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SOME PROPERTIES OF THE CUMULATIVE RESIDUAL ENTROPY OF COHERENT AND MIXED SYSTEMS

A. TOOMAJ,* *Gonbad Kavous University*
S. M. SUNOJ,** *Cochin University of Science and Technology*
J. NAVARRO,*** *Universidad de Murcia*

Abstract

Recently, Rao *et al.* (2004) introduced an alternative measure of uncertainty known as the cumulative residual entropy (CRE). It is based on the survival (reliability) function $\bar{F}$ instead of the probability density function f used in classical Shannon entropy. In reliability based system design, the performance characteristics of the coherent systems are of great importance. Accordingly, in this paper, we study the CRE for coherent and mixed systems when the component lifetimes are identically distributed. Bounds for the CRE of the system lifetime are obtained. We use these results to propose a measure to study if a system is close to series and parallel systems of the same size. Our results suggest that the CRE can be viewed as an alternative entropy (dispersion) measure to classical Shannon entropy.

Keywords: Coherent system; cumulative residual entropy; stochastic order; system signature

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1. Introduction

One of the most important measures of uncertainty is Shannon entropy which plays an important role in various areas of sciences such as probability and statistics, financial analysis, engineering, and information theory; see, e.g. Cover and Thomas [6]. Let X be an absolutely continuous nonnegative random variable with cumulative distribution function (CDF) F and probability density function (PDF) f. The Shannon entropy of X is defined as

$$H(X) = -\int_0^\infty f(x)\log f(x)\,\mathrm{d}x, \tag{1}$$

where 'log' stands for the natural logarithm and where, by convention, $0\log 0 = 0$. The Shannon entropy (1) represents the predictability and information on X. Some recent properties can be seen in [2], [8], [9], and [13]. Other measures of uncertainty were developed in various disciplines and contexts.

Recently, Rao *et al.* [22] introduced an alternative measure of uncertainty called the *cumulative residual entropy* (CRE) which is based on the survival (reliability) function $\bar{F}(x) = 1-F(x)$

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* Postal address: Department of Statistics, Gonbad Kavous University, Gonbad Kavous, Iran.
Email address: ab.toomaj@gonbad.ac.ir
** Postal address: Department of Statistics, Cochin University of Science and Technology, Cochin 682 022, Kerala, India. Email address: smsunoj@cusat.ac.in
*** Postal address: Facultad de Matematicas, Universidad de Murcia, 30100 Murcia, Spain.
Email address: jorgenav@um.es

instead of the PDF $f(x)$ used in Shannon entropy (1). For a nonnegative random variable X with survival function $\bar{F} = 1 - F$, the CRE of X is

$$\mathcal{E}(X) = -\int_0^\infty \bar{F}(x) \log \bar{F}(x)\,\mathrm{d}x. \tag{2}$$

This measure has been applied in reliability engineering and in computer vision; see, e.g. [22] and [29]. Recent papers have obtained several properties of this alternative entropy but its meaning and interpretation have not yet been made sufficiently clear.

In this paper we study general properties of $\mathcal{E}(X)$ and properties of $\mathcal{E}(T)$ for a coherent or mixed system with lifetime T. We show that the CRE is closely related to the standard deviation and that it can be used to define a measure to see if a system is close to the series system (the worst system) or to the parallel system (the best system). Our results shed light upon $\mathcal{E}(X)$ as an alternative measure of entropy (dispersion).

The rest of the paper is organized as follows. In Section 2 we study basic properties of the CRE and provide some recent references for additional properties and practical applications. The general properties of the CRE in coherent and mixed systems are studied in Section 3. Bounds are obtained in Section 4 and a measure for comparing systems based on the CRE is proposed in Section 5.

2. General properties of the CRE

It is clear that (2) can be used for both continuous and discrete distributions. Moreover, for an atom distribution F_X for which $X = c$ (almost surely), we have $\mathcal{E}(X) = 0$. This is, of course, the most informative case (with the minimum uncertainty possible). Then $\mathcal{E}(X)$ can be used to measure if the X is close to an atom distribution (i.e. it is a dispersion measure). However, for continuous distributions, Shannon entropy is a measure of disparity of the PDF $f(x)$ from the uniform distribution. Some examples of the CRE are provided in Table 1. In Figure 1 we show that, in these models, there is a close relationship between the standard deviation and the cumulative residual entropy. Furthermore, the CRE has the following property: $\mathcal{E}(aX + b) = a\mathcal{E}(X)$ for all $a > 0$ and $b \geq 0$. This is similar to the well-known property $\sigma(aX+b) = a\sigma(X)$ for the standard deviation. Moreover, the CRE has an interesting connection with another dispersion measure, the Gini mean difference $D_{\mathrm{G}}(X) = \int_0^\infty 2\bar{F}(x)(1 - \bar{F}(x))\,\mathrm{d}x$. Since $0 \leq x(1-x) \leq -x \log x$ for $x \in [0, 1]$, then we have $2\mathcal{E}(X) \geq D_{\mathrm{G}}(X)$. It is evident from (2) that $\mathcal{E}(X) \geq 0$, while the Shannon entropy may take negative values when F is absolutely continuous. For an absolutely continuous nonnegative random variable X with survival function $\bar{F} = 1 - F$, by applying the probability integral transformation $U = F(X)$, the CRE may be

TABLE 1: The cumulative residual entropy and the standard deviation for some models.

Model	$\bar{F}(x)$	Support	$\mathcal{E}(X)$	$\sigma(X)$
Uniform	$\dfrac{\beta - x}{\beta}$	$0 \leq x \leq \beta$	$\dfrac{\beta}{4}$	$\dfrac{\beta}{2\sqrt{3}}$
Pareto	$\left(\dfrac{\beta}{\beta + x}\right)^\alpha$	$x \geq 0$	$\dfrac{\alpha\beta}{(\alpha-1)^2},\ \alpha > 1$	$\dfrac{\beta}{\alpha - 1}\sqrt{\dfrac{\alpha}{\alpha - 2}},\ \alpha > 2$
Weibull	$\mathrm{e}^{-(\lambda x)^\alpha}$	$x \geq 0$	$\dfrac{\Gamma(1 + 1/\alpha)}{\alpha\lambda}$	$\sqrt{\dfrac{\Gamma(1 + 2/\alpha)}{\lambda^2} - \left(\dfrac{\Gamma(1 + 1/\alpha)}{\lambda}\right)^2}$

