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Nagoya Math. J.
Vol. 39 (1970), 29–37

ON HOLOMORPHIC FAMILIES OF HOLOMORPHIC MAPS

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Let D be the unit disk $\{z : |z| < 1\}$ in the complex plane $\boldsymbol{C}$ with boundary ∂D and closure $\bar{D}$, and denote by $\boldsymbol{R}$ the image of the canonical embedding $r \to r + iO$ of the real line into $\boldsymbol{C}$. The symbol ε will be used throughout to denote a complex parameter; the unit disk in the complex ε-plane will be denoted by D_p. A C^{1+a} map $\mathscr{C} : \partial D \times D_p \to \boldsymbol{D}$ $(0 < a < 1)$ is called a *holomorphic family of* C^{1+a} *curves* if

1° $\mathscr{C}_\varepsilon = \mathscr{C} | \partial D \times \{\varepsilon\}$ is a C^{1+a} Jordan curve in $\boldsymbol{C}$ for every $\varepsilon \in D_p$;

2° $\mathscr{C}_t = \mathscr{C} | \{t\} \times D_p$ is a holomorphic function for every $t \in \partial D$;

3° $\dfrac{\partial \mathscr{C}(t, \varepsilon)}{\partial t}$ is continuous in t and ε.

Denote by Ω_ε the simply-connected region in $\boldsymbol{C}$ bounded by $\mathscr{C}(\partial D \times \{\varepsilon\})$.

We are interested in the existence of holomorphic maps $f : D \times D_p \to \boldsymbol{C}$ which map $D \times \{\varepsilon\}$ conformally onto Ω_ε for every $\varepsilon \in D_p$ (f is then said to be *associated with* $\mathscr{C}$). The following theorem will be proved.

THEOREM 1. *Let $\mathscr{C} : \partial D \times D_p \to \boldsymbol{C}$ be a holomorphic family of C^{1+a} curves. If f is a holomorphic map associated with $\mathscr{C}$, then there exists a C^{1+a} homeomorphism $g : \partial D \to \partial D$ for which*

$$(*) \qquad \mathscr{C}(t, \varepsilon) = f(g(t), \varepsilon)$$

for all $(t, \varepsilon) \in \partial D \times D_p$, where f on the right hand side denotes the continuous extension of f to $\bar{D} \times D_p$.

Now $\mathscr{C}$ can always be normalized by the condition that for some $\varepsilon_0 \in D_p$, $\mathscr{C}_{\varepsilon_0}$ is the boundary value of a conformal map of $D \times \{\varepsilon_0\}$ onto Ω_{ε_0} (for let g_{ε_0} be such a conformal map, the existence of which is ensured by

Received January 8, 1969.

1) This research was partially supported by Contract Nonr 2216 (28) (NR–043–332). The author is currently an NSF postdoctoral fellow.

https://doi.org/10.1017/S0027763000013684 Published online by Cambridge University Press

the Riemann mapping theorem, and replace $\mathscr{C}(t,\varepsilon)$ with $\mathscr{C}(\pi\circ(\mathscr{C}_{\varepsilon_0})^{-1}\circ g_{\varepsilon_0}(t),\varepsilon)$ with projection $\pi:\partial D\times D_p\to\partial D$). If $\mathscr{C}$ is normalized in this sense, setting $\varepsilon=\varepsilon_0$ in (*) shows that $g:\partial D\to\partial D$ is the boundary value of a conformal map cf D onto itself. Consequently, we have

COROLLARY 1. *Let $\mathscr{C}:\partial D\times D_p\to \boldsymbol{C}$ be a normalized holomorphic family of $C^{1+\alpha}$ curves. Then there is a holomorphic map $f:D\times D_p\to\boldsymbol{C}$ associated with $\mathscr{C}$ if and only if $\mathscr{C}$ itself is the boundary value of a holomorphic map associated with $\mathscr{C}$.*

We may write

$$\mathscr{C}(t,\varepsilon)=\sum_{k=0}^{\infty}c_k(t)\varepsilon^k,$$

where if $\mathscr{C}$ is normalized at $\varepsilon=0$, $c_0(t)$ is the boundary value of a conformal map of D onto Ω_0.

