Macroeconomic Dynamics, **7**, 2003, 424–472. Printed in the United States of America.
DOI: 10.1017.S1365100502020060

MD SURVEY

INTERACTIVE-AGENT ECONOMIES: AN ELUCIDATIVE FRAMEWORK AND SURVEY OF RESULTS

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This paper is helpful for understanding interactive-agent economies. It presents a user-friendly framework for exploring the implications of economic interaction, and demonstrates its wide applicability. Agents repeatedly face binary choices and receive i.i.d. shocks; payoffs depend upon one's own action and shocks, as well as upon the actions of a subset of other agents. Simulations explore how properties change as the number of agents, the number of neighbors, and the degree of interaction change. Key results in this framework include the following: The law of large numbers can readily be postponed or defeated, stationarity and ergodicity are trivial to deduce, *any* equilibrium distribution may be attained in "globally interactive" economies, continuous- and discrete-time analogues generally behave very similarly, dynamics can be sensitive to details of the interaction structure (implying that popular mean-field approximations are not always appropriate), and substantial persistence is typical.

Keywords: Interacting Particle Systems, Contagion, Nonergodicity

1. INTRODUCTION

Most of the multiagent systems that economists study feature heterogeneous agents who interact *directly*; that is, agents' strategies are functions other agents' actions. Ford and General Motors compete vigorously; foreign exchange traders observe each others' bid-ask rates (and respond accordingly); little Johnny learns from his bright peers. Yet once such interaction is modeled, the implications are often startling. Convergence rates and equilibrium sets can be dramatically altered. Upon

Previous versions of this paper have circulated under the titles "Interaction in Finite Economies" and "A Framework for Studying Economic Interactions." I thank the Santa Fe Institute for its gracious hospitality during the 1992 Complex Systems Summer School, where I initially began thinking about these models; also, thanks go to Rick Ashley, Russell Cooper, Richard Cothren, John Duffy, Tim Erickson, Hans Haller, Michael T.K. Horvath, Boyan Jovanovic, Roger Lagunoff, Rob McClelland, Ted To, Ping Wang, Peter Zadrozny, seminar participants at Virginia Commonwealth University, Virginia Tech, the 1996 SEDC conference in Mexico City, and the 1998 NBER macroeconomic complementarities workshop. Comments from anonymous referees and an associate editor helped to substantially improve this paper. Needless to say, I am entirely responsible for all errors. All views expressed in this paper are those of the author and do not reflect the views or policies of the Bureau of Labor Statistics or the views of other BLS staff members. Address correspondence to: Randal Verbrugge, 132 Whitmoor Terrace, Silver Spring, MD 20901, USA; e-mail: verbrugge_r@bls.gov.

 1365-1005/03 $12.00

aggregation, the macromodels' properties *typically* diverge from those of their component micromodels: Phenomena emerge at the macro level that cannot be understood from micro elements studied in isolation, "nice" properties vanish, and responses to shocks become very complex.[1] Some view this as a curse. Yet those macro properties are often appealing, clearing up empirical puzzles or explaining, for example, persistence or striking shifts in aggregate outcomes over space and time. For these and other reasons, Kirman (1992, p. 119) suggests that "the way to develop appropriate microfoundations for macroeconomics . . . rests in an essential way on studying the aggregate activity resulting from direct interaction between different individuals."

A growing body of work explores the intriguing implications of direct interactions between heterogeneous agents for macroeconomics, especially in finance,[2] fluctuations,[3] information or human capital spillovers,[4] and technological adoption and growth.[5] Yet much of this work appears to be very technical—which could "scare away" prospective researchers—and is relatively unknown outside of particular subfields. Furthermore, much of it uses specialized assumptions tailored to specific contexts, raising questions about the generality of the results. Gray (1986) notes that once one deviates slightly from the small set of well-studied models, virtually all the known facts about such systems become open questions. Thus, little is known about the generic properties of interactions models.

This paper is a focused and user-friendly introduction to the burgeoning interactive-agent literature. It presents a simple general model that is a framework for understanding and describing much of the work in interactive-agent systems. The model is widely applicable and easy to use, and allows several key dynamic issues to be resolved with little effort. The paper gives the reader some insight about what is going on in local interactions models, what their "typical" dynamics are like, and why these models produce interesting results. It surveys analytical results that are general and presents two special-case results that have broad implications. It illustrates the model in a number of simple but suggestive examples [richer examples may be found in, e.g., Verbrugge (2000a,b)]. My goal is to encourage others to further explore this fascinating area.

Four analytical results are noteworthy. First, ergodicity and stationarity are easy to determine. Second, essentially *any* long-run distribution of states is possible. Third, endogenous variables can violate the law of large numbers (LLN) and/or display nonlinear behavior; in fact, the underlying probability laws giving rise to highly volatile/nonlinear behavior are not unusual, implying that aggregate volatility may be typical of interactive systems. Finally, aggregate shocks might have nonlinear effects on the economy, and temporary shocks might have indefinitely persistent effects.

The models investigated here generally feature complex dynamics; outside of special cases, many questions cannot be resolved analytically. One noteworthy simulation result is that, though discrete- and continuous-time versions of the same finite economy *can* behave differently, in practice they often do not. This suggests that convenience may dictate the choice between discrete- and continuous-time modeling.

The simulations also demonstrate that several other properties depend, in intuitive ways, upon the dynamic specification. Economies whose agent-level dynamics are governed by *linear* decision (or updating) rules have characteristic behavior: Shocks are amplified and volatility increases in the level of interaction, but the LLN still holds. The degree of persistence is independent of the degree of interaction. The qualitative behavior varies smoothly with changes in parameters; thus, analytical results that rely upon the simplifying assumption of global interaction (everyone interacts with everyone) probably carry over to other interaction structures.

Economies with *nonlinear* decision rules display more complicated behavior. Near particular parameter combinations ("critical points"), they become *highly* variable and persistent. As such parameter values are approached, persistence rises and the LLN is greatly postponed—variance does not fall proportionally with $1/N$. At *lower* levels of interaction, the economy has a unique equilibrium. At *higher* levels of interaction, the economy possesses multiple quasi-stable equilibria or modes, between which infrequent switches occur. The greater the degree of interaction, the less frequent are mode switches, and the more the economy behaves as if it were nonergodic and path dependent. Economies with occasional mode-switches display a number of interesting properties, such as fat-tails and persistence. In addition, they possess the "small shocks, large shocks" property highlighted by Kelly (1994): Small aggregate shocks often have an impact on the economy which is transient, but large shocks can have long-lasting impacts if they move the economy to a different mode. The dynamic properties of any particular economy with a nonlinear decision rule are, however, sensitive to changes in interaction specifications. Thus, interaction must be modeled carefully, and popular mean-field approximation techniques are not always appropriate.

Naturally, there are numerous closely related studies. Key early studies are by Föllmer (1974), Weidlich (1974), Allen (1982a,b), Weidlich and Haag (1985), Ceccatto and Huberman (1989), Brock (1992, 1993), Blume (1993), Durlauf (1993), and Aoki (1994). The mean-field models of Brock and Durlauf (2001a,b) are essentially static models in which interactions occur via (rational-expectations) *beliefs* about the behavior of other agents in one's reference group. These models parametrically restrict the return function and the error term, which allows one to obtain analytic results relating to conditions for multiple equilibria, to conduct welfare analysis, and to undertake econometric analysis.[6] The models investigated by Glaeser and Scheinkman (2001a) are also static. In these models, individuals' interactions are somewhat restricted, and individual choice depends on the *realization* (rather than the expectation) of the choices of others (so that the system is a Markov random field). In contrast to most of the literature, choice variables are continuous, price variables are explicitly introduced, and only weak parametric restrictions are imposed. Glaeser and Scheinkman (2001a) focus on establishing conditions for unique Nash equilibria and social multipliers. They extensively discuss econometric applications; see also Glaeser and Scheinkman (2001b). The models of Kandori et al. (1993), Young (1993), and, especially, of Blume (1993, 1995a, 1997) and Kelly (1994), are closer predecessors to this paper—in fact, they can be mapped into its framework. The models are explicitly dynamic and call for repeated state revisions on the part of each agent, based upon current states of neighbors.[7] Blume and Kelly both parametrically restrict the return function and error term in order to obtain analytic results. This practice is largely the rule in this literature, but it is not innocuous. Such restrictions can be *essential*; see Bergin and Lipman (1994), Blume (1995a, 1997), and Haller and Outkin (1998). The work of Aoki (1994; 1995a,b; 1996; 1998; 2000; 2001a; 2002) is very closely related, with its focus on dynamics, Markov processes, metastable equilibria, critical phenomena, the multiplicity of states compatible with one macro state ("degeneracy"), persistence, and so on. Aoki (1996) presents analytical results and techniques applicable to continuous-time special cases of the framework presented here; Aoki (2001b) is more focused and has closer ties to the economics literature.[8]

Aside from previously mentioned work, there are other related studies that bear mentioning (though there are too many papers to list completely). Fagiolo (1998) is an earlier, focused review that is similar in spirit to this paper. Hors and Lordon (1997) make some of the same points as Aoki (1996) and others, independently. Lohmann (2000) uses an interactions model to study social upheaval and regime collapse; Irwin and Bockstael (1998) use another to study land-use patterns. There is a growing literature building closely related models that feature "self-organized criticality." Economies with this trait continually evolve to "critical" configurations where minor shocks cause large bursts of activity. In contrast to most local interactions models, this self-organizing behavior is fairly robust to changes in parameter values (though not necessarily to other perturbations of the model). Volatility of aggregates in such economies can remain constant as $N \to \infty$. [See Bak et al. (1993), Focardi et al. (1999), Andergassen (2001), Arenas et al. (2001), and Bertrand (2001).] Durlauf and Young (2001) survey some of the recent work in social dynamics. Models in the "agent-based" literature [see, e.g., Arthur et al. (1997), LeBaron (2000, 2001a,b,c), Tesfatsion (2000)] typically posit much larger strategy spaces in very rich environments, with agents repeatedly choosing strategies using learning rules. Such models must generally be analyzed by simulation. Weisbuch et al. (2001) model a *continuous* choice variable representing opinion; opinion formation is also studied in Ianni and Corradi (2000, 2002). Other related studies include locally interdependent preferences [Bell (1997), Cowan et al. (1998)], temporally ordered interaction leading to lock-in, herd behavior, informational cascades, and the like [see, e.g., Arthur (1990), Banerjee (1992), Bikchandani et al. (1992), Arthur and Lane (1993)], and random linkages between agents and network formation models [Kirman (1983), Kirman et al. (1986), Ioannides (1990, 2001), Durlauf (1993), Haller and Sarangi (2000), and Bala and Goyal (2002)]. Some of the conventions literature is related, as it seeks to explain the evolution (and possible coexistence) of conventions; see, for example, Boyer and Orléan (1992), Young (1993), Sugden (1995), and Berninghaus and Schwalbe (1996). Morris (1997) describes an equivalence between local interaction games, random matching games and incomplete information games, and characterizes the equilibria of binary action coordination games, with many players. Morris (2000) extends these results; working with an infinite economy with best-response dynamics devoid of noise, he characterizes when contagion is possible for arbitrary interaction structures.

2. SIMPLE MODEL FOR STUDYING INTERACTION

2.1. Statement of the Model

We model in discrete time (the extension to continuous time is in Section 5). There are $N \leq \infty$ agents, each of whom must be playing one of two strategies, such as corrupt or honest. Identify each agent with an integer $i = 1, \ldots, N$. Label the states $+1$ and -1, and label the strategy of agent i at time t by $\eta_i(t)$. The current state of the economy is denoted $\boldsymbol{\eta}(t) := [\eta_1(t), \ldots, \eta_N(t)]'$.

Associated with each agent i is a "neighbor set" (or peer group), a subset of the N agents, whose members are called the neighbors of x. If j is a neighbor of i, then agent i will take j's state into account at her next strategy-revision opportunity—that is, j's state at time $t-1$ is an argument in i's decision rule at time t. The neighbor set relations in the economy are described by a matrix W, with entry $w_{ij} \neq 0$ if agent j is a neighbor of agent i. If $w_{ik} = 0$, then agent k is not a neighbor of agent i, and thus her state at $t-1$ does not influence agent i's strategy choice at t. (Though k does not *directly* influence i, there will be a network of interdependencies among all agents if neighbor sets are not disjoint.) The matrix W is normalized so that the absolute values of the entries in each row sum to 1—that is, $\sum_{j=1}^{N} |w_{ij}| = 1$. Then,

$$\theta_i(t-1) := \sum_j w_{ij}\eta_j(t-1)$$

is interpreted as the weighted-average action, in period $t-1$, of i's neighbors. Interactions are restricted in that i's decision rule may depend only upon $\theta_i(t-1)$, rather than upon individual strategies of neighbors.

In particular, each agent i has a stochastic reaction function, of a threshold type, whose only arguments are an idiosyncratic shock and $\theta_i(t-1)$. Agent i's state evolves according to

$$P[\eta_i(t) = 1 \mid \boldsymbol{\eta}(t-1)] = P[\eta_i(t) = 1 \mid \theta_i(t-1)] =: G(\theta_i(t-1)). \qquad \textbf{(1)}$$

The function G: $[-1, 1] \to [0, 1]$ is the same for every agent, and θ_i may be termed i's "local field." Typically, $G(\cdot)$ will derive from a maximization problem on the part of an agent, as follows: Let $\varepsilon_i(t)$ be a random variable that is i.i.d. across agents and time. Let $U[\eta_i, \theta_i(t-1), \varepsilon_i(t)]$ be the return to agent i at time t. Then, define

$$\eta_i^*(t) := \operatorname*{arg\,max}_{\eta_i \in \{-1,+1\}} U[\eta_i, \theta_i(t-1), \varepsilon_i(t)].$$

The solution, which is a threshold-type decision rule, immediately leads to

$$G(\theta_i(t-1)) := P[\eta_i^*(t) = 1 \mid \theta_i(t-1)].$$

This construction is demonstrated in Example 1. Whether or not $G(\cdot)$ derives from explicit optimization, one can often express the behavior of agent i as

$$G(\theta_i(t-1)) = P\{\varepsilon_i(t) < T[\theta_i(t-1)]\}, \tag{2}$$

where $T:[-1,1] \to \mathbf{R}$, a threshold-type function which is the same across agents, describes the way an agent is sensitive to the average behavior of his neighbors, and $\varepsilon_i(t)$ is a random variable that is i.i.d. across both agents and time. Example 3 illustrates such a case. Note that increasing the variance of $\varepsilon_i(t)$ corresponds to reducing the sensitivity of $G(\cdot)$ to $\theta_i(t)$.

In words, the underlying dynamic (1) says that an agent's probability of choosing strategy 1 at period t depends upon what his neighbors were doing, on average, in period $t-1$. Such dependence could result, for example, from intertemporal interdependencies, or through an expectations-formation process. Note that interaction is *direct*: Each agent's decision depends upon other agents' *actual* states [cf. Brock and Durlauf (2001a,b)]. Given the weights matrix W, and the initial state of the population, (1) determines, stochastically, the time path of the economy. Since the only thing that matters for transitions is the previous-period state of the economy—$G(\cdot)$ is a Markov rule—the economy is a Markov chain. [Note that conditional on $\boldsymbol{\eta}(t-1)$, each agent moves independently, so the joint conditional probability measure is a product of the individual conditional probability measures (1).]