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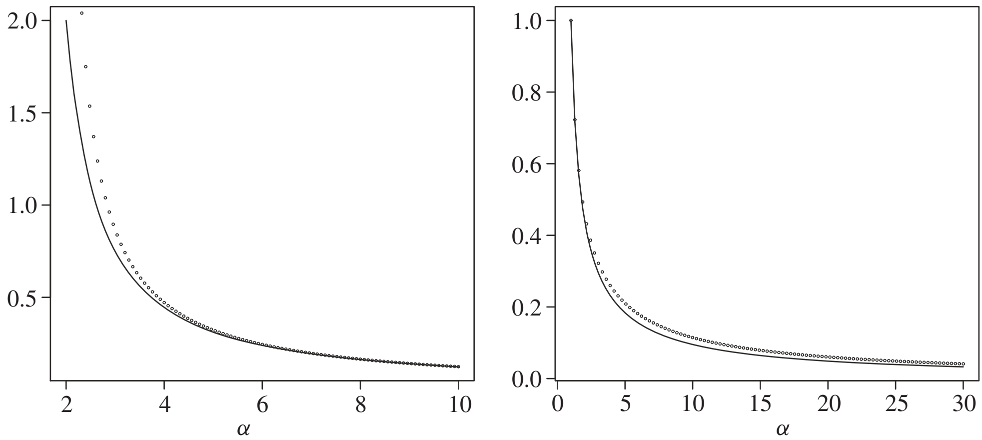

FIGURE 1: Comparative plots between the standard deviation (*circles*) and the CRE (*solid line*) for Pareto (*left*) and Weibull (*right*) distributions with $\beta = 1$ and $\lambda = 1$.

rewritten as

$$\mathcal{E}(X) = \int_0^1 \frac{\psi(u)}{f(\bar{F}^{-1}(u))}\,\mathrm{d}u, \tag{3}$$

where $\psi(u) = -u\log u$, $\psi(0) = \psi(1) = 0$, and where the function $\bar{F}^{-1}(u) = \sup\{x : \bar{F}(x) \geq u\}$ is known as the quantile function of $\bar{F}$.

It is well known that some probability models can be characterized by maximizing the Shannon entropy under given conditions (see, e.g. [2] and the references therein). For example, for a fixed mean $\mu > 0$, the model with support $(0, \infty)$ and maximum Shannon entropy is the exponential distribution. The same techniques can be applied to the CRE to obtain the more dispersed models. As $\mu = \int_0^\infty \bar{F}(x)\,\mathrm{d}x$, we can define the (decreasing) PDF $g(x) = \bar{F}(x)/\mu$ for $x \geq 0$. Actually, this is the PDF of the equilibrium distribution in a renewal process. Thus, maximizing $\mathcal{E}(X)$ for a fixed μ is equivalent to maximizing the Shannon entropy of g given by

$$H(g) = -\int_0^\infty g(x)\log g(x)\,\mathrm{d}x = -\int_0^\infty \frac{\bar{F}(x)}{\mu}\log\frac{\bar{F}(x)}{\mu}\,\mathrm{d}x.$$

This problem was called the maximize equilibrium distribution entropy (MEDE) in [2, Section 3]. There they proved that the exponential distribution is the MEDE model when we fix $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$.

The dispersive ordering is defined as follows: X is smaller than Y in the dispersive order (denoted by $X \leq_{\mathrm{d}} Y$) if $\bar{F}^{-1}(u) - \bar{F}^{-1}(v) \leq \bar{G}^{-1}(u) - \bar{G}^{-1}(v)$, $0 < u \leq v < 1$. If X and Y are absolutely continuous with PDFs f and g, respectively, the preceding condition is equivalent (see [25, Equation (3.B.11)]) to

$$g(\bar{G}^{-1}(v)) \leq f(\bar{F}^{-1}(v)), \qquad 0 < v < 1. \tag{4}$$

Then, from (3), $X \leq_{\mathrm{d}} Y$ implies $\mathcal{E}(X) \leq \mathcal{E}(Y)$. Hence, $\mathcal{E}(X) \leq \mathcal{E}(Y)$ is a necessary condition for the dispersive ordering $X \leq_{\mathrm{d}} Y$ and we can consider the following order.

Definition 1. We say that X is smaller than Y in the cumulative residual entropy order (denoted by $X \leq_{\mathrm{CRE}} Y$) if $\mathcal{E}(X) \leq \mathcal{E}(Y)$.

Note that $X \overset{\text{CRE}}{=} Y$ does not imply that X and Y have the same distribution. Moreover, if $Y = \phi(X)$ for a strictly increasing function ϕ on the support of X then

$$\begin{aligned}\mathcal{E}(Y) &= -\int_0^\infty \bar{F}_Y(y)\log \bar{F}_Y(y)\,\mathrm{d}y \\ &= -\int_0^\infty \bar{F}_X(\phi^{-1}(x))\log \bar{F}_X(\phi^{-1}(x))\,\mathrm{d}x \\ &= -\int_0^\infty \phi'(u)\bar{F}_X(u)\log \bar{F}_X(u)\,\mathrm{d}u \\ &= \mathcal{E}(X) - \int_0^\infty (\phi'(u)-1)\bar{F}_X(u)\log \bar{F}_X(u)\,\mathrm{d}u.\end{aligned}$$

Therefore, if $\phi'(u) \geq 1$ then $X \leq_{\text{CRE}} Y$. This property is similar to that obtained in Theorem 1 of [10] for Shannon entropy.

The CRE is particularly suitable to describe the information (dispersion) in problems related to ageing properties of reliability theory; see, e.g. [1], [3], [4], [16], [20], [21], and the references therein. For example, Asadi and Zohrevand [1] showed that the CRE is the expected value of the mean residual lifetime function $m(x) = \mathbb{E}(X - x \mid X > x)$, i.e. $\mathcal{E}(X) = \mathbb{E}(m(X))$. We have obtained a similar expression in terms of the cumulative hazard function $\Lambda(x) = -\log \bar{F}(x)$. As

$$\mathcal{E}(X) = -\int_0^\infty \log \bar{F}(x) \int_x^\infty f(z)\,\mathrm{d}z\,\mathrm{d}x = \int_0^\infty f(z)\int_0^z \Lambda(x)\,\mathrm{d}x\,\mathrm{d}z$$

then we have $\mathcal{E}(X) = \mathbb{E}(v(X))$, where $v(z) = \int_0^z \Lambda(x)\,\mathrm{d}x$. Moreover, as $v(z)$ is increasing and convex, $X \leq_{\text{icx}} Y$ implies $X \leq_{\text{CRE}} Y$, where '$\leq_{\text{icx}}$' denotes the increasing convex order; see [25]. Hence, we have

$$X \leq_{\text{d}} Y \quad \Longrightarrow \quad X \leq_{\text{st}} Y \quad \Longrightarrow \quad X \leq_{\text{icx}} Y \quad \Longrightarrow \quad X \leq_{\text{CRE}} Y,$$

where $\leq_{\text{st}}$ denotes the usual stochastic order (see [25]).

