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A DUAL CHARACTERIZATION OF BANACH SPACES WITH THE CONVEX POINT-OF-CONTINUITY PROPERTY

BY

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ABSTRACT. We introduce a new type of differentiability, called *cofinite Fréchet differentiability*. We show that the convex point-of-continuity property of Banach spaces is dual to the cofinite Fréchet differentiability of all equivalent norms. A corresponding result for dual spaces with the weak* convex point-of-continuity property is also established.

All Banach spaces considered in this note are over the real field, and are infinite dimensional unless otherwise specified. For unexplained terms and notation, see [5].

The following theorem, which was the culmination of several years of effort by many mathematicians, is the starting point of the work presented here:

THEOREM 1. *Let X be a Banach space. Then: (a) X^* has the Radon-Nikodym Property (RNP; every closed, bounded subset is dentable) if and only if X is an Asplund space (every continuous, convex $\varphi : X \longrightarrow R$ is densely Fréchet differentiable). (b) X has RNP if and only if X^* is a weak*-Asplund space (every continuous, convex, dual function $\varphi : X^* \longrightarrow \mathbf{R}$ is densely Fréchet differentiable).*

In [2], Bourgain introduced a variation of the *RNP*, which he called *Property* (*), but which is now known as the *Convex Point-of-Continuity Property (CPCP): X* has *CPCP* if every closed, bounded, convex subset of X has relatively weakly open subsets of arbitrarily small (norm) diameter. More recently, the dualized version, called C*PCP, was introduced in [6]: X^* has *C*PCP* if every weak*-compact, convex subset of X^* has relatively weak*-open subsets of arbitrarily small diameter. These properties are coming to be viewed as important tools in the study of the geometry of Banach spaces.

Building on the theme of Theorem 1, Deville et al. proved the following result in [4]:

THEOREM 2. *Let X be a separable Banach space. Then: (a) X^* has C*PCP if and only if X is a Phelps space (every continuous, convex, Gâteaux differentiable $\varphi : X \longrightarrow R$ is densely Fréchet differentiable). (b) If X^* is also separable, then X has CPCP if and only if X^* is a weak*-Phelps space (every continuous, convex, Gâteaux*

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differentiable, dual function $\varphi : X^* \longrightarrow \mathbf{R}$ *is densely Fréchet differentiable).*

The proof of Theorem 2, as presented in [4], depends heavily on the separability assumptions, and thus Theorem 2 falls short of providing a complete dual characterization of Banach spaces which have *CPCP* or whose duals have C^*PCP. The purpose of this note is to establish such a complete characterization.

The idea for our result comes from an observation made by Namioka and Phelps in [9]: A Banach space X is an Asplund space if and only if every equivalent norm on X is densely Fréchet differentiable. (A similar observation concerning Phelps spaces is made in [4]). This suggests that a dual characterization of $CPCP/C^*PCP$ spaces might be obtained through properties of norms rather than arbitrary continuous, convex functions. To this end, we introduce the following notion of differentiability:

DEFINITION 3. *Let* X *be a Banach space. A norm,* $\|\cdot\|$, *on* X *will be called* ϵ*-Fréchet differentiable at a point* $x \in X$, *where* $\epsilon > 0$, *if*

$$\limsup_{h \longrightarrow 0} \frac{\|x+h\| + \|x-h\| - 2\|x\|}{\|h\|} < \epsilon.$$

(Recall that the norm is Fréchet differentiable at x *if and only if it is* ϵ*-Fréchet differentiable at* x *for every* $\epsilon > 0$.)

The norm, $\|\cdot\|$, *will be called cofinitely Fréchet differentiable at* $x \in X$ *if for every* $\epsilon > 0$ *there is a finite dimensional* $F \leqq X$ *such that the quotient norm,* $\|\cdot\|_{X/F}$, *is* ϵ*—Fréchet differentiable at* $\overline{x}$, *where* $\overline{x}$ *denotes the equivalence class of* x *in* X/F. *Observe that necessarily* $x \notin F$.

As examples, it is straightforward to show that the usual norm of the space c_0 is cofinitely Fréchet differentiable everywhere (it is not Fréchet differentiable everywhere) and that the usual norm on l_1 is cofinitely Fréchet differentiable nowhere. Also, it is immediate that a Fréchet differentiable norm is cofinitely Fréchet differentiable.

