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SUBGROUPS OF ONE-RELATOR FUCHSIAN GROUPS

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Let Γ be the Fuchsian group with canonical presentation:

Generators: $x_1, x_2, \ldots, x_r, a_1, b_1, a_2, b_2, \ldots, a_g, b_g, p_1, p_2, \ldots, p_s$

$$\text{Relations: } x_1^{m_1} = x_2^{m_2} = \ldots = x_r^{m_r} = \prod_{i=1}^{r} x_i \prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1} \prod_{k=1}^{s} p_k = 1,$$

where $M(\Gamma) = 2(g-1) + r + s - \sum_{i=1}^{r} 1/m_i > 0$. The *genus* g, the *period partition* $\{m_1, m_2, \ldots, m_r\}$ and the number s are invariants of Γ. The symbol $(g|m_1, m_2, \ldots, m_r|s)$ is called the *signature* of Γ and it determines Γ up to isomorphism. We shall sometimes speak loosely of the group

$$(g|m_1, m_2, \ldots, m_r|s).$$

If Γ_1 is a subgroup of index N in Γ then Γ_1 is a Fuchsian group and we have the Riemann-Hurwitz formula $N = M(\Gamma_1)/M(\Gamma)$. Further, by considering the action of Γ on the left cosets of Γ_1, Singerman proves the following result in [**1**].

THEOREM 1. *The group Γ with signature $(g|m_1, \ldots, m_r|s)$ contains a subgroup Γ_1 of index N with signature*

$$(g'|n_{11}, n_{12}, \ldots, n_{1\rho_1}, \ldots, n_{r_1}, n_{r_2}, \ldots, n_{r\rho_r}|s')$$

if and only if

(a) *there exists a transitive permutation group G on N points and an epimorphism $\theta : \Gamma \to G$ satisfying the following conditions:*

(i) *The permutation $\theta(x_i)$ has precisely ρ_i cycles of lengths different from m_i, the lengths of these cycles being $m_i/n_{i1}, m_i/n_{i2}, \ldots, m_i/n_{i\rho_i}$,*

(ii) *If we denote the number of cycles in the permutation $\theta(\gamma)$ by $\delta(\gamma)$, then $s' = \sum_{k=1}^{s} \delta(p_k)$.*

(b) $N = M(\Gamma_1)/M(\Gamma)$.

Amongst the Fuchsian groups are two families of one-relator groups, namely the groups with no periods ($r = 0$) and the groups with a single period ($r = 1$). Our objective in this paper is to determine all groups which are subgroups of finite index in one-relator Fuchsian groups. In the case of $r = 0$, Γ having no periods is a torsion-free group, and in this event any subgroup of finite index in Γ, being likewise a torsion-free Fuchsian group, is a one-relator group of the same type. Our problem is therefore trivially solved for this case and it remains to determine subgroups of the singly periodic groups $(g|n|s)$.

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If $(g'|m_1, m_2, \ldots, m_r|s')$ is a subgroup of $(g|n|s)$ then, (as can be seen from Theorem 1, for example), $s' = 0$ if and only if $s = 0$. This leads us to consider separately the two cases $s = 0$ and $s > 0$.

Let $\{m_1, m_2, \ldots, m_r\}$ be any period partition. We first determine all values of the genus g' such that $(g'|m_1, \ldots, m_r|0)$ is a subgroup of some $(g|n|0)$. In the light of Theorem 1 we consider transitive permutation representations of $(g|n|0)$ in which the element $\prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1}$ maps to a permutation with cycle structure $(n/m_1)(n/m_2) \ldots . (n/m_r)(n)^d$, for some $d \geqq 0$. Letting L denote the least common multiple of the periods $\{m_1, m_2, \ldots, m_r\}$ and $\sigma = \sum_{i=1}^{r} 1/m_i$, we observe the following facts concerning such a representation:

(i) n must be a multiple of L, say $n = \lambda L$, some $\lambda \geqq 1$;

(ii) N = degree of representation (= index of subgroup) $= \lambda L/m_1 + \lambda L/m_2 + \ldots + \lambda L/m_r + \lambda L d = \lambda L(\sigma + d)$;

(iii) since $\prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1}$ must map to an even permutation, (being a product of commutators), $\lambda L(\sigma + d) - (d + r)$ is an even integer.