2.2. Commentary

Since the economy is a Markov chain, many of its properties follow straightforwardly from standard Markov process theory. In particular, when N is finite, the economy is simply a finite-state Markov chain, whose transition matrix P (of dimension $2^N \times 2^N$) is implicitly defined by $G(\cdot)$ and W. Hence many theoretical results—such as the existence of a stationary distribution and upper bounds on the transition time to the stationary measure (if it exists)—are readily deduced. Furthermore, if a finite-state Markov chain is irreducible, the stationary distribution is the ergodic distribution.[9]

The typical state space of dynamics in this type of model can be described as one or more valleys (basins of attraction) separated by barriers of varying heights; local "metastable" equilibria (or "modes") are located in valley floors. The "landscape"—and hence, the dynamics—may be very sensitive to parameter values and to the pattern of weights. Economies wander around in valleys, and, if the barriers are not too high, occasionally jump over them and move into neighboring valleys. This mode-switching behavior is often of interest—in financial market applications, for example, it may imply bull vs. bear markets. (Mode-switching could also be induced by a sufficiently large aggregate shock.) Because the economy's behavior is different in different valleys, a particular sequence of shocks could well have a very different impact on the economy, depending upon the valley in which the economy is located. Here is a simple example, building upon (2). Suppose $T(-1) = \delta$ and $T(1) = 1 - \delta$, with $\delta < 1/2$. Consider two different "positively interacting" economies (i.e., all weights w_{ij} nonnegative) that differ only in their initial conditions: economy *neg* starts with all agents in state -1, and economy *pos* starts with all agents in state 1. Now, suppose all agents happen to receive the same shock, of size $\delta/2$. Recall that the shock must lie below $T(\theta)$ in order for the agent to move to 1. This aggregate shock has no impact on economy *pos*, yet induces all agents in economy *neg* to switch from state -1 to state 1. However, if the shock had instead been of size $1/2$, no agent would have switched states. This demonstrates that both the global state -1 and the global state 1 are consistent with the same realization of shocks.

If an economy has multiple equilibria, how does it select the equilibrium to which it moves next? As demonstrated in the *pos*$-$*neg* example, selection implicitly conforms to Cooper's (1987, 1994) history-dependent selection criterion—which, roughly speaking, implies that agents expect the current equilibrium to continue, unless it is no longer a possible equilibrium—but in *this* context, such dynamics need not result from beliefs coordination.

These history-dependent dynamics can help generate substantial persistence of aggregate variables. Infrequent shifts between modes that have different average states increase measured persistence even more—so much so that conventional tests on aggregate measurements may have difficulty rejecting a unit root. Thus, there is substantial scope for interacting models to endogenously generate sluggish aggregate movements, even though the driving forces are i.i.d. shocks.[10]

The framework here is quite general, both in the wide class of situations to which it can be applied and in the nature and patterns of interaction that it allows. Here are four reasons why.

Binary restriction. The model's binary-state restriction is not as restrictive as it first appears. First, there are numerous interesting economic decisions that are effectively binary—e.g., strategy choices (chartist or fundamentalist) remain active or withdraw from market—in addition to situations in which the binary variable is a literal state (e.g., informed or uninformed). Brock and Durlauf (2001a) list a number of interesting binary choices. More generally, the binary state could refer to one of two possible *equilibria*—such as collusive or competitive—if the "agents" are, for example, coalitions, industries, or physical locations; see Verbrugge (2000c) for an example and further discussion. Second, for many substantive applications, a binary state is a reasonable "first-order" approximation. Many interesting economic situations involve optimal adjustment behavior by microeconomic units that is not continuous or small, with decision rules that are of the threshold type; or they involve what amounts to binary states or decisions, e.g., lumpy factor adjustment or bank runs. The framework itself should be taken as illustrative of economic situations in which there are both interactions and nonlinear or lumpy adjustments; see Aoki (1996) for a long list of such instances. Third, most of the results in this paper could be readily extended to allow a finite number of states, at the cost of expositional clarity.

$G(\cdot)$ flexibility. The form of $G(\cdot)$ is determined by the economics of the situation. In the literature [e.g., Blume (1993)], $G(\cdot)$ is often a best-response strategy rule in a game-theoretic situation. Such rules may be justified in several ways, depending upon the context. Interacting-agent economies often feature complex dynamics, and it may be unreasonable to suppose that agents can fully solve the dynamic games they are playing. In other game-theoretic applications, $G(\cdot)$ can describe "mixing" over strategies, or it can describe "trembling"—a setup in which agents sometimes make "mistakes" and choose the wrong strategy with some probability. $G(\cdot)$ can incorporate forward-looking behavior [see, e.g., Blume (1995b)] as well as boundedly rational behavior. However, the framework is not restricted to strategic situations. For example, agents' states might not be choice variables—e.g., the state might be "infected" or "not infected"—with $G(\cdot)$ determining the evolution of those state variables [see Carroll (2002)]. Or, as noted earlier, each "agent" might be an entity such as an industry, and $G(\cdot)$ might describe transitions between one of two possible equilibria, such as price slashing or colluding. The only restriction is that (1) must hold: "Agent" transition probabilities must depend upon the *current* (or just prior) *weighted-average* state of neighbors. [Note that optimal responses to Markovian rules are often themselves Markovian; Bhaskar and Vega-Redondo (2000) provide a theoretical justification for the use of Markov strategies in repeated games.] Modest extensions would make the framework even more flexible; besides a larger strategy set, $G(\cdot)$ could differ across agents, or have other arguments, such as multiple "fields" or aggregate variables such as prices [Glaeser and Scheinkman (2001a)]. Additionally, the function $G(\cdot)$ might depend upon one's own state, which is one way to model "stickiness" (another being to set w_{ii} close to 1).

In fact, many of the models in the literature feature a $G(\cdot)$ which varies with the state agent x is in—i.e., $P[\eta_i(t) = 1 \mid \boldsymbol{\eta}(t-1)] = P[\eta_i(t) = 1 \mid \eta_i(t-1), \theta_i(t-1)]$. [Most commonly used is the "log-linear response model," which assumes that $P[\eta_i(t) = 1 \mid \eta_i(t-1), \theta_i(t-1)] = e^{\beta U(\cdot)}$, where $U(\cdot)$ is a return function which depends upon $\eta_i(t-1)$, and β is a "sensitivity parameter" in that larger values of β imply stronger interaction.] Thus, the probability that agent x moves from state -1 to state 1, given a particular value of $\theta_i(t-1)$, is given by $G_{-1}[\theta_i(t-1)]$; while the probability that x *stays* in state 1, given the same $\theta_i(t-1)$, is given by $G_1(\theta_i(t-1)) \neq G_{-1}(\theta_i(t-1))$. Since this alternative assumption is so common, I will note which analytic results still follow when it is imposed.

$G(\cdot)$ functions that are constant imply that agents behave independently. $G(\cdot)$ functions that lie below the line $y = 1/2$ for values of $\theta < 0$, and conversely for $\theta > 0$, imply something like strategic complementarity—a tendency to conform to the average play of one's neighbors. The greater the slope, the stronger the interaction—the more likely is the agent to conform to the majority (as measured by the weighted average θ). It will become evident below that in most contexts,

https://doi.org/10.1017/S1365100502020060 Published online by Cambridge University Press

something like this is required for multiple modes to exist. However, one can generate multiple modes by careful selection of neighborhoods combined with strategic substitutability—see Glaeser and Scheinkman (2001a).

Neighbor sets. The framework also allows a fair amount of flexibility in neighborhoods. For instance, one can readily model the common interaction patterns used in the literature. In studying *local* interactions, researchers have often defined neighbor sets by placing agents on a line, a two-dimensional grid, or some higher-dimensional lattice, and then positing the neighbors of x as those agents immediately adjacent to x (e.g., in two dimensions, the agents at the points directly north, south, east, and west). To ensure symmetric neighborhoods, typically the "edges" of lattices are folded over to meet the far side—thus, a line becomes a circle, a two-dimensional grid becomes a torus (doughnut), etc.[11] Usually, all neighbors are weighted equally ($w_{ij} = w_{ik} \forall\, j, k, \forall\, i$); this is termed "uniform local interaction." The other interaction pattern that receives wide use in the literature is "uniform *global* interaction" (UGI), which simply implies that every agent's field (θ) is the economywide average. "Global interaction" implies that all elements of W are positive; that is, $w_{ij} > 0\ \forall\, i, \forall\, j$. The assumption of global interaction, and particularly UGI, lets one apply more specialized techniques and obtain sharper results; see Aoki (1996) and Lux (1997). Independent agents may be modeled by setting $w_{ij} = 0,\ i \neq j, \forall\, i$, or by setting $G(\cdot)$ to a constant.

However, this framework allows much more flexibility in determining interaction patterns. Neighbor relations need not be symmetric, reflexive, or similar across agents; even "negative" interactions between agents are allowed by choosing negative weights. Such flexibility is important. The specific pattern of interaction (or "topology of interaction") can matter a great deal [see below and, e.g., Ellison (1993), Ioannides (2001), and Page (2001)]. We would like to be able to investigate implications of stochastic neighborhoods—in which the w_{ij} are random variables [for discussion and some results, see Föllmer (1974), Kirman (1983), Kirman et al. (1986), Ioannides (1990), and Haller (1990)]—as well as "leader–follower" or "upstream–downstream" relationships. Such hierarchical relationships may help generate self-organized criticality. Randomly *evolving* neighborhoods [see Schelling (1971), Haller (1990), David and Foray (1992), and Kirman (1997)] might facilitate its generation [see Stauffer and Sornette (1999)].

In Sections 3 and 5, I derive some general analytical results about the behavior of such systems, most of which follow from standard Markov process theory. However, these results do not encompass all of the relevant questions in any given situation. An economist studying corruption might wish to know whether persistence is a typical feature of such models, a growth theorist might want to know how transition times relate to scale, and so on. Without some general feel about how such models typically behave, the theorist might be reluctant to embark upon the modeling endeavor. Since many such questions are not answerable analytically for the general case, a number of these issues are investigated via a simulation study in Section 6.

3. ANALYTIC RESULTS

3.1. Multiple Equilibria and Path Dependence

Stationarity and ergodicity are properties that are of central interest in many models [e.g., Durlauf (1993), Verbrugge (2000a,b)]. If the Markov process describing the economy's dynamics is ergodic and stationary, then the economy has a unique stationary equilibrium to which it converges regardless of the initial conditions. This is often interpreted to mean that "history does not matter." Alternatively, if there are multiple trapping sets (multiple "infinitely deep valleys"), and initial conditions combined with the sample path of shocks determine which outcome eventually transpires, this is often taken to mean that "history matters." A necessary condition for such "path dependence" to hold is that the Markov process have multiple trapping sets—i.e., that it is nonergodic. Lemma 1 shows how to easily determine if a finite economy is stationary and ergodic; both properties hold if $G(\cdot)$ lies strictly between 0 and 1. Lemma 2 is a partial converse: It presents conditions on $G(\cdot)$ under which finite economies are *not* ergodic.

LEMMA 1: Sufficient conditions for ergodicity. *Suppose* $N < \infty$. *If* $0 < G(\theta) < 1 \forall \theta$, *then the Markov chain is aperiodic, ergodic, and stationary. The invariant distribution will be attained in the limit, regardless of the initial distribution of the process, and convergence to this distribution is at an exponential rate.*

One proves this as follows. Since $G(\cdot)$ is never 0 or 1, there is always nonzero probability that *any* configuration can be attained in one step from *any other* configuration. Hence the chain is irreducible and aperiodic; since the economy is finite, the chain is positive recurrent. Hence the Markov chain has a unique stationary distribution, it is ergodic, and the convergence rate is exponential. ■

There are three points to note. First, this result will still hold when $G_{-1}(\cdot) \neq G_1(\cdot)$, as long as $0 < G_i(\cdot) < 1$ for $i = -1, 1$ and for all θ, using essentially the same proof. Second, there is no assumption about the neighborhood structure. Third, Lemma 1 can be strengthened—it is not absolutely necessary that $G(\cdot)$ lie between 0 and 1 everywhere. As long as any state can be reached in a finite number of transitions from any other state (i.e., the chain is irreducible), and there is *some* state that leads back to itself with some positive probability, the result will still follow.

Lemma 2 gives the simplest sufficient conditions for the existence of multiple trapping sets.

LEMMA 2: Sufficient conditions for nonergodicity. *Suppose* $N < \infty$. *If* $w_{ij} \geq 0 \forall j, \forall i, G(-1) = 0$, *and* $G(1) = 1$, *then the economy possesses multiple trapping sets and is nonergodic.*

If $G(-1) = 0$ and all w_{ij} are nonnegative, then the global state $-\mathbf{1}$ (i.e., all agents in state -1) is a trapping set—if it is ever attained, there is no chance of any

https://doi.org/10.1017/S1365100502020060 Published online by Cambridge University Press

agent ever changing her mind—since, as all w_{ij} are nonnegative, $\theta_i = -1$ for all i. Similarly, if $G(1) = 1$ and all w_{ij} are nonnegative, then the global state **1** (i.e., all agents in state 1) is a trapping set. ■

There are three points to note. First, Lemma 2 will still hold as long as $G_{-1}(-1) = 0$ and $G_1(1) = 1$; the same proof, with the obvious substitutions, would apply. Second, if the economy is nonergodic, the economy's eventual equilibrium will at least partly depend upon the initial conditions. However, whether or not it is *path dependent* depends upon whether there exist initial conditions such that the ensuing history of shocks determine that eventual equilibrium. The economy will not be path dependent if, for example, there are two trapping sets, but one of them can only be attained if the economy starts there.

Third, Lemma 2 does *not* state that the economy can only converge to one of the states $-\mathbf{1}$ and $\mathbf{1}$; depending on the structure of interaction, other long-run distributions may be possible. In fact, a necessary condition for $-\mathbf{1}$ and $\mathbf{1}$ to be the *only* trapping sets—see Lemma 3—is that the neighborhood structure "fully communicates," that is, each agent directly or indirectly (via an extended chain of interactions of neighbors with neighbors) interacts with every other agent. This rules out "coalitions" which are not influenced by outsiders—since each such coalition would form its own little independent economy, which itself could converge to $-\mathbf{1}$ and $\mathbf{1}$ independently of the rest of the economy. Furthermore, whether the total number of agents is even or odd can also influence the number and type of trapping sets. For example, suppose agents are arranged on a two-dimensional lattice embedded on a torus, as described in Section 2, and the conditions in Lemma 2 hold. If the number of agents is even, there is an equilibrium in which an oscillating "checkerboard" pattern of strategies is played—at even dates, red-square players (for example) play strategy -1, and black-square players play strategy 1; and at odd dates, all players reverse their strategies. (Since the economy returns to its original starting point after two periods, this is called a two-cycle.) However, this trapping set does not exist when there are an odd number of players. This dependence upon the pattern of interaction is a hint that oft-advocated mean-field approximations (which, among other things, assume a particular pattern of interactions) could well be inappropriate whenever the economy is nonergodic—or even *effectively* nonergodic (see Section 6).