With this definition of differentiability, we can state our main result:

THEOREM 4. *Let* X *be a Banach space. Then:* (*a*) X^* *has* C^*PCP *if and only if every equivalent norm on* X *is cofinitely Fréchet differentiable everywhere, if and only if every equivalent norm on* X *is cofinitely Fréchet differentiable somewhere.* (*b*) X *has CPCP if and only if every equivalent dual norm on* X^* *is cofinitely Fréchet differentiable everywhere, if and only if every equivalent dual norm on* X^* *is cofinitely Fréchet differentiable somewhere.*

An application of this theorem to *RNP* spaces is the following result:

COROLLARY 5. *Let* X *be a Banach space.* (*a*) *If* X^* *has RNP then every equivalent norm on* X *is* 1*-cofinitely Fréchet differentiable everywhere* (*meaning that for every* $x \in X\backslash\{0\}$ *there is a* 1*-dimensional subspace* $F \leqq X$ *such that the quotient norm* $\|\cdot\|_{X/F}$ *is Fréchet differentiable at* $\overline{x}$). (*b*) *If* X *has RNP then every equivalent dual norm on* X^* *is* 1*-cofinitely Fréchet differentiable everywhere.*

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Thus, suppose X is a Banach space such that X^* has *RNP*, and let $\|\cdot\|$ be an equivalent norm on X. By Theorem 1, $\|\cdot\|$ is densely Fréchet differentiable. Corollary 5 then characterizes those points where $\|\cdot\|$ fails to be Fréchet differentiable as points of 1-cofinite Fréchet differentiability.

Theorem 4 and Corollary 5 will be proven after we fix our notation and terminology.

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As a norm can never be differentiable, at the origin, in any sense, the expression "differentiable everywhere," when applied to a norm, means everywhere except at 0, which is equivalent to everywhere on $\mathcal{S}_X$, the unit sphere of X.

The open and closed unit balls of X are denoted by $\mathcal{U}_X$ and $\mathcal{B}_X$, respectively.

The action of $f \in X^*$ on $x \in X$ will be denoted by $\langle f, x\rangle$. For purposes of clarity, the dual on X^* to a norm, $\|\cdot\|$, on X will often by indicated by $\|\cdot\|^*$.

The notation $F \leqq X$ means that F is a closed, linear subspace of X. The *annihilator* of F is the set

$$F^{\perp} = \{f \in X^* : \langle f, F\rangle = 0\}.$$

A *slice* of a non-empty set $C \subset X$ is a set of the form

$$Sl(C, f, \alpha) = \{x \in C : \langle f, x\rangle > \sup\langle f, C\rangle - \alpha\},$$

where $f \in X^*$ and $\alpha > 0$. Recall that C is *dentable* if it has slices of arbitrarily small diameter. Note that a slice is always non-empty.

When discussing dual spaces, the modifier *weak** indicates that the corresponding functional(s) are to be taken from the predual, and not from the second dual. For example, the *weak*-annihilator* of a subspace $F \leqq X^*$ is the set

$$F_{\perp} = \{x \in X : \langle x, F\rangle = 0\}.$$

In preparation for the proof of Theorem 4, we collect here some necessary basic results. The first two, Lemmas 6 and 7, are due to Bourgain [2]:

LEMMA 6. *Let X be a Banach space with norm $\|\cdot\|$. Then: (a) X has CPCP if and only if for every closed, bounded, convex $C \subset X$ the identity map id: $(C, wk) \longrightarrow (C, \|\cdot\|)$ has a point of continuity. (Such a point is called a point of weak-to-norm continuity of C.) (b) X^* has C^*PCP if and only if for every weak*-compact, convex $C \subset X^*$ the identity map id: $(C, w^*) \longrightarrow (C, \|\cdot\|^*)$ has a point of continuity. (Such a point is called a point of weak*-to-norm continuity of C.)*

LEMMA 7. *Let X be a Banach space. (a) If X does not have CPCP, then there is an equivalent norm, $\|\cdot\|$, on X and an $\epsilon > 0$ such that if $\|x\| < 1, F \leqq X$ is finite codimensional, and V is a weak-open neighbourhood of x, then* diam $\mathcal{B}_X \cap (x + F) \cap V \geqq \epsilon$*, where "$\mathcal{B}_X$" and "diameter" both refer to the norm $\|\cdot\|$. (b) If X^* does not have C^*PCP, then there is an equivalent dual norm, $\|\cdot\|^*$, on X^* and an $\epsilon > 0$*

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such that if $\|f\|^* < 1, F \leqq X^*$ *is finite codimensional and weak*-closed, and* V *is a weak*-open neighbourhood of* f, *then* $\operatorname{diam} \mathcal{B}_{X^*} \cap (f+F) \cap V \geqq \epsilon$, *where "*$\mathcal{B}_{X^*}$*" and "diameter" both refer to the norm* $\|\cdot\|^*$.

