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Semi-invariant Submersions from Almost Hermitian Manifolds

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Abstract. We introduce semi-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds. We give examples, investigate the geometry of foliations that arise from the definition of a Riemannian submersion, and find necessary sufficient conditions for total manifold to be a locally product Riemannian manifold. We also find necessary and sufficient conditions for a semi-invariant submersion to be totally geodesic. Moreover, we obtain a classification for semi-invariant submersions with totally umbilical fibers and show that such submersions put some restrictions on total manifolds.

1 Introduction

A Riemannian submersion is a smooth submersion $F\colon M_1 \to M_2$ between two Riemannian manifolds (M_1, g_1) and (M_2, g_2) with the property that at any point $p \in M_1$,

$$g_{1p}(x, y) = g_{2F(p)}\big(F_*(x), F_*(y)\big)$$

for any x, y in the tangent space T_pM_1 to M_1 at $p \in M_1$ that are perpendicular to the kernel of F_*.

Riemannian submersions between Riemannian manifolds were studied by O'Neill [9] and Gray [6]. Later such submersions have been studied widely in differential geometry; for details, see [5]. Riemannian submersions between Riemannian manifolds equipped with an additional structure of almost complex type was first studied by Watson in [11]. Watson defined almost Hermitian submersions between almost Hermitian manifolds, and he showed that the base manifold and each fiber have the same kind of structure as the total space, in most cases. More precisely, let M_1 be a complex $m-$dimensional almost Hermitian manifold with Hermitian metric g_1 and almost complex structure J_1, and let M_2 be a complex $n-$dimensional almost Hermitian manifold with Hermitian metric g_2 and almost complex structure J_2. A Riemannian submersion $F\colon M_1 \to M_2$ is called an almost Hermitian submersion if F is an almost complex mapping, *i.e.*, $F_* J_1 = J_2 F_*$. The main result of this notion is that the vertical and horizontal distributions are J_1-invariant. On the other hand, Escobales [4] studied Riemannian submersions from complex projective space onto a Riemannian manifold under the assumption that the fibers are connected, complex, totally geodesic submanifolds. In fact, this assumption also implies that the vertical distribution is invariant with respect to the almost complex structure. We note that

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almost Hermitian submersions have been extended to the almost contact manifolds [3], locally conformal Kähler manifolds [8], and quaternion Kähler manifolds [7].

All these submersions mentioned above have one common property. In these submersions vertical and horizontal distributions are invariant. Therefore, recently we have introduced the notion of anti-invariant Riemannian submersions, which are Riemannian submersions from almost Hermitian manifolds such that their vertical distribution is anti-invariant under the almost complex structure of the total manifold ([10]).

In this paper, we introduce semi-invariant Riemannian submersions as a generalization of anti-invariant Riemannian submersions and almost Hermitian submersions when the base manifold is an almost Hermitian manifold. We show that such submersions are useful to investigate the geometry of the total manifold of the submersion.

The paper is organized as follows. In Section 2, we give brief information about almost Hermitian manifolds, Riemannian submersions, and distributions that are defined by the Riemannian submersion. In Section 3, we define semi-invariant Riemannian submersion, give examples, and investigate the geometry of its leaves. Then we use these results to obtain decomposition theorems for the total manifold. We also find necessary and sufficient conditions for semi-invariant submersions to be totally geodesic. In Section 4, we first show that the notion of semi-invariant submersions puts some restrictions on the sectional curvature of the total manifold when it is a complex space form. Then we obtain a classification theorem of semi-invariant submersions with totally umbilical fibers.

2 Preliminaries

In this section, we define almost Hermitian manifolds, recall the notion of Riemannian submersions between Riemannian manifolds and give a brief review of basic facts of Riemannian submersions.

Let $(\overline{M}, g)$ be an almost Hermitian manifold. This means that $\overline{M}$ admits a tensor field J of type $(1, 1)$ on $\overline{M}$ such that, $\forall X, Y \in \Gamma(T\overline{M})$, we have ([13])

$$J^2 = -I, \quad g(X, Y) = g(JX, JY). \tag{2.1}$$

An almost Hermitian manifold $\overline{M}$ is called a Kähler manifold if

$$(\overline{\nabla}_X J)Y = 0, \forall X, Y \in \Gamma(T\overline{M}), \tag{2.2}$$

where $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$.

Let (M_1^m, g_1) and (M_2^n, g_2) be Riemannian manifolds, where $\dim(M_1) = m$, $\dim(M_2) = n$, and $m > n$. A Riemannian submersion $F\colon M_1 \to M_2$ is a map from M_1 onto M_2 satisfying the following axioms:

(S1) F has maximal rank.