When the economy is stationary and ergodic, is history truly irrelevant? Not necessarily. Even given exponential convergence to a unique stationary equilibrium, it may still take so long for the process to make its way from one "valley" to another that, for all practical purposes, this transition will never occur. Thus, the stationary distribution may be economically irrelevant [see Blume (1997)]. In this case, history—i.e., the path of shocks, which selects the initial valley to which the process moves—will still matter. (I term this "effective nonergodicity.") Even the expected first-passage times between modes may tell us little about the sample path behavior [again, see Blume (1997)]. Likewise, nonergodicity may be economically irrelevant: Transient states may lie in a "deep" valley. Numerical methods may be required to study such issues adequately. See Section 6 and the conclusion for further discussion.

Previous discussion notwithstanding, Lemma 2 is useful: See Verbrugge (2000b)—which studies how ex-ante identical regions can diverge in levels of political corruption—and the following Example 1.

Example 1: Trapping or take off [inspired by Durlauf (1993) and by Aoki (1995a,b; 1996)]. At any time, there are N firms. Each exists for one period and is subsequently replaced the next period. Each of the N firms produces a unique perishable good and faces a demand schedule that is given by $p_t = a y_t^{-b}$, where $a \geq 0$, $1 \geq b \geq 0$. At the beginning of the period, each firm must decide whether to use technology B (which is nonstochastic) or pay a fixed cost of C and use technology A (which has a random return). Let k denote the fraction of firms who used technology 1 last period, and let $\theta = 2k - 1$. (This function maps $[0, 1]$ into $[-1, 1]$.) Let $\gamma(\theta_{t-1})$, labor productivity, be a random variable, observed at the beginning of period. Technology A is given by $y_t = \gamma(\theta_{t-1}) L_t$, where L_t is the labor input. Technology B is given by $y_t = L_t$. Wage is the numeraire. The optimal profit associated with using technology A is $\pi_1(\theta_{t-1}) = \kappa\gamma(\theta_{t-1})^{\alpha} - C$ (where $\kappa := a^{1/b}(1-b)^{(1-b)/b} b$), whereas the optimal profit associated with using technology B is κ; hence, technology A is more profitable if $\gamma(\theta_{t-1})^{(1-b)/b} > 1 + C/\kappa$. The distribution of γ is given by

$$P\left[\gamma(\theta_{t-1})^{(1-b)/b} > 1 + C/\kappa\right] = \frac{1}{2}\left[\sin\left(\frac{\pi\theta_{t-1}}{2}\right) + 1\right].$$

This distribution captures the idea that prior use of technology A, via a learning-by-experience mechanism, makes more productive draws more likely. Since

$$G(\theta_i(t-1)) := P[\eta_i(t) = A \mid \theta_i(t-1)] = \frac{1}{2}\left[\sin\left(\frac{\pi\theta_{t-1}}{2}\right) + 1\right],$$

so that $G(-1) = 0$ and $G(1) = 1$, Lemma 2 implies that the economy has two potential limit sets: all firms forever using technology A or all firms forever using technology B. If the economy starts with random initial conditions, the sample path of shocks will determine the long-run level of output in the economy: The economy is path dependent. If many firms draw high productivities early on, the economy could converge to the good equilibrium ("take off"), but this outcome is far from certain.

Lemma 3 restricts the types of trapping sets in finite, nonergodic, globally interacting economies. Interestingly enough, this result does not hold in infinite economies—see Proposition 1.

LEMMA 3: Limit configurations. *Suppose* $N < \infty$, $G(-1) = 0$, $G(1) = 1$, *and* $G(\theta) \in (0, 1)\ \forall \theta \in (-1, 1)$. *If* $w_{ij} > 0\ \forall j, \forall i$, *then the only trapping sets are the configurations* $-\mathbf{1}$ *and* $\mathbf{1}$.

Proof. Suppose the contrary; that is, suppose that there is a trapping set in which a group of agents is in state 1, whereas the rest of the economy is in state -1.

Since $w_{ij} > 0 \forall j, \forall i$, then $\theta_i \in (-1, 1) \forall i$. Since $G(\theta) \in (0, 1) \forall \theta \in (-1, 1)$, $0 < G(\theta_i) < 1$ for each agent x—meaning that, for example, there is positive probability that *every* agent will move to state 1 next period. Hence the original configuration cannot have been a trapping set. Conversely, consider the configuration $-\mathbf{1}$. Here, $\theta_i = -1$ for every agent, and since $G(-1) = 0$, every agent will stay in -1 next period. Sooner or later, a jump to this extreme (or the other extreme) will occur. This result will still hold when $G_{-1}(-1) = 0$, $G_1(1) = 1$, and $G_i(\theta) \in (0, 1) \forall \theta \in (-1, 1)$ for $i = 1, -1$, using the same proof (with the obvious substitutions). ■

This lemma could be strengthened—what is necessary is that there are no subpopulations that are disconnected and who could thus find opposite extremes. However, one could get other trapping sets if, for some values of θ *other than* -1 and 1, $G(\cdot)$ "hit" 0 and 1. For example, consider four agents (labeled 1, 2, 3, and 4), such that odd agents weight the current state of each odd agent at 1/3 (and each even agent at 1/6), while even agents weight the current state of each even agent at 1/3 (and each odd agent at 1/6). Suppose 1 and 3 are in state -1, and 2 and 4 are in state 1; θ_1 and θ_3 both equal $-1/3$, and θ_2 and θ_4 both equal 1/3. If $G(-1/3) = 0$ and $G(1/3) = 1$, then this configuration is a trapping set.

3.2. Limit Behavior as N Approaches Infinity

It is commonly believed that idiosyncratic shocks will have minimal effects on aggregate dynamics in big economies, due to the LLN. Lemma 4 presents one set of conditions under which such averaging-out of shocks will occur. It states that the economywide average $\bar{\eta}(t)$ must converge to a constant as $N \to \infty$ in economies devoid of interaction ($w_{ij} = 0 \forall j \neq i, \forall i$). It thus emphasizes that LLN arguments about the aggregate unimportance of idiosyncratic shocks [see Lucas (1977)] rely upon sufficient independence across agents.[12] Interactions models typically feature strong, nonlinear dependence across agents. Hence, in such models, one *cannot* assume that laws of large numbers will generally hold—see Proposition 1, below.

LEMMA 4: Lucas averaging-out case. *If* $W = I$, *the identity matrix, then* $\bar{\eta}(t) \underset{t\to\infty, N\to\infty}{\longrightarrow} K$ *with probability* 1, *where* $K \in [-1, 1]$.

Proof. Each agent's behavior is now an independent two-state Markov chain, even if $G_{-1}(\cdot) \neq G_1(\cdot)$, since $\theta_i = \eta_i(t-1)$. As t goes to infinity, this chain must converge to a stationary, possibly degenerate, distribution. Because each agent's state is an independent bounded random variable, the strong law of large numbers implies that the average state converges to a constant as $N \to \infty$. ■

Proposition 1, a statement about the dynamic properties of infinite UGI economies, demonstrates that such averaging-out need not hold in interacting economies. Moreover, the proof of this result yields valuable intuition about the dynamic properties of these models in general. The key insight is that in infinite UGI economies, despite the uncertainty at the micro level, the *fraction* of agents who move to state 1 is deterministic—namely $G(\bar{\eta}(t))$. Hence, the aggregate moves deterministically: $\bar{\eta}(t+1) = 1/2[G(\bar{\eta}(t)) + 1]$, a simple difference equation. Many dynamic properties can thus be deduced by the shape of $G(\cdot)$; Proposition 1 spells out some implications.

PROPOSITION 1. *Steady states and dynamics in infinite globally interacting economies. Consider an economy with $N = \infty$ such that $\theta_i(t) = \bar{\eta}(t)\ \forall\, i$. The average $\bar{\eta}(t)$ evolves according to the difference equation $\bar{\eta}(t+1) = 1/2[G(\bar{\eta}(t)) + 1]$. Every $\bar{\eta}'$ that satisfies $G(\bar{\eta}') = 1/2(\bar{\eta}' + 1)$ is a fixed point or steady state for the economy. Initial conditions will determine the limit to which $\bar{\eta}$ converges. The stability condition for any fixed point $\bar{\eta}'$ is $|G'(\bar{\eta}')| < 1/2$. If $G(\cdot)$ is monotonically increasing, the economy will converge to a fixed point. If $G(\cdot)$ is not monotonic, nonlinear behavior, cycles, and/or chaotic behavior can occur.*

This result is not new; the physics literature has been aware of it for some time [see, e.g., Grinstein et al. (1985)], and it has been previously reported (in part) by Föllmer (1994) and Aoki and Shirai (2000). One can use an analogue to the 45-deg-line technique (with the line $y = (\theta + 1)/2$ playing the role of the 45-deg line) to prove this proposition and to deduce many important properties of the dynamics. Figure 1 and the ensuing discussion illustrate the use of this technique.

To trace out the dynamics of $\bar{\eta}$, proceed as follows. Start from an arbitrary point $\bar{\eta}(t) = x$. Then, $\bar{\eta}(t+1)$ is given by $\bar{\eta}(t+1) = 1/2[G(x) + 1]$, found graphically by moving from $G(x)$ horizontally to the line $y = (x+1)/2$. [If $G(x) > (x+1)/2$, one moves to the right, and if $G(x) < (x+1)/2$, one moves to the left.] One can trace out the *orbit* of x by iterating on this procedure. *Fixed points* $\bar{\eta}'$ lie at the intersection of $G(\cdot)$ and the line $y = (\theta + 1)/2$. A *trapping set* is a set T such that if $\bar{\eta}(t)$ is in T, then its orbit is in T. An *attracting set* A is a trapping set with the property that there is some open trapping set $U \subset A$ such that orbits beginning in U approach A as $t \to \infty$. An *attractor* is a set to which "typical" orbits approach.[13] The *basin of attraction* of an attracting set A is the set of all points that approach A as $t \to \infty$.

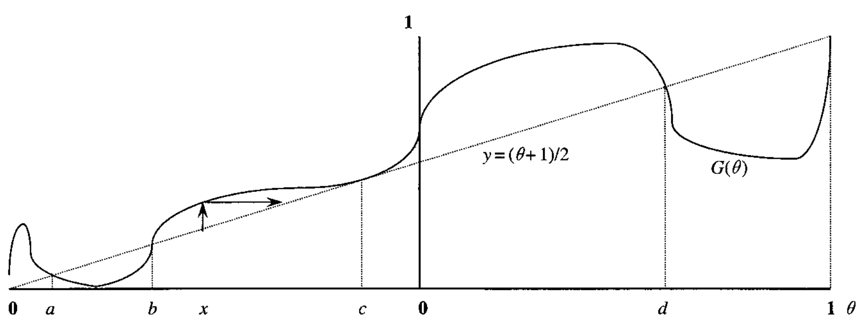

FIGURE 1. Dynamics of infinite UGI economies.

https://doi.org/10.1017/S1365100502020060 Published online by Cambridge University Press

It is easily deduced upon inspection of Figure 1 that if $G(\cdot)$ is continuous, a sufficient condition for multiple steady states is that the slope of $G(\cdot)$ exceeds $1/2$ at some steady state.[14] There are five steady-state equilibria in Figure 1: a, b, c, d, and 1. Not all of these are stable. Assume $G'(a) > -1/2$. Then, a is stable and is an attractor. (In other words, a lies at the bottom of a "valley.") Because $G'(a) < 0$, orbits converging to a will be oscillatory: If $\bar{\eta}(t) < a$ and $\bar{\eta}(t) \in A$, then $\bar{\eta}(t+1) > a$, $\bar{\eta}(t+2) < a$, and so on. Steady states b and 1 are unstable; neither would be attained unless $\bar{\eta}$ started there. Point c, a steady state that is neither stable nor unstable, will attract orbits that begin in the set $(b, c]$. Point d is unstable since $G'(d) < -1/2$. However, note that $(c, 1)$ is a trapping set; this set will likely possess a subset that is an attracting set. Dynamics in $(c, 1)$ will involve oscillations around d and orbits could well be periodic[15] or chaotic. If this is the case, the variance of the aggregate does not fall to zero—the microshocks are *not* "canceling out," even though they are independent and subject to the LLN.

Thus, interesting dynamics accompany nonmonotonic $G(\cdot)$ functions—but are such functions reasonable? Certainly. Recall that when $G(\theta)$ lies above $1/2$ for $\theta > 0$ (and below $1/2$ for $\theta < 0$), one may loosely interpret this as a form of "strategic complementarity"—a tendency to conform to the average behavior of one's neighbors. A nonmonotonic $G(\theta)$ function, then, is nothing extraordinary: the strength of the "strategic complementarity" simply declines over some θ. For example, there may be coexisting (and competing) positive and negative externalities—increasing returns of some sort, in conjunction with congestion externalities—with the negative externality gaining in dominance for extreme values of θ.

If $G_{-1}(\cdot) \neq G_1(\cdot)$, Proposition 1 still follows, but the proof is slightly altered. The process still moves deterministically: Given θ, the fraction of "1" agents who remain in state 1 is deterministic, as is the fraction of "-1" agents who move to state 1. Hence, one could map a new function $H(\cdot)$ which could be used (as above) to determine the system's evolution.

There are three important things we can learn about more general cases from this special-case proposition. First, the orbits of the infinite case will be a respectable approximation to the sample paths of large economies that feature sufficient interaction. For example, in large economies with a dense network of interdependencies and with the $G(\cdot)$ function depicted in Figure 1, we would expect the stationary distribution of $\bar{\eta}$ to have a great deal of mass in the vicinity of d (corresponding to long-run average behavior in the attracting set), a small peak at a, and perhaps a very small peak at point c. We would not expect the peaks to be dirac delta functions (spikes), since random fluctuations would constantly move the economy away from steady states—though the larger the economy, and the more closely it approximates uniform global interaction, the more concentrated would be the distribution. Second, the proof yields intuition about what generates persistence in these models. When $G(\cdot)$ lies near the line $y = (\theta + 1)/2$, even in finite systems and even absent aggregate interaction, $\bar{\eta}(t+1)$ is unlikely to differ much from $\bar{\eta}(t)$. [When $G(\cdot)$ lies *on* the line, of course, there is a continuum of equilibria, and no natural tendency for the economy to move in a particular direction.] Last, the proof makes clear that the economy's response to an aggregate shock might be highly nonlinear [see Potter (2000)]. How? An aggregate shock would likely shift or alter the $G(\cdot)$ function, which could dramatically alter dynamics in an attracting set—or even eliminate it. A temporary shift of $G(\cdot)$, even a small one, could have profound effects on the economy, sending it to another region of the state space where completely different behavior might be typical. This is the "small shocks, large shocks" property highlighted by Kelly (1994).

It is worth reemphasizing the fact that, under conditions that generate cycles and/or chaos, the *aggregate* level of activity is fluctuating. This implies that, in such economies, the variance of aggregate variables will *not* converge to 0 as N goes to infinity . . . despite the fact that each individual agent receives an *independent* disturbance. Thus, the endogenous data will not obey the law of large numbers, though the only driving forces are i.i.d. infinitesimal shocks. Put differently, idiosyncratic shocks will *not* be canceled out by aggregation as N goes to infinity [cf. Forni and Lippi (1997)]. In Example 2, I apply this result to a modification of Gale's (1996) delay- and-cycles model to produce a model of aggregate fluctuations without aggregate shocks.

