \frac{x^j}{j!}\,E(Y^{\alpha+n-j})\,[E(X^{\beta+n-j})]^j\,G_{n-j}(0|E(X^{\beta+i}),\, i \geq 0), \quad n \geq 0. \qquad \textbf{(3.8)}$$

PROOF: We first establish (3.8) when $Y = 1$ a.s. Denote by $R_n(x)$ the left-hand side (l.h.s.) of (3.8), that is,

$$R_n(x) = E[(\Pi_n)^\beta G_n(x|\Pi_i,\, i \geq 0)], \quad n \geq 0. \qquad \textbf{(3.9)}$$

By (2.10), $R_0(x) = 1$ and $R_n(1) = 0$ when $n \geq 1$. From (2.9) and (2.7), we get

$$G_n(x|\Pi_i,\, i \geq 0) = \int_1^x G_{n-1}(y|\Pi_i,\, i \geq 1)dy$$
$$= (X_1)^{n-1} \int_1^x G_{n-1}(y/X_1|\tilde{\Pi}_i,\, i \geq 0)dy \;\text{ a.s.},$$

where $\tilde{\Pi}_0 = 1$ and $\tilde{\Pi}_i = X_2 \ldots X_{i+1}$ for $i \geq 1$. Thus, (3.9) can be rewritten as

$$R_n(x) = E\left[(X_1)^{\beta+n-1} \int_1^x (\tilde{\Pi}_{n-1})^\beta\, G_{n-1}(y/X_1|\tilde{\Pi}_i,\, i \geq 0)dy\right].$$

As the X_i's are i.i.d., the products $\tilde{\Pi}_i$ and Π_i are equidistributed. Using $E(.) = E[E(.|X_1)]$, we then obtain

$$R_n(x) = E\left[X^{\beta+n-1} \int_1^x R_{n-1}(y/X)dy\right],$$

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by virtue of (3.9) for $R_{n-1}(x/X)$. This shows that $R_n(x)$, $n \geq 1$, is a polynomial in x of degree n whose derivative satisfies

$$R_n'(x) = E\left[X^{\beta+n-1} R_{n-1}(x/X)\right]. \tag{3.10}$$

Let us now express $R_n(x)$ under the form

$$R_n(x) = \sum_{j=0}^{n} a_{n,j} \frac{x^{n-j}}{(n-j)!}, \quad n \geq 0. \tag{3.11}$$

For $n = 0$, (3.11) gives $a_{0,0} = 1$. For $n \geq 1$, substituting (3.11) into (3.10) and identifying the coefficients of $x^{n-j-1}/(n-j-1)!$ yields

$$\begin{aligned} a_{n,j} &= E[X^{\beta+n-1} a_{n-1,j} (1/X)^{n-j-1}] \\ &= a_{n-1,j} E(X^{\beta+j}) = \cdots = a_{j,j} [E(X^{\beta+j})]^{n-j}, \quad 0 \leq j \leq n. \end{aligned} \tag{3.12}$$

It remains to determine the coefficients $a_{j,j}$ using the condition $R_n(1) = \delta_{n,0}$, that is,

$$\sum_{j=0}^{n} a_{j,j} \frac{[E(X^{\beta+j})]^{n-j}}{(n-j)!} = 0, \quad n \geq 1,$$

with $a_{0,0} = 1$. In fact, from the Abelian expansion (2.15) at point $x = 0$ in which $u_i = E(X^{\beta+i})$, $i \geq 0$, we see that

$$\sum_{j=0}^{n} \frac{[E(X^{\beta+j})]^{n-j}}{(n-j)!} G_j(0|E(X^{\beta+i}), i \geq 0) = 0, \quad n \geq 1,$$

which implies

$$a_{j,j} = G_j(0|E(X^{\beta+i}), i \geq 0), \quad j \geq 0. \tag{3.13}$$

Inserting (3.12) and (3.13) into (3.11), we deduce that $R_n(x)$ is given by the r.h.s. of (3.8) when $Y = 1$.

Finally, let us examine the effect of the factor Y. Denote by $R_n^c(x)$ the l.h.s. of (3.8). Using (2.7) and $E(.) = E[E(.|Y)]$ with Y independent of the X_i's, we can write

$$R_n^c(x) = E[Y^{\alpha+n} R_n(x/Y)].$$

From (3.11), we then obtain

$$\begin{aligned} R_n^c(x) &= \sum_{j=0}^{n} E\left[Y^{\alpha+n} a_{n,j} \frac{(x/Y)^{n-j}}{(n-j)!}\right] \\ &= \sum_{j=0}^{n} E(Y^{\alpha+j}) a_{n,j} \frac{x^{n-j}}{(n-j)!}, \end{aligned}$$

which provides the r.h.s. of (3.8) using (3.12) and (3.13). ■

The identity (3.8) generalizes Theorem 4.1 of Picard and Lefèvre [40]. Moreover, the method of proof followed here has the advantage to be constructive and not to rely on an argument by induction.

4. INSURANCE MODELS

4.1. Ordered Arrivals

Recently, Lefèvre, and Picard [30,31] investigated a risk model with ordered claim arrivals defined on a finite-time interval $[0,t]$. The key assumption is that the claim arrival process $\{N_t(s),\ 0 \leq s \leq t\}$ satisfies an order statistic property (see e.g., Puri [41]). Specifically, the total number of claims during $[0,t]$, $N(t)$ say, has any given distribution, and given $N(t) = n \geq 1$, the claim arrival times $(T_1, \ldots, T_n)$ are distributed as the order statistics $(R_{(1:n)}, \ldots, R_{(n:n)})$ associated with a set of n i.i.d. random variables, with a continuous distribution function F_t on $[0,t]$. We notice that a similar claim arrival process is examined by Sendova and Zitikis [45].