We will require the following

LEMMA. *In the symmetric group S_N, every even permutation can be expressed as a product of two N-cycles, and hence as a commutator $ACA^{-1}C^{-1}$ where A is an N-cycle.*

Proof. Let $(a_1)(a_2) \ldots . (a_t)$ be the cycle structure of any even permutation in S_N, where $1 \leqq a_1 \leqq a_2 \leqq \ldots . \leqq a_t$. Then $\sum_{i=1}^{t} a_i = N$ and $N - t$ is even. Define increasing integers α_i, $0 \leqq i \leqq t + 1$, by $\alpha_0 = 0$, $\alpha_1 = 1$ $\alpha_i = 2a_{i-1} + \alpha_{i-2}$ $(2 \leqq i \leqq t)$, and $\alpha_{t+1} = N$. Let A and B be the N-cycles:

$$A = (1, 2, 3, \ldots \ldots \ldots, N - 1, N)$$

$$B = (1, \alpha_2, \alpha_2 - 1, \alpha_2 - 2, \ldots . , \alpha_1 + 1, \alpha_3, \alpha_3 - 1, \ldots . , \alpha_2 + 1, \alpha_4, \alpha_4 - 1, \ldots, \alpha_3 + 1, \ldots . , \alpha_{t-1} + 1, \alpha_{t+1}, \alpha_{t+1} - 1, \alpha_{t+1} - 2, \ldots . . , \alpha_t + 1).$$

Then AB^{-1} has disjoint cycles

$$(\alpha_i + 1, \alpha_i + 3, \alpha_i + 5, \ldots . , \alpha_{i+1}, \alpha_{i+1} + 2, \ldots, \alpha_{i+2} - 1)$$

of length $\frac{1}{2}(\alpha_{i+2} - \alpha_i) = a_{i+1}$, for $0 \leqq i \leqq t - 2$, together with the cycle

$$(\alpha_{t-1} + 1, \alpha_{t-1} + 3, \ldots . , \alpha_t, \alpha_t + 2, \alpha_t + 4, \ldots . , \alpha_{t+1}, \alpha_t + 1, \alpha_t + 3, \ldots . , \alpha_{t+1} - 1)$$

of length $\alpha_{t+1} - \frac{1}{2}(\alpha_t + \alpha_{t-1} - 1) = a_t$. That is, AB^{-1} has the required cycle structure.

It follows immediately from the lemma that for all $\lambda \geqq 1$ and $d \geqq 0$ such that $\lambda L(\sigma + d) - (d + r)$ is even the group $(1|\lambda L|0)$ can be mapped epimorphically onto a transitive subgroup of $S_{\lambda L(\sigma+d)}$ with $a_1b_1a_1^{-1}b_1^{-1}$ mapping to an even permutation with cycle structure

$$(\lambda L/m_1)(\lambda L/m_2)\ldots(\lambda L/m_r)(\lambda L)^d.$$

Furthermore (by mapping all additional generators to the identity permutation), for all $g \geqq 1$ the group $(g|\lambda L|0)$ can be mapped epimorphically onto the same transitive subgroup of $S_{\lambda L(\sigma+d)}$ with $\prod_{j=1}^{g} a_jb_ja_j^{-1}b_j^{-1}$ mapping to the same prescribed even permutation. In all cases the corresponding subgroup of index $\lambda L(\sigma + d)$ in $(g|\lambda L|0)$ has the required period partition $\{m_1, m_2, \ldots, m_r\}$ and genus g' determined by the Riemann-Hurwitz formula:

$$2(g' - 1) + r - \sigma = \lambda L(\sigma + d)[2(g - 1) + 1 - 1/\lambda L],$$

giving

$$g' = \tfrac{1}{2}[\lambda L(\sigma + d)(2g - 1) - (r + d)] + 1.$$

If $s > 0$ the Fuchsian group with signature $(g|m_1, \ldots, m_r|s)$ is abstractly a free product of cyclic groups

$$Z_{m_1} * Z_{m_2} * \ldots * Z_{m_r} * (* Z)^{2g+s-1}$$

(where we use $(* Z)^\alpha$ to denote the free product of α infinite cyclic groups). Two such groups are isomorphic if and only if they have the same period partition and the same free rank $2g + s - 1$. Conversely, if

$$\alpha + \textstyle\sum_{i=1}^{r} (1 - 1/m_i) > 1 \quad \text{then} \quad Z_{m_1} * Z_{m_2} * \ldots * Z_{m_r} * (* Z)^\alpha$$

can be represented as a Fuchsian group with $s > 0$. Again, for the given period partition $\{m_1, \ldots, m_r\}$ we determine all possible values of the free rank α for which this group is a subgroup of some one-relator Fuchsian group $Z_n * (* Z)^\beta$ $(\beta \geqq 1)$. Up to isomorphism, we may assume without loss of generality that this latter group is given by the presentation:

$$\begin{aligned}&\text{Generators: } x_1, p_1, p_2, \ldots, p_{\beta+1}\\&\text{Relations : } x_1^n = x_1p_1p_2\ldots p_{\beta+1} = 1.\end{aligned}$$

Using Theorem 1, we look for transitive representations of this group in which the element $p_1p_2\ldots p_{\beta+1}$ maps to a permutation with cycle structure

$$(n/m_1)(n/m_2)\ldots(n/m_r)(n)^d.$$

Again we have $n = \lambda L$ and the degree given by $N = \lambda L(\sigma + d)$. Since $\beta + 1 \geqq 2$ we can certainly achieve such a representation for any $\lambda \geqq 1$ and for any $d \geqq 0$ by simply mapping p_1 to an N-cycle (to ensure transitivity), p_1p_2 to the required permutation and $p_j (3 \leqq j \leqq \beta + 1)$ to the identity permutation. Applying the Riemann-Hurwitz formula:

$$\alpha - 1 + r - \sigma = \lambda L(\sigma + d)[\beta - 1/\lambda L],$$

giving $\alpha = \lambda L(\sigma + d)\beta - (r + d) + 1$.

We have proved the following

THEOREM 2. *Let* $\{m_1, m_2, \ldots, m_r\}$ *be any period partition. The only Fuchsian groups with this period partition which are subgroups of a one-relator Fuchsian group are given by:*

(i) *For all* $\lambda \geqq 1$ *and* $d \geqq 0$ *such that* $\lambda L(\sigma + d) - (d + r)$ *is even, and for all* $g \geqq 1$, $(g'|m_1, m_2, \ldots, m_r|0)$ *is a subgroup of index* $\lambda L(\sigma + d)$ *in* $(g|\lambda L|0)$, *where*

$$g' = \tfrac{1}{2}[\lambda L(\sigma + d)(2g - 1) - (r + d)] + 1.$$

(ii) *For all* $\lambda \geqq 1$, $d \geqq 0$ *and* $\beta \geqq 1$, $Z_{m_1} * Z_{m_2} * \ldots * Z_{m_r} * (* Z)^\alpha$ *is a subgroup of index* $\lambda L(\sigma + d)$ *in* $Z_{\lambda L} * (* Z)^\beta$, *where*

$$\alpha = \lambda L(\sigma + d)\beta - (d + r) + 1.$$

REFERENCE

1. D. Singerman, *Subgroups of Fuchsian groups and finite permutation groups*, Bull. London Math. Soc. *2* (1970), 319–323.

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