(S2) The differential F_* preserves the lenghts of horizontal vectors.

For each $q \in M_2$, $F^{-1}(q)$ is an $(m - n)$ dimensional submanifold of M_1. The submanifolds $F^{-1}(q)$, $q \in M_2$ are called fibers. A vector field on M_1 is called vertical if

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it is always tangent to fibers. A vector field on M_1 is called horizontal if it is always orthogonal to fibers. A vector field X on M_1 is called basic if X is horizontal and F-related to a vector field X_* on M_2, *i.e.*, $F_*X_p = X_{*F(p)}$ for all $p \in M_1$. Note that we denote the projection morphisms on the distributions $\ker F_*$ and $(\ker F_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively.

We recall the following lemma from O'Neill [9].

Lemma 2.1 *Let $F\colon M_1 \to M_2$ be a Riemannian submersion between Riemannian manifolds and X, Y be basic vector fields of M_1. Then we have*

(a) $g_1(X,Y) = g_2(X_*, Y_*) \circ F$;
(b) *the horizontal part $[X,Y]^{\mathcal{H}}$ of $[X,Y]$ is a basic vector field and corresponds to* $[X_*, Y_*]$, i.e., $F_*([X,Y]^{\mathcal{H}}) = [X_*, Y_*]$;
(c) $[V,X]$ *is vertical for any vector field V of* $\ker F_*$;
(d) $(\overset{1}{\nabla}_X Y)^{\mathcal{H}}$ *is the basic vector field corresponding to $\overset{2}{\nabla}_{X_*} Y_*$, where $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ are the Levi-Civita connections of g_1 and g_2, respectively.*

The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ defined for vector fields E, F on M_1 by

$$\mathcal{A}_E F = \mathcal{H}\overset{1}{\nabla}_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\overset{1}{\nabla}_{\mathcal{H}E}\mathcal{H}F, \tag{2.3}$$

$$\mathcal{T}_E F = \mathcal{H}\overset{1}{\nabla}_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\overset{1}{\nabla}_{\mathcal{V}E}\mathcal{H}F. \tag{2.4}$$

It is easy to see that a Riemannian submersion $F\colon M_1 \to M_2$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. For any $E \in \Gamma(TM_1)$, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on $(\Gamma(TM_1), g)$ reversing the horizontal and the vertical distributions. It is also easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$, and $\mathcal{A}_E$ is horizontal, $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E}$. We note that the tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy

$$\mathcal{T}_U W = \mathcal{T}_W U, \ \forall U, W \in \Gamma(\ker F_*)$$
$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \tfrac{1}{2}\mathcal{V}[X,Y], \ \forall X, Y \in \Gamma\big((\ker F_*)^\perp\big).$$

On the other hand, from (2.3) and (2.4) we have

$$\overset{1}{\nabla}_V W = \mathcal{T}_V W + \widehat{\nabla}_V W, \tag{2.5}$$

$$\overset{1}{\nabla}_V X = \mathcal{H}\overset{1}{\nabla}_V X + \mathcal{T}_V X, \tag{2.6}$$

$$\overset{1}{\nabla}_X V = \mathcal{A}_X V + \mathcal{V}\overset{1}{\nabla}_X V, \tag{2.7}$$

$$\overset{1}{\nabla}_X Y = \mathcal{H}\overset{1}{\nabla}_X Y + \mathcal{A}_X Y \tag{2.8}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$, where $\widehat{\nabla}_V W = \mathcal{V}\overset{1}{\nabla}_V W$. If X is basic, then $\mathcal{H}\overset{1}{\nabla}_V X = \mathcal{A}_X V$.

Finally, we recall the notion of the second fundamental form of a map between Riemannian manifolds. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds and suppose that $\varphi\colon M_1 \to M_2$ is a smooth map between them. Then the differential φ_* of φ can be viewed as a section of the bundle $\mathrm{Hom}(TM_1)\varphi^{-1}TM_2) \to M_1$, where $\varphi^{-1}TM_2$ is the pullback bundle that has fibers $(\varphi^{-1}TM_2)_p = T_{\varphi(p)}M_2$, $p \in M_1$. $\mathrm{Hom}(TM_1, \varphi^{-1}TM_2)$ has a connection ∇ induced from the Levi-Civita connection $\overset{1}{\nabla}$ and the pullback connection. Then the second fundamental form of φ is given by

$$(\nabla\varphi_*)(X,Y) = \nabla^{\varphi}_X\varphi_*(Y) - \varphi_*(\overset{1}{\nabla}_X Y) \tag{2.9}$$

for $X, Y \in \Gamma(TM_1)$, where ∇^{φ} is the pullback connection. It is known that the second fundamental form is symmetric.