4. BUSINESS CYCLES WITHOUT AGGREGATE SHOCKS

Numerous studies have been undertaken investigating the importance of delay in amplifying the effects of cycles in economic activity. The prominent study by Gale (1996) demonstrated how endogenous delay in investment could amplify "the natural cyclical tendencies of the economy." In his model, agents with innovations may use an innovation for only one period, and the key problem is deciding when one should begin this utilization.

A feature of Gale's model—shared by the vast majority of business-cycle models—is that it must rely upon aggregate shocks to generate aggregate volatility. However, some minor modifications of Gale's framework give rise to an environment in which aggregate fluctuations will arise in infinite economies, even though the *only* driving forces are infinitesimal, idiosyncratic shocks. Following Gale, the modified model features interactions that are intertemporal (ruling out the possibility of perfect coordination and self-generating bursts of activity) and global (in the sense that each agent interacts with *all* previously active agents, rather than a subset of them).

Example 2. In the model, time is discrete. At each date, a countable infinity of agents enters the economy and lives for two periods. The utility function of agent i born at date t is given by $U_i(c_{i,t},\ c_{i,t+1}) = c_{i,t+1}$. While young, agents have the option to search for innovations that may be converted, next period, into profitable investment opportunities. Search is costless, thus all young agents search; each discovers an innovation with probability 1. Innovations are not equally profitable; the gross return $r_{i,t}$ on the innovation discovered by agent i is an i.i.d. uniform [0, 1] random variable.

Following the full depreciation case in Gale (1996), entrepreneurs benefit from innovations for only one period. Here, entrepreneurs with innovations in hand have no incentive to delay. Instead, they decide whether to exploit the opportunity ("invest"), or forgo it (and earn a rate-of-return 0). The profitability of production is a function of both the gross return on the innovation and, following Gale, the general level of prior investment in the economy.[16] Let m_t denote the fraction of agents who invested in period t. Then, $\pi_{i,t+1,}$, the date $t+1$ profit accruing to innovator i who exploits his opportunity, is given by

$$\pi_{i,t+1} = r_{it} - C(m_t),$$

where $C(\cdot)$ is a continuous cost function that lies between 0 and 1. $C(m_t)$ is assumed to be "low" if many agents invested last period, and "high" if few agents invested last period. More precisely, $C(\cdot)$ is assumed to lie above $1/2$ for $m \in (0, 1/2)$, and to lie below $1/2$ for $m \in (1/2, 1)$. Hence, there is a weak form of strategic complementarity present: If the majority of agents invested last period, costs are "low" and more projects are profitable this period, and the reverse holds if only a minority invested last period.

This setup maps into the framework above, with $\theta(t) = 2m_t - 1$ and $G(\theta) = 1 - C\{[\theta(t) + 1]/2\}$. Proposition 1 demonstrates that this economy will display deterministic dynamics: Despite the microlevel uncertainty, there is no macro uncertainty, since the fraction of agents who receive innovations and whose return exceeds $C(m_t)$ is given (with probability 1) by $[1 - C(m_t)]$, which in turn determines m_{t+1}. If $G(\cdot)$ intersects the line $y = (\theta + 1)/2$ several times, the economy will possess multiple steady states, and initial conditions will determine the limit. Aggregate fluctuations are possible, despite the absence of aggregate shocks, if $C(\cdot)$ is not monotonic. For example, if $C(\cdot)$ has a minimum at fraction $1/2 < m' < 1$ and rises sharply enough after that (reflecting, say, intertemporal congestion effects such as increasing costs in the construction industry), this economy may exhibit nonlinear (possibly chaotic) behavior. Hence, the LLN need not apply, in the sense that the economy is composed of an infinite number of agents—each receiving independent shocks—yet the aggregate is volatile.

5. CONTINUOUS-TIME ECONOMIES: MARKOV JUMP PROCESSES

The discrete-time economies described earlier have continuous-time analogues, which differ in the timing of agents' strategy revision opportunities. In discrete time, all agents move simultaneously at each integer time t. However, in the continuous-time case, agents move asynchronously. Agents may only update their choices at random times, the occurrences of which are governed by identical but independent Poisson processes. (Thus, the stochastically independent times between agent i's kth and $k-1$st strategy revision opportunities are distributed exponentially.) Given this independence, the probability that two or more agents will be updating simultaneously is zero. Aside from this asynchronous updating, the determinants of i's movements are essentially the same; i moves according to (1′):

$$\text{Prob}[\eta_i(t) = 1 \mid \theta_i(t_-)] = G(\theta_i(t_-)), \tag{1'}$$

where $\theta_i(t_-) := \Sigma_j w_{ij}\eta_j(t_-)$, and $\eta_j(t_-)$ refers to j's state *just before* agent i gets to move at time t. Note that agent i's strategy is now "locked in" until her next random update time. The process described is a Markov jump process [see Lamperti (1977)]—in fact, a birth–death process. Thus, many properties of these systems are readily deduced. Stochastic Ising models[17] are examples of these processes.

Continuous-time modeling poses advantages in some contexts. For instance, dynamic programming/forward-looking problems may be easier to solve [see Blume (1995b)]. Or, the economics of the situation might suggest that a continuous-time formulation is more natural, in that the particular neighbor strategies that agent i is "reacting to" remain in place after i moves. Furthermore, some results (e.g., Proposition 2) are much easier to prove in continuous time because the continuous-time economies are birth–death processes. [Aoki (1996) collects a number of results related to such economies.] Lemmas 1 through 4 can be extended to continuous-time—of particular importance, the conditions for ergodicity and nonergodicity are unchanged.

Is the choice between discrete- and continuous-time innocuous? Continuous-time models *can* behave quite differently than their discrete-time analogues. For example, infinite UGI continuous-time economies typically will not display cycles or chaos; to obtain such behavior in infinite (or even large) economies, one would need a higher-dimensional state space, or a shifting $G(\cdot)$ function.[18] Furthermore, one cannot presuppose that in ergodic cases, discrete- and continuous-time analogues have the same invariant distribution. To see why, consider the *embedded chain* of the continuous-time process $\eta(t)$, which is constructed as follows: Let τ_k denote the time of the kth jump of $\eta(t)$. Then, $\eta(\tau_k)$, a discrete-time Markov chain, is the embedded chain of $\eta(t)$, and shares its invariant distribution. However, the discrete-time analogue $\eta'(t)$ of $\eta(t)$ does *not* have the same transition matrix as $\eta(\tau_k)$ because the transition matrix of $\eta(\tau_k)$ implies that only *one* agent can move at each time period (the matrix has a lot of zeros).

A relevant question is thus: Do continuous- and discrete-time analogues *generally* behave similarly? The simulations in Section 6 indicate that they often do.

The following is an example of a continuous-time model built using this framework.

Example 3: Bank runs (or thick market externalities).[19] The economy consists of N risk-neutral agents and evolves in continuous time. Each agent owns one unit of capital, which can neither grow nor shrink. This capital may either be used in a home investment project or deposited in a financial institution. The decision is irreversible: Once tied up in an investment, capital yields a constant flow of returns but cannot be withdrawn until the project expires. Project duration times are random variables. The distribution of these period lengths is known to be exponential (independent across agents and projects) with parameter λ.

Suppose agent i's project expires at date t. Then he must decide whether to lend his asset to the financial institution or use it at home. Each agent is risk-neutral, so his choice depends solely upon expected returns. The rate of return on agent i's home production process at t is governed by a random variable v_{it} that is i.i.d. and uniform [0, 1] across agents, and revealed at t. The rate of return on loans to the bank is also stochastic, but unknown at t. However, its *expected* return depends positively upon the current aggregate amount of capital the bank has available at t, as follows: Define ρ_t to be the fraction of the economy's capital that is on deposit at time t in the financial institution. Then, the expected return to a new deposit in the institution is given by $1/2 + (\alpha - 1/2)(2\rho_t - 1)^{1/3}$ with $1/2 \leq \alpha \leq 1$.

Agent i does not observe the aggregate fraction ρ_t directly. However, each agent *does* observe the behavior of a subset of the agents; that is, i can observe the current asset decisions of $n_i < N$ other agents in the economy. These are i's neighbors. Agent i does not know who his neighbors' neighbors are; in particular, he does not know if his neighbors observe him. Agent i forms his expectations about the return of the bank loan on the basis of his maximum likelihood estimate of ρ_t. Letting ψ_{it} denote the fraction of i's neighbors whose deposits are currently in the financial intermediary, his estimate of ρ_t is ψ_{it}.

Optimal behavior on the part of agent i can be summarized by Figure 2. The vertical axis represents v_{it}, the random home productivity shock. On the horizontal axis is ψ_{it}, i's estimate of ρ_t. If (ψ_{it}, v_{it}) lies below the dark curved line, the optimal decision on the part of the agent is to invest in the financial institution. For example, consider the pair $(\frac{3}{4}, \frac{1}{2})$. The return on the home production technology is moderate, but the expected return on investing in the financial institution is high; hence it is optimal to invest in the financial institution.

Dynamics of this economy are investigated by simulation in Section 6. (The mapping into the framework is accomplished by setting $\theta_i(t) = 2\psi_{it} - 1$.) For appropriately chosen α, this economy will spend long periods of time during which most agents invest their funds in the financial institution, interspersed with long periods of time during which the majority of agents have pulled their funds out of the financial institution (bank runs). Given this "mean-switching" behavior, conventional tests for long memory in the time series of ρ will likely not reject it [see Hiemstra and Jones (1997)]. Positing the high-intermediation case to be Pareto superior, the economy features dynamic coordination failure and spends

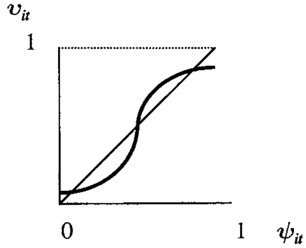

FIGURE 2. When to invest at home.

https://doi.org/10.1017/S1365100502020060 Published online by Cambridge University Press

long periods of time in a Pareto-inferior state. If $\alpha = 1$, this economy is nonergodic: With probability 1, the economy will converge to one of two potential long-run (stochastic) equilibria, one in which all agents always lend their assets to the financial institution, and the other in which none ever do so.

Proposition 2, reminiscent of a theorem in Jovanovic (1987), is a statement about the flexibility of these models. Suppose a researcher requires a particular distribution μ to be the economy's stationary distribution—for example, she has a desired long-run distribution of play she wishes to generate, or requires a fat-tailed distribution of asset returns. Is there a $G(\cdot)$ that would generate this distribution? Subject to some restrictions, yes. For any appropriate stationary distribution μ, there exists a $G(\cdot)$ function such that the stationary distribution of a UGI continuous-time economy with this $G(\cdot)$ will be μ. In fact, there is a *continuum* of appropriate $G(\cdot)$ functions, any member of which generates μ. Induced dynamics will differ along other dimensions, such as average times between mode switches.

This result is not general: It is a proposition about the steady-state distribution (and not about *dynamics* within that steady state), it restricts the class of distributions μ, and it applies to UGI continuous-time economies. However, it does indicate the substantial scope for interacting systems that are driven purely by idiosyncratic noise to display economically interesting behavior—and to match up to data. Since an infinity of $G(\cdot)$ functions will generate the desired stationary distribution, there is some scope to choose $G(\cdot)$ both to match μ and to accomplish other goals—for example, matching *additional* data characteristics such as the *dynamics* [i.e., the *conditional predictive distribution*, see Gallant et al. (1993)], or satisfying theoretical considerations such as finding the smallest externality necessary to generate the required distribution. Finally, the result is more general than it appears at first blush; Section 6 suggests that economies with less interaction or evolving in discrete time could well have similar properties.

To prove the proposition, it is convenient to keep track of a different state variable, the number k of agents who are in state 1. Similarly, the domain of $G(\cdot)$ will be k/N, the *fraction* of agents in state 1. In UGI economies, k completely characterizes the state, thus analytics are greatly simplified [see Aoki (1996)]. With k as the state, the dynamic evolution of the process obeys the following transition function:

$$Q(k,l) = \begin{cases} \left(1-\frac{k}{N}\right)G\left(\frac{k}{N}\right) & l = k+1 \\ \left(1-\frac{k}{N}\right)\left[1-G\left(\frac{k}{N}\right)\right] + \frac{k}{N}G\left(\frac{k}{N}\right) & l = k \\ \frac{k}{N}\left[1-G\left(\frac{k}{N}\right)\right] & l = k-1 \\ 0 & \text{else} \end{cases}, \tag{3}$$

where $0 < G(k/N) < 1\,\forall\, k$, which ensures that the Markov process is stationary and ergodic (see Lemma 1).

Does transition function (3) make sense? Consider the case where k might rise by one step. This can happen only if an agent currently in state -1 moves to state 1. The probability that an agent in state -1 is chosen (i.e., that a "-1" agent's alarm goes off) is given by $(1 - k/N)$. The probability that this agent will go to state 1 is given by $G(k/N)$. Similarly, for k to remain constant, chance must either select a "-1" agent *and* cause the agent to stay at -1, or select a "1" agent and cause the agent to stay at 1. For k to fall, a "1" agent must be selected, and that agent must go to -1.

Since univariate birth–death processes are reversible, the equilibrium distribution μ_k satisfies

$$\lambda Q(k,l)\mu_k = \lambda Q(l,k)\mu_l. \tag{4}$$

It follows that

$$\mu_k = \frac{(\mu_1/\mu_0)\ldots(\mu_k/\mu_{k-1})}{1+\sum_{j=1}^{N}(\mu_1/\mu_0)\ldots(\mu_j/\mu_{j-1})},$$

where

$$\frac{\mu_{k+1}}{\mu_k} = \frac{Q(k,k+1)}{Q(k+1,k)} = \frac{\left(1-\dfrac{k}{N}\right)G\left(\dfrac{k}{N}\right)}{\left(\dfrac{k+1}{N}\right)\left(1-G\left(\dfrac{k+1}{N}\right)\right)}.$$

The proposition is given below.

PROPOSITION 2. *Let $N < \infty$, and let $w_{ij} = 1/N\,\forall\, i, j$. Let μ be any distribution with $\mu_k > \varepsilon\,\forall\, k$ for some $\varepsilon > 0$ such that $\mu_k/\mu_{k-1} \le (k+1)/(N-k)$ for $k = 0, \ldots, N-1$. Then there exists a probability transition law $G(k/N)$ with $0 < G(\cdot) < 1\,\forall\, k$ such that the continuous-time UGI process with transition law $G(\cdot)$ has the stationary distribution μ. The $G(\cdot)$ is constructed as follows: Set $G(N/N) = G(1)$ arbitrarily between 0 and 1. Then, let*

$$G(k/N) = \frac{\mu_{k+1}}{\mu_k}\left(\frac{k+1}{N-k}\right)\left(1-G\left(\frac{k+1}{N}\right)\right) \quad \text{for} \quad k = 0, \ldots, N-1.$$

Proof. See Appendix. ■

Intuitively, why does this proposition work? Any distribution $(\mu_0, \mu_1, \ldots, \mu_N)$ is a collection of $N+1$ numbers, and we get to pick $N+1$ numbers $G(0/N)$, $G(1/N), \ldots, G(N/N)$. It turns out that this is a problem of N equations in $N+1$ unknowns, and one can readily show that it has a continuum of solutions. [Note that, by using (4) and the fact that $\Sigma_i \mu_i = 1$, one can easily compute the equilibrium distribution of a globally interactive continuous-time process (with finite N), once a particular $G(\cdot)$ is specified.]