It is well known that Shannon entropy was used to define the Kullback–Leibler divergence measure. In a similar way, Baratpour and Rad [4] used the CRE to define the cumulative Kullback–Leibler (CKL) distance as

$$\begin{aligned}\text{CE}(X, Y) &= \mathbb{E}(Y) - \mathbb{E}(X) + \int_0^\infty \bar{F}(t)\log\frac{\bar{F}(t)}{\bar{G}(t)}\,\mathrm{d}t \\ &= \mathbb{E}(Y) - \mathbb{E}(X) + \mathcal{E}(X, Y) - \mathcal{E}(X) \\ &\geq 0 \end{aligned} \tag{5}$$

for nonnegative random variables X and Y with survival functions $\bar{F}$ and $\bar{G}$ and finite means $\mathbb{E}(X)$ and $\mathbb{E}(Y)$, respectively, provided that $\bar{F}(t) = 0$ whenever $\bar{G}(t) = 0$, where

$$\mathcal{E}(X, Y) = -\int_0^\infty \bar{F}(t)\log \bar{G}(t)\,\mathrm{d}t \tag{6}$$

is known as the cumulative residual inaccuracy (CRI) of X and Y (see [26]). Of course, we have $\text{CE}(X, Y) = 0$ when $F = G$. Note that $\text{CE}(X, Y)$ is not necessarily equal to $\text{CE}(Y, X)$.

This measure will be more useful in situations where the survival function has a more simple form than the PDF. We will see that this measure is appropriate to investigate the properties of a system by using its representation based on signatures.

Finally, we provide some references for the reader interested in a deep discussion on the interest and possible applications of the CRE in practice. A survey was given in [11] where a general extension of the CRE was considered. This includes applications in five different disciplines (fuzzy set theory, generalized maximum entropy principle, theory of dispersion of ordered categorial variables, uncertainty theory, and reliability theory); see the references therein for detailed applications. They also suggest that these measures can be seen as dispersion measures. Some applications in risk theory were given in [30]. Toomaj and Doostparast [27], [28] obtained an expression for the system's entropy as well as the bounds for the entropy of the system's lifetime. They also provided expressions for the Kullback–Leibler discrimination information of system and component lifetimes. Recently, Park and Kim [19] obtained some recurrence relations for the CRE of order statistics (k-out-of-n systems).

3. Cumulative residual entropy for systems

A system is said to be coherent if it does not have any irrelevant components and its structure function is monotone (see [5]). A special case of coherent systems is the k-out-of-n system, where the system fails when the kth component failure occurs. A mixed system is a stochastic mixture of coherent systems. Hence, any coherent system is a (degenerated) mixed system (see, e.g. [23]). Let T denote the lifetime of a mixed system consisting of n independent and identically distributed (i.i.d.) components with lifetimes $X_1, \ldots, X_n$ having an absolutely continuous CDF F. Then it follows (see, e.g. [23]) that its survival function $\bar{F}_T$ satisfies

$$\bar{F}(t) = \mathbb{P}(T > t) = \sum_{i=1}^{n} s_i \bar{F}_{i:n}(t), \tag{7}$$

where $\bar{F}_{i:n}(t) = \sum_{j=0}^{i-1} \binom{n}{j} [F(t)]^j [\bar{F}(t)]^{n-j}$ for $i = 1, \ldots, n$ are the survival functions of the ordered component lifetimes $X_{1:n}, \ldots, X_{n:n}$ (i.e. the lifetimes of k-out-of-n systems). The vector of coefficients $\boldsymbol{s} = (s_1, \ldots, s_n)$ in (7) is called the *system signature*, where $s_i = \mathbb{P}(T = X_{i:n})$ is the probability that the ith failure causes the system failure. Note that $s_1, \ldots, s_n$ are nonnegative real numbers which do not depend on the common CDF F and such that $\sum_{i=1}^{n} s_i = 1$. One can see that the survival function of a mixed system is a mixture of the survival functions of the i-out-of-n systems with weights s_i. Recent results on system signatures can be seen in [7], [24].

First we provide an expression for the CRE of a given mixed system with signature $\boldsymbol{s} = (s_1, \ldots, s_n)$ consisting of n i.i.d. component lifetimes $X_1, \ldots, X_n$ having a common CDF F. If we use the probability integral transformation $U = \bar{F}(X)$ then $U_i = \bar{F}(X_i)$ is uniformly distributed in [0, 1] and $W_{i:n} = \bar{F}(X_{i:n})$ has a beta distribution with parameters $n - i + 1$ and i and with the following distribution function:

$$G_{i:n}(w) = \sum_{j=0}^{i-1} \binom{n}{j} (1-w)^j w^{n-j}, \qquad 0 \le w \le 1,$$

for $i = 1, \ldots, n$. The transformation $V = \bar{F}(T)$ has the distribution function

$$G_V(v) = \sum_{i=1}^{n} s_i G_{i:n}(v), \qquad 0 \le v \le 1.$$

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From (2) and the transforms mentioned earlier, we have $\bar{F}_T(t) = G_V(\bar{F}(t))$ and

$$\mathcal{E}(T) = -\int_0^\infty \bar{F}_T(t) \log \bar{F}_T(t)\,\mathrm{d}t = \int_0^1 \frac{\psi(G_V(v))}{f(\bar{F}^{-1}(v))}\,\mathrm{d}v, \tag{8}$$

where the second integral is obtained with the variable change $v = \bar{F}(t)$.

Navarro *et al.* [17] proved that the survival function $\bar{F}_T$ of a coherent (or mixed) system with identically distributed (i.d.) components can be written as