3 Semi-invariant Riemannian Submersions

In this section, we define semi-invariant Riemannian submersions from an almost Hermitian manifold onto a Riemannian manifold, investigate the integrability of distributions and obtain a necessary and sufficient condition for such submersions to be totally geodesic map. We also obtain two decomposition theorems for the total manifolds of such submersions.

Definition 3.1 Let M_1 be a complex m-dimensional almost Hermitian manifold with Hermitian metric g_1 and almost complex structure J, and let M_2 be a Riemannian manifold with Riemannian metric g_2. A Riemannian submersion $F\colon M_1 \to M_2$ is called a *semi-invariant Riemannian submersion* if there is a distribution $\mathcal{D}_1 \subseteq \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2 \quad \text{and} \quad J(\mathcal{D}_1) = \mathcal{D}_1,\ J(\mathcal{D}_2) \subseteq (\ker F_*)^{\perp},$$

where $\mathcal{D}_2$ is orthogonal complementary to $\mathcal{D}_1$ in $\ker F_*$.

We note that it is known that the distribution $\ker F_*$ is integrable. Hence, Definition 3.1 implies that the integral manifold (fiber) $F^{-1}(q)$, $q \in M_2$, of $\ker F_*$ is a CR-submanifold of M_1. For CR-submanifolds, see [1, 2, 12]. We now give some examples of semi-invariant Riemannian submersions.

Example 1 Every anti-invariant Riemannian submersion from an almost Hermitian manifold onto a Riemannian manifold is a semi-invariant Riemannian submersion with $\mathcal{D}_1 = \{0\}$.

Example 2 Every Hermitian submersion from an almost Hermitian manifold onto an almost Hermitian manifold is a semi-invariant submersion with $\mathcal{D}_2 = \{0\}$.

Example 3 Let F be a submersion defined by

$$\begin{array}{cccc} F: & R^6 & \longrightarrow & R^3 \\ & (x_1, x_2, x_3, x_4, x_5, x_6) & \longmapsto & (\frac{x_1+x_2}{\sqrt{2}}, \frac{x_3+x_5}{\sqrt{2}}, \frac{x_4+x_6}{\sqrt{2}}) \end{array}.$$

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Then it follows that

$$\ker F_* = \operatorname{span}\{V_1 = -\partial x_1 + \partial x_2, V_2 = -\partial x_3 + \partial x_5, V_3 = -\partial x_4 + \partial x_6\}$$

and

$$(\ker F_*)^\perp = \operatorname{span}\{X_1 = \partial x_1 + \partial x_2, X_2 = \partial x_3 + \partial x_5, X_3 = \partial x_4 + \partial x_6\}.$$

Hence we have $JV_2 = V_3$ and $JV_1 = -X_1$. Thus it follows that $\mathcal{D}_1 = \operatorname{span}\{V_2, V_3\}$ and $\mathcal{D}_2 = \operatorname{span}\{V_1\}$. Moreover one can see that $\mu = \operatorname{span}\{X_2, X_3\}$. By direct computations, we also have

$$g_{R^6}(JV_1, JV_1) = g_{R^3}\big(F_*(JV_1), F_*(JV_1)\big),$$

$$g_{R^6}(X_2, X_2) = g_{R^3}\big(F_*(X_2), F_*(X_2)\big) \quad g_{R^6}(X_3, X_3) = g_{R^3}\big(F_*(X_3), F_*(X_3)\big),$$

which shows that F is a Riemannian submersion. Thus F is a semi-invariant Riemannian submersion.

We now investigate the integrability of the distributions $\mathcal{D}_1$ and $\mathcal{D}_2$. Since fibers of semi-invariant submersions from Kähler manifolds are CR-submanifolds and $\mathcal{T}$ is the second fundamental form of the fibers, the following results can be deduced from [1, Theorem 1.1, p. 39].