Numerical example. Let $N = 3$, and suppose the desired stationary distribution is $(1/8, 3/8, 3/8, 1/8)$—that is, on average, $1/8$ of the time all three agents will

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be in the -1 state, 3/8 of the time exactly one of the three agents will be in the $+1$ state, and so on. Let $G(1) = 0.9$. Then, $G(2/3) = 0.1$, $G(1/3) = 0.9$, and $G(0) = 0.1$. This $G(\cdot)$ generates the distribution (1/8, 3/8, 3/8, 1/8). [Another $G(\cdot)$ that generates the same distribution is $G(1) = 0$, $G(2/3) = \frac{1}{3}$, $G(1/3) = \frac{2}{3}$, $G(0) = 1$.]

Whenever detailed balance (4) holds (and the process is irreducible), the equilibrium distribution μ is of the Gibbs–Boltzmann form. In some contexts, this fact might be useful for deriving other results [see Aoki (1996), Blume (1997), and Haller and Outkin (1998)]. For those systems whose neighborhood structures are associated with d-dimensional lattices, the class of $G(\cdot)$ that generates detailed balance can be characterized; see Grinstein et al. (1985) for details.

In the next section, some of the interesting (but analytically intractable) open questions alluded to in the introduction are addressed via simulation. The simulations also serve to give the reader some intuition about how these economies typically behave.

6. FURTHER RESULTS ON LOCALLY INTERACTIVE SYSTEMS BASED UPON SIMULATION

Relatively little is known about "generic" properties of interacting systems. Once one moves away from the handful of particular processes for which analytic results are readily obtainable, one cannot really say much about the details of the dynamical behavior. This is obviously problematic for applied users, who wish to know if such models are likely to be useful in their work. Such users might wish to know, for example, if dynamics are likely to hinge critically on the choice of discrete versus continuous time, or if they should expect their model to possess high persistence, or if they are relatively safe in assuming mean-field or global interaction. This section of the paper attempts to rectify this deficiency in the literature by investigating via simulation the "typical" behavior of such models. (Of course, simulation evidence is merely suggestive, not definitive.)

For the simulation study, two particular $G(\cdot)$ functions are considered, both parameterized by α which satisfies $1/2 < \alpha \leq 1$. The first $G(\cdot)$ function is linear[20]: on $\theta > 0$ $G(\theta) = 1/2 + (\alpha - 1/2)\theta$. The second $G(\cdot)$ considered is nonlinear: $G(\theta) = 1/2 + (\alpha - 1/2)\theta^{1/3}$. I refer to the latter function as the "effectively nonergodic" (ENE) $G(\cdot)$ function because ergodic systems with this law may *behave* nonergodically over shorter time scales. These $G(\cdot)$ functions are depicted in Figure 3. In both cases, higher values of α increase the degree of dependence across agents. From Lemmas 1 and 2, when $\alpha < 1$, both yield ergodic processes, and when $\alpha = 1$, both yield nonergodic processes. Note that the neighborhood size restricts the relevant domain of $G(\cdot)$—e.g., for neighborhoods of size 4, θ_i can potentially take on only five values.

Ten independent simulation runs are conducted for every parameter set (though 50 runs are conducted when the standard deviation across runs is large). Each begins with random initial conditions. Then the process is run for 400 time periods

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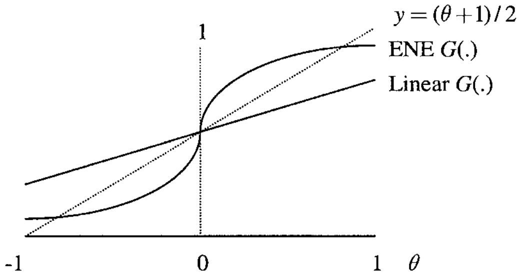

FIGURE 3. Linear and ENE $G(\cdot)$ functions.

to shed the effects of initial conditions. After that, measurements on the process are recorded for the next 1,000 time periods. For each simulation run, the variance of the mean, the variance of first differences of the mean, and the sample first autocorrelation of the mean are computed; furthermore, the histogram of the sample is constructed. The "parameter" set consists of N, α, the temporal structure (discrete vs. continuous-time), and the neighborhood structure.

Two neighborhood structures are studied. A *lattice* neighborhood structure associates each agent i with a unique lattice point on a d-dimensional integer lattice on a torus (as described in Section 2). Her neighbors are those agents that are a distance 1 away, with $w_{ij} = 1/2d \, \forall \, ij$. *Stochastic* neighborhood structures are constructed as follows: For each agent j, a random draw from an exponential distribution (truncated at $\min\{N, 40\}$) selects the number n_j of neighbor agents. Then the n_j neighbors are selected by drawing from a uniform distribution over the set of all agents. Each neighbor gets a weight $w_{jk} = 1/n_j$. Selection is with replacement: If x is selected s times to be j's neighbor, $w_{jx} = s/n_j$. In this structure, being a neighbor is not reflexive: b might be a's neighbor, but not vice versa.[21] Further, in this structure there is positive probability that the economy (or "network") is not fully connected: There may be sets of agents such that each neighbor of each member of the set lies within the set. In such cases, each group's behavior is independent of the rest of the economy.

I illustrate the results in a representative series of graphs; discussion will be mostly informal.

6.1. Systems with a Linear $G(\cdot)$ Function

When $\alpha < 1$, systems with the linear $G(\cdot)$ have a characteristic behavior that varies smoothly as N or neighborhood size or temporal structure changes. Figures 4–7 depict results on discrete-time processes; discrete- and continuous-time analogues are compared in Figure 8. Figure 4 illustrates how the variance, the variance of first differences, and first autocorrelation of this process change as one increases the number of agents N, holding both the neighborhood structure and the parameter α fixed. To better illustrate how variance measures change as N rises, the first two panels in Figure 4 plot the product $kN\sigma_N^2$, where σ_N^2 refers either to the variance

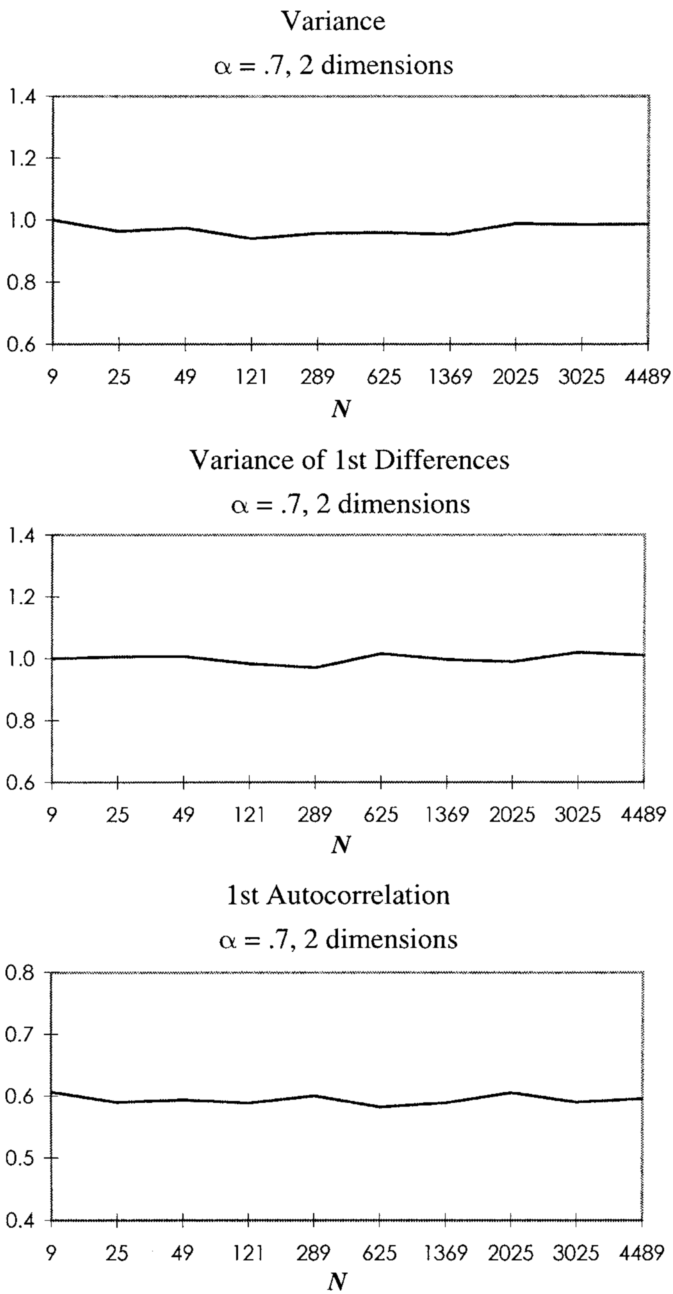

FIGURE 4. Linear $G(\cdot)$ does not postpone LLN.

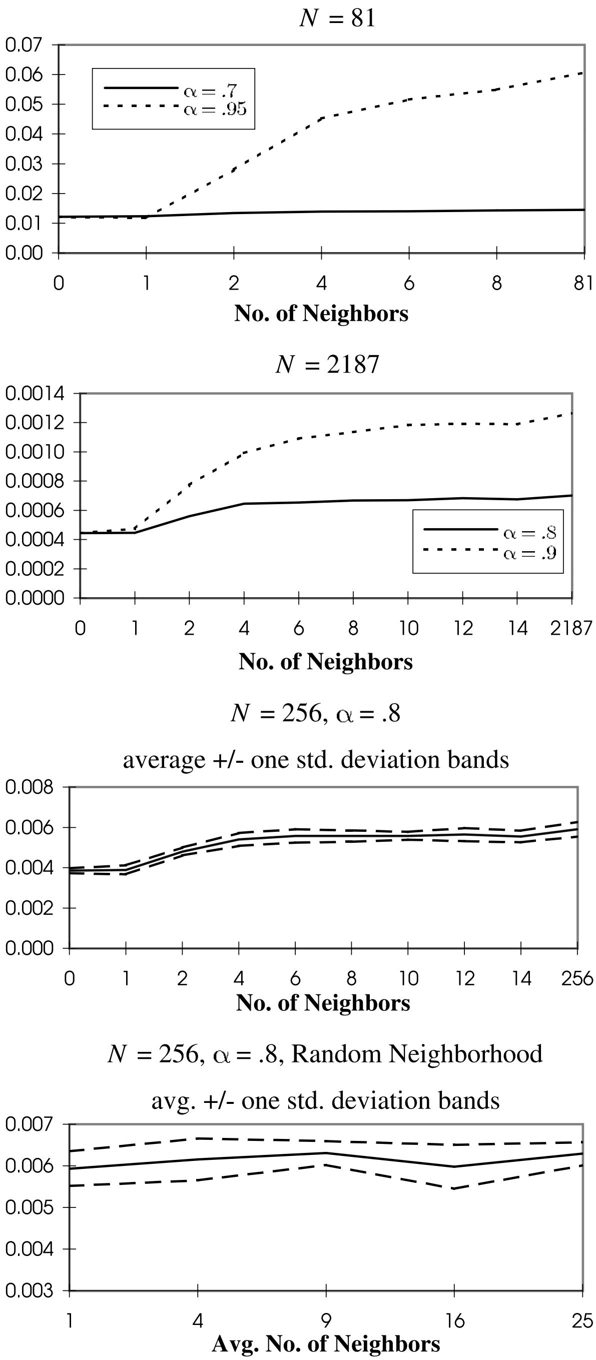

FIGURE 5. Variance with linear $G(\cdot)$.

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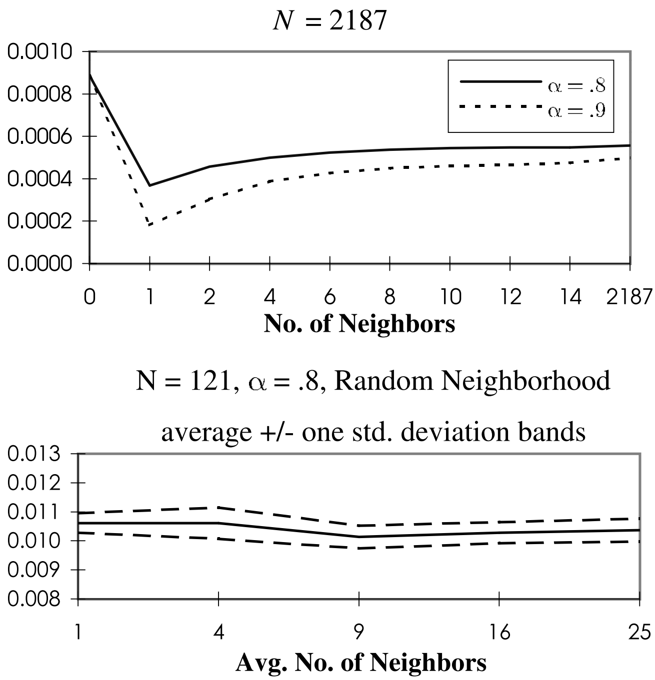

FIGURE 6. Variance of first differences with linear $G(\cdot)$.

(first panel) or the variance of first differences (second panel), N is the number of agents, and k is a constant that is chosen so that the initial value is 1. This transformation, used only here and in Figures 15 and 16, makes it easy to see at a glance whether the system "obeys the LLN," that is, whether its variance is inversely proportional to N. The graph of an "LLN-obedient" system will have zero slope, the graph of a system with LLN *postponement* will have *positive* slope, and the graph of a system with LLN *failure* will have a slope greater than or equal to 1. The first two panels in Figure 4 illustrate that systems with a linear $G(\cdot)$ obey the LLN. Though interaction generally implies *amplification* (which reduces k), this does not imply LLN *postponement*. Figure 4's final panel indicates that persistence does *not* change with N. Even large UGI systems display high autocorrelation, in stark contrast to noninteractive systems. The intuition: In finite systems, it is always possible for the system to move away from $1/2$. After it has done so, it will *not* immediately jump back to $1/2$; instead, given the closeness of the $G(\cdot)$ function to the 45-deg line, it will move back gradually. A noninteractive system always tends to jump back to $1/2$ immediately, regardless of its prior position.