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t)), \tag{9}$$

where $\bar{F}$ is the common survival function of the components and $\bar{q}$ is an increasing continuous function in [0, 1] such that $\bar{q}(0) = 0$ and $\bar{q}(1) = 1$. The *distortion function* $\bar{q}$ depends only on the structure of the system and on the copula of the random vector $(X_1 \ldots, X_n)$. In particular, if $(X_1 \ldots, X_n)$ is exchangeable (i.e. its joint distribution does not change if we permute the components) then

$$\bar{q}(v) = \sum_{i=1}^n a_i K(\boldsymbol{v}_i), \tag{10}$$

where $\boldsymbol{v}_i = (u_1, \ldots, u_n)$ with $u_1 = \cdots = u_i = v$ and $u_{i+1} = \cdots = u_n = 1$, K is the exchangeable survival copula of $(X_1, \ldots, X_n)$, and $(a_1, \ldots, a_n)$ is the minimal signature of the system (see, e.g. [16]). In particular, if the components are i.i.d. then

$$\bar{q}(v) = G_V(v) = \sum_{i=1}^n a_i v^i. \tag{11}$$

Thus, representation (8) can be extended to the mixed systems with (possibly dependent) i.d. components. In the general i.d. case, from (9),

$$\mathcal{E}(T) = -\int_0^\infty \bar{F}_T(t) \log \bar{F}_T(t)\,\mathrm{d}t = \int_0^1 \frac{\psi(\bar{q}(v))}{f(\bar{F}^{-1}(v))}\,\mathrm{d}v \tag{12}$$

holds. If the components are exchangeable or i.i.d. then $\bar{q}$ can be replaced with the expressions given in (10) or (11), respectively. In particular, if the components are i.i.d. with a common exponential distribution with mean μ then

$$\mathcal{E}(T) = -\mu \sum_{i=1}^n a_i \int_0^1 v^{i-1} \log\biggl(\sum_{i=1}^n a_i v^i\biggr)\,\mathrm{d}v. \tag{13}$$

As an application of (8), (12), and (13), we have the following example.

Example 1. Let $\boldsymbol{s} = (0, \frac{2}{3}, \frac{1}{3}, 0)$ be the signature of the coherent system with lifetime $T = \max\{\min\{X_1, X_2\}, \min\{X_3, X_4\}\}$ and i.i.d. components having the common exponential survival function $\bar{F}(t) = \exp(-t/\mu)$ for $\mu > 0$ and $t \geq 0$. It is easy to see that $f(\bar{F}^{-1}(v)) = v/\mu$ and, hence, from (8), we obtain $\mathcal{E}(T) \approx 0.5569\mu$. One can see that the CRE is increasing with respect to μ, i.e. the system's uncertainty in terms of the CRE increases with increasing the scale (dispersion) parameter μ. The minimal signature of the system is $(0, 2, 0, -1)$ and so, from (13), integrating by parts (with $x = \log(2v^2 - v^4)$ and $\mathrm{d}y = 2v - v^3$), we obtain

$$\mathcal{E}(T) = -\mu \int_0^1 \frac{2v^2 - v^4}{v} \log(2v^2 - v^4)\,\mathrm{d}v = \biggl(\frac{5}{4} - \log 2\biggr)\mu \approx 0.5569\mu.$$

If the system have dependent i.d. exponential components with an exchangeable survival copula K, then we have

$$\mathcal{E}(T) = -\mu \int_0^1 \frac{2K(v, v, 1, 1) - K(v, v, v, v)}{v} \log(2K(v, v, 1, 1) - K(v, v, v, v))\, \mathrm{d}v.$$

For example, if the components have the following Farlie–Gumbel–Morgenstern copula:

$$K(u_1, u_2, u_3, u_4) = u_1 u_2 u_3 u_4 (1 + \alpha(1 - u_1)(1 - u_2)(1 - u_3)(1 - u_4))$$

for $\alpha \in [-1, 1]$, then the system survival function is

$$\bar{F}_T(t) = 2\bar{F}_{1:2}(t) - \bar{F}_{1:4}(t) = \bar{q}(\bar{F}(t)),$$

where $\bar{q}(v) = 2K(v, v, 1, 1) - K(v, v, v, v) = 2v^2 - v^4(1 + \alpha(1 - v)^4)$. If the components have a common exponential distribution and $\alpha = \frac{1}{2}$, then

$$\mathcal{E}(T) = -\mu \int_0^1 \frac{2v^2 - v^4(1 + (1 - v)^4/2)}{v} \log\big(2v^2 - v^4\Big(1 + \frac{1}{2}(1 - v)^4\Big)\Big)\, \mathrm{d}v = 0.5565\mu.$$

Numerically, we see that $\mathcal{E}(T)$ decreases when the dependence parameter α increases.

The minimal signatures of systems with 1–5 components were computed in [15]. So it is easy to compute the values of $\mathcal{E}(T)$. In Table 2 we give these values for systems with 1–4 i.i.d. exponential components.The system lifetimes can be seen in Table 1 of [18]. In Table 2 we see that the values of $\mathcal{E}(T)$ are well approximated to that of the respective standard deviations. An interesting application of (12) is the comparison of the CRE of mixed systems when two systems have the same structure with different i.d. component lifetimes. Equation (12) gives the following theorem.

Theorem 1. *Let T^X and T^Y be the lifetimes of two mixed systems with the same structure, based respectively on i.d. component lifetimes $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ with the same copula and CDFs F and G and PDFs f and g.*

(i) *If $X \le_{\mathrm{d}} Y$ then $T^X \le_{\mathrm{CRE}} T^Y$.*

(ii) *If $X \le_{\mathrm{CRE}} Y$ and*

$$\inf_{v \in A_1} \frac{\psi(\bar{q}(v))}{\psi(v)} \ge \sup_{v \in A_2} \frac{\psi(\bar{q}(v))}{\psi(v)} \tag{14}$$

for $A_1 = \{v \in [0, 1]\colon f(\bar{F}^{-1}(v)) > g(\bar{G}^{-1}(v))\}$ and $A_2 = \{v \in [0, 1]\colon f(\bar{F}^{-1}(v)) \le g(\bar{G}^{-1}(v))\}$, then $T^X \le_{\mathrm{CRE}} T^Y$.

Proof. (i) As the systems have the same structure and the same copula, then they have a common distortion function $\bar{q}$. Moreover, since $X \le_{\mathrm{d}} Y$, from (4),

$$\frac{\psi(\bar{q}(v))}{f(\bar{F}^{-1}(v))} \le \frac{\psi(\bar{q}(v))}{g(\bar{G}^{-1}(v))}$$

holds for $0 < v < 1$, where $\psi(\bar{q}(v)) \ge 0$. Hence, (12) completes the proof.

TABLE 2: Signature $\boldsymbol{s}$, minimal signature $\boldsymbol{a}$, CRE, standard deviation, and distance symmetric measure (DSM) for all the coherent systems with 1–4 i.i.d. components and a standard exponential distribution.