Lemma 3.2 *Let F be a semi-invariant Riemannian submersion from a Kähler manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). Then*

(i) *the distribution $\mathcal{D}_2$ is always integrable, and*
(ii) *The distribution $\mathcal{D}_1$ is integrable if and only if $g_1(T_X JY - T_Y JX, JZ) = 0$ for $X, Y \in \Gamma(\mathcal{D}_1)$ and $Z \in \Gamma(\mathcal{D}_2)$.*

Let F be a semi-invariant Riemannian submersion from a Kähler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). We denote the complementary distribution to $J\mathcal{D}_2$ in $(\ker F_*)^\perp$ by μ. Then for $V \in \Gamma(\ker F_*)$, we write

$$JV = \phi V + \omega V, \tag{3.1}$$

where $\phi V \in \Gamma(\mathcal{D}_1)$ and $\omega V \in \Gamma(J\mathcal{D}_2)$. Also for $X \in \Gamma((\ker F_*)^\perp)$, we have

$$JX = \mathcal{B}X + \mathcal{C}X, \tag{3.2}$$

where $\mathcal{B}X \in \Gamma(\mathcal{D}_2)$ and $\mathcal{C}X \in \Gamma(\mu)$. Then, by using (3.1), (3.2), (2.5), and (2.6) we get

$$(\nabla_V \phi)W = \mathcal{B}\mathcal{T}_V W - \mathcal{T}_V \omega W, \quad (\nabla_V \omega)W = \mathcal{C}\mathcal{T}_V W - \mathcal{T}_V \phi W,$$

for $V, W \in \Gamma(\ker F_*)$, where

$$(\nabla_V \phi)W = \widehat{\nabla}_V \phi W - \phi \widehat{\nabla}_V W \quad \text{and} \quad (\nabla_V \omega)W = \mathcal{H}\overset{1}{\nabla}_V \omega W - \omega \widehat{\nabla}_V W.$$

The proof of the following proposition can be deduced from [1, Theorem 5.1, p. 63].

Proposition 3.3 *Let F be a semi-invariant Riemannian submersion from a Kähler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then the fibers of F are locally product Riemannian manifolds if and only if $(\nabla_V \phi)W = 0$ for $V, W \in \Gamma(\ker F_*)$.*

We now obtain necessary and sufficient conditions for a semi-invariant submersion to be totally geodesic. We recall that a differentiable map F between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called a totally geodesic map if $(\nabla F_*)(X, Y) = 0$ for all $X, Y \in \Gamma(TM_1)$.

Theorem 3.4 *Let F be a semi-invariant submersion from a Kähler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then F is a totally geodesic map if and only if*

(a) $\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y$ *and* $\widehat{\nabla}_X \mathcal{B}Z + \mathcal{T}_X \mathcal{C}Z$ *belong to* $\mathcal{D}_1$;
(b) $\mathcal{H}\overset{1}{\nabla}_X \omega Y + T_X \phi Y$ *and* $\mathcal{T}_X \mathcal{B}Z + \mathcal{H}\overset{1}{\nabla}_X \mathcal{C}Z$ *belong to* $J\mathcal{D}_2$

for $X, Y \in \Gamma(\ker F_*)$ *and* $Z \in \Gamma((\ker F_*)^\perp)$.

Proof First of all, since F is a Riemannian submersion, we have

$$(\nabla F_*)(Z_1, Z_2) = 0, \forall Z_1, Z_2 \in \Gamma((\ker F_*)^\perp). \tag{3.3}$$

For $X, Y \in \Gamma(\ker F_*)$, we get $(\nabla F_*)(X, Y) = -F_*(\overset{1}{\nabla}_X Y)$. Then from (2.2) we get $(\nabla F_*)(X, Y) = F_*(J\overset{1}{\nabla}_X JY)$. Using (3.1) we have $(\nabla F_*)(X, Y) = F_*(J\overset{1}{\nabla}_X \phi Y + J\overset{1}{\nabla}_X \omega Y)$. Then from (2.5) and (2.6) we arrive at

$$(\nabla F_*)(X, Y) = F_*\big(J(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \phi Y + \mathcal{H}\overset{1}{\nabla}_X \omega Y + \mathcal{T}_X \omega Y)\big).$$

Using (3.1) and (3.2) in above equation we obtain

$$\begin{aligned}(\nabla F_*)(X, Y) = F_*\big(&\phi\widehat{\nabla}_X \phi Y + \omega\widehat{\nabla}_X \phi Y + \mathcal{B}\mathcal{T}_X \phi Y \\ &+ \mathcal{C}\mathcal{T}_X \phi Y + \mathcal{B}\mathcal{H}\overset{1}{\nabla}_X \omega Y + \mathcal{C}\mathcal{H}\overset{1}{\nabla}_X \omega Y + \phi\mathcal{T}_X \omega Y + \omega\mathcal{T}_X \omega Y\big).\end{aligned}$$