Our next preliminary result was proven in [4]. We sketch a proof here, suggested to us by V. Zizler, which yields an improvement on the quantitative estimates (namely, there is no growth in the diameter control, ϵ).

LEMMA 8. *Let* X *be a Banach space. (a) Let* $C \subset X$ *be closed, bounded and convex, let* $f \in X^*$, *and let* $\alpha \in \mathbf{R}$ *such that* $\inf\langle f, C\rangle < \alpha < \sup\langle f, C\rangle$. *Then* $x \in C \cap f^{-1}(\alpha)$ *is a point of weak-to-norm continuity of* $C \cap f^{-1}(\alpha)$ *if and only if* x *is a point of weak-to-norm continuity of* C. *(b) Let* $C \subset X^*$ *be weak*-compact and convex, let* $x \in X$, *and let* $\alpha \in \mathbf{R}$ *such that* $\inf\langle x, C\rangle < \alpha < \sup\langle x, C\rangle$. *Then* $f \in C \cap x^{-1}(\alpha)$ *is a point of weak*-to-norm continuity of* $C \cap x^{-1}(\alpha)$ *if and only if* f *is a point of weak*-to-norm continuity of* C.

PROOF. Sufficiency in both (a) and (b) is immediate. We prove necessity for part (a). The proof for part (b) is similar.

By translating C, if necessary, we may assume $\alpha = 0$. Let $\epsilon > 0$ and let V be an elementary (hence, convex) neighbourhood of x such that diam $V \cap C \cap f^{-1}(0) < \epsilon$. For $\beta > 0$, let

$$U_\beta = V \cap f^{-1}(-\beta, \beta).$$

Let $x_1, x_2 \in V \cap C$ such that $\langle f, x_1\rangle = -\langle f, x_2\rangle > 0$. For $i = 1, 2$, let K_i be the positive cone generated by x_i and $V \cap C \cap f^{-1}(0)$, i.e.,

$$K_i = \{x_i + t(y - x_i) : t \geqq 0, y \in V \cap C \cap f^{-1}(0)\}.$$

Then, by convexity, $V \cap C \subset K_1 \cup K_2$, so for $\beta > 0$,

$$U_\beta \cap C \subset (K_1 \cup K_2) \cap f^{-1}(-\beta, \beta).$$

It is now a straightforward homothety argument to show that a $\beta > 0$ can be chosen sufficiently small so that

$$\operatorname{diam}(K_1 \cup K_2) \cap f^{-1}(-\beta, \beta) < \epsilon,$$

from which the result clearly follows. □

Note that Lemma 7 follows easily from Lemma 8 (this proof is substantially different from Bourgain's own proof in [2]).

Next we have a result which is essentially due to John and Zizler [8]:

PROPOSITION 9. *Let* X *be a Banach space with norm* $\|\cdot\|, \epsilon > 0, x \in X$ *and* $f \in X^*$. *Then: (a)* $\|\cdot\|$ *is* ϵ*-Fréchet differentiable at* x *if and only if there exists* $\alpha > 0$ *such that* diam $Sl(\mathcal{B}_{X^*}, x, \alpha) < \epsilon$. *(b)* $\|\cdot\|^*$ *is* ϵ*-Fréchet differentiable at* f *if and only if there exists* $\alpha > 0$ *such that* diam $Sl(\mathcal{B}_X, f, \alpha) < \epsilon$.

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COROLLARY 10. *Let X be a Banach space with norm $\|\cdot\|, x \in X$, and $f \in X^*$. (a) The norm, $\|\cdot\|$, is cofinitely Fréchet differentiable at x if and only if for every $\epsilon > 0$ there is a finite dimensional $F \leqq X$ and an $\alpha > 0$ such that* diam $Sl(\mathcal{B}_{F^\perp}, \bar{x}, \alpha) < \epsilon$. *(b) The dual norm, $\|\cdot\|^*$, is cofinitely Fréchet differentiable at f if and only if for every $\epsilon > 0$ there is a finite dimensional $F \leqq X^*$ and an $\alpha > 0$ such that* diam$Sl(\mathcal{B}_{F_\perp}, \bar{f}, \alpha) < \epsilon$.

PROOF. The proof of (a) follows from Proposition 9(a), using the fact that $F^\perp$, the annihilator of F in X^*, is isometrically isomorphic to the dual of the quotient space, X/F.

The proof of (b) is similar. □

PROOF OF THOREM 4. (a) Suppose X^* does not have C^*PCP. By Lemma 7(b), there is an $\epsilon > 0$ and an equivalent norm, $\|\cdot\|$, on X such that if $\|f\|^* < 1$, $F \leqq X$ is finite dimensional, and V is a weak*-open neighbourhood of f in X^*, then diam$\mathcal{B}_{X^*} \cap (f + F^\perp) \cap V \geqq \epsilon$.