COROLLARY 2. *Let $\mathscr{C}:\partial D\times D_p\to\boldsymbol{C}$ be a holomorphic family of $C^{1+\alpha}$ curves normalized at $\varepsilon=0$. If there is a holomorphic map f associated with $\mathscr{C}$, then necessarily each coefficient $c_k(t)$, $k\geqslant 0$, in the above expansion of $\mathscr{C}$ is the boundary value of a holomorphic function on D.*

EXAMPLE 1. For $|\varepsilon|$ sufficiently small, $\mathscr{C}(t,\varepsilon)=t+\varepsilon\bar{t}$ is a holomorphic family of $C^{1+\alpha}$ curves normalized at $\varepsilon=0$, where $\bar{t}$ is the complex conjugate of t. By corollary 2. there is no holomorphic map associated with $\mathscr{C}$.

S.E. Warschawski [6] has proved a general perturbation theorem which yields the following related result. If we restrict our attention to $\varepsilon\in\boldsymbol{R}$ and replace condition 2° on $\mathscr{C}$ with

> 2′° both $\mathscr{C}(t,\varepsilon)$ and $\dfrac{\partial\mathscr{C}(t,\varepsilon)}{\partial t}$ have "Taylor" expansions at $\varepsilon=0$ of order m,

then there always exists a continuous map $f:D\times(D_p\cap\boldsymbol{R})\to\boldsymbol{C}$ which maps $D\times\{\varepsilon\}$ conformally onto Ω_ε for every ε and which has a "Taylor" expansion at $\varepsilon=0$ of order m. In particular, if $\mathscr{C}$ depends real analytically on the parameter ε then there exists real analytic f associated with $\mathscr{C}$. This real analytic case was also proved by D. Zeitlin [7] (there are minor differences between these two results of a technical nature). His method involves proving that the solution $F(t,\varepsilon)$ of a certain extension of the well-known

https://doi.org/10.1017/S0027763000013684 Published online by Cambridge University Press

Gershgorin integral equation into the complex domain is a holomorphic function in ε for $\varepsilon\in U$, U being a certain open neighborhood of 0 in D_p. For every $\varepsilon\in U\cap \boldsymbol{R}$, $F(t,\varepsilon)$ gives the mapping function of Ω_ε onto D in the usual manner. An open question is the relationship of F on all of U to, when it exists, a holomorphic map f associated with the holomorphic family of curves whose restriction to $\partial D\times(D_p\cap \boldsymbol{R})$ is the given real analytic family of curves.

The proof of theorem 1 goes as follows. Let $\mathscr{C}:\partial D\times D_p\to \boldsymbol{C}$ be a holomorphic family of $C^{1+\alpha}$ curves and $f: D\times D_p\to \boldsymbol{C}$ a holomorphic map associated with $\mathscr{C}$. Define $\tilde{\mathscr{C}}:\partial D\times D_p\to \boldsymbol{C}^2$ and $\tilde{f}: D\times D_p\to \boldsymbol{C}^2$ by the rules $\tilde{\mathscr{C}}(t,\varepsilon)=(\mathscr{C}(t,\varepsilon),\varepsilon)$ and $f(t,\varepsilon)=(f(t,\varepsilon),\varepsilon)$. Let $\Omega=\{(z,\varepsilon): z\in\Omega_\varepsilon,\ \varepsilon\in D_p\}$. Then $\tilde{f}: D\times D_p\to\Omega$ is a biholomorphic map and $(\tilde{f}^{-1}|\Omega_\varepsilon\times\{\varepsilon\})(z,\varepsilon)=(f_\varepsilon^{-1}(z),\varepsilon)$, where $f_\varepsilon=f|D\times\{\varepsilon\}$. As is wellknown, $\tilde{f}_\varepsilon=\tilde{f}|D\times\{\varepsilon\}$ has a homeomorphic extension to $\bar{D}\times\{\varepsilon\}$ for every $\varepsilon\in D_p$. It will be shown (lemma 2.) that $\tilde{f}^{-1}\circ\tilde{\mathscr{C}}_t:\{t\}\times D_p\to\partial D\times D_p$ is a holomorphic map for every $t\times\partial D$. This is the central point in the proof of the theorem, for now write