Since $\phi\widehat{\nabla}_X \phi Y + \mathcal{B}\mathcal{T}_X \phi Y + \phi\mathcal{T}_X \omega Y + \mathcal{B}\mathcal{H}\overset{1}{\nabla}_X \omega Y \in \Gamma(\ker F_*)$, we derive

$$(\nabla F_*)(X, Y) = F_*(\omega\widehat{\nabla}_X \phi Y + \mathcal{C}\mathcal{T}_X \phi Y + \mathcal{C}\mathcal{H}\overset{1}{\nabla}_X \omega Y + \omega\mathcal{T}_X \omega Y).$$

Then, since F is a linear isometry between $(\ker F_*)^\perp$ and TM_2, $(\nabla F_*)(X, Y) = 0$ if and only if $\omega\widehat{\nabla}_X \phi Y + \mathcal{C}\mathcal{T}_X \phi Y + \mathcal{C}\mathcal{H}\overset{1}{\nabla}_X \omega Y + \omega\mathcal{T}_X \omega Y = 0$. Thus $(\nabla F_*)(X, Y) = 0$ if and only if

$$\omega(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y) = 0, \quad \mathcal{C}(\mathcal{T}_X \phi Y + \mathcal{H}\overset{1}{\nabla}_X \omega Y) = 0. \tag{3.4}$$

In a similar way for $X \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$, $(\nabla F_*)(X, Z) = 0$ if and only if

$$\omega(\widehat{\nabla}_X \mathcal{B}Z + \mathcal{T}_X \mathcal{C}Z) = 0, \quad \mathcal{C}(\mathcal{T}_X \mathcal{B}Z + \mathcal{H}\overset{1}{\nabla}_X \mathcal{C}Z) = 0. \tag{3.5}$$

The proof follows from (3.3)–(3.5). ∎

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We now investigate the geometry of leaves of the distribution $(\ker F_*)^\perp$.

Proposition 3.5 *Let F be a semi-invariant submersion from a Kähler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then the distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation if and only if*

$$\mathcal{A}_{Z_1}\mathcal{B}Z_2 + \mathcal{H}\overset{1}{\nabla}_{Z_1}\mathcal{C}Z_2 \in \Gamma(\mu), \quad \mathcal{A}_{Z_1}\mathcal{C}Z_2 + \mathcal{V}\overset{1}{\nabla}_{Z_1}Z_2 \in \Gamma(\mathcal{D}_2)$$

for $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$.

Proof From (2.1) and (2.2) we have $\overset{1}{\nabla}_{Z_1}Z_2 = -J\overset{1}{\nabla}_{Z_1}JZ_2$ for $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$. Using (3.2), (2.7), and (2.8) we obtain

$$\overset{1}{\nabla}_{Z_1}Z_2 = -J(A_{Z_1}\mathcal{B}Z_2 + \mathcal{V}\overset{1}{\nabla}_{Z_1}\mathcal{B}Z_2) - J(\mathcal{H}\overset{1}{\nabla}_{Z_1}\mathcal{C}Z_2 + \mathcal{A}_{Z_1}\mathcal{C}Z_2).$$

Then by using (3.1) and (3.2) we get

$$\begin{aligned}\overset{1}{\nabla}_{Z_1}Z_2 = -\mathcal{B}A_{Z_1}\mathcal{B}Z_2 - \mathcal{C}A_{Z_1}\mathcal{B}Z_2 + \phi\mathcal{V}\overset{1}{\nabla}_{Z_1}\mathcal{B}Z_2 - \omega\mathcal{V}\overset{1}{\nabla}_{Z_1}\mathcal{B}Z_2 - \mathcal{B}\mathcal{H}\overset{1}{\nabla}_{Z_1}\mathcal{C}Z_2 \\ - \mathcal{C}\mathcal{H}\overset{1}{\nabla}_{Z_1}\mathcal{C}Z_2 - \phi\mathcal{A}_{Z_1}\mathcal{C}Z_2 - \omega\mathcal{A}_{Z_1}\mathcal{C}Z_2.\end{aligned}$$

Hence, we have $\overset{1}{\nabla}_{Z_1}Z_2 \in \Gamma((\ker F_*)^\perp)$ if and only if

$$-\mathcal{B}A_{Z_1}\mathcal{B}Z_2 - \phi\mathcal{V}\overset{1}{\nabla}_{Z_1}\mathcal{B}Z_2 - \mathcal{B}\mathcal{H}\overset{1}{\nabla}_{Z_1}\mathcal{C}Z_2 - \phi\mathcal{A}_{Z_1}\mathcal{C}Z_2 = 0.$$