We will show that this norm is nowhere cofinitely Fréchet differentiable.

Let $x_0 \in X, F \leqq X$ with dim$F < \infty$ and $x_0 \notin F$. Without loss of generality, $\|\bar{x}_0\|_{X/F} = 1$. Let $0 < \alpha < 1$, and choose $f \in \mathcal{U}_{F^\perp}$ such that $\langle f, \bar{x}_0\rangle = 1 - \alpha/2$. Let $\{e_k\}_1^m$ be a basis for F, and let $x_k = x_0 + e_k, k = 1, \ldots, m$. Let $V = \{g \in X^* : |\langle g - f, x_k\rangle| < \alpha/2, k = 0, \ldots, m\}$. Then, by the choice of the norm, we have diam$\mathcal{B}_{X^*} \cap (f + F^\perp) \cap V \geqq \epsilon$. Now $f \in F^\perp$, so $f + F^\perp = F^\perp$. If $g \in F^\perp$, then $\langle g, x_k\rangle = \langle g, x_0\rangle \equiv \langle g, \bar{x}_0\rangle$, thus $\mathcal{B}_{X^*} \cap (f + F^\perp) \cap V = \{g \in \mathcal{B}_{F^\perp} : |\langle g - f, \bar{x}_0\rangle| < \alpha/2\}$. But $|\langle g - f, \bar{x}_0\rangle| < \alpha/2$ if and only if $1 - \alpha = \langle f, \bar{x}_0\rangle - \alpha/2 < \langle g, \bar{x}_0\rangle < \langle f, \bar{x}_0\rangle + \alpha/2 = 1$. Since $\|\bar{x}_0\| = 1$, the first inequality says that $\mathcal{B}_{X^*} \cap (f + F^\perp) \cap V \subset Sl(\mathcal{B}_{F^\perp}, \bar{x}_0, \alpha)$, and so diam $Sl(\mathcal{B}_{F^\perp}, \bar{x}_0, \alpha) \geqq \epsilon$. Since $0 < \alpha < 1$ was arbitrary, it follows from Corollary 10(a) that this norm cannot be cofinitely Fréchet differentiable at x_0.

Now let $\|\cdot\|$ be an equivalent norm on X, and assume that X^* has C^*PCP. Let $x_0 \in \mathcal{S}_X, \epsilon > 0$, and $0 < \alpha < 1$. Then $\mathcal{B}_{X^*} \cap x_0^{-1}(\alpha)$ is weak*-compact and convex, so by Lemma 8, there is an $f \in \mathcal{B}_{X^*} \cap x_0^{-1}(\alpha)$ which is a point of weak*-to-norm continuity of $\mathcal{B}_{X^*}$.

Let V be an elementary weak*-open neighbourhood of f with diam$\mathcal{B}_{X^*} \cap V < \epsilon/2$, say $V = \{g \in X^* : |\langle g - f, x_k\rangle| < \delta, k = 1, \ldots, m\}$, where $\{x_k\}_1^m \subset X$ and $\delta > 0$. Let $F =$ span$\{\langle f, x_0\rangle x_k - \langle f, x_k\rangle x_0 : k = 1, \ldots, m\}$. Note that $f \in F^\perp$and $\|\bar{x}_0\| \geqq \langle f, \bar{x}_0\rangle \equiv \langle f, x_0\rangle = \alpha$.

Let $M = \max_{0 \leqq k \leqq m} |\langle f, x_k\rangle| > \alpha$. We claim that diam$Sl(\mathcal{B}_{F^\perp}, \bar{x}_0, \alpha\delta M^{-1}) < \epsilon$, where δ is as in the definition of V. By Corollary 10(a), this implies that this norm is cofinitely Fréchet differentiable at x_0, completing the proof of part (a).

To see this, consider the sets

$$W_1 = \{g \in B_{F^\perp} : \langle g, x_0\rangle > \alpha - \alpha\delta M^{-1}\}$$
$$W_2 = \{g \in W_1 : \langle g, x_0\rangle \leqq \alpha\}$$

Note that since $\|\bar{x}_0\| \geqq \alpha, Sl(\mathcal{B}_{F^\perp}, \bar{x}_0, \alpha\delta M^{-1}) \subset W_1$, so it suffices to show that diam$W_1 < \epsilon$. Note further that diam$W_1 < 2$ diamW_2. For suppose $g_1 \in W_1 \backslash W_2$.