$$\tilde{f}^{-1}\circ\tilde{\mathscr{C}}(t,\varepsilon)=(f'(t,\varepsilon),\varepsilon),$$

where $f':\partial D\times D_p\to\partial D$ is a continuous map; such an f' clearly exists. According to lemma 2., $(f'|\{t\}\times D_p):\{t\}\times D_p\to\partial D$ is a holomorphic map for every $t\in\partial D$ and is consequently constant in ε for every $t\in\partial D$. Therefore there is a homeomorphism $g:\partial D\to\partial D$ for which $f'(t,\varepsilon)=g(t)$, which implies that $\tilde{\mathscr{C}}(t,\varepsilon)=\tilde{f}(g(t),\varepsilon)$, and so $\mathscr{C}(t,\varepsilon)=f(g(t),\varepsilon)$. That g is a $C^{1+\alpha}$ map follows from Kellogg's theorem by normalizing $\mathscr{C}$ at some $\varepsilon_0\in D_p$, and the proof of theorem 1. is complete.

It should be clear from the local nature of lemma 2. that theorem 1. admits readily to generalizations. A few of these are presented after the proofs of lemmas 1. and 2.

§1. Choose any point $\tilde{\mathscr{C}}(t_0,\varepsilon_0)\in$ *bdy* Ω and let $n=n(t_0,\varepsilon_0)$ be the inward normal to $\mathscr{C}_{\varepsilon_0}$ at $\mathscr{C}(t_0,\varepsilon_0)$, i.e. $n\subseteq\Omega_{\varepsilon_0}$. Denote by $W(\alpha,r)=W_{t_0,\varepsilon_0}(\alpha,r)$ the wedge in Ω_{ε_0} with radius r and interior angle α at the vertex $\mathscr{C}(t_0,\varepsilon)$, and which is symmetric about the normal n, i.e. $W(\alpha,r)=\{z\in\Omega_{\varepsilon_0}: \mathrm{dist}(z,n)\leqslant|z-\mathscr{C}(t_0,\varepsilon_0)|\sin(\alpha/2)$ and $0<|z-\mathscr{C}(t_0,\varepsilon_0)|<r\}$. Also, for $z\in\Omega_{\varepsilon_0}$ denote by $\tilde{\mathscr{C}}_z=\tilde{\mathscr{C}}_{z,t_0,\varepsilon_0}: D_p\to\boldsymbol{C}^2$ the holomorphic map $(\mathscr{C}_{t_0}(\varepsilon)-\mathscr{C}(t_0,\varepsilon_0)+z,\varepsilon)$. Clearly, for each $z\in\Omega_{\varepsilon_0}$ there is an open neighborhood $U=U_z$ of ε_0 in D_p

such that $\tilde{\mathscr{C}}_z(U) \subseteq \Omega$. Lemma 1. will show that there are wedges $W(\alpha, r)$ for which U_z may be chosen independent of $z \in W(\alpha, r)$.

LEMMA 1. *For every* $\alpha (0 < \alpha < \pi)$ *there is an* $r > 0$ *and an open neighborhood* U *of* ε_0 *in* D_p *such that* $\tilde{\mathscr{C}}_z(U) \subseteq \Omega$ *for all* $z \in W(\alpha, r)$.