N	$\boldsymbol{s}$	$\boldsymbol{a}$	$\mathcal{E}(T)$	$\sigma(T)$	DSM(T)
1	(1)	(1)	1.0000	1.0000	
2	$(1, 0)$	$(0, 1)$	0.5000	0.5000	−1.0000
3	$(0, 1)$	$(2, -1)$	1.1137	1.1180	1.0000
4	$(1, 0, 0)$	$(0, 0, 1)$	0.3333	0.3333	−1.0000
5	$(\frac{1}{3}, \frac{2}{3}, 0)$	$(0, 2, -1)$	0.5758	0.5773	−0.3333
6	$(0, 1, 0)$	$(0, 3, -2)$	0.5974	0.6009	−0.1807
7	$(0, \frac{2}{3}, \frac{1}{3})$	$(1, 1, -1)$	0.9566	0.9574	0.3079
8	$(0, 0, 1)$	$(3, -3, 1)$	1.1580	1.1667	1.0000
9	$(1, 0, 0, 0)$	$(0, 0, 0, 1)$	0.2500	0.2500	−1.0000
10	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$	$(0, 0, 2, -1)$	0.3814	0.3818	−0.5923
11	$(\frac{1}{4}, \frac{3}{4}, 0, 0)$	$(0, 0, 3, -2)$	0.4064	0.4082	−0.5147
12	$(\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0)$	$(0, 1, 1, -1)$	0.5061	0.5069	−0.3256
13	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$	$(0, 3, -3, 1)$	0.6255	0.6291	−0.1505
14	$(0, 1, 0, 0)$	$(0, 0, 4, -3)$	0.4139	0.4166	−0.4522
15	$(0, \frac{5}{6}, \frac{1}{6}, 0)$	$(0, 1, 2, -2)$	0.4984	0.5000	−0.2840
16, 17	$(0, \frac{2}{3}, \frac{1}{3}, 0)$	$(0, 2, 0, -1)$	0.5568	0.5590	−0.1918
18, 19	$(0, \frac{1}{2}, \frac{1}{2}, 0)$	$(0, 3, -2, 0)$	0.5974	0.6009	−0.1171
20, 21	$(0, \frac{1}{3}, \frac{2}{3}, 0)$	$(0, 4, -4, 1)$	0.6238	0.6291	−0.0515
22	$(0, \frac{1}{6}, \frac{5}{6}, 0)$	$(0, 5, -6, 2)$	0.6385	0.6455	0.0084
23	$(0, 0, 1, 0)$	$(0, 6, -8, 3)$	0.6431	0.6508	0.0645
24	$(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$	$(1, 0, 1, -1)$	0.9607	0.9610	0.2042
25	$(0, \frac{1}{6}, \frac{7}{12}, \frac{1}{4})$	$(1, 2, -3, 1)$	0.9446	0.9465	0.3011
26	$(0, 0, \frac{3}{4}, \frac{1}{4})$	$(1, 3, -5, 2)$	0.9255	0.9279	0.3476
27	$(0, 0, \frac{1}{2}, \frac{1}{2})$	$(2, 0, -2, 1)$	1.0793	1.0833	0.5763
28	$(0, 0, 0, 1)$	$(4, -6, 4, -1)$	1.1815	1.1932	1.0000

(ii) Since $X \le_{\mathrm{CRE}} Y$, from (2), we have

$$\mathcal{E}(Y) - \mathcal{E}(X) = \int_0^1 \biggl(\frac{\psi(v)}{g(\bar{G}^{-1}(v))} - \frac{\psi(v)}{f(\bar{F}^{-1}(v))}\biggr)\,\mathrm{d}v \ge 0.$$

Moreover, if $\Delta(u) = \psi(v)/g(\bar{G}^{-1}(v)) - \psi(v)/f(\bar{F}^{-1}(v))$, we have

$$\begin{aligned}
\mathcal{E}(T^Y) - \mathcal{E}(T^X) &= \int_0^1 \frac{\psi(\bar{q}(v))}{\psi(v)}\Delta(u)\,\mathrm{d}v \qquad \text{(by (12))}\\
&= \int_{A_1} \frac{\psi(\bar{q}(v))}{\psi(v)}\Delta(u)\,\mathrm{d}v + \int_{A_2} \frac{\psi(\bar{q}(v))}{\psi(v)}\Delta(u)\,\mathrm{d}v \qquad (\text{since } A_1 \cup A_2 = [0,1])\\
&\ge \inf_{v\in A_1} \frac{\psi(\bar{q}(v))}{\psi(v)} \int_{A_1} \Delta(u)\,\mathrm{d}v + \sup_{v\in A_2} \frac{\psi(\bar{q}(v))}{\psi(v)} \int_{A_2} \Delta(u)\,\mathrm{d}v\\
&\qquad (\text{since } \Delta(u) \ge 0 \text{ in } A_1 \text{ and } \Delta(u) \le 0 \text{ in } A_2)\\
&\ge \sup_{v\in A_2} \frac{\psi(\bar{q}(v))}{\psi(v)} \int_0^1 \Delta(u)\,\mathrm{d}v \qquad \text{(by (14))}\\
&\ge 0 \qquad \text{(by (3))}.
\end{aligned}$$

Then $T^X \le_{\mathrm{CRE}} T^Y$ holds. □

Under the assumptions of the preceding theorem, if $\bar{q}$ is strictly increasing in (0, 1) then Theorem 2.9 of [17] proved that $X \leq_{\mathrm{d}} Y$ is equivalent to $T^X \leq_{\mathrm{d}} T^Y$. If the components are i.i.d. then $\bar{q}$ is always strictly increasing in (0, 1) (since it is a polynomial) and so this equivalence holds.

4. Bounds for the CRE of mixed systems

In this section we provide bounds for the CRE of mixed systems by using the properties obtained in the preceding section. Generally, it is not easy to compute the CRE of the system's lifetime, especially when the number of components of the system is large or the system has a complicated structure. This is a common situation in practice. Hence, in such situations it is important to have bounds to approximate the behavior of the CRE of the system's lifetime. In the first result, the CRE of the system is bounded in terms of the common CRE of the components.

Proposition 1. *Let T be the lifetime of a mixed system with i.d. component lifetimes $X_1, \ldots, X_n$. Let $\bar{q}$ be the associated distortion function. Then*

$$B_1 \mathcal{E}(X_1) \leq \mathcal{E}(T) \leq B_2 \mathcal{E}(X_1),$$

where $B_1 = \inf_{v \in (0,1)} \psi(\bar{q}(v))/\psi(v)$, $B_2 = \sup_{v \in (0,1)} \psi(\bar{q}(v))/\psi(v)$, and $\psi(u) = -u \log(u)$.

Proof. The upper bound can be obtained from (12) as

$$\begin{aligned}
\mathcal{E}(T) &= \int_0^1 \frac{\psi(\bar{q}(v))}{f(\bar{F}^{-1}(v))} \,\mathrm{d}v \\
&= \int_0^1 \frac{\psi(\bar{q}(v))}{\psi(v)} \frac{\psi(v)}{f(\bar{F}^{-1}(v))} \,\mathrm{d}v \\
&\leq \sup_{v \in (0,1)} \frac{\psi(\bar{q}(v))}{\psi(v)} \int_0^1 \frac{\psi(v)}{f(\bar{F}^{-1}(v))} \,\mathrm{d}v \\
&= B_2 \mathcal{E}(X_1).
\end{aligned}$$

The lower bound can be obtained in a similar way. □

Other simple and interesting bounds, which depend on the extremes of the PDF and the distortion function of the system, are given in the following proposition.