For discrete-time systems with a linear $G(\cdot)$, Figures 5–7 illustrate how the variance, variance of first differences, and first autocorrelation vary, for fixed N, as the number of neighbors changes. Some figures include one standard deviation bands. As shown in Figure 5, for systems with nonstochastic neighborhood structures,

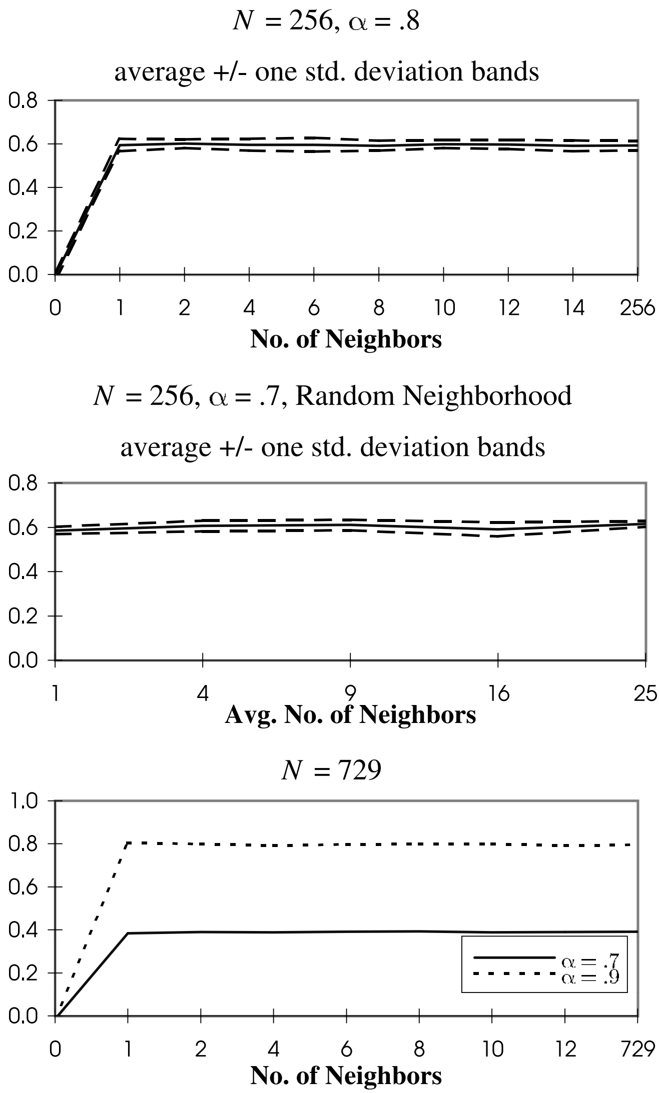

FIGURE 7. First autocorrelation with linear $G(\cdot)$.

variance typically increases as neighborhood size increases, which represents amplification relative to the i.i.d. case of zero neighbors. Increasing α unambiguously increases volatility. In stochastic neighborhood systems, average variance does *not* appear to be systematically affected by changing the average number of neighbors. In Figure 6, the variance of first differences is maximized in noninteractive systems, there being no tendency for agents to synchronize over time. For systems with nonstochastic neighborhoods, it is minimized in systems with minimal

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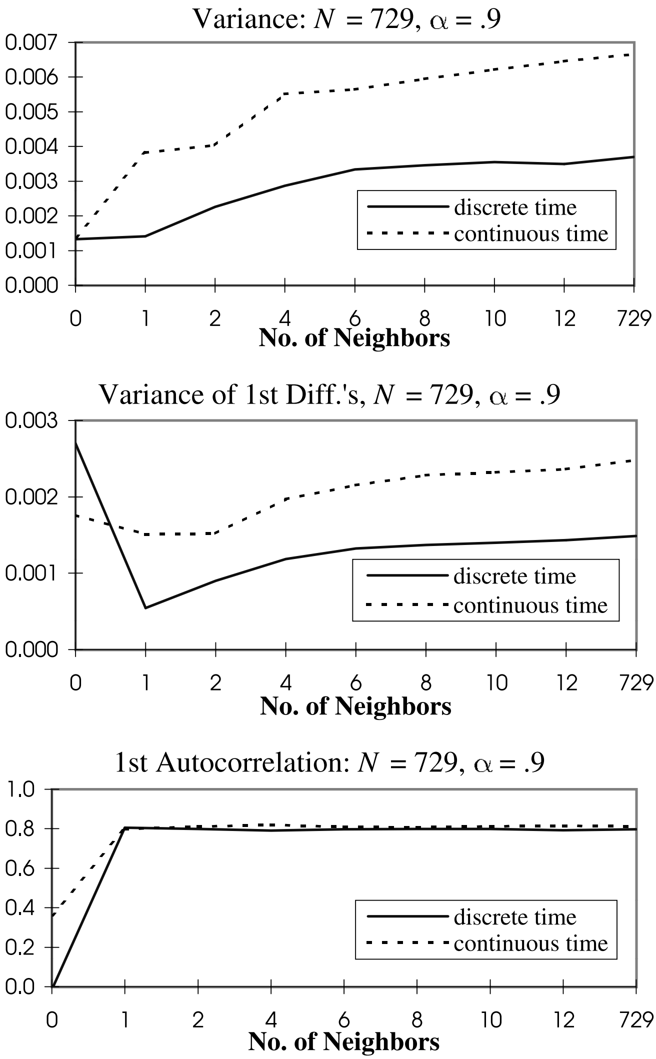

FIGURE 8. With linear $G(\cdot)$, discrete- and continuous-time similar.

interaction (one neighbor). In Figure 7, first autocorrelation is apparently unaffected by varying the number of neighbors, once even minimal interaction occurs. Persistence is a robust consequence of positive interaction.

A key finding is that, for this class of models, the behaviors of the discrete-time and continuous-time processes appear surprisingly similar[22]; however, the continuous-time processes are generally more volatile.

The first panel in Figure 9 depicts histograms of the mean of two continuous-time UGI linear $G(\cdot)$ processes ($\alpha = 0.95$ and $\alpha = 0.995$). The histograms cover

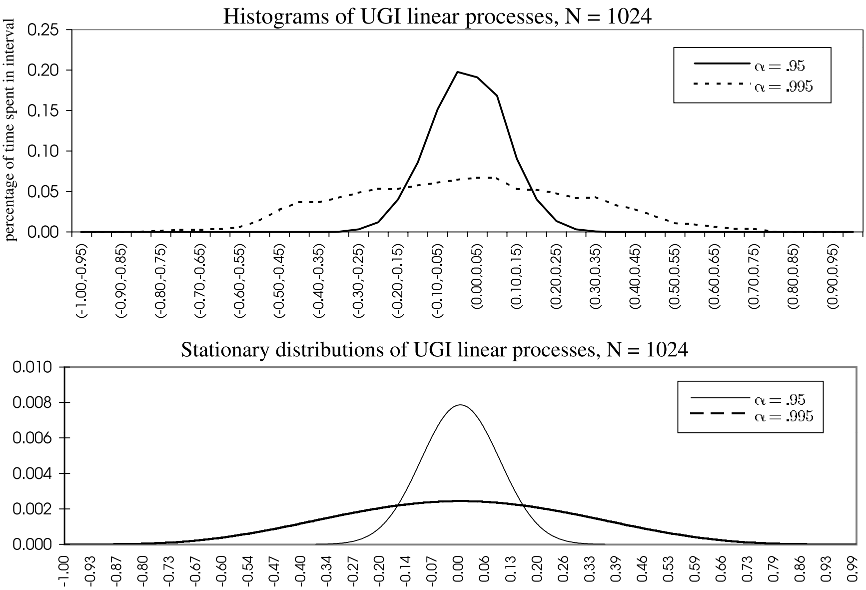

FIGURE 9. Histograms and stationary distributions, linear $G(\cdot)$.

10,000 iterations; histogram bins are intervals of size 0.05. These linear processes are both ergodic and effectively ergodic. However, the $\alpha = 0.995$ process ranges much more widely. The second panel in Figure 9 depicts the corresponding stationary distributions of these systems; these are virtually identical to the measured distributions, which suggests that these systems converge to their stationary distributions rapidly.

When α increases from 0.99 to 1.0, the process becomes nonergodic. Is it economically relevant? Yes. Even for $N = 1{,}024$, such nonergodic processes will quickly converge to either $\mathbf{1}$ or $-\mathbf{1}$. However, Figure 9 makes it clear that there exist $G(\cdot)$ functions such that nonergodicity would be effectively irrelevant. Consider, for example, a $G(\cdot)$ of the following form:

$$G(\theta) = \begin{cases} 5\theta + 5 & \theta \in [-0.9, -1] \\ 5\theta - 4 & \theta \in [0.9, 1] \\ 0.5 & \text{else} \end{cases} .$$

Even after 10,000 time units, such a process would be unlikely to converge to either $\mathbf{1}$ or $-\mathbf{1}$.

Thus, we have a reasonably complete picture of the qualitative behavior of these linear-$G(\cdot)$ systems. The behavior of the system with the nonlinear ENE $G(\cdot)$ is significantly different, however.

6.2. Systems with an ENE $G(\cdot)$ Function

Proposition 1 implies that infinite UGI economies with the ENE $G(\cdot)$ would possess two stable steady states (which are attractors); denote these *I-UGI attractors*. Their existence suggests that, in sufficiently interdependent large economies, the I-UGI attractors would correspond to metastable states—whose basins of attraction would be observationally equivalent, over finite time scales, to trapping sets. Hence one might expect that the ENE $G(\cdot)$ will give rise to qualitatively different behavior than the linear $G(\cdot)$, such as the appearance of effective nonergodicity.

Indeed, this is the case. Even small systems can be effectively nonergodic. The qualitative change in behavior is apparent upon comparison of the graphs of the variance and persistence of ENE processes—shown in Figures 10 and 12—with those of the linear processes shown in Figures 5 and 7. Figures 10 and 12 illustrate that ENE systems have "critical points"—specific combinations of N, α, and neighborhood size at which the behavior of the system changes markedly, exhibiting substantially more variance and persistence. At these parameter settings, strategic complementarities begin overpowering the idiosyncratic shocks, inducing more synchronization and drawing the average level of activity in the economy toward I-UGI attractors. Upon *further* increase in the level of interaction (more neighbors or larger α), these systems then effectively behave nonergodically: They become "trapped" in either a relatively high, or low, activity mode. The $N = 81$ and $\alpha = 0.8$ case, depicted in the first panel in Figure 10, illustrates this. This economy's variance is maximized in the four-dimensional case (eight neighbors); *increasing* the level of interaction moves the economy to a metastable mode with lower variance. In contrast, for $\alpha = 0.7$, variance (*and* persistence, see Figure 12's first panel) is maximized in the UGI case. Figure 10 also indicates that continuous- and discrete-time analogues behave in a basically similar fashion for this $G(\cdot)$ as well, though the continuous-time processes are more volatile and their critical points may differ slightly under random neighborhood structures.[23]

Interestingly, behavior is not completely simple to describe; details of the interaction structure matter [see also, e.g., Blume (1997), Ioannides (2001), Page (2001)]. For example, the $N = 729$ case in Figure 10 demonstrates that variance graphs can have more than one local maximum; on some runs, the system jumps from the I-UGI attractor basin to a different attractor basin nearby. Indeed, different interaction topologies admit different attracting sets. For instance, consider the $N = 256$, $\alpha = 0.9$, two-dimensional case (undepicted). On some runs, the system average $\bar{\eta}$ stays far from zero, "trapped" in either a high- or low-activity mode. However, on other runs—despite the identical and high level of interaction—$\bar{\eta}$ stays close to zero, seemingly untrapped. The existence of the even-N "checkerboard" mode mentioned in Section 3 accounts for the difference; $\bar{\eta}$ will remain close to zero when a system is trapped in this mode. No attempt was made to exhaustively study this or any other particular case, since the point of this simulation study is to examine general behavior.

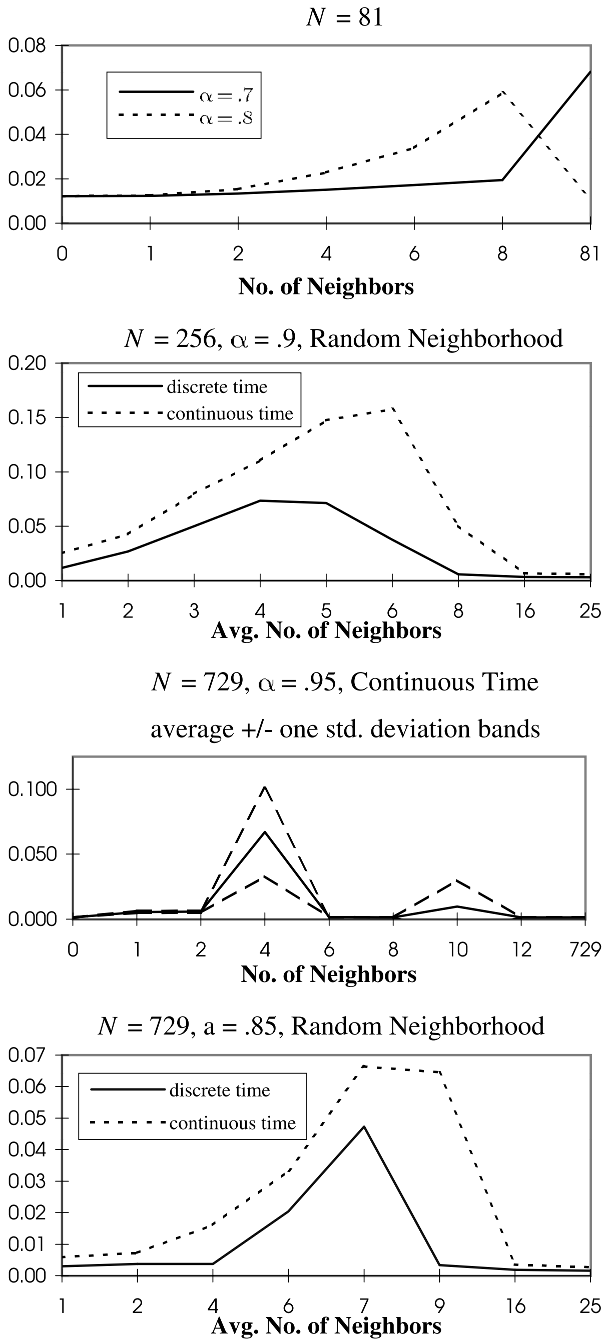

FIGURE 10. Variance with ENE $G(\cdot)$.

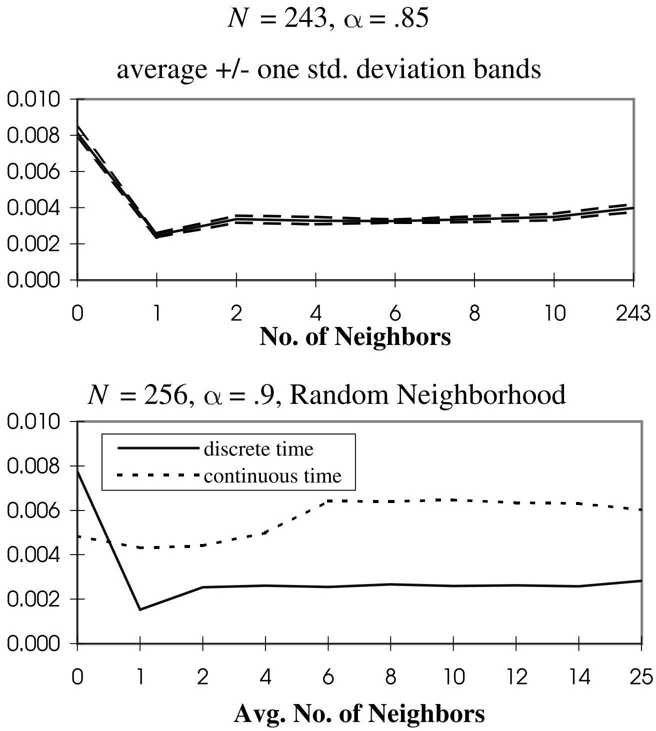

Figure 11. Variance of first differences with ENE $G(\cdot)$.