Proof. It is well-known [4] that since $\mathscr{C}_{\varepsilon_0}$ is a $C^{1+\alpha}$ curve, for every β $(0 < \beta < \pi/2)$ there is a connected subarc $\Gamma = \Gamma_\beta$ of ∂D containing t_0 in its interior such that the chord joining $\mathscr{C}(t_0, \varepsilon_0)$ and $\mathscr{C}(t, \varepsilon_0)$ makes an angle smaller than β with the tangent line to $\mathscr{C}_{\varepsilon_0}(\partial D)$ at $\mathscr{C}(t_0, \varepsilon_0)$ for every $t \in \Gamma$. It follows from the conditions on the map $\mathscr{C}$ that there is an open neighborhood U_1 of ε_0 such that the same is true for every $\mathscr{C}_\varepsilon$ with $\varepsilon \in U_1$ when β is replaced by 2β. Choose β so that $\pi - 4\beta > \alpha$.

Now it is also known that $r > 0$ may be chosen so that $|\mathscr{C}(t, \varepsilon_0) - \mathscr{C}(t_0, \varepsilon_0)| > 2r$ for every $t \in \partial D \backslash \Gamma$, and it follows again from the conditions on $\mathscr{C}$ that there is an open neighborhood U_2 of ε_0 such that $|\mathscr{C}(t, \varepsilon) - \mathscr{C}(t_0, \varepsilon)| > r$ for every $t \in \partial D \backslash \Gamma$ and every $\varepsilon \in U_2$.

Let $U = U_1 \cap U_2$. If $\tilde{\mathscr{C}}_z(\varepsilon) \in bdy\, \Omega$ for some $\varepsilon \in U$ there must be a $t \in \partial D$ such that $\tilde{\mathscr{C}}_z(\varepsilon) = \tilde{\mathscr{C}}_t(\varepsilon)$, or equivalently $\mathscr{C}_{t_0}(\varepsilon) - \mathscr{C}(t, \varepsilon_0) + z = \mathscr{C}_t(\varepsilon)$. Since $\mathscr{C}(t, \varepsilon) = \mathscr{C}_t(\varepsilon) = \mathscr{C}_\varepsilon(t)$, we have

$$(**) \qquad \tilde{\mathscr{C}}_z(\varepsilon) \in bdy\, \Omega \Longleftrightarrow \mathscr{C}(t_0, \varepsilon) - \mathscr{C}(t_0, \varepsilon_0) + z = \mathscr{C}(t, \varepsilon).$$

Suppose that $t \in \Gamma$. By the choice of Γ, since $\varepsilon \in U$, and since from $(**)$ it follows that $z - \mathscr{C}(t_0, \varepsilon_0) = \mathscr{C}(t, \varepsilon) - \mathscr{C}(t_0, \varepsilon)$, we have $\operatorname{dist}(z, n) > |z - \mathscr{C}(t_0, \varepsilon_0)| \sin(\pi/2 - 2\beta)$. But $\pi/2 - 2\beta > \alpha/2$, and so $\operatorname{dist}(z, n) > |z - \mathscr{C}(t_0, \varepsilon_0)| \sin(\alpha/2)$. Therefore $z \notin W(\alpha, r)$. Now suppose that $t \notin \Gamma$. Then by the choice of r, since $\varepsilon \in U$, and by $(**)$ as above we have $|z - \mathscr{C}(t_0, \varepsilon_0)| > r$, and so $z \notin W(\alpha, r)$. Consequently, $\tilde{\mathscr{C}}_z(\varepsilon) \in bdy\, \Omega$ for some $\varepsilon \in U$ implies that $z \notin W(\alpha, r)$, and the lemma is proved.

LEMMA 2. $\tilde{f}^{-1} \circ \tilde{\mathscr{C}}_t : \{t\} \times D_p \to \partial D \times D_p$ *is a holomorphic map for every* $t \in \partial D$.