On the other hand, the successive claim amounts X_j, $j \geq 1$, are nonnegative random variables, possibly dependent but independent of the claim arrival times. For each X_j, the claim amount covered by the insurance is $f(X_j)$ where f is some positive nondecreasing function. Furthermore, the company begins with initial reserves $u \geq 0$ and receives premiums at a determistic rate. The cumulated premium until time s (including initial reserves) is the nondecreasing function $h(s)$ where $h(0) = u$.

Let $\{U_t(s),\ 0 \leq s \leq t\}$ be the surplus process over the period $[0,t]$. The aggregate claim amount until time s is $S_t(s) = \sum_{j=1}^{N_t(s)} f(X_j)$, so that $U_t(s) = h(s) - S_t(s)$. Ruin occurs at a claim instant T when the aggregate claim amount cannot be covered by the aggregate premium income. In other words, the non-ruin probability until time $t > 0$ is given by

$$\phi(t) \equiv P(T > t) = P[S_t(s) \leq h(s), \quad \text{for } 0 \leq s \leq t].$$

Suppose that $N(t) = n \geq 1$, and consider the conditional non-ruin probability until time t, $\phi(t|n)$ say. Of course, $\phi(t) = E[\phi(t|N(t))]$. To derive $\phi(t|n)$, it will be useful to introduce the instant where the total premium h allows us to cover any given effective claim amount $y > 0$. Evidently, this instant corresponds to $h^{-1}(y)$ (and reduces to 0 if $y \leq u$). Write $S_j = f(X_1) + \cdots + f(X_j)$ for the total effective claim amount caused by the first j claims, $j \geq 1$.

PROPOSITION 4.1: *For $t > 0$,*

$$\phi(t|n) = n!\, E\left\{A_n(1|V_1, \ldots, V_n)\, \mathbf{1}\left[S_n \leq h(t)\right]\right\}, \tag{4.1}$$

where $\mathbf{1}(A)$ is the indicator of the event A, and

$$V_i = F_t(h^{-1}(S_i)), \quad 1 \leq i \leq n. \tag{4.2}$$

PROOF: Suppose for the moment that $S_1 = s_1 \leq \cdots \leq S_n = s_n$ are fixed. By definition, if $s_n > h(t)$, then $\phi(t|n) = 0$. Otherwise, that is, when $s_n \leq h(t)$, we have

$$\begin{aligned} \phi(t|n) &= P[h(T_1) \geq s_1, \ldots, h(T_n) \geq s_n] \\ &= P[T_1 \geq h^{-1}(s_1), \ldots, T_n \geq h^{-1}(s_n)]. \end{aligned}$$

By the order statistics property, this can be rewritten as

$$\phi(t|n) = P[R_{(1:n)} \geq h^{-1}(s_1), \ldots, R_{(n:n)} \geq h^{-1}(s_n)],$$

and therefore,

$$\begin{aligned} \phi(t|n) &= P[U_{(1:n)} \geq F_t(h^{-1}(s_1)), \ldots, U_{(n:n)} \geq F_t(h^{-1}(s_n))] \\ &= n!\, A_n(1|F_t(h^{-1}(s_1)), \ldots, F_t(h^{-1}(s_n))), \end{aligned} \tag{4.3}$$

using (3.1). Taking the expectation of (4.3), valid provided $S_n \leq h(t)$, with respect to the S_j's and using the notation (4.2), we then deduce the formula (4.1). ■

A result similar to (4.1) is derived in Lefèvre and Picard [30]. In Lefèvre and Picard [31], the theory of Appell polynomials is exploited to obtain explicit expressions for the finite and ultimate non-ruin probabilities when the premium function h is linear, the claim amounts are i.i.d. variables and the distribution function F_t is of linear or exponential form.

We limit ourselves here to reexamine the particular case where there are no initial reserves, the premium rate is a constant $c > 0$, the claim amounts are exchangeable variables and $F_t(s) = s/t$, $0 \leq s \leq t$, which occurs when claims arrive according to a (mixed) Poisson process. The following result, classical, can be proved by different methods (e.g., using the ballot theorem).

Proposition 4.2 (e.g., Takács [47]): *Under these assumptions,*

$$\phi(t|n) = E\left[\left(1 - \frac{S_n}{ct}\right)_+\right]. \quad \textbf{(4.4)}$$

Proof: We first observe that $h(s) = cs$, $s \geq 0$, so that $h^{-1}(y) = y/c$, $y > 0$. Now, from (4.1) and (4.2), by conditioning on S_n and using (2.7), we have

$$\begin{aligned}\phi(t|n) &= n!\, E\left[A_n\left(1|\frac{S_1}{ct}, \ldots, \frac{S_n}{ct}\right) \mathbf{1}(S_n \leq ct)\right] \\ &= \frac{n!}{(ct)^n}\, E\left\{\mathbf{1}(S_n \leq ct)\, E[A_n(ct|S_1, \ldots, S_n)|S_n]\right\}.\end{aligned}$$

From (3.4), we then obtain

$$\phi(t|n) = \frac{n!}{(ct)^n}\, E\left[\mathbf{1}(S_n \leq ct)\, \frac{(ct)^{n-1}}{(n-1)!}\, (\frac{ct}{n} - \frac{S_n}{n})\right],$$

hence the formula (4.4). ■

4.2. Schur-Constant Arrivals

Let us consider a variant of this risk model that is studied by following the reserves process until the arrival time of the nth claim (and not over a fixed period of time $[0, t]$ as before). The probability of non-ruin over the period $[0, T_n]$ is denoted by ϕ_n (instead of $\phi(t|n)$ above).

