The Journal of Symbolic Logic
Volume 85, Number 2, June 2020

THE EXPONENTIAL DIOPHANTINE PROBLEM FOR $\mathbb{Q}$

MIHAI PRUNESCU

Abstract. We show that the set of natural numbers has an exponential diophantine definition in the rationals. It follows that the corresponding decision problem is undecidable.

There were two steps leading to the negative solution of Hilbert's Tenth Problem. First Martin Davis, Hilary Putnam and Julia Robinson proven in [2] that it is undecidable if exponential diophantine equations have solutions in $\mathbb{N}$. Later Yuri Matiyasevich shown that the exponential relation (x, y, x^y) is diophantine in $\mathbb{N}$ and concluded that it is undecidable if diophantine equations have solutions in $\mathbb{N}$, see [3].

In the present note is shown that the Davis–Putnam–Robinson step for $\mathbb{Q}$ reduces easily to Hilbert's Tenth Problem for $\mathbb{N}$.

We study here systems of exponential diophantine equations, which are expressions of the shape:

$$\bigwedge_{i=1}^{m} P_i(t_1, \dots, t_n) = 0,$$

where $P_i \in \mathbb{Z}[t_1, \dots, t_n]$ are polynomials and the terms t_j have the shape α^β where α and β are allowed to be variables or integers. We observe that an exponential expression x^y with $x, y \in \mathbb{Q}$ does not have necessarily real values, and is in general multivalued. We know, however, that the values belong to the algebraic closure $\overline{\mathbb{Q}}$. A disjunction of exponential diophantine equations over $\mathbb{Q}$ can be always represented as an exponential diophantine equation, because $P = 0 \vee Q = 0 \leftrightarrow PQ = 0$. This cannot be said anymore about conjunctions, at least in an apparent way, and probably is not true about complements.

The following result has been already known to Euler, see Dickson's History [1]:

Lemma. *The system $x^y = y^x \wedge 0 < x < y$ has no other solutions in $\mathbb{Q}$ as all the pairs*:

$$x_n = \left(1 + \frac{1}{n}\right)^n, \;\; y_n = \left(1 + \frac{1}{n}\right)^{n+1},$$

where $n \in \mathbb{N} \setminus \{0\}$. ⊣

Received March 8, 2019.
2020 *Mathematics Subject Classification.* 11D61, 14G05.
Key words and phrases. exponential diophantine equations, rational numbers, decision problems, undecidability.

0022-4812/20/8502-0007
DOI:10.1017/jsl.2020.18

$$\mathbb{Q} \models \ \alpha > 0 \ \leftrightarrow \ \exists x_1, \ldots, x_5 \ \alpha = x_1^2 + \cdots + x_4^2 \ \wedge \ \alpha x_5 = 1.$$

It follows that in $\mathbb{Q}$ the set $\mathbb{N}$ of natural numbers is exponential diophantine as one has:

$$\mathbb{Q} \models \ w \in \mathbb{N} \ \leftrightarrow \ w = 0 \ \vee \ \exists x, y (x > 0 \ \wedge \ y - x > 0 \ \wedge \ x^y = y^x \ \wedge \ wy = (w+1)x.)$$

Indeed, if w satisfies the right side and $w \neq 0$, one gets $y = (1 + \frac{1}{w})x$. If we apply the Lemma, we see that for some $n \in \mathbb{N} \setminus \{0\}$,

$$\left(1 + \frac{1}{n}\right)^{n+1} = \left(1 + \frac{1}{w}\right)\left(1 + \frac{1}{n}\right)^n,$$

with unique solution $w = n$. On the other hand for some natural number $w = n \in \mathbb{N} \setminus \{0\}$ we construct the pair (x_n, y_n) as defined in the lemma. This pair satisfies the right side. ⊣

Using this definition, for every diophantine equation we can effectively construct an exponential diophantine system of equations, such that the later has solutions in $\mathbb{Q}$ if and only if the former has solutions in $\mathbb{N}$. This implies that the decision problem for exponential diophantine systems of equations over $\mathbb{Q}$ is undecidable.

REFERENCES

[1] L. E. Dickson, ***History of the Theory of Numbers***, vol. 2, Carnegie Institute, Washington, 1919; reprinted by Chelsea, New York, 1966.

[2] M. Davis, H. Putnam and J. Robinson, *The decision problem for exponential diophantine equations*. ***Annals of Mathematics***, vol. 74 (1961), no. 3, pp. 425–436.

[3] Y. Matiyasevich, ***Hilbert's Tenth Problem***, MIT Press, Cambridge, MA, 1993.

FACULTY OF MATHEMATICS AND INFORMATICS
UNIVERSITY OF BUCHAREST,
BUCHAREST, ROMANIA
and
SIMION STOILOW INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY
RESEARCH UNIT 5, P.O. BOX 1–764
RO-014700 BUCHAREST, ROMANIA
E-mail: mihai.prunescu@imar.ro
E-mail: mihai.prunescu@gmail.com