Proof. Choose $(t_0, \varepsilon_0) \in \partial D \times D_p$; by lemma 1. there is a sequence of points $\{z_k : k = 1, 2, \cdots\} \subseteq \Omega_{\varepsilon_0}$ and an open neighborhood U of ε_0 for which $\tilde{\mathscr{C}}_{z_k}(U) \subseteq \Omega$ while $z_k \to \mathscr{C}(t_0, \varepsilon_0)$ as $k \to \infty$. Therefore $\tilde{f}^{-1} \circ \tilde{\mathscr{C}}_{z_k} : U \to D \times D_p$ is a well-defined holomorphic map for every $k=1,2,\cdots$. Since $\tilde{f}|\bar{D}\times\{\varepsilon\}$ is a homeomorphism for every $\varepsilon\in D_p$, the map $f^{-1}\circ\tilde{\mathscr{C}}_{t_0}$ is also defined; clearly the sequence $\{\tilde{f}^{-1}\circ\tilde{\mathscr{C}}_{z_k}: k=1,2,\cdots\}$ converges pointwise to $\tilde{f}^{-1}\circ\tilde{\mathscr{C}}_{t_0}$ on U. The lemma now follows from Vitali's theorem.

§ 2. **Generalizations.** (Ahlfors [1] has shown the existence of a holomorphic map f from a bordered Riemann surface with finite genus and a finite number of boundary components onto a full covering surface $S\xrightarrow{\pi}D$ of the unit disk. N. Alling [2] has shown that $\pi\circ f|U$ is a covering map of D near ∂D for some open neighborhood U of ∂X. Theorems 2.-4. can be thought of as concerning holomorphic families of such maps.)

Let X and Y be open Riemann surfaces such that X has a $C^{1+\alpha}$ boundary ∂X, and let V be a connected analytic set in some open set in $\boldsymbol{C}^n$. Let $\mathscr{C}:\partial X\times V\to Y$ be a $C^{1+\alpha}$ map satisfying

1° for every local coordinate t on ∂X for which t^{-1} describes ∂X locally as a $C^{1+\alpha}$ curve, $\mathscr{C}\circ t^{-1}$ is a holomorphic family of $C^{1+\alpha}$ curves on Y ($\mathscr{C}$ is then said to be locally a holomorphic family of $C^{1+\alpha}$ curves on Y);

2° for every $\mathscr{C}_\varepsilon=\mathscr{C}|\partial X\times\{\varepsilon\}$, $\mathscr{C}_\varepsilon(\partial X\times\{\varepsilon\})$ is the boundary of an open Riemann surface Ω_ε.

Theorem 2. is the most straightforward generalization of theorem 1. which can be proved.

THEOREM 2. *Denote the set* $\{(y,\varepsilon): y\in\Omega_\varepsilon,\ \varepsilon\in V\}$ *by* Ω. *There exists a holomorphic map* $f:\Omega\to X$ *which maps* $\partial\Omega_\varepsilon\times\{\varepsilon\}$ *into* ∂X *for every* $\varepsilon\in V$ *if and only if there is a* $C^{1+\alpha}$ *map* $g:\partial X\to\partial X$ *for which*

$$f\circ\tilde{\mathscr{C}}(x,\varepsilon)=g(x)$$

for all $x\in\partial X$ *and all* $\varepsilon\in V$.

More generally, one has

THEOREM 3. *Let* C *be an arc on* ∂X, $\mathscr{C}:C\times V\to Y$ *locally a holomorphic family of* $C^{1+\alpha}$ *curves on* Y, *and* $\Omega_\varepsilon\subseteq Y$ *a bordered Riemann surface with* $\mathscr{C}(C\times\{\varepsilon\})\subseteq\partial\Omega_\varepsilon$ *for every* $\varepsilon\in V$. *Define* Ω *as in theorem* 2. *There is a holomorphic map* $f:\Omega\to X$ *which maps* $\mathscr{C}(C\times\{\varepsilon\})\times\{\varepsilon\}$ *into* ∂X *for every* $\varepsilon\in V$ *if and only if there is a* $C^{1+\alpha}$ *map* $g:C\to\partial X$ *for which*

$$f\circ\tilde{\mathscr{C}}(x,\varepsilon)=g(x)$$

https://doi.org/10.1017/S0027763000013684 Published online by Cambridge University Press

for all $x \in C$ *and all* $\varepsilon \in V$.