One obtains further intuition about critical behavior upon examining histograms of ENE systems as neighborhood size changes, for fixed N and α—see Figure 13. Each run covers 10,000 time periods. As the neighborhood size grows, the mean of the system ranges more and more widely. However, past the critical threshold level of interaction (four dimensions), the system begins to behave nonergodically, "trapped" (here) in a high-activity state. Figure 14 depicts the stationary distributions of two globally interactive continuous-time processes; this tells the same story.

Figures 15–17 illustrate how volatility and first autocorrelation change as N increases, holding α and the neighborhood structure fixed. As in Figure 2, to illustrate how variance (or variance of first differences) changes with N, Figures 15 and 16 plot the value $kN\sigma_N^2$, where σ_N^2 refers to variance or to variance of first differences, and k is a constant chosen in each case to make the initial value 1.

The systems depicted in Figure 15 all display some "LLN postponement"—variance does not fall proportionally with N. The impact of an increase in N is, however, not straightforwardly described. First consider economies whose neighborhood size and structure are fixed. In such economies, the average distance between agents, as measured by the number of intermediate neighbors separating two agents, grows with N. *Ceteris paribus*, longer extended chains of neighbor

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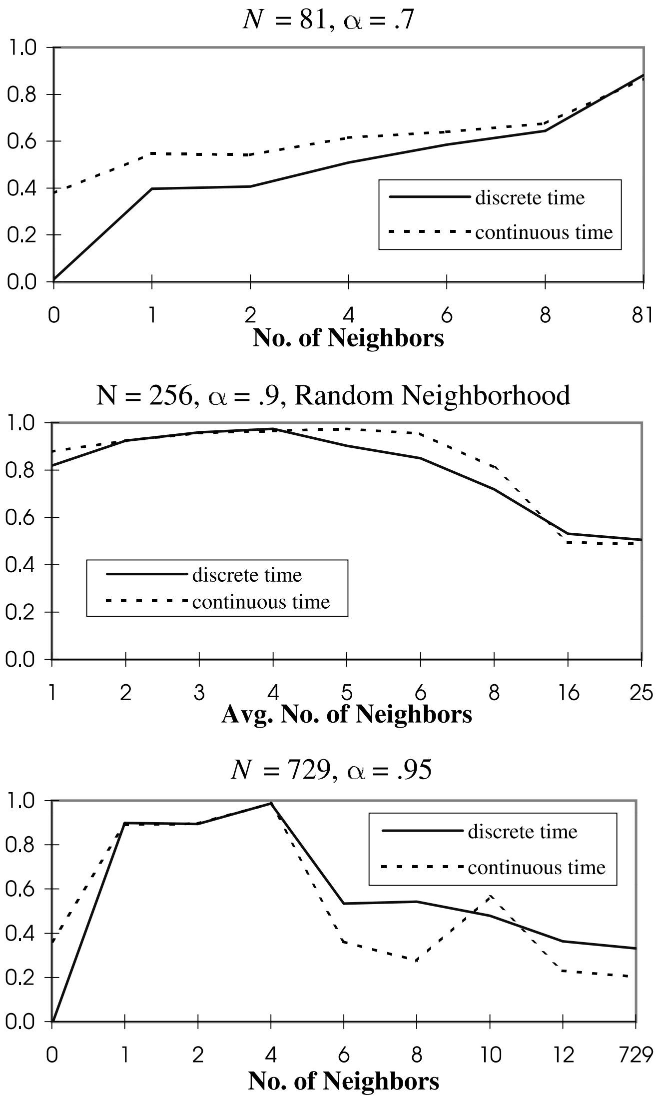

FIGURE 12. First autocorrelation with ENE $G(\cdot)$.

relations imply less dependence—to influence m, i must first influence j, who must then influence k, and so on. Thus, as N grows, one might expect diminishing dependence between agents, and hence one would expect the economy to increasingly resemble a noninteracting economy. However, this is mitigated by the nature of the interactions. Agents' behaviors are highly synchronized *locally*, analogous to a situation in which agents are placed on small islands—the average behavior of which is itself dependent on the average behavior of nearby islands.

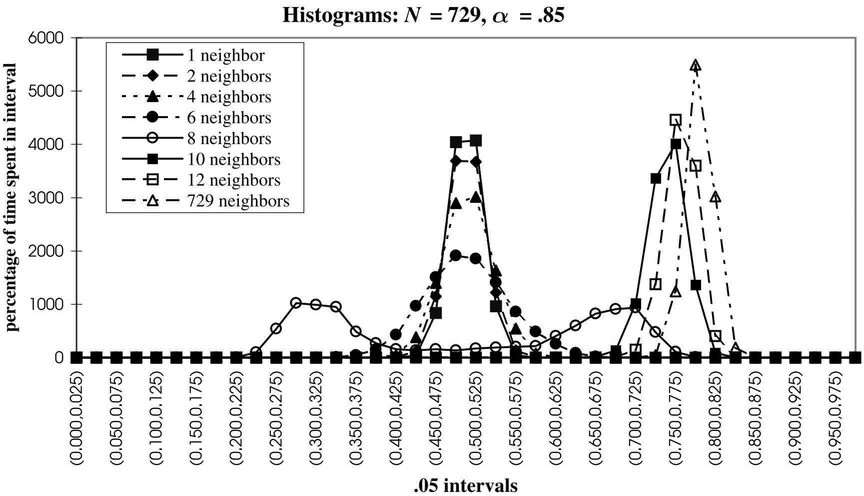

FIGURE 13. Histograms: Effective nonergodicity as interaction increases.

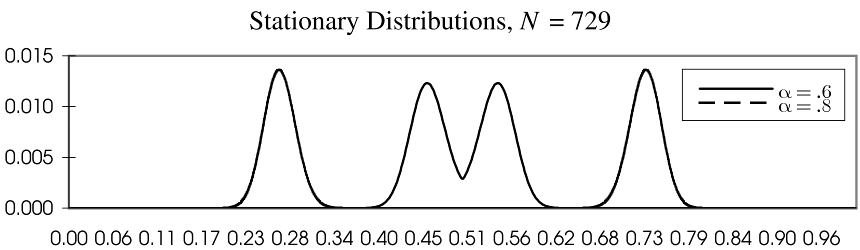

FIGURE 14. Stationary distributions of UGI systems.

Furthermore, dependence between agents does not fall rapidly because, although average distance between neighbors rises, the *number* of extended neighborhood relations by which i can indirectly influence m also grows with N.

Next consider UGI economies. There, neighborhoods implicitly change as N rises: The weight on each *particular* agent *falls*, while the *number* of neighbors *grows*. It is unclear a priori how average dependence will be affected. In the case studied here, dependence *rises*: Increases in N increase synchronization. Past a critical point at $N = 128$, the system behaves nonergodically.

The behavior of the other measurements as N rises is straightforward. Variance of first differences falls at rate N regardless—see Figure 16. The first autocorrelation of these processes is related to critical points—see Figure 17. At such points, autocorrelation is high, but if economies become effectively nonergodic, autocorrelation can diminish.

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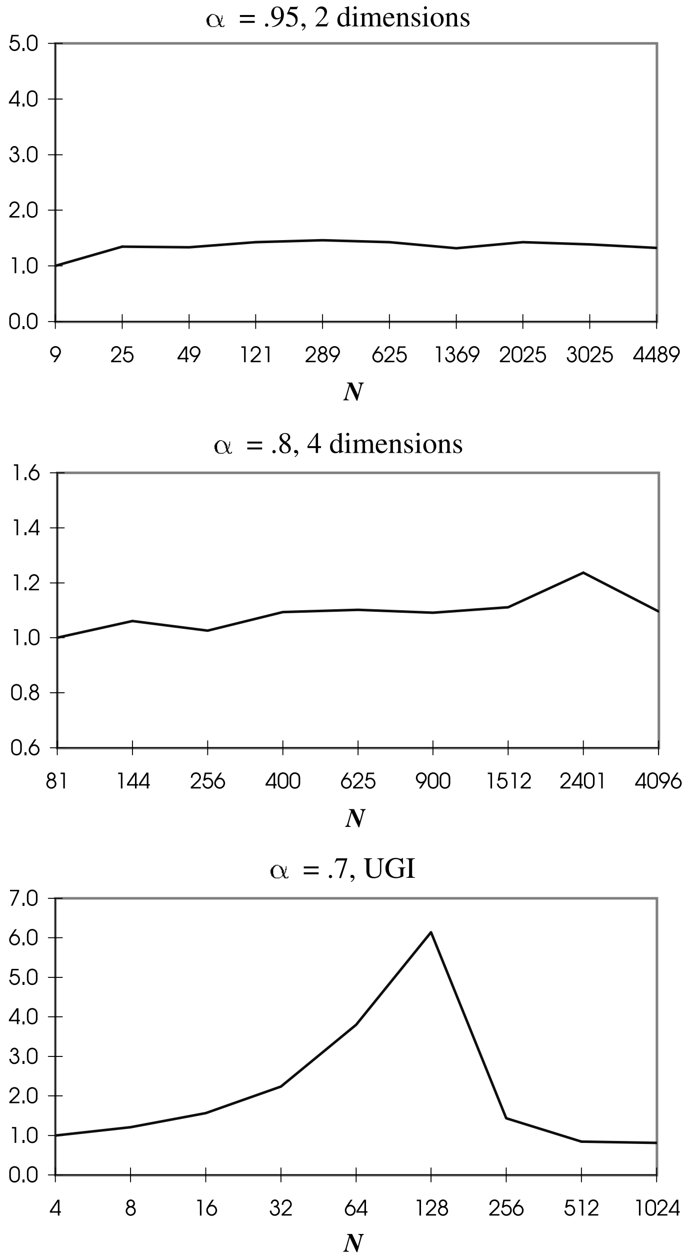

FIGURE 15. ENE $G(\cdot)$ postpones LLN on variance.

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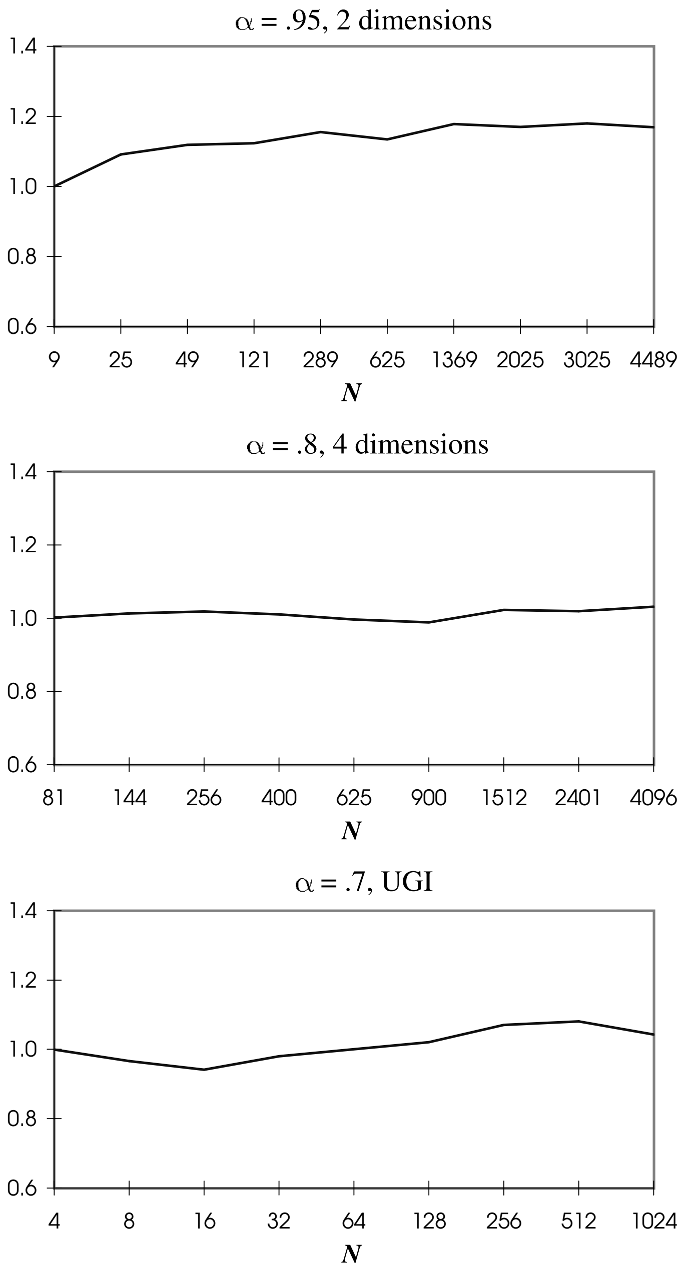

FIGURE 16. ENE $G(\cdot)$ does not postpone LLN on variance of first differences.

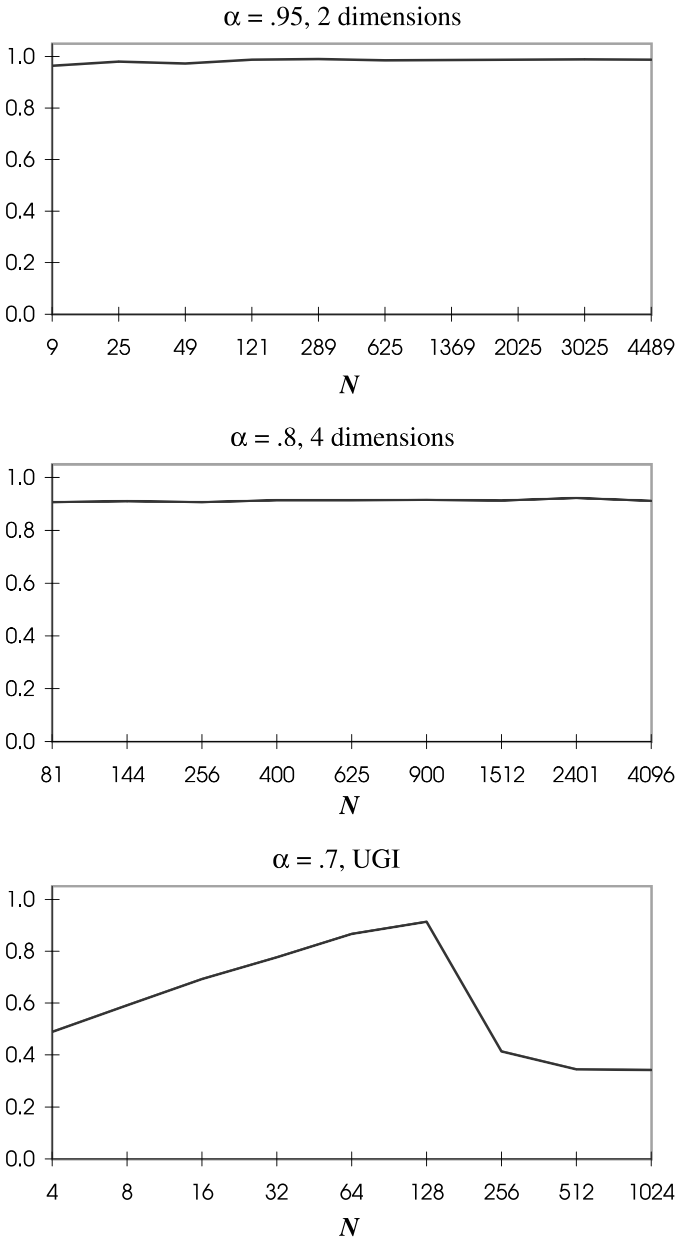

FIGURE 17. First autocorrelation scaling with ENE $G(\cdot)$.