Thus $\overset{1}{\nabla}_{Z_1}Z_2 \in \Gamma((\ker F_*)^\perp)$ if and only if

$$\mathcal{B}(A_{Z_1}\mathcal{B}Z_2 + \mathcal{H}\overset{1}{\nabla}_{Z_1}\mathcal{C}Z_2) = 0, \quad \phi(\mathcal{V}\overset{1}{\nabla}_{Z_1}\mathcal{B}Z_2 + \mathcal{A}_{Z_1}\mathcal{C}Z_2) = 0$$

which completes proof. ■

In a similar way, we have the following result.

Proposition 3.6 *Let F be a semi-invariant Riemannian submersion from a Kähler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then the distribution* $\ker F_*$ *defines a totally geodesic foliation if and only if*

$$\mathcal{T}_{X_1}\phi X_2 + \mathcal{H}\overset{1}{\nabla}_{X_1}\omega X_2 \in \Gamma(J\mathcal{D}_2), \widehat{\nabla}_{X_1}\phi X_2 + \mathcal{T}_{X_1}\omega X_2 \in \Gamma(\mathcal{D}_1)$$

for $X_1, X_2 \in \Gamma(\ker F_*)$.

From Proposition 3.6, we have the following result.

Corollary 3.7 *Let F be a semi-invariant Riemannian submersion from a Kähler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then the distribution* $\ker F_*$ *defines a totally geodesic foliation if and only if*

$$\begin{aligned} g_2\big((\nabla F_*)(X_1, X_2), F_*(JZ)\big) &= 0, \\ g_2\big((\nabla F_*)(X_1, \omega X_2), F_*(W)\big) &= -g_1(\mathcal{T}_{X_1}W, \phi X_2) \end{aligned}$$

for $X_1, X_2 \in \Gamma(\ker F_*)$, $Z \in \Gamma(\mathcal{D}_2)$, and $W \in \Gamma(\mu)$.

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Proof For $X_1, X_2 \in \Gamma(\ker F_*)$, $\widehat{\nabla}_{X_1}\phi X_2 + \mathcal{T}_{X_1}\omega X_2 \in \Gamma(\mathcal{D}_1)$ if and only if $g_1(\widehat{\nabla}_{X_1}\phi X_2 + \mathcal{T}_{X_1}\omega X_2, Z) = 0$ for $Z \in \Gamma(\mathcal{D}_2)$. Skew-symmetric $\mathcal{T}$ and (2.5) imply that

$$g_1(\widehat{\nabla}_{X_1}\phi X_2 + \mathcal{T}_{X_1}\omega X_2, Z) = g_1(\overset{1}{\nabla}_{X_1}\phi X_2, Z) - g_1(\omega X_2, \mathcal{T}_{X_1}Z).$$

Hence we have

$$g_1(\widehat{\nabla}_{X_1}\phi X_2 + \mathcal{T}_{X_1}\omega X_2, Z) = -g_1(\phi X_2, \overset{1}{\nabla}_{X_1}Z) - g_1(\omega X_2, \mathcal{T}_{X_1}Z).$$

Using (2.5) again we get

$$g_1(\widehat{\nabla}_{X_1}\phi X_2 + \mathcal{T}_{X_1}\omega X_2, Z) = -g_1(JX_2, \widehat{\nabla}_{X_1}Z) - g_1(\omega X_2, \mathcal{T}_{X_1}Z).$$

Hence we have

$$g_1(\widehat{\nabla}_{X_1}\phi X_2 + \mathcal{T}_{X_1}\omega X_2, Z) = -g_1(JX_2, \overset{1}{\nabla}_{X_1}Z).$$

Then from (2.2) we derive

$$g_1(\widehat{\nabla}_{X_1}\phi X_2 + \mathcal{T}_{X_1}\omega X_2, Z) = g_1(X_2, \overset{1}{\nabla}_{X_1}JZ).$$

Thus we have

$$g_1(\widehat{\nabla}_{X_1}\phi X_2 + \mathcal{T}_{X_1}\omega X_2, Z) = -g_1(\overset{1}{\nabla}_{X_1}X_2, JZ).$$

Then Riemannian submersion F implies that

$$g_1(\widehat{\nabla}_{X_1}\phi X_2 + \mathcal{T}_{X_1}\omega X_2, Z) = -g_2(F_*(\overset{1}{\nabla}_{X_1}X_2), F_*(JZ)).$$