Proof. All that must be shown is that theorem 1. remains true when D_p is replaced with the connected analytic set V. First of all, lemmas 1. and 2. carry over just as they were presented when D_p is replaced by a polydisk in $\boldsymbol{C}^k$. This means that theorem 1. is true when D_p is replaced by a connected component V_i of the set of regular points of V; let g_i be the map of theorem 1. for V_i. In fact $(*)$ holds on $Cl_V V_i$ and the usual continuity argument shows that $g_i = g_j$ when $Cl_V V_i \cap Cl_V V_j \neq \emptyset$. The theorem is therefore proved since V is connected and $V = U\{Cl_V V_i : i \in I\}$.

THEOREM 4. *Let* X, Y, $\mathscr{C} : C \times V \to Y$ *and* $\{\Omega_\varepsilon : \varepsilon \in V\}$ *be given as in theorem* 3. *If* $f : X \times V \to Y$ *is a holomorphic map satisfying*

a) $f(\partial X \times \{\varepsilon\}) \subseteq \partial \Omega_\varepsilon$ *for every* $\varepsilon \in V$;

b) $f_\varepsilon = f | X \times \{\varepsilon\}$ *is a covering map of* Ω_ε *near* $\partial \Omega_\varepsilon$ *for some open neighborhood of* $\partial X \times \{\varepsilon\}$ *in* $X \times \{\varepsilon\}$, *again for every* $\varepsilon \in V$, *then there exists a* $C^{1+\alpha}$ *map* $g : C \to \partial X$ *for which*

$$\mathscr{C}(x, \varepsilon) = f(g(x), \varepsilon)$$

for all $x \in C$ *and all* $\varepsilon \in V$.

By viewing theorems 1.-4. from another point of view one gets mapping theorems for complex manifolds. Theorem 5. below is one such result, although clearly not the most general one.

Let $\boldsymbol{P}$ be a polydisk in $\boldsymbol{C}^{n-1} (n > 1)$ and let C be a subarc of ∂D. Given a holomorphic family of $C^{1+\alpha}$ curves $\mathscr{C}' : C \times \boldsymbol{P} \to \boldsymbol{C}$ and holomorphic maps $\mathscr{C}_\nu : \boldsymbol{P} \to \boldsymbol{C}$ for each $\nu = 2, \cdots, m (m > n)$, define $\mathscr{C} : C \times \boldsymbol{P} \to \boldsymbol{C}^m$ by the rule

$$\mathscr{C}(t, \varepsilon) = (\mathscr{C}'(t, \varepsilon), \ \mathscr{C}_2(\varepsilon), \ \cdots, \ \mathscr{C}_m(\varepsilon))$$

for all $t \in C$ and all $\varepsilon \in \boldsymbol{P}$. We may assume without loss in generality that $\mathscr{C} | C \times \{0\}$ is the boundary value of a holomorphic function on $U \cap D$ for some open set $U \subseteq \boldsymbol{C}$. Let Ω be a domain in $\boldsymbol{C}^m$ for which $\mathscr{C}(C \times \boldsymbol{P}) \subseteq \partial \Omega$.

THEOREM 5. *If there is a holomorphic map* $f : \Omega \to D \times \boldsymbol{P}$ *for which* $f(\mathscr{C}(C \times \boldsymbol{P})) \subseteq \partial D \times \boldsymbol{P}$, *then necessarily* $\mathscr{C}$ *is the boundary value of a holomorphic map* $\mathscr{C} : U \cap D \times \boldsymbol{P} \to \Omega$ *for some open set* $U \subseteq \boldsymbol{C}^n$.

Proof. This theorem is a straightforward generalization of corollary 1.

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In view of this theorem one may ask for conditions on $\partial\Omega$ of a given domain Ω under which there exists a subarc C of ∂D and a map $\mathscr{C}: C\times \boldsymbol{P}\to\partial\Omega$ like the one described above. In this direction we have a Levi-type condition.