Specifically, the same (general) assumptions are made on the claim amounts, but the claim arrivals here follow a different model. Let $\tau_i = T_i - T_{i-1}$, $1 \leq i \leq n$, be the n successive claim interarrival times. We suppose that the vector $(\tau_1, \ldots, \tau_n)$ forms a continuous Schur-constant model (as defined e.g., in Caramellino and Spizzichino [7], Nelsen [34], Chi, Yang, and Qi [8]). By definition, this means that

$$P(\tau_1 > x_1, \ldots, \tau_n > x_n) = G(x_1 + \cdots + x_n), \quad x_1, \ldots, x_n \geq 0,$$

for some adequate function G. Under this assumption, it is known that

(i) $(\tau_1/T_n, \ldots, \tau_n/T_n)$ is independent of T_n,

(ii) $(T_1/T_n, \ldots, T_{n-1}/T_n)$ is distributed as the order statistics $(U_{(1:n-1)}, \ldots, U_{(n-1:n-1)})$ associated with $(n-1)$ independent $(0,1)$-uniforms,

(iii) the distribution function of T_n is given by

$$F_{T_n}(x) = 1 + \sum_{k=0}^{n-1} (-1)^{k+1} \frac{t^k}{k!} G^{(k)}(x), \quad x > 0.$$

Moreover, if the Schur-constant property holds for all $n \geq 1$, the τ_i are necessarily mixed exponentials, that is, the occurrence of claims follows a mixed Poisson process.

PROPOSITION 4.3: *For $t > 0$,*

$$\phi_n = (n-1)!\, E\left\{ A_{n-1}(1|W_{n,1}, \ldots, W_{n,n-1})\, \mathbf{1}\left[S_n \leq h(T_n)\right]\right\}, \tag{4.5}$$

where

$$W_{n,i} = \frac{h^{-1}(S_i)}{T_n}, \quad 1 \leq i \leq n. \tag{4.6}$$

PROOF: Let us fix $S_1 = s_1 \leq \cdots \leq S_n = s_n$ and $T_n = t_n$. Evidently,

$$\begin{aligned} \phi_n &= P[h(T_1) \geq s_1, \ldots, h(T_{n-1}) \geq s_{n-1}]\, \mathbf{1}[h(t_n) \geq s_n] \\ &= P\left[\frac{T_1}{t_n} \geq \frac{h^{-1}(s_1)}{t_n}, \ldots, \frac{T_{n-1}}{t_n} \geq \frac{h^{-1}(s_{n-1})}{t_n}\right] \mathbf{1}[h(t_n) \geq s_n]. \end{aligned}$$

As the n claim interarrival times are Schur-constant, we can write

$$\phi_n = P\left[U_{(1:n-1)} \geq \frac{h^{-1}(s_1)}{t_n}, \ldots, U_{(n-1:n-1)} \geq \frac{h^{-1}(s_{n-1})}{t_n}\right] \mathbf{1}[h(t_n) \geq s_n],$$

and by virtue of (3.1),

$$\phi_n = (n-1)!\, A_{n-1}\left[1|\frac{h^{-1}(s_1)}{t_n}, \ldots, \frac{h^{-1}(s_{n-1})}{t_n}\right] \mathbf{1}[h(t_n) \geq s_n]. \tag{4.7}$$

It remains to take the expectation with respect to the S_j's and T_n, which gives formula (4.5) using (4.6). ■

Let us examine the particular case where there are no initial reserves, the premium rate is a constant $c > 0$ and the claim amounts are exchangeable variables.

PROPOSITION 4.4: *Under these assumptions,*

$$\phi_n = E\left[\left(1 - \frac{n-1}{n}\frac{S_n}{cT_n}\right) \mathbf{1}(S_n \leq cT_n)\right]. \tag{4.8}$$

PROOF: From (4.5) and (4.6) and the assumptions, we have

$$\phi_n = (n-1)!\, E\left[A_{n-1}\left(1|\frac{S_1}{cT_n}, \ldots, \frac{S_{n-1}}{cT_n}\right) \mathbf{1}(S_n \leq cT_n)\right].$$

Using (2.7) and conditioning on (S_n, T_n) then yields

$$\phi_n = (n-1)!\, E\left\{\mathbf{1}(S_n \leq cT_n)\, \frac{1}{(cT_n)^{n-1}}\, E\left[A_{n-1}(cT_n|S_1, \ldots, S_{n-1})|S_n, T_n\right]\right\}.$$

By formula (3.4), we deduce that

$$\phi_n = (n-1)!\, E\left[\mathbf{1}(S_n \leq cT_n)\, \frac{1}{(cT_n)^{n-1}}\, \frac{(cT_n)^{n-2}}{(n-2)!} \left(\frac{cT_n}{n-1} - \frac{S_n}{n}\right)\right],$$

which reduces to (4.8). ■

4.3. Schur-Constant Claims

The Schur-constant property can also be applied to the claim amounts. So, consider a risk model with initial reserves u, a constant premium rate $c > 0$ and a Poisson claim arrival process. We now suppose that the n claims are totally covered (i.e., $f(X_j) = X_j$) and their amounts $(X_1, \ldots, X_n)$ form a continuous Schur-constant model. As in Section 4.1, let $\phi(t|n)$ be the non-ruin probability until time t given that $N(t) = n$.

PROPOSITION 4.5: *For $t > 0$,*

$$\phi(t|n) = 1 - \sum_{j=1}^{n} E\left[\left(\frac{u + cT_j}{S_j}\right)^{j-2}\left(\frac{u + cT_j/j}{S_j}\right)\mathbf{1}(S_j > u + cT_j, T_j \le t)\right], \tag{4.9}$$

where $(T_1, \ldots, T_n)$ is distributed as the order statistics of n independent $(0,t)$-uniforms.