Using (2.9) we get

$$g_1(\widehat{\nabla}_{X_1}\phi X_2 + \mathcal{T}_{X_1}\omega X_2, Z) = g_2\big((\nabla F_*)(X_1, X_2), F_*(JZ)\big). \tag{3.6}$$

On the other hand, for $X_1, X_2 \in \Gamma(\ker F_*)$, $\mathcal{T}_{X_1}\phi X_2 + \mathcal{H}\overset{1}{\nabla}_{X_1}\omega X_2 \in \Gamma(J\mathcal{D}_2)$ if and only if $g_1(\mathcal{T}_{X_1}\phi X_2 + \mathcal{H}\overset{1}{\nabla}_{X_1}\omega X_2, W) = 0$ for $W \in \Gamma(\mu)$. Since $\mathcal{T}$ is skew-symmetric, we have

$$g_1(\mathcal{T}_{X_1}\phi X_2 + \mathcal{H}\overset{1}{\nabla}_{X_1}\omega X_2, W) = -g_1(\phi X_2, \mathcal{T}_{X_1}W) + g_1(\overset{1}{\nabla}_{X_1}\omega X_2, W).$$

Since F is a Riemannian submersion, we get

$$g_1(\mathcal{T}_{X_1}\phi X_2 + \mathcal{H}\overset{1}{\nabla}_{X_1}\omega X_2, W) = -g_1(\phi X_2, \mathcal{T}_{X_1}W) + g_2(F_*(\overset{1}{\nabla}_{X_1}\omega X_2), F_*W).$$

Then from (2.9) we arrive at

$$\begin{aligned} g_1(\mathcal{T}_{X_1}\phi X_2 + \mathcal{H}\overset{1}{\nabla}_{X_1}\omega X_2, W) = \\ - g_1(\phi X_2, \mathcal{T}_{X_1}W) + g_2\big(-(\nabla F_*)(X_1, \omega X_2), F_*W\big). \end{aligned} \tag{3.7}$$

Thus proof follows from (3.6), (3.7), and Proposition 3.6 ■

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From Proposition 3.3 and Proposition 3.5 we have the following theorem.

Theorem 3.8 *Let F be a semi-invariant submersion from a Kähler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then M_1 is locally a product Riemannian manifold $M_{\mathcal{D}_1} \times M_{\mathcal{D}_2} \times M_{(\ker F_*)^\perp}$ if and only if $(\nabla\phi) = 0$ on $\ker F_*$ and*

$$\mathcal{A}_{Z_1}\mathcal{B}Z_2 + \mathcal{H}\overset{1}{\nabla}_{Z_1}\mathcal{C}Z_2 \in \Gamma(\mu), \mathcal{A}_{Z_1}\mathcal{C}Z_2 + \mathcal{V}\overset{1}{\nabla}_{Z_1}Z_2 \in \Gamma(\mathcal{D}_2)$$

for $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$, where $M_{\mathcal{D}_1}$, $M_{\mathcal{D}_2}$ and $M_{(\ker F_*)^\perp}$ are integral manifolds of the distributions $\mathcal{D}_1$, $\mathcal{D}_2$ and $(\ker F_*)^\perp$.

Also from Corollary 3.7 and Proposition 3.5, we have the following result.

Theorem 3.9 *Let F be a semi-invariant submersion from a Kähler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then M_1 is locally a product Riemannian manifold $M_{\ker F_*} \times M_{(\ker F_*)^\perp}$ if and only if*

$$g_2\big((\nabla F_*)(X_1, X_2), F_*(JZ)\big) = 0,$$
$$g_2\big((\nabla F_*)(X_1, \omega X_2), F_*(W)\big) = -g_1(\mathcal{T}_{X_1}W, \phi X_2)$$

and

$$\mathcal{A}_{Z_1}\mathcal{B}Z_2 + \mathcal{H}\overset{1}{\nabla}_{Z_1}\mathcal{C}Z_2 \in \Gamma(\mu), \mathcal{A}_{Z_1}\mathcal{C}Z_2 + \mathcal{V}\overset{1}{\nabla}_{Z_1}Z_2 \in \Gamma(\mathcal{D}_2)$$

for $X_1, X_2 \in \Gamma(\ker F_*)$, $W \in \Gamma(\mu)$, $Z \in \Gamma(\mathcal{D}_2)$ and $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$, where $M_{\ker F_*}$, and $M_{(\ker F_*)^\perp}$ are integral manifolds of the distributions $\ker F_*$ and $(\ker F_*)^\perp$.