PROPOSITION 1. *Let Ω be an open domain in $\boldsymbol{C}^n (n>1)$ and suppose that $(z_0, \varepsilon^0)\in\partial\Omega$, where $\varepsilon^0 = (\varepsilon_2^0, \cdots, \varepsilon_m^0)\in \boldsymbol{C}^{n-1}$. In order that there exist an open neighborhood U of (z_0, ε^0), a polydisc $\boldsymbol{P}\subseteq\boldsymbol{C}^{n-1}$, a subarc C of ∂D and an injective $C^{2+\alpha}$ map $\mathscr{C}: C\times\boldsymbol{P}\to\partial\Omega\cap U$ satisfying the conditions in theorem 5. it is necessary that there exist an open neighborhood U' of (z_0, ε^0) and a C^2 map $\varphi: U'\to\boldsymbol{R}$ such that*

1° $\{(z, \varepsilon): \varphi(z,\varepsilon)=0\} = U'\cap\partial\Omega$;

2° *grad* $\varphi\neq 0$ *on* $U'\cap\partial\Omega$;

3° *denoting* $(z,\varepsilon) = (z, \varepsilon_2, \cdots, \varepsilon_n)$ *by* $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$, *then* $\sum_{i=1}^{n}\frac{\partial\varphi}{\partial\varepsilon_i}w_i = 0$ *at* $(\varepsilon_1, \cdots, \varepsilon_n)\in U'\cap\partial\Omega$ *implies that*

$$\sum_{i,j=1}^{n}\frac{\partial^2\varphi}{\partial\varepsilon_i\partial\bar\varepsilon_j}w_i w_j = 0.$$

Proof. Given injective $\mathscr{C}$, let ρ denote the coordinate function of $\mathscr{C}^{-1}$; it is known [5] that there is an open neighborhood V of (z_0, ε^0) and holomorphic functions $f_1: V\cap\Omega\to\boldsymbol{C}$, $f_2: V\cap(\boldsymbol{C}^n\backslash\bar\Omega)\to\boldsymbol{C}$ with C^2 extensions to $\partial\Omega\cap V$ such that $\rho(z,\varepsilon) = f_1(z,\varepsilon)f_2(z,\varepsilon)$ for all $(z,\varepsilon)\in\partial\Omega\cap V$. The differentiability properties of f_1 and f_2 on $\partial\Omega\cap V$ allow us to choose C^2 functions $\tilde f_1$ and $\tilde f_2$ on V for which $\tilde f_1|V\cap\Omega = f_1$ and $\tilde f_2|V\cap(\boldsymbol{C}^n\backslash\bar\Omega) = f_2$. Define the extension of ρ into V to be $f_1(z,\varepsilon)f_2(z,\varepsilon)$ and $\varphi: V\to\boldsymbol{R}$ by the rule $\varphi(z,\varepsilon) = |\rho(z,\varepsilon)|^2-1$. $\tilde f_1$ and $\tilde f_2$ can clearly be chosen so that 1° is satisfied, while 2° clearly holds for any choice of $\tilde f_1$ and $\tilde f_2$; 3° is the result of a straightforward computation.

The next result has to do with "extending" differentiable families of complex manifolds to holomorphic families. It will follow from theorem 1. in a manner similar to that for theorems 1.-4. except that X instead of V is to be viewed as the parameter space.

Let $\mathscr{V}\xrightarrow{\omega} X$ be a differentiable (i.e. C^∞) family of complex structures on the complex manifold V in the sense of Kodaira and Spencer [3], where X is an open Riemann surface. This means that for every point $v\in\mathscr{V}$ there is an open neighborhood U of v and a diffeomorphism $\Psi_U: U\to W\times\omega(U)$ for some open set W in $\boldsymbol{C}^n$ such that

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1° $\omega = pr^2 \circ \Psi_U$ (pr^2 is the canonical projection $W \times \omega(U) \to \omega(U)$);

2° $\Psi_U | \Psi_U^{-1}(W \times \{x\})$ is biholomorphic for every $x \in \omega(U)$. If ω is a holomorphic map $\mathscr{V} \xrightarrow{\omega} X$ is called a holomorphic family of complex structures on V.