PROOF: We will argue with $\psi(t|n) = 1 - \phi(t|n)$, the conditional probability of ruin before time t. First, let us fix $T_1 = t_1 \le \cdots \le T_n = t_n \le t$. By definition,

$$\psi(t|n) = \sum_{j=1}^{n} P[S_1 \le h(t_1), \ldots, S_{j-1} \le h(t_{j-1}), S_j > h(t_j)].$$

Let us also fix $S_j = s_j$ in each probability above. Then,

$$\psi(t|n) = \sum_{j=1}^{n} P\left[\frac{S_1}{s_j} \le \frac{h(t_1)}{s_j}, \ldots, \frac{S_{j-1}}{s_j} \le \frac{h(t_{j-1})}{s_j}\right]\mathbf{1}[s_j > h(t_j)],$$

and as the n claim amounts are Schur-constant,

$$\psi(t|n) = \sum_{j=1}^{n} P\left[U_{(1:j-1)} \le \frac{h(t_1)}{s_j}, \ldots, U_{(j-1:j-1)} \le \frac{h(t_{j-1})}{s_j}\right]\mathbf{1}[s_j > h(t_j)].$$

By virtue of (3.2), this becomes

$$\psi(t|n) = \sum_{j=1}^{n} (j-1)!(-1)^{j-1} G_{j-1}\left[0|\frac{h(t_1)}{s_j}, \ldots, \frac{h(t_{j-1})}{s_j}\right]\mathbf{1}[s_j > h(t_j)].$$

Finally, taking $h(s) = u + cs$ and using (2.7), we obtain

$$\psi(t|n) = \sum_{j=1}^{n} (j-1)!(-1)^{j-1} G_{j-1}\left(-\frac{u}{c}|t_1, \ldots, t_{j-1}\right)\left(\frac{c}{s_j}\right)^{j-1}\mathbf{1}(s_j > u + ct_j). \tag{4.10}$$

Remember that (4.10) holds insofar as $t_j \le t$ in each jth term of this sum.

Now, let us take the expectation in (4.10) with respect to the T_j's and S_j's. Given $N(t) = n$, the arrival times $(T_1, \ldots, T_n)$ are distributed as the order statistics of n

independent $(0,t)$-uniforms. Thus, each T_i is the sum of i interarrivals periods τ_i that are exchangeable random variables. By (2.12) and (2.7), we can write

$$\begin{aligned} G_{j-1}\left(-\frac{u}{c}|T_1,\ldots,T_{j-1}\right) &= A_{j-1}\left(-\frac{u}{c}|T_{j-1},\ldots,T_1\right) \\ &= A_{j-1}\left(-\frac{u}{c}-T_j|T_{j-1}-T_j,\ldots,T_1-T_j\right) \\ &= (-1)^{j-1}A_{j-1}\left(\frac{u}{c}+T_j|\tau_j,\tau_{j-1}+\tau_j,\ldots,\tau_2+\cdots+\tau_j\right) \text{ a.s.} \end{aligned}$$

By substitution in the expectation of (4.10), we then get

$$\psi(t|n) = \sum_{j=1}^{n}(j-1)!\,E\left[A_{j-1}\left(\frac{u}{c}+T_j|\tau_j,\ldots,\tau_2+\cdots+\tau_j\right)\left(\frac{c}{S_j}\right)^{j-1}\mathbf{1}(S_j > u+cT_j, T_j \le t)\right].$$

Conditioning each jth expectation above on T_j yields

$$\begin{aligned} \psi(t|n) = \sum_{j=1}^{n}(j-1)!\,E\Bigg\{&\left(\frac{c}{S_j}\right)^{j-1}\mathbf{1}(S_j > u+cT_j, T_j \le t) \\ &\times E\left[A_{j-1}\left(\frac{u}{c}+T_j|\tau_j,\ldots,\tau_2+\cdots+\tau_j\right)|T_j\right]\Bigg\}. \end{aligned}$$

Since the τ_j's are exchangeable, we finally obtain by (3.4) that

$$\psi(t|n) = \sum_{j=1}^{n}(j-1)!\,E\left[\left(\frac{c}{S_j}\right)^{j-1}\mathbf{1}(S_j > u+cT_j, T_j \le t)\,\frac{(u/c+T_j)^{j-2}}{(j-2)!}\left(\frac{u/c+T_j}{j-1}-\frac{T_j}{j}\right)\right].$$

The announced formula (4.9) follows after some simplifications. ■

5. EPIDEMIC MODELS

5.1. Final Size

Consider a closed homogeneous population with initially n susceptibles and m infectives. The spread of the epidemic is of the Susceptible → Infective → Removed schema (see e.g., Andersson and Britton [1]). Any infective, j say, remains infectious during a random period of time X_j. The variables X_j are i.i.d. ($=_d X$). During its infectious period, the infective j exerts on each susceptible an infection pressure measured by $f(X_j)$ where f is a positive nondecreasing function. Any susceptible, i say, has a random resistance to infection R_i. The variables R_i are i.i.d. ($=_d R$), with a continuous distribution function F. The susceptible i is not infected by the (only) infective j if $R_i > f(X_j)$; it remains uninfected if R_i exceeds the sum of the infection pressures exerted by the infectives present. The epidemic terminates at time T when all the infectives, initial and subsequent, are removed. The random variable under study is the final size of the epidemic, F_T, which counts the new infected cases among the n initial susceptibles.