Proposition 2. *Let T denote the lifetime of a mixed system with i.d. components with a common PDF f and distortion function $\bar{q}$. If S is the support of f, $m = \inf_{x \in S} f(x)$ and $M = \sup_{x \in S} f(x)$, then*

$$\frac{1}{M} I_q \leq \mathcal{E}(T) \leq \frac{1}{m} I_q, \tag{15}$$

where $I_q = \int_0^1 \psi(\bar{q}(v)) \,\mathrm{d}v$ and $\psi(v) = -v \log(v)$.

Proof. Since $m \leq f(\bar{F}^{-1}(v)) \leq M$, $0 < v < 1$, from (8), we have

$$\mathcal{E}(T) = \int_0^1 \frac{\psi(\bar{q}(v))}{f(\bar{F}^{-1}(v))} \,\mathrm{d}v \geq \frac{1}{M} \int_0^1 \psi(\bar{q}(v)) \,\mathrm{d}v.$$

The upper bound can be obtained in a similar way. □

Note that

$$I_q = \int_0^1 \psi(\bar{q}(v))\,\mathrm{d}v = \int_0^1 \psi(\bar{q}(1-v))\,\mathrm{d}v = \mathcal{E}(T_{\mathrm{U}}),$$

where $T_{\mathrm{U}} = F(T)$ is the lifetime of a system with the same structure as T, the same survival copula K, and i.d. components having a common uniform distribution over (0, 1). Hence, we also have $I_q = \mathcal{E}(V)$, where $V = \bar{F}(T)$. Of course, I_q depends only on $\bar{q}$ (i.e. on the system structure and on the survival copula). In particular, in the i.i.d. case, it depends only on the system signature. Note that the bounds in (15) also depend on the extremes of the PDF f. If $m = 0$ then we do not have an upper bound and if $M = \infty$ then we do not have a lower bound.

Example 2. For the system in Example 1 with i.i.d. components, $I_q = 0.1993$.

(i) For the exponential distribution with mean μ, $m = 0$, and $M = 1/\mu$, and, hence, from the preceding proposition, we obtain $\mathcal{E}(T) \geq 0.1993\mu$. From Proposition 1, we obtain the bounds $0 \leq \mathcal{E}(T) \leq B_2\mathcal{E}(X_1) = 1.0452\mu$ (since $B_1 = 0$). The exact value of $\mathcal{E}(T)$ was obtained in Example 1.

(ii) If X is uniformly distributed over the interval $[\alpha, \beta]$, $m = M = (\beta-\alpha)^{-1}$, and, hence $\mathcal{E}(T) = 0.1993(\beta-\alpha)$. From Proposition 1, we obtain the bound $\mathcal{E}(T) \leq B_2\mathcal{E}(X_1) = 1.0452 \times 0.25(\beta-\alpha) = 0.2613(\beta-\alpha)$ which, in this case, is not useful.

(iii) If X has a Pareto type II distribution with the survival function given in Table 1, then $m = 0$ and $M = \alpha\beta^{\alpha}$. Therefore, $\mathcal{E}(T) \geq 0.1993/(\alpha\beta^{\alpha})$. From Proposition 1, we obtain the upper bound $\mathcal{E}(T) \leq B_2\mathcal{E}(X_1) = 1.0452\alpha\beta/(\alpha-1)^2$ whenever $\alpha > 1$.

Proceeding as in the preceding example, we can obtain the bounds given in Table 3 for all the coherent systems with 1–4 i.i.d. components having a continuous distribution F. From the results in Table 3, we see that $B_1 = 0$ for all the coherent systems with 1–4 i.i.d. components different from X_1 (row 1). This is a general property for coherent systems with i.i.d. components (see the next proposition). However, it is not true for mixed systems. For example, if we consider the mixed system with signature $(\frac{5}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$, a straightforward calculation shows that $B_1 = \frac{1}{2}$ and $B_2 = \frac{5}{2}$. In any case, note that, if the PDF of the components is bounded then we can use I_q to obtain a lower bound for $\mathcal{E}(T)$.

Proposition 3. *The lower bound B_1 in Proposition 1 is 0 for all the mixed system with i.i.d. components and signature $(s_1, \dots, s_n)$ satisfying $s_1 = 0$ or $s_n = 0$. In particular, it is 0 for all the coherent systems with $n > 1$ i.i.d. components.*

Proof. If $s_n = 0$ then from (7), we have $a_1 = 0$. Hence, a straightforward calculation proves that $B_1 = \psi(\bar{q}(0))/\psi(0) = 0$. If $s_1 = 0$ then $s_n^* = 0$ and $a_1^* = 0$ for the dual system whose distortion function is $\bar{q}^*(u) = 1 - \bar{q}(1-u) = \sum_{i=2}^n a_i^* u^i$. Hence, from L'Hôpital's rule, $B_1 = \lim_{u\to 1^-} \psi(\bar{q}(u))/\psi(u) = \bar{q}'(1) = 0$ since $\bar{q}'(1-u) = \sum_{i=2}^n i a_i^* u^{i-1}$.

From [7], we know that if $n > 1$ then $s_1 = 0$ or $s_n = 0$. So $B_1 = 0$. □

Now, we obtain general lower and upper bounds for the CRE of T, which can be obtained from the system signature and the CRE of k-out-of-n systems.

Proposition 4. *If T denotes the lifetime of a mixed system with signature $(s_1, \dots, s_n)$ based on n i.i.d. components, then*

$$\mathcal{E}_{\mathrm{L}}(T) \leq \mathcal{E}(T) \leq \mathcal{E}_{\mathrm{U}}(T), \tag{16}$$

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TABLE 3: Bounds for $\mathcal{E}(T)$ for the system given in Table 2 obtained from Propositions 1, 2, and 4. In Proposition 4 we assume a standard exponential distribution. We also provide the optimal index $j^\star$ in (17) used to determine $\mathcal{E}_{\mathrm{U}}(T)$.