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Choose $0 < \beta < 1$ so that $\beta g_1 \in W_2$. Let K be the positive cone generated by g_1 and the set

$$W_3 = \{g \in W_2 : \langle g, x_0\rangle = \langle \beta g_1, x_0\rangle\}.$$

Then, by homothety and the convexity of $\mathcal{B}_{F^\perp}$,

$$\|g_1 - \beta g_1\| \leqq \frac{\|g_1 - \beta g_1\|}{\|g_1\|} = \frac{\operatorname{diam} W_3}{\operatorname{diam}(K \cap x_0^{-1}(0))} \leqq \frac{\operatorname{diam} W_2}{2},$$

and so $\operatorname{diam} W_1 \leqq \frac{3}{2} \operatorname{diam} W_2 < 2 \operatorname{diam} W_2$.

Thus we must show that $\operatorname{diam} W_2 < \epsilon/2$. Suppose $g \in W_2$. By the definition of F, we have that for $k = 1, \ldots, m$,

$$\begin{aligned} |\langle f - g, x_k\rangle| &= |\langle f, x_k\rangle - \langle g, x_0\rangle\langle f, x_k\rangle / \langle f, x_0\rangle| \\ &= |\langle f, x_k\rangle|(1 - \langle g, x_0\rangle/\alpha) \\ &< M(\delta M^{-1}) \\ &= \delta, \end{aligned}$$

and so $g \in \mathcal{B}_{X^*} \cap V$, and we are done.

(b) The proof of part (b) is essentially the same as that for part (a), using Corollary 10(b), where necessary. □

REMARK. Observe that in the second part of the proof of Theorem 4(a), the α chosen can be arbitrarily close to 1. This means that if $\|\cdot\|$ is cofinitely Fréchet differentiable at $x \in \mathcal{S}_X$, then for every $\epsilon > 0$ and $\delta > 0$ there is a finite dimensional $F \leqq X$ such that $\|\cdot\|_{X/F}$ is ϵ-Fréchet differentiable at $\bar{x}$ and $\|\bar{x}\| > 1 - \delta = \|x\| - \delta$. This latter condition implies that the translate $x + F$ of the subspace F is nearly tangent to $\mathcal{B}_X$ at x.

Indeed, if there is an $f \in \mathcal{B}_{X^*}$ such that $\langle f, x\rangle = 1$ and f is a point of weak*-to-norm continuity of $\mathcal{B}_{X^*}$, then we can choose F in Theorem 4(a) so that $x + F$ is tangent to $\mathcal{B}_X$ at x, and so $\|\bar{x}\| = \|x\|$. This requires f to be in both the set of points of weak*-to-norm continuity of $\mathcal{B}_{X^*}$ and the set of functionals which attain their norm in X. The former set is a weak*- dense $\mathcal{G}_\delta$ subset of $\mathcal{S}_{X^*}$ (see, e.g., [2]), while the latter set is norm dense, by the Bishop-Phelps Theorem. It is unknown if these two sets have non-trivial intersection.

COROLLARY 11. *Every equivalent norm on the James Tree space* (*JT*) *is cofinitely Fréchet differentiable everywhere.*

PROOF. It is shown in [6] that the dual space, JT^*, has C^*PCP. Now apply Theorem 4. □

PROOF OF COROLLARY 5. (a) Suppose X^* has *RNP*, and let $\|\cdot\|$ be an equivalent norm on X. Let $x_0 \in X$ and let $0 < \alpha < 1$. Then the set $\mathcal{B}_{X^*} \cap x_0^{-1}(\alpha)$ is weak*-compact and convex, hence has a weak*-strongly exposed point, say f [10]. Let x_1 be the corresponding strongly exposing functional, so that

$$\lim_{\delta \to 0} \operatorname{diam} Sl(\mathcal{B}_{X^*} \cap x_0^{-1}(\alpha), x_1, \delta) = 0.$$

Then the proof of Lemma 8 shows that

$$\lim_{\delta \to 0} \operatorname{diam}\{g \in \mathcal{B}_{X^*} : |\langle g - f, x_k \rangle| < \delta, k = 0, 1\} = 0.$$

That is, f is a point of weak*-to-norm continuity of $\mathcal{B}_{X^*}$ of a very special form, namely, f has arbitrarily small relative weak*- neighbourhoods in $\mathcal{B}_{X^*}$ determined by just the functionals x_0 and x_1.

The proof of Theorem 4(a) then shows that the quotient norm, $\|\cdot\|_{X/F}$, for the space $F = \operatorname{span}\{\langle f, x_0 \rangle x_1 - \langle f, x_1 \rangle x_0\}$ is Fréchet differentiable at $\bar{x}_0$.

The proof of part (b) is similar. □

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