As long as the final size is concerned, an equivalent representation of the epidemic consists in following the set of n susceptibles after the infection first by the m initial infectives and then by each new infected case (e.g., Lefèvre and Utev [32]). In this framework, a final size of at least k individuals, $1 \le k \le n$, means that the k weakest susceptibles have not been able to resist to the infection pressure exerted by the m initial infectives and the first k new infectives. That is the starting point for deriving formula (5.1) below. Denote by $S_j = f(X_1) + \cdots + f(X_j)$ be the total infection pressure exerted by the first j infectives, $j \geq 1$.

PROPOSITION 5.1:

$$P(F_T = k) = n_{[k]}\, E\left\{(V_{m+k})^{n-k} G_k(1|V_m, \ldots, V_{m+k-1})\right\}, \quad 0 \leq k \leq n, \tag{5.1}$$

where

$$V_{m+i} = 1 - F(S_{m+i}), \quad 0 \leq i \leq n. \tag{5.2}$$

PROOF: Let $(R_{(1:n)}, \ldots, R_{(n:n)})$ be the order statistics associated with the n resistances $(R_1, \ldots, R_n)$. As explained above, we can assert that, for $1 \leq k \leq n$,

$$P(F_T \geq k) = P[R_{(1:n)} \leq S_m, R_{(2:n)} \leq S_{m+1}, \ldots, R_{(k:n)} \leq S_{m+k-1}].$$

Therefore, we have, for $0 \leq k \leq n$,

$$P(F_T = k) = P[R_{(1:n)} \leq S_m, R_{(2:n)} \leq S_{m+1}, \ldots, R_{(k:n)} \leq S_{m+k-1}, R_{(k+1:n)} > S_{m+k}]. \tag{5.3}$$

Suppose for the moment that $S_{m+j} = s_{m+j}$ are fixed, $j \geq 0$. As the R_i's are i.i.d., we see by a simple probabilistic argument that (5.3) can be rewritten as

$$P(F_T = k) = \binom{n}{k}[P(R > s_{m+k})]^{n-k} P[R_{(1:k)} \leq s_m, R_{(2:k)} \leq s_{m+1}, \ldots, R_{(k:k)} \leq s_{m+k-1}], \tag{5.4}$$

where the order statistics are associated with the set of k variables $(R_1, \ldots, R_k)$ only.

Since $(R_{(1:k)}, \ldots, R_{(k:k)})$ is distributed as $[F^{-1}(U_{(1:k)}), \ldots, F^{-1}(U_{(k:k)})]$, (5.4) is equivalent to

$$P(F_T = k) = \binom{n}{k}[P(R > s_{m+k})]^{n-k}\, P[U_{(1:k)} \leq F(s_m), \ldots, U_{(k:k)} \leq F(s_{m+k-1})],$$

By virtue of (3.2), we then obtain

$$P(F_T = k) = \binom{n}{k}[1 - F(s_{m+k})]^{n-k}\, k!(-1)^k\, G_k(0|F(s_m), \ldots, F(s_{m+k-1})). \tag{5.5}$$

Moreover, we know by (2.7) that

$$\begin{aligned}(-1)^k\, G_k(0|F(s_m), \ldots, F(s_{m+k-1})) &= G_k(0|-F(s_m), \ldots, -F(s_{m+k-1})) \\ &= G_k(1|1 - F(s_m), \ldots, 1 - F(s_{m+k-1})).\end{aligned}$$

Thus, (5.5) becomes

$$P(F_T = k) = n_{[k]}(v_{m+k})^{n-k} G_k(1|v_m, \ldots, v_{m+k-1}), \tag{5.6}$$

using the notation $v_{m+i} = 1 - F(s_{m+i})$ (see (5.2)). Finally, taking the expectation of (5.6) to remove the conditioning on the S_{m+j}'s yields the announced formula (5.1). ■

An analogous result is obtained in Ball and O'Neill [5] and Lefèvre and Picard [29]. The formula is rather easy to handle when the S_{m+j}'s are fixed, that is, when all the infectious

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periods are of constant length (not necessarily equal). Indeed, the expectation in (5.1) is then superfluous, and the distribution of F_T is expressed in terms of the A–G polynomials which can be computed recursively using (2.15). The situation, however, becomes more complicated if the infectious periods are random.

5.2. Exponential Resistances

A special case for which calculations remain simple is when the resistances of the susceptibles are exponentially distributed, that is, if $F(x) = 1 - e^{-rx}$, $x > 0$ $(r > 0)$.

Proposition 5.2:

$$P(F_T = k) = n_{[k]} \sum_{j=0}^{k} \frac{1}{j!} [q(n-j)]^{m+j} G_{k-j}(0|q(n-k+i), i \geq 0), \quad 0 \leq k \leq n, \qquad \textbf{(5.7)}$$

where the parameters $q(i)$ *are defined by*

$$q(i) = E(e^{-rif(X)}), \quad i \geq 0. \qquad \textbf{(5.8)}$$

Proof: By (5.1) and (5.2) with $F(x) = 1 - e^{-rx}$, we have

$$P(F_T = k) = n_{[k]} E\left\{ e^{-rS_{m+k}(n-k)} G_k(1|e^{-rS_m}, e^{-rS_{m+1}}, \ldots, e^{-rS_{m+k-1}}) \right\},$$

for $0 \leq k \leq n$. Put $Y = e^{-rS_m}$ and $\Pi_j = Z_1 \ldots Z_j$ where $Z_j = e^{-rf(X_{m+j})}$, $j \geq 1$. Then, this probability can be rewritten as

$$P(F_T = k) = n_{[k]} E\left\{ Y^{n-k} (\Pi_k)^{n-k} G_k(1|Y, Y\Pi_1, \ldots, Y\Pi_{k-1}) \right\}. \qquad \textbf{(5.9)}$$