N	B_1	B_2	I_q	$\mathcal{E}_{\mathrm{L}}(T)$	$\mathcal{E}_{\mathrm{U}}(T)$	$j^\star$
1	1	1.0000	0.2500	1.0000	1.0000	1
2	0	2.0000	0.2222	0.5000	0.5000	1
3	0	2.0000	0.1869	1.1137	1.1137	2
4	0	3.0000	0.1875	0.3333	0.3333	1
5	0	1.1509	0.2216	0.5094	0.5974	2
6	0	1.0121	0.1980	0.5974	0.5974	2
7	0	1.0564	0.2075	0.7843	1.1579	3
8	0	3.0000	0.1464	1.1580	1.1580	3
9	0	4.0000	0.1600	0.2500	0.2500	1
10	0	2.0000	0.1976	0.3320	0.4139	2
11	0	1.3437	0.1957	0.3729	0.4139	2
12	0	1.2493	0.2128	0.4111	0.5177	2
13	0	1.0572	0.2245	0.4876	0.6963	3
14	0	1.1591	0.1845	0.4139	0.4139	2
15	0	1.0967	0.1954	0.4521	0.5177	2
16, 17	0	1.0453	0.1993	0.4903	0.6216	2
18, 19	0	1.0121	0.1980	0.5285	0.6431	3
20, 21	0	1.0002	0.1924	0.5667	0.6431	3
22	0	1.0052	0.1830	0.6049	0.6431	3
23	0	1.0202	0.1703	0.6431	0.6431	3
24	0	1.0125	0.2196	0.6631	1.0609	3
25	0	1.0982	0.1924	0.7395	1.0609	3
26	0	1.1367	0.1746	0.7777	1.0609	3
27	0	2.0000	0.1653	0.9123	1.1815	4
28	0	4.0000	0.1198	1.1815	1.1815	4

where $\mathcal{E}_{\mathrm{L}}(T) = \sum_{i=1}^{n} s_i \mathcal{E}(X_{i:n})$, $\mathcal{E}_{\mathrm{U}}(T) = \min_{1\le j\le n}\{\mathcal{E}(T, X_{j:n}) + \mathbb{E}(X_{j:n}) - \mathbb{E}(T)\}$, *and* $\mathcal{E}(T, X_{j:n})$ *is the CRI defined in (6) of* T *and the* j*th order statistic* $X_{j:n}$.

Proof. From Samaniego's representation, we have $\bar{q} = \sum_{i=1}^{n} s_i \bar{q}_{i:n}$, where $\bar{q}_{i:n} = G_{i:n}$ is the distortion function associated to $X_{i:n}$. Then from (12) and the concavity of ψ, the lower bound can be obtained as

$$\mathcal{E}(T) = \int_0^1 \frac{\psi(\bar{q}(v))}{f(\bar{F}^{-1}(v))}\,\mathrm{d}v \ge \int_0^1 \frac{\sum_{i=1}^{n} s_i \psi(\bar{q}_{i:n}(v))}{f(\bar{F}^{-1}(v))}\,\mathrm{d}v = \sum_{i=1}^{n} s_i \mathcal{E}(X_{i:n}).$$

To provide an upper bound, we use the CKL expression given in (5) to obtain

$$\mathrm{CE}(T, X_{j:n}) = \mathbb{E}(X_{j:n}) - \mathbb{E}(T) + \mathcal{E}(T, X_{j:n}) - \mathcal{E}(T) \ge 0 \quad \text{for } 1 \le j \le n,$$

provided that $\bar{F}_T(t) = 0$ whenever $\bar{F}_{j:n}(t) = 0$. Then the upper bound is derived by finding the minimum of the above terms for $1 \le j \le n$. □

From (7), the upper bound can be rewritten as

$$\min_j(\mathcal{E}(T, X_{j:n}) + \mathbb{E}(X_{j:n}) - \mathbb{E}(T)) = \mathcal{E}_{\mathrm{L}}(T) + \min_j\bigg(\sum_{i=1}^{n} s_i \mathrm{CE}(X_{i:n}, X_{j:n})\bigg). \tag{17}$$

Let the optimal index, say $j^\star$, satisfy

$$\mathcal{E}(T, X_{j^\star:n}) + \mathbb{E}(X_{j^\star:n}) - \mathbb{E}(T) = \min_{1\le j\le n}\{\mathcal{E}(T, X_{j:n}) + \mathbb{E}(X_{j:n}) - \mathbb{E}(T)\}.$$

Then $j^\star$ can be obtained through

$$c_T(j^\star) = \sum_{i=1}^{n} s_i \mathrm{CE}(X_{i:n}, X_{j^\star:n}) = \min_{1\le j\le n}\sum_{i=1}^{n} s_i \mathrm{CE}(X_{i:n}, X_{j:n}). \tag{18}$$

One can see that the optimal index $j^\star$ depends on the system signature and on the parent distribution function F. In Table 3 the bounds and the optimal index $j^\star$ for all coherent systems with 1–4 components are presented when the lifetimes of components follow the exponential distribution with mean unity. Recalling (17), the upper bound can be computed as the sum of two terms, the lower bound given by Proposition 4 and (18).

5. A CRE-based ordering of systems

The physical nature of certain system structures often restricts the use of the usual stochastic ordering for pairwise comparison. For instance, the systems numbers 13 and 17 in Table 2, are not comparable with the usual stochastic order (see [12], [14] or [18]). It is well known that the survival function of any arbitrary mixed system is located between that of the series and parallel systems, i.e. $X_{1:n} \le_{\mathrm{st}} T \le_{\mathrm{st}} X_{n:n}$ for any system lifetime T. Therefore, instead of pairwise comparison of systems, one can find a system in which its structure (or distribution) is closer to the distribution of the parallel system or the series system. In other words, using these systems, we seek an answer to the following question: which of these systems is similar (or closer) to the configuration of the parallel system and far from the configuration of the series system? We will use the CRE to answer this question.

To proceed with our results, we define the following measure of distance between two distributions considered by Baratpour and Rad [4]. It is a symmetric version of the CKL divergence $\mathrm{CE}(X, Y)$ defined in Section 2.

Definition 2. If X and Y are two nonnegative random variables with a common support and CDFs F and G, respectively, then the symmetric CKL is defined as:

$$\mathrm{SCE}(X, Y) = \mathrm{CE}(X, Y) + \mathrm{CE}(Y, X) = \int_0^\infty [\bar{F}(x) - \bar{G}(x)] \log\frac{\bar{F}(x)}{\bar{G}(x)}\,\mathrm{d}x. \tag{19}$$

The proposed measure (19) is nonnegative, symmetric, and $\mathrm{SCE}(X, Y) = 0$ if and only if $\bar{F}(x) = \bar{G}(x)$ almost everywhere; see [3]. We also have the following properties.

Lemma 1. *Let X, Y, and Z be random variables with CDFs F, G, and H, respectively. If $X \le_{\mathrm{st}} Y \le_{\mathrm{st}} Z$ then* $\mathrm{SCE}(X, Y) \le \mathrm{SCE}(X, Z)$ *and* $\mathrm{SCE}(Y, Z) \le \mathrm{SCE}(X, Z)$.