4 Semi-invariant Submersions with Totally Umbilical Fibers

In this section we give two theorems on semi-invariant submersions with totally umbilical fibers. The first result shows that a semi-invariant submersion puts some restrictions on total manifolds. Also we obtain a classification for such submersions. Let F be a Riemannian submersion from a Riemannian manifold onto a Riemannian manifold (M_2, g_2). Recall that a Riemannian submersion is called a Riemannian submersion with totally umbilical fibers if

$$\mathcal{T}_X Y = g_1(X, Y)H \tag{4.1}$$

for $X, Y \in \Gamma(\ker F_*)$, where H is the mean curvature vector field of the fiber. We also recall that a simply connected complete Kähler manifold of constant sectional curvature c is called a complex space-form, denoted by $M(c)$. The curvature tensor of $M(c)$ is

$$R(X, Y)Z = \frac{c}{4}\big[g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ\big] \tag{4.2}$$

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for $X, Y, Z \in \Gamma(TM)$. Moreover, from [9] we have the following relation for a Riemannian submersion:

$$g_1\big(R^1(X_1,X_2)X_3,Z\big) = g_1\big((\nabla_{X_2}\mathcal{T})_{X_1}X_3,Z)\big) - g_1\big((\nabla_{X_1}\mathcal{T})_{X_2}X_3,Z\big) \tag{4.3}$$

for $X_1, X_2, X_3 \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^{\perp})$, where R^1 is the curvature tensor field of M_1 and $(\nabla\mathcal{T})$ is the covariant derivative of $\mathcal{T}$.

By using (4.1), (4.2), and (4.3), as in CR-submanifolds, see [1, Theorem 1.2, p. 78], we have the following result.

Theorem 4.1 *Let F be a semi-invariant submersion with totally umbilical fibers from a complex space form $(M_1(c), g_1, J)$ onto a Riemannian manifold (M_2, g_2). Then $c = 0$.*

We now give a classification theorem for semi-invariant Riemannian submersions with totally umbilical fibers. But we need the following result, which shows that the mean curvature vector field of semi-invariant Riemannian submersions has special form.

Lemma 4.2 *Let F be a semi-invariant submersion with totally umbilical fibers from a Kähler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then $H \in \Gamma(J\mathcal{D}_2)$.*

Proof Using (2.1), (2.2), (2.5), (3.1), and (3.2) we get

$$\mathcal{T}_{X_1}JX_2 + \widehat{\nabla}_{X_1}JX_2 = \mathcal{B}\mathcal{T}_{X_1}X_2 + \mathcal{C}\mathcal{T}_{X_1}X_2 + \phi\widehat{\nabla}_{X_1}X_2 + \omega\widehat{\nabla}_{X_1}X_2$$

for $X_1, X_2 \in \Gamma(\mathcal{D}_1)$. Thus, for $W \in \Gamma(\mu)$, we obtain

$$g_1(\mathcal{T}_{X_1}JX_2, W) = g_1(\mathcal{C}\mathcal{T}_{X_1}X_2, W).$$

Using (4.1) we derive

$$g_1(X_1, JX_2)g_1(H, W) = g_1(J\mathcal{T}_{X_1}X_2, W).$$

Hence we have

$$g_1(X_1, JX_2)g_1(H, W) = -g_1(\mathcal{T}_{X_1}X_2, JW).$$

Using (4.1) again we arrive at

$$g_1(X_1, JX_2)g_1(H, W) = -g_1(X_1, X_2)g_1(H, JW). \tag{4.4}$$

Interchanging the role of X_1 and X_2, we obtain

$$g_1(X_2, JX_1)g_1(H, W) = -g_1(X_2, X_1)g_1(H, JW). \tag{4.5}$$

Thus from (4.4) and (4.5) we derive $g_1(X_1, X_2)g_1(H, JW) = 0$, which shows that $H \in \Gamma(J\mathcal{D}_2)$ due to μ is invariant distribution. ∎

We now give a classification theorem for a semi-invariant submersion with totally umbilical fibers, which is similar to that [12, Theorem 6.1, p. 96], and therefore we omit its proof. We note that Lemma 4.2 implies that one can use the method that was used in the proof of [12, Theorem 6.1].

Theorem 4.3 *Let F be a semi-invariant submersion with totally umbilical fibers from a Kähler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then either $\mathcal{D}_2$ is one dimensional or the fibers are totally geodesic.*

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