Applying the identity (3.8) to the expectation in (5.9) now yields

$$P(F_T = k) = n_{[k]} \sum_{j=0}^{k} \frac{1}{j!} E(Y^{n-j}) [E(Z^{n-j})]^j G_{k-j}(0|E(Z^{n-k+i}), i \geq 0), \quad n \geq 0. \qquad \textbf{(5.10)}$$

By definition of Y and Z and the notation (5.8),

$$E(Y^{n-j}) = [q(n-j)]^m \;\text{ and }\; E(Z^{n-j}) = q(n-j), \quad j \geq 0,$$

so that (5.10) reduces to the desired formula (5.7). ■

The model with exponential resistances is analogous to the randomized Reed–Frost process studied by for example, Ball [4], Martin-Löf [33], Picard and Lefèvre [37], Ball and O'Neill [5], Clancy [10]. If the infectious periods are of constant length c, the model corresponds to the classical Reed–Frost process for which, by (5.8),

$$q(i) = q^i \text{ where } q = e^{-rf(c)}, \quad i \geq 0.$$

For such epidemics, it is well-known that the distribution of F_T is also given as the solution of a triangular system of linear equations. We rederive this result below from formula (5.7).

Corollary 5.3:

$$E\left\{\binom{n-F_T}{l}\frac{1}{[q(l)]^{F_T}}\right\}=\binom{n}{l}[q(l)]^m,\quad 1\le l\le n. \tag{5.11}$$

Proof: Let E_l denote the l.h.s. of (5.11). From (5.7), we have

$$E_l=\frac{n!}{l!}\sum_{k=0}^{n-l}\frac{1}{(n-l-k)!}\,\frac{1}{[q(l)]^k}\sum_{j=0}^{k}\frac{1}{j!}\,[q(n-j)]^{m+j}\,G_{k-j}(0|q(n-k+i),\,i\ge 0).$$

After permutation of the two sums and insertion of a factor $[q(l)]^{n-l}$, we get

$$E_l=\frac{n!}{l!}\sum_{j=0}^{n-l}\frac{1}{j!}\,[q(n-j)]^{m+j}\,\frac{1}{[q(l)]^{n-l}}\,F_{l,j}, \tag{5.12}$$

where

$$F_{l,j}\equiv\sum_{k=j}^{n-l}\frac{1}{(n-l-k)!}\,[q(l)]^{n-l-k}\,G_{k-j}(0|q(n-k+i),\,i\ge 0),\quad 0\le j\le n-l.$$

Putting k for $k-j$ in $F_{l,j}$ gives

$$F_{l,j}\equiv\sum_{k=0}^{n-l-j}\frac{1}{(n-l-j-k)!}\,[q(l)]^{n-l-j-k}\,G_k(0|q(n-k-j+i),\,i\ge 0), \tag{5.13}$$

in which we can write, by virtue of (2.9),

$$G_k(0|q(n-k-j+i),\,i\ge 0)=G_{n-l-j}^{(n-l-j-k)}(0|q(l+i),\,i\ge 0).$$

Applying Taylor's formula, we obtain from (5.13) that

$$F_{l,j}=G_{n-l-j}(q(l)|q(l+i),\,i\ge 0),\quad 0\le j\le n-l.$$

By (2.10), we then deduce that $F_{l,n-l}=1$, and $F_{l,j}=0$ for $0\le j\le n-l-1$. Therefore, the l.h.s. E_l reexpressed by (5.12) reduces to $\binom{n}{l}[q(l)]^m$, that is, the r.h.s. of (5.11). ■

Furthermore, a closed expression for the probability generating function of $S_T=n-F_T$ easily follows from (5.7).

Corollary 5.4: *For x real,*

$$E(x^{S_T})=\sum_{k=0}^{n}n_{[k]}[q(k)]^{m+n-k}G_k(x|q(i),\,i\ge 0). \tag{5.14}$$

PROOF: Inserting (5.7), we get

$$E(x^{S_T}) = \sum_{k=0}^{n} x^{n-k} n_{[k]} \sum_{j=0}^{k} \frac{1}{j!} [q(n-j)]^{m+j} G_{k-j}(0|q(n-k+i), i \geq 0)$$
$$= \sum_{j=0}^{n} \frac{1}{j!} [q(n-j)]^{m+j} n_{[j]} \sum_{k=j}^{n} x^{n-k} (n-j)_{[k-j]} G_{k-j}(0|q(n-k+i), i \geq 0)$$
$$= \sum_{j=0}^{n} \binom{n}{j} [q(n-j)]^{m+j} \sum_{k=0}^{n-j} x^{n-j-k} (n-j)_{[k]} G_k(0|q(n-j-k+i), i \geq 0).$$

Since by (2.9),

$$G_{n-j}^{(n-j-k)}(0|q(i), i \geq 0) = G_k(0|q(n-j-k+i), i \geq 0),$$

Taylor's formula gives

$$G_{n-j}(x|q(i), i \geq 0) = \sum_{k=0}^{n-j} \frac{x^{n-j-k}}{(n-j-k)!} G_k(0|q(n-j-k+i), i \geq 0).$$

By substitution, we then obtain

$$E(x^{S_T}) = \sum_{j=0}^{n} \binom{n}{j} [q(n-j)]^{m+j} (n-j)! \, G_{n-j}(x|q(i), i \geq 0),$$

which corresponds to formula (5.14). ■

Remark: An alternative approach to derive these results is by exhibiting and exploiting a family of martingales (see e.g., Picard [36], Lefèvre and Picard [28], O'Neill [35], Clancy [9]). We briefly show in the Appendix how to apply that method to the present model.