As parameters approach critical points, the interesting changes in the dynamics of ENE systems result from significant alterations in the state-space "landscape." For weak levels of interaction, idiosyncratic shocks dominate the behavior of each agent, and the process average is constantly "thrown back" to zero. As α or the number of neighbors increases, *ceteris paribus*, the resultant increase in interagent dependency generates and deepens "valleys," thereby inducing systems toward effectively nonergodic and path-dependent behavior. The relationship between increases in N and changes in the landscape is much less obvious, as noted earlier; interagent dependence has a complicated relationship to changes in N. Uniform global interaction is an interesting special case. There, the dynamics of the system average become more closely tied to $G(\cdot)$ as N rises—LLN's can actually *aid* the system average in moving away from zero, as in Proposition 1. Ultimately, the network of interagent interactions determines how system dynamics change as N rises, and these simulations provide another demonstration of the principle that "network structure matters" [Page (2001)]. An implication is that simplifying assumptions regarding interaction patterns, such as mean-field approximations, must be used with great caution when studying systems with critical points—precisely those systems whose dynamics are most interesting.

The qualitative behavior of the ENE processes noted above—e.g., growth in variance as the process approaches a critical point, and effective "trapping" of the process once past this point—is consistent with special-case analytic results in the literature. For example, Lux (1997) demonstrates much of this analytically for a specific process;[24] and it is known that the two-dimensional infinite Ising model displays *infinite* variance exactly at its critical point. Hence, a fairly safe conjecture is that the variance of interactions processes is maximized "on the edge" of effective nonergodicity. Indeed, LLN failure is likely attainable in other versions of the model than those considered in Proposition 1, if one makes the strength of interaction a function of N [see Jovanovic (1987)]. This is not unrealistic: Marshall (1920, book 4, ch. 8) asserts that as the economy grows, specialization *and interdependence* increase.

7. CONCLUSION

Aggregation and interaction are key issues in macroeconomics, and a number of recent studies have demonstrated their critical implications for aggregate dynamics. However, a wide range of questions and issues have remained largely unexplored, in some part due both to the intractability of current tools and to uncertainty about the generality of known results. This paper introduces a tractable and intuitive framework for studying economic interaction, and summarizes some key analytical and numerical results. Perhaps most importantly, it presents valuable intuition about the behavior of these models, and demonstrates the potential of such models to generate economically interesting behavior, including: persistence; high volatility generated by small independent shocks, including the possibility of LLN failure; nonergodicity; existence of "critical" points; attainability of any equilibrium distribution; and multiple equilibria and nonlinear behavior. In addition, it highlights the underlying model structure (degree of interaction, form of probability law, neighborhood type) that are necessary to generate this.

The simulations in this study provide some intuition about the dynamic properties of interactions models. They suggest that persistence is typical, and reveal the extent to which interaction amplifies shocks and postpones the LLN. Somewhat surprisingly, they suggest that the choice of discrete versus continuous time is unlikely to matter much in many contexts. However, they also inform that details can matter. The probability law that summarizes microinteraction, the neighborhood structure, and the number of agents can all have a profound influence upon aggregate dynamics. Thus, a theorist must choose them with care. Interacting-agent economies with linear probability laws have behavior that varies smoothly with changes in parameters, number of neighbors, or N; but economies with nonlinear probability laws often possess critical points, which implies that behavior can change markedly given relatively small changes in interaction strength or neighborhood.

Although the tools were applied to a small number of examples, specifically "take-off" to development, business cycles, and bank runs, a large number of other topics can be addressed using this methodology, such as nonergodic economic growth, peer effects on crime, stock price fluctuations, corruption, regional economic activity, nonlinear aggregate responses to economic policy, local thick market externalities—the list goes on and on.[25] Indeed, it is probably safe to say that the research agenda advocated by Kirman (1992) will—indeed, must—receive an increasing amount of attention by macroeconomists, and thus that the sorts of models studied here will become increasingly important.

Since the class of models for which analytical results are readily obtained is rather small, a good deal of this research will likely rely in part upon simulation studies—a practice that is routine in other scientific fields, such as physics and mathematical biology. Ideally, there should be a give-and-take relationship between simulation and analytical investigation. Both forms of analysis are open to the criticism that conclusions are not general, but each is limited along different dimensions. Analytical results often hinge on functional forms or sharp assumptions. The assumptions necessary to obtain analytical results may omit *essential* aspects of the economic situation, and tractable special cases need not be representative of the general theory. Furthermore, a weakness of theoretical analysis is that identifying what is important, and what is not, can be difficult [see Judd (1998)]; for example, one cannot generally know if one's result is quantitatively (i.e., economically) important, or if it is robust. Finally, theory must often limit its investigation to unicausal analysis, a limitation in the complex environments macroeconomists are usually interested in, for which the whole is not the sum of its parts [again, see Judd (1998)]. Computational analysis allows one to relax assumptions and study richer and more realistic environments, along with wider classes of functions. (Such analysis is *essential* in real-world applications such as the conduct of monetary policy, where one cannot afford to investigate only one interesting feature of economic interaction in isolation.) Numerical studies can also allow one to assess quantitative importance, and to yield evidence about relevant time scales of analytical results. Furthermore, simulations can reveal patterns that hint at the existence of general theories. However, such studies are in turn limited in that only a finite number of parameterizations can be investigated,[26] exceedingly rare but interesting events may never be observed, and the underlying causes of various occurrences may be more difficult to determine (though careful and thorough simulation studies that include sensitivity analyses are quite helpful in this regard).

NOTES

1. This is true even absent direct interaction. See, e.g., Durlauf (1991), Kirman (1992), Ramsey (1996), Forni and Lippi (1997), Fratantoni and Schuh (2001), and Abadir and Talmain (2002).

2. For investigations of herd behavior/piling-up phenomena (and hyperinflation/speculative bubble-type phenomena), see Orléan (1990), Brock (1993), Lux (1995), and Kaizoji (2001). Fat tails, excess and clustered volatility, and long memory are studied by Arthur et al. (1997), Bak et al. (1997), Brock (1997), Brock and Hommes (1997, 1998, 1999), Lux (1997, 1998), Lux and Marchesi (2000), LeBaron et al. (1999), Gaunersdorfer (2000), Goeree and Hommes (2000), Kirman and Teyssière (2000), LeBaron (2000, 2001a), Brock et al. (2001), Hommes (2001), and Stauffer (2001).

3. If microeconomic agents interact, macrovariables of large economies can possess significant variance even if the only shocks are microlevel, i.i.d. shocks [Jovanovic (1987), Durlauf (1996b), Aoki (1998), Horvath (1998, 2000), Cont and Bouchaud (2000), Verbrugge (2000c), Arenas et al. (2001)]. U.S. data do not appear greatly at variance with this hypothesis, either theoretically [Horvath (1998)] or empirically [Horvath and Verbrugge (1999)]. This is especially interesting because a growing empirical literature casts doubt on the hypothesis that aggregate fluctuations are caused by persistent *aggregate* shocks hitting representative agents [e.g., Long and Plosser (1987), Cooper and Haltiwanger (1990, 1996)]. Other interactions models of fluctuations are presented by Aoki (1995a, 1998, 2001b), Gaffeo (1999), Aoki and Shirai (2000), Delli Gatti et al. (2001), Aoki and Yoshikawa (2002), and Nirei (2002).

4. See Montgomery (1991), Aoki (1996, 2000), Benabou (1996a,b), Durlauf (1996a,c), Bala and Sorger (1998, 2001), and Cooper (1998).

5. For investigations of technology adoption, nonergodicity and path-dependent growth, see David (1985), Katz and Shapiro (1986), Arthur (1989), David and Foray (1992), Durlauf (1993, 1994), An and Kiefer (1995), Dalle (1987), Cowan and Miller (1998), Dalle (1998), Ardeni and Gallegati (1999), Fagiolo (1999), Verbrugge (2000a,b,c), Arenas et al. (2001), Corradi and Ianni (2001), Cowan and Cowan (2001), Fagiolo and Dosi (2001), and Kelly (2001).

6. Ioannides (2001) extends these models to richer interaction topologies; this alters some of the results. Weisbuch et al. (2000) use the mean-field approximation in another way in a model of trading relationships.

7. Some of these papers fall into the evolutionary game literature, which typically posits Darwinian or best-response play (of normal-form games) with learning rules perturbed by noise, and generally focus upon the limiting distribution as noise vanishes. "Population games" feature pairwise random matching, myopia plus noise, and random arrivals of choice opportunities.

8. Among other things, Aoki (1996) develops new aggregation procedures to help understand and analyze macrodynamics, relates methods for computing mean first-passage times, and reports large-deviation techniques that are useful in characterizing particular aspects of the distribution (e.g., predicting how frequently the economy will perform 33% poorer than average, and bounding the probability of rare events like severe depressions). These kinds of questions may not be answerable

using simulation techniques because they may occur too rarely. Aoki and Shirai (2000) apply these to a Diamond search-model variant.

9. Irreducible means that with nonzero probability, every state may be eventually reached starting from any other state. Ergodicity means that every population configuration is endlessly revisited such that the average amount of time spent in any configuration corresponds to the stationary distribution of the Markov chain—the "time mean" coincides with the "space mean."

10. See Durlauf (1991), Kelly (1994), Kirman and Teyssière (2000), and Verbrugge (2000c), on this point.

11. Systems with neighborhood structures tied to d-dimensional lattices are stochastic (or probabilistic) cellular automata, developed by Ulam and von Neumann in the 1940's [see von Neumann (1966)]; systems with more general neighborhood structures (which are allowed here) may be interpreted as neural networks [see Haller and Outkin (1998)]. A handful of prior studies have modeled particular economic applications using cellular automata (or their stochastic analogues); see Schelling (1969, 1971), Albin (1975), Bhargava and Mukherjee (1994), Hegselmann (1996), and Page (1997) for their use in economic contexts. Keenan and O'Brien (1993) and Föllmer (1994) are closer predecessors to this paper, in that they explicitly avoid the common practice of simply positing behavioral functions and payoffs, and then exploring their implications.

12. For noninteracting linear models, Forni and Lippi (1997) present conditions under which idiosyncratic shocks—which may be partially dependent—cancel in the large-economy limit.

13. It is relatively difficult to give a mathematically precise definition of an attractor; see Ruelle (1981).

14. Recall the condition for multiple symmetric Nash equilibria with strategic complementarity given by Cooper and John (1988).

15. Point y has a periodic orbit (with period z) if, starting from y, the economy will return to y after z time periods.

16. This crucial assumption may be justified on both empirical and theoretical grounds. Cooper and Johri (1997) locate evidence for such intertemporal strategic complementarities in the U.S. economy. Several recent studies have highlighted intertemporal strategic complementarities related to investment [Gale (1996), Acemoglu and Scott (1997), and Ruiz (1998)]; those authors (and others) suggest numerous sources for such complementarities, such as R&D spillovers or various forms of learning by doing.

17. The infinitesimal transition rates of the stochastic Ising model are specifically chosen to generate equilibria that are Gibbs states. Most of what is known about this dynamic model derives from known properties of such states. As noted earlier, little is known about such models once one deviates from the small set of appropriate transition rates [see Gray (1986)].

18. See Lux (1995, 1997, 1998). In these financial-market studies, agents respond both to other agents' actions *and* to a price variable; the latter, in effect, shifts $G(\cdot)$ around. This amplifies interactions. The economy can exhibit nonlinear behavior.

19. Blume (1997) presents a related model. See also Iori and Jafarey (2001) and Aleksiejuk et al. (2002).

20. A linear function is not as natural as one might think. To obtain such a model on the basis of (5), for example, one must posit a particular distribution of shocks tailored to $T(\cdot)$. With a linear $T(\cdot)$, one must assume that shocks are uniformly distributed.

21. Such neighborhood patterns might well give rise to quite different behavior [see, e.g., Page (2001)]. See also Evstigneev and Taksar (2001), who extend the theory of random fields on *directed* graphs and apply it to economics.

22. Of course, this is not a proof of a general statement! Behavior in some game-theoretic models is very different under simultaneous (discrete-time) versus asynchronous (continuous-time) updating; see, e.g., Huberman and Glance (1993). However, results here suggest that in many contexts, continuous-time economies will behave similarly to discrete-time analogues.

23. The processes may differ in other ways as well. For example, the secondary variance "hump" exhibited by Figure 10's continuous-time $N = 729$ case occurs at 10 neighbors; the secondary variance

hump in the corresponding discrete-time case occurs at 8 neighbors (undepicted). The autocorrelation pattern in the $N = 729$ panel in Figure 12 indicates this discrepancy as well.

24. Unfortunately, the power-series approximation method outlined there, as well as the approximation of Kubo—both outlined by Aoki (1996)—are intractable for general processes, given the size of the state space. Indeed, both break down near critical points.

25. Verbrugge (2000a,b) are two direct applications of this framework; see also Kelly (1994).

26. This constraint becomes less binding as increased computational power enables wider searches.

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APPENDIX: PROOF OF PROPOSITION 2

Equation (4) implies that

$$Q(k,l)\mu_k = Q(l,k)\mu_l.$$

Writing this out (denoting $g_k := G(k/N)$ to save on notation), one obtains

$$\begin{aligned}
\mu_0(N/N)g_0 &= \mu_1(1/N)(1-g_1)\\
\mu_1[(N-1)/N]g_1 &= \mu_2(2/N)(1-g_2)\\
&\vdots\\
\mu_{N-1}(1/N)g_{N-1} &= \mu_N(N/N)(1-g_N).
\end{aligned}$$

Given the $N+1$ numbers $\mu_0, \mu_1, \ldots, \mu_N$, this is a system of N equations, but there are $N+1$ unknowns, namely, the g_i. The N equations imply N relations of the form

$$g_k = \frac{\mu_{k+1}}{\mu_k}\left(\frac{k+1}{N-k}\right)(1-g_{k+1}).$$

https://doi.org/10.1017/S1365100502020060 Published online by Cambridge University Press

Since $\frac{\mu_k}{\mu_{k+1}} \leq \frac{k+1}{N-k}$, the product of the first two terms on the right-hand side of the equation is, at most, 1 (and greater than 0); thus, g_k lies between 0 and 1 as long as g_{k+l} does. Hence, g_N may be chosen *arbitrarily* between 0 and 1, and the remaining g_k may be chosen by iterating on the above equation.

By construction, the resulting set of g_k will give rise to μ as the stationary distribution. This is a standard result in birth–death chains, and verifying this is a bit tedious, but a sketch of how it is done goes as follows. Note that

$$\mu_1 = N \frac{g_0}{(1 - g_1)} \mu_0,$$

that

$$\mu_2 = \frac{\left(\frac{N-1}{N}\right)}{\left(\frac{2}{N}\right)} \frac{g_1}{(1 - g_2)} \mu_1 = \frac{\left(\frac{N-1}{N}\right)}{\left(\frac{2}{N}\right)} \frac{g_1}{(1 - g_2)} \left[N \frac{g_0}{(1 - g_1)} \mu_0 \right],$$

and so on. A final restriction is $\Sigma_i \mu_i = 1$. Substituting into this final restriction, one can then solve for μ_0, and the rest of the probabilities follow. ■