Two differentiable families of complex structures on V, say $\omega_1 : \mathscr{V} \to X$ and $\omega_2 : \mathscr{V}_2 \to X$, are said to be *equivalent* if there is a diffeomorphism $\varphi : \mathscr{V}_1 \to \mathscr{V}_2$ satisfying

a) $\omega_1 = \omega_2 \circ \varphi$;

b) $\varphi \circ \Psi_U | \Psi_U^{-1}(W \times \{x\})$ is biholomorphic for every $x \in \omega_1(U)$ and every pair U, Ψ_U of open neighborhoods and diffeomorphisms respectively for $\omega_1 : \mathscr{V}_1 \to X$ as described above.

Let $X_0 \subseteq \bar{X}_0 \subseteq X$ be an open Riemann surface with differentiable boundary ∂X_0. $\mathscr{V} \xrightarrow{\omega} X$ induces by way of the canonical injections $X_0 \to X$, $\bar{X}_0 \to X$, and $\partial X_0 \to X$ differentiable families $\omega_0 : \mathscr{V}_0 \to X_0$, $\bar{\omega}_0 : \mathscr{V}_0 \to \bar{X}_0$ and $\omega_0^\partial : \mathscr{V}_0^\partial \to \partial X_0$ of complex structures on V which are called the *restrictions* of the family $\omega : \mathscr{V} \to X$ to X_0, $\bar{X}_0$, and ∂X_0 respectively.

THEOREM 6. *Let X_0 and X be open Riemann surfaces with $\bar{X}_0 \subseteq X$ and $\omega : \mathscr{V} \to X$, $\tilde{\omega} : \tilde{\mathscr{V}} \to X$ differentiable families of complex structures on a complex manifold V for which*

1) *the restriction $\tilde{\omega}_0 : \tilde{\mathscr{V}}_0 \to X_0$ is a holomorphic family of complex structures on V;*

2) *the restrictions $\omega_0^\partial : \mathscr{V}_0^\partial \to \partial X_0$ and $\tilde{\omega}_0^\partial : \tilde{\mathscr{V}}_0^\partial \to \partial X_0$ are equivalent.*

Then there is a differentiable map $g : \partial X_0 \to \partial X_0$ such that

$$\tilde{\omega}_0^\partial = g \circ \omega_0^\partial .$$

Lemmas 1. and 2. yield two other kinds of results. The first concerns boundary values of holomorphic functions of one variable. For example, every injective $C^{1+\alpha}$ map $h : \partial D \to \boldsymbol{C}$ can be embedded in a normalized holomorphic family of $C^{1+\alpha}$ curves $\mathscr{C} : \partial D \times D_p \to \boldsymbol{C}$; then the property that h^{-1} is the boundary value of a holomorphic function on the bounded domain with boundary $h(\partial D)$ is equivalent to the existence of a holomorphic map of Ω onto D, where Ω is defined for $\mathscr{C}$ as before. The second result concerns partial differential equations.

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THEOREM 7. *Let* $\mathscr{C} : \partial D \times D_p \to \boldsymbol{C}$ *be a holomorphic family of* $C^{1+\alpha}$ *curves and* $\Omega \subseteq \boldsymbol{C}^2$ *the domain described by* $\mathscr{C}$ *as before. Let* f_1, f_2 *be complexvalued,* C^∞ *functions on* Ω *with compact support in* $\bar{\Omega} \backslash \mathscr{C}(\partial D \times D_p)$. *If* u *is a* C^∞ *solution of the system*

$$\frac{\partial u}{\partial z} = f_1, \qquad \frac{\partial u}{\partial \varepsilon} = f_2$$

(in which case u *will have a continuous extension to* $\Omega \cup \mathscr{C}(\partial D \times D_p))$ *satisfying the boundary condition that the topological dimension of* $u \circ \mathscr{C}(\{t\} \times D_p)$ *is smaller than or equal to* 1 *for every* $t \in \partial D$, *then on* $\mathscr{C}(\partial D \times D_p)$ u *is necessarily of the form*

$$u(w) = g \circ \tilde{\mathscr{C}}^{-1}(w),$$

where $g : \partial D \to \boldsymbol{C}$ *is a* $C^{1+\alpha}$ *function.*

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