5.3. Final Severity

The final severity A_T is defined as the sum of the infectious periods until time T. It represents an important component of the cost of the epidemic (e.g., Gani and Jerwood [19]). The previous calculations can be easily adapted to incorporate A_T in the analysis. Let $A_j = X_1 + \cdots + X_j$ denote the sum of the first j infectious periods, $j \geq 1$.

PROPOSITION 5.5: *For $\theta \geq 0$,*

$$E\left(1_{(F_T=k)}\, e^{-\theta A_T}\right) = n_{[k]} E\left\{(V_{m+k})^{n-k}\, e^{-\theta A_{m+k}}\, G_k(1|V_m, \ldots, V_{m+k-1})\right\}, \quad 0 \leq k \leq n.$$

When the resistances are exponentially distributed,

$$E\left(1_{(F_T=k)}\, e^{-\theta A_T}\right) = n_{[k]} \sum_{j=0}^{k} \frac{1}{j!} [q(n-j,\theta)]^{m+j} G_{k-j}(0|q(n-k+i,\theta), i \geq 0), \quad 0 \leq k \leq n,$$

where the parameters $q(i,\theta)$ are defined by

$$q(i,\theta) = E(e^{-rif(X)-\theta X}), \quad i \geq 0.$$

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As a consequence,

$$E\left\{\binom{n-F_T}{l}\frac{1}{[q(l,\theta)]^{m+F_T}}\,e^{-\theta A_T}\right\}=\binom{n}{l},\quad 1\le l\le n,$$

and for x real,

$$E(x^{S_T}e^{-\theta A_T})=\sum_{k=0}^{n}n_{[k]}[q(k,\theta)]^{m+n-k}G_k(x|q(i,\theta),\,i\ge 0).$$

The proofs are omitted for brevity reasons. A small generalization of formula (3.8) is needed to deal with the exponential special case; it is stated in the Remark below.

Remark: Let $\tilde{\Pi}_j=\tilde{X}_1\ldots\tilde{X}_j$, $j\ge 1$, where the $\tilde{X}_j$'s are i.i.d. random variables (distributed as $\tilde{X}$), with each $\tilde{X}_j$ allowed us to depend on X_j. Let Z be a random variable independent of the X_j and $\tilde{X}_j$'s, but allowed us to depend on Y. Then, (3.8) can be extended as follows: for any reals α,β,x,

$$E[Y^\alpha Z(\Pi_n)^\beta\tilde{\Pi}_nG_n(x|Y\Pi_i,\,i\ge 0)]=$$
$$\sum_{j=0}^{n}\frac{x^j}{j!}\,E(Y^{\alpha+n-j}Z)\,[E(X^{\beta+n-j}\tilde{X})]^j\,G_{n-j}(0|E(X^{\beta+i}\tilde{X}),\,i\ge 0),\quad n\ge 0,$$

where all the involved moments are assumed to be finite.

5.4. Dependent Uniform Resistances

The maximum infection that can be generated in the population is of amount S_{m+n}. A simple assumption consists in considering that the resistances of the susceptibles are uniformly distributed on the interval $(0,S_{m+n})$, that is,

$$R_i=_d S_{m+n}U_i,\quad 1\le i\le n,$$

where the U_i's are independent $(0,1)$-uniform random variables. Observe that the R_i's are now dependent through the common factor S_{m+n}. We also note that by a well-known theorem (Khintchine [26]), the marginal density of R_i is then a nonincreasing function, which seems to be realistic for many situations.

Proposition 5.6:

$$P(F_T=k)=\binom{n}{k}\frac{m}{m+k}\,E\left[\left(1-\frac{S_{m+k}}{S_{m+n}}\right)^{n-k}\left(\frac{S_{m+k}}{S_{m+n}}\right)^k\right],\quad 0\le k\le n.\qquad \textbf{(5.15)}$$

Proof: Let us first proceed as for (5.1). Conditionally on $S_{m+j}=s_{m+j}$, $j\ge 0$, the p.m.f. of F_T is still given by the formula (5.6). Moreover, $F(x)=x/s_{m+n}$ in the present case.

Thus, we get by (2.7)

$$P(F_T = k) = n_{[k]}\, E\left[\left(1 - \frac{S_{m+k}}{S_{m+n}}\right)^{n-k} G_k\left(1|1 - \frac{S_m}{S_{m+n}}, \ldots, 1 - \frac{S_{m+k-1}}{S_{m+n}}\right)\right]$$
$$= n_{[k]}\, E\left[\frac{(S_{m+n} - S_{m+k})^{n-k}}{(S_{m+n})^n} G_k(0| - S_m, \ldots, -S_{m+k-1})\right],$$

for $0 \le k \le n$. Using $E(.) = E[E(.|S_{m+k}, S_{m+n})]$ then yields

$$P(F_T = k) = n_{[k]}\, E\left\{\frac{(S_{m+n} - S_{m+k})^{n-k}}{(S_{m+n})^n} E[G_k(0| - S_m, \ldots, -S_{m+k-1})|S_{m+k}]\right\}. \quad \mathbf{(5.16)}$$

It remains to evaluate the inner conditional expectation. By (2.12) and (2.7),

$$E[G_k(0| - S_m, \ldots, -S_{m+k-1})|S_{m+k}] = E[A_k(0| - S_{m+k-1}, \ldots, -S_m)|S_{m+k}]$$
$$= E\{A_k[S_{m+k}|f(X_{m+k}), \ldots, f(X_{m+k}) + \cdots + f(X_{m+1})]|S_{m+k}\}. \quad \mathbf{(5.17)}$$