Proof. As the function $(x-1)\log(x)$ is decreasing in $(0,1)$ and increasing in $(1,\infty)$, condition $\bar F(t) \le \bar G(t) \le \bar H(t)$ for $t>0$, implies that

$$[\bar G(t) - \bar F(t)]\log\frac{\bar G(t)}{\bar F(t)} \le [\bar H(t) - \bar F(t)]\log\frac{\bar H(t)}{\bar F(t)}, \qquad t>0,$$
$$0 \le [\bar G(t) - \bar H(t)]\log\frac{\bar G(t)}{\bar H(t)} \le [\bar F(t) - \bar H(t)]\log\frac{\bar F(t)}{\bar H(t)}, \qquad t>0.$$

Integrating both sides of the above descriptions, the desired results follow. □

As $X_{1:n} \le_{\mathrm{st}} T \le_{\mathrm{st}} X_{n:n}$ holds for any system T, we have the following result.

Proposition 5. *If T is the lifetime of a mixed system based on $X_1,\dots,X_n$, then* $\mathrm{SCE}(T, X_{i:n}) \le \mathrm{SCE}(X_{1:n}, X_{n:n})$ *for $i=1,n$.*

Thus, we propose the following distance symmetric measure (DSM) for T:

$$\mathrm{DSM}(T) = \frac{\mathrm{SCE}(T, X_{1:n}) - \mathrm{SCE}(T, X_{n:n})}{\mathrm{SCE}(X_{1:n}, X_{n:n})}.$$

From Proposition 5, we have $-1 \le \mathrm{DS}(T) \le 1$. One can see that $\mathrm{DS}(T)=1$ if and only if $T \overset{\mathrm{st}}{=} X_{n:n}$ and $\mathrm{DS}(T)=-1$ if and only if $T \overset{\mathrm{st}}{=} X_{1:n}$. In other words, one can say that if $\mathrm{DS}(T)$ is closer to 1, the distribution of T is closer to the distribution of the parallel system, and if $\mathrm{DS}(T)$ is closer to -1, the distribution of T is closer to the distribution of the series system. Now, we propose the following definition.

Definition 3. Let T_1 and T_2 be the lifetimes of two mixed systems. Then T_2 is more preferable than T_1 in the DSM, denoted by $T_1 \le_{\mathrm{DSM}} T_2$, if $\mathrm{DSM}(T_1) \le \mathrm{DSM}(T_2)$.

We should note here that $\mathrm{DSM}(T_1) = \mathrm{DSM}(T_2)$ does not imply $T_1 \overset{\mathrm{st}}{=} T_2$. Under the conditions of Definition 3, we define $\mathrm{DDS}(T) = \mathrm{SCE}(T, X_{1:n}) - \mathrm{SCE}(T, X_{n:n})$. If the components are i.d., (19) and the above transformations imply that

$$\mathrm{SCE}(T, X_{i:n}) = \int_0^1 \frac{[\bar q(v) - \bar q_{i:n}(v)]}{f(\bar F^{-1}(v))}\log\frac{\bar q(v)}{\bar q_{i:n}(v)}\,\mathrm{d}v \quad \text{for } i=1,n. \tag{20}$$

Then, from (20), we obtain

$$\mathrm{DDS}(T) = \int_0^1 \frac{[\bar q(v) - \bar q_{1:n}(v)]}{f(\bar F^{-1}(v))}\log\frac{\bar q(v)}{\bar q_{1:n}(v)}\,\mathrm{d}v - \int_0^1 \frac{[\bar q(v) - \bar q_{n:n}(v)]}{f(\bar F^{-1}(v))}\log\frac{\bar q(v)}{\bar q_{n:n}(v)}\,\mathrm{d}v,$$
$$\mathrm{SCE}(X_{1:n}, X_{n:n}) = \int_0^1 \frac{[\bar q_{n:n}(v) - \bar q_{1:n}(v)]}{f(\bar F^{-1}(v))}\log\frac{\bar q_{n:n}(v)}{\bar q_{1:n}(v)}\,\mathrm{d}v.$$

If the components are i.i.d. then $\bar q_{1:n}(v) = v^n$ and $\bar q_{n:n}(v) = 1-(1-v)^n$. As an application of the proposed measure, consider the following example.

Example 3. Consider the two coherent systems numbers 13 and 17 in Table 3. These systems are not comparable with the usual stochastic order (see [14]). If we suppose that component lifetimes are i.i.d. and have a standard exponential distribution, we have $\mathrm{DSM}(T_{13}) = -0.1505$ and $\mathrm{DSM}(T_{17}) = -0.1918$ and, hence, $T_{17} \le_{\mathrm{DSM}} T_{13}$. Therefore, the system with lifetime T_{13} is closer to the distribution of the parallel system and hence to be preferred to T_{17}.

Theorem 2. *Let T_1 and T_2 be the lifetimes of two mixed systems with signatures $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ based on n i.i.d. (or exchangeable) components with the common CDF F (and the common copula K). If $\boldsymbol{s}_1 \le_{\mathrm{st}} \boldsymbol{s}_2$ then $T_1 \le_{\mathrm{DSM}} T_2$.*

Proof. Since $\boldsymbol{s}_1 \le_{\mathrm{st}} \boldsymbol{s}_2$ then $X_{1:n} \le_{\mathrm{st}} T_1 \le_{\mathrm{st}} T_2 \le_{\mathrm{st}} X_{n:n}$ by Theorem 2.1 in Navarro *et al.* [18]. Hence, from Lemma 1, we have $\mathrm{SCE}(T_1, X_{1:n}) \le \mathrm{SCE}(T_2, X_{1:n})$ and $\mathrm{SCE}(T_1, X_{n:n}) \ge \mathrm{SCE}(T_2, X_{n:n})$. Therefore, the desired result follows. □

In a similar way it can be proved that if T_1 and T_2 are two coherent (or mixed) systems based on the component lifetimes $X_1, \dots, X_n$, $T_1 \le_{\mathrm{st}} T_2$ implies $T_1 \le_{DSM} T_2$. Hence, the DSM comparison can be seen as a necessary condition for the usual stochastic order. Thus, the DSM order can be used to compare systems which cannot be compared by using the usual stochastic order. In particular, $T_1 \overset{\mathrm{st}}{=} T_2$ implies $T_1 \overset{\mathrm{DSM}}{=} T_2$. The values of $\mathrm{DSM}(T)$ for the coherent systems with 1–4 i.i.d. components and a common standard exponential distribution are given in Table 2.

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