Put $\tilde{X}_i = X_{m+k-i+1}$ and $\tilde{S}_i = f(\tilde{X}_1) + \cdots + f(\tilde{X}_i)$, $1 \le i \le m+k$; in particular, $\tilde{S}_{m+k} = S_{m+k}$. Then, (5.17) can be rewritten as

$$E[A_k(\tilde{S}_{m+k}|\tilde{S}_1, \ldots, \tilde{S}_k)|\tilde{S}_{m+k}],$$

and by (3.4), this expectation is equal to

$$\frac{(\tilde{S}_{m+k})^{k-1}}{(k-1)!}\left(\frac{\tilde{S}_{m+k}}{k} - \frac{\tilde{S}_{m+k}}{m+k}\right) = \frac{(S_{m+k})^{k-1}}{(k-1)!}\left(\frac{S_{m+k}}{k} - \frac{S_{m+k}}{m+k}\right) = \frac{(S_{m+k})^k}{k!}\frac{m}{m+k}. \quad \mathbf{(5.18)}$$

Substituting (5.18) for (5.17) and inserting in (5.16) then provides the formula (5.15). ■

If the infectious periods are of constant length c, (5.15) gives

$$P(F_T = k) = \binom{n}{k}\frac{m}{m+k}\left(1 - \frac{m+k}{m+n}\right)^{n-k}\left(\frac{m+k}{m+n}\right)^k,$$

independently of c and the function f. Putting $p_1 = m/(m+n)$ and $p_2 = 1/(m+n)$, this can be rewritten as

$$P(F_T = k) = \binom{n}{k} p_1\, (p_1 + p_2 k)^{k-1}(1 - p_1 - p_2 k)^{n-k}, \quad 0 \le k \le n, \quad \mathbf{(5.19)}$$

which shows that F_T has a quasi-binomial distribution (Consul [11]). We note that the same distribution is derived by Islam, O'Shaughnessy and Smith [23] for the final size of household infections in a random graph model. We also refer to Dobson, Carreras, and Newman [18] and Lefèvre [27] for the total number of failures in a cascading failure model.

Remark: A similar argument allows us to treat the more general case in which $R_i =_d S_{m+n} U_i^{(\rho)}$, $1 \le i \le n$, where the $U_i^{(\rho)}$'s are independent $(\rho, 1)$-uniform random variables $(0 < \rho < 1)$. Such an extension is relevant when the susceptibles possess a positive minimum level of resistance to infection.

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APPENDIX

An alternative approach to the final epidemic size is by using martingale theory. This is briefly shown below in the special case of exponential resistances.

The spread of infection is represented through the time scale described in Section 5.1. Denote by F_1 the number of new infections caused by the m initial infectives, and let F_t, $t \geq 2$, be the number of new infections caused by the m initial infectives and the first $t-1$ new infectives (if ever). Here too, the $q(i)$'s are the parameters defined by (5.8).

LEMMA A..1. *For each $1 \leq l \leq n$,*

$$E\left[\binom{n-F_1}{l}\right] = \binom{n}{l}[q(l)]^m, \tag{A.1}$$

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and conditionally on $n - F_1$,

$$\left\{ \binom{n-F_t}{l} \frac{1}{[q(l)]^t},\ t \geq 1 \right\} \text{ is a martingale.} \tag{A.2}$$

Proof: The susceptibles present at time $t \geq 0$ are in number $n - F_t$ $(F_0 = 0)$. Consider all the subsets of l individuals among these $n - F_t$ susceptibles, for any $1 \leq l \leq n$. Clearly, we can write that

$$\binom{n-F_{t+1}}{l} = \sum_{\alpha=1}^{\binom{n-F_t}{l}} \mathbf{1}(\alpha), \quad t \geq 0,$$

where $\mathbf{1}(\alpha)$ is the indicator of the event [the group of l susceptibles at time t labelled α remains uninfected at time $t+1$]. Remember that $F(x) = e^{-rx}$ by assumption. When $t = 0$, we then have

$$E\left[\binom{n-F_1}{l} | S_m = s_m\right] = \binom{n}{l} [P(R > s_m)]^l = \binom{n}{l} e^{-rlsS_m},$$

so that taking the expectation with respect to S_m gives (A.1). For $t \geq 1$, we get

$$\begin{aligned} &E\left[\binom{n-F_{t+1}}{l} | F_t, S_{m+t} = s_{m+t}, S_{m+t-1} = s_{m+t-1}\right] \\ &= \binom{n-F_t}{l} [P(R > s_{m+t} | R > s_{m+t-1}]^l \\ &= \binom{n-F_t}{l} \frac{[1-F(s_{m+t})]^l}{[1-F(s_{m+t-1})]^l} = \binom{n-F_t}{l} e^{-rlf(x_{m+t})} \text{ a.s.}, \end{aligned}$$

where $x_{m+t} = s_{m+t} - s_{m+t-1}$. Taking again the expectation then gives

$$E\left[\binom{n-F_{t+1}}{l} | F_t\right] = \binom{n-F_t}{l} q(l), \quad t \geq 1,$$

hence the assertion (A.2). ■

Corollary A..2: *The distribution of* F_T *satisfies the* n *relations (5.11).*

Proof: The epidemic terminates at the first time T where there are no more infectives present. Applying the martingale stopping theorem to (A.1), (A.2), we obtain

$$\begin{aligned} E\left[\binom{n-F_T}{l} \frac{1}{[q(l)]^T}\right] &= E\left[\binom{n-F_1}{l} \frac{1}{q(l)}\right] \\ &= \binom{n}{l} [q(l)]^{m-1}. \end{aligned} \tag{A.3}$$

Now, we see that $T = 1, 2, 3, \ldots$, means $F_1 = 0, F_2 = 1, F_3 = 2, \ldots$, respectively. In fact, the identity $T - 1 = F_T$ holds true. Inserting this in (A.3) yields the relations (5.11). ■