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Maximal Inequalities of Noncommutative Martingale Transforms

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Abstract. In this paper, we investigate noncommutative symmetric and asymmetric maximal inequalities associated with martingale transforms and fractional integrals. Our proofs depend on some recent advances on algebraic atomic decomposition and the noncommutative Gundy decomposition. We also prove several fractional maximal inequalities.

1 Introduction

Let $f = (f_n)_{n\geq 1}$ be a martingale on a probability space $(\Omega, \mathbb{P}, (\mathcal{F}_n)_{n\geq 1}, \mathcal{F})$. Burkholder [5] introduced the martingale transform

$$(T_\xi f)_n = \sum_{k=1}^{n} \xi_{k-1}(f_k - f_{k-1}),$$

where $\xi = (\xi_n)_{n\geq 1}$, ξ_n is measurable with respect to $\mathcal{F}_n$ for each n, and $\sup_n \|\xi_n\|_\infty < \infty$. The following well-known weak type maximal inequality was proved in [5]:

$$\Big\| \sup_{n\geq 1} |(T_\xi f)_n| \Big\|_{L_{1,\infty}(\Omega)} \leq C \sup_{n\geq 1} \|f_n\|_{L_1(\Omega)}. \tag{1.1}$$

Nowadays, martingale transforms have been proven to be a very powerful tool not only in probabilistic situation but also in harmonic analysis (see *e.g.*, [1, 40] and the references therein).

In this paper, we mainly consider noncommutative martingale inequalities, more precisely, we focus on noncommutative maximal inequalities associated with martingale transforms. The development of the noncommutative martingale inequalities began with the establishment of the noncommutative Burkholder–Gundy inequality (see (2.2) and (2.3)) by Pisier and Xu [34]. Since then, many of the classical results about martingales have been extended to the noncommutative setting; see, for instance, [2, 3, 9, 15–19, 23, 25, 32, 36, 38].

Similar to the classical case, the noncommutative Doob maximal inequalities play an important role in the theory of noncommutative martingales and harmonic analysis. We recall some results. In the sequel, let $(\mathcal{M}, \tau)$ be a noncommutative probability space and let $\mathcal{E}_n$ denote the conditional expectation associated with a given weak-$*$

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dense filtration $(\mathcal{M}_n)_{n\geq 1}$. By constructing special projections (see Lemma 3.1), Cuculescu [7] established the following weak type Doob maximal inequality: if $x = (x_n)_{n\geq 1} \in L_1(\mathcal{M})$ is a martingale, then

$$\|(x_n)_{n\geq 1}\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty)} \leq \sup_n \|x_n\|_1,$$

where for $0 < p < \infty$, $\Lambda_{p,\infty}(\mathcal{M},\ell_\infty)$ (this notation was first introduced in [12, p. 997]) is defined as the space of all sequences $(x_n)_{n\geq 1}$ in $L_{p,\infty}(\mathcal{M})$ with quasi-norm

$$\|(x_n)_{n\geq 1}\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty)} = \sup_{\lambda>0} \inf_{e\in\mathcal{P}(\mathcal{M})} \left\{\lambda\tau(1-e)^{\frac{1}{p}} : \|ex_ne\|_\infty \leq \lambda, \forall n\geq 1\right\} < \infty, \tag{1.2}$$

where $\mathcal{P}(\mathcal{M})$ stands for the projection lattice in $\mathcal{M}$. The strong type of noncommutative form of Doob maximal inequality is due to Junge [21]: for $1 < p \leq \infty$, if $x \in L_p(\mathcal{M})$, then there exist a, b and w_n such that for each $n \geq 1$,

$$\mathcal{E}_n(x) = aw_nb \quad \text{and} \quad \|a\|_{2p}\Big(\sup_n \|w_n\|_\infty\Big)\|b\|_{2p} \leq C_p\|x\|_p. \tag{1.3}$$

Our first main objective is to extend (1.1) to a noncommutative setting for general noncommutative martingale transforms T_ξ^c and T_ξ^r introduced by Hong et al. [14, p. 1254].

Definition 1.1 Let $\xi = (\xi_n)_{n\geq 0} \subset \mathcal{M}$ be an adapted sequence (for each $n \geq 1$, ξ_n is measurable with respect to $\mathcal{M}_n$; for convenience, $\xi_0 \in \mathcal{M}_1$) such that $\sup_n \|\xi_n\|_\infty \leq 1$. The noncommutative martingale transforms T_ξ^c and T_ξ^r of a martingale $x = (x_n)_{n\geq 1}$ are defined by setting

$$(T_\xi^c x)_n = \sum_{k=1}^n \xi_{k-1}d_kx \quad \text{and} \quad (T_\xi^r x)_n = \sum_{k=1}^n (d_kx)\xi_{k-1}, \quad n \geq 1,$$

where $d_1x = x_1$ and $d_kx = x_k - x_{k-1}$ is the martingale difference for each $k \geq 2$.

If in addition ξ_{n-1} commutes with $\mathcal{M}_n$ for every $n \geq 1$, we denote T_ξ^c and T_ξ^r by T_ξ. We always use $T_\xi^c x$ (resp. $T_\xi^r x$) to denote the sequence $((T_\xi^c x)_n)_{n\geq 1}$ (resp. $((T_\xi^r x)_n)_{n\geq 1}$).

Noncommutative martingale transforms have been studied by several authors. In [34, Remark 2.4], for $1 < p < \infty$, the strong type (p,p) inequality of the noncommutative transform T_ξ with $\xi_n = \pm 1$ for every $n \geq 1$ was deduced from the Burkholder–Gundy inequality. The result was strengthened by Randrianantoanina (see [36, Theorem 3.1]) who proved that the martingale transform T_ξ with $\xi_n = \pm 1$ for each $n \geq 1$ is of weak type $(1,1)$; that is,

$$\sup_{n\geq 1} \|(T_\xi x)_n\|_{L_{1,\infty}} \leq C \sup_{n\geq 1} \|x_n\|_1. \tag{1.4}$$

In addition, the weak type $(1,1)$ boundedness of T_ξ was given as an application of noncommutative Gundy's decomposition by Parcet and Randrianantoanina [31, Theorem 3.1]. By standard interpolation and dual argument, T_ξ is then of strong type

(p,p) for $1 < p < \infty$ (see *e.g.*, [?, Remark 3.3.4]); that is,

$$\|T_\xi x\|_p \le C_p \|x\|_p, \quad 1 < p < \infty. \tag{1.5}$$

It should be noted that $\Lambda_{1,\infty}(\mathcal{M}, \ell_\infty) \subset L_{1,\infty}(\mathcal{M})$ (see Remark 3.4). Osękowski [29] improved (1.4) (see [29, Theorem 1 and Lemma 3]) by showing that there exists a constant $C > 0$ such that

$$\left\| \big((T_\xi x)_n \big)_{n\ge1} \right\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty)} \le C \sup_{n\ge1} \|x_n\|_1 \quad \text{provided } \xi_{n-1} \text{ commutes with } \mathcal{M}_n. \tag{1.6}$$

An important tool in the study of noncommutative martingale transform is the noncommutative Calderón-Zygmund decomposition developed by Parcet [30]. With the help of such decomposition, Hong et al. [14, Theorem C] obtained a weak type inequality for general noncommutative martingale transforms T_ξ^c and T_ξ^r: if the filtration $(\mathcal{M}_n)_{n\ge1}$ is regular (see (2.1)), then there exists a decomposition $x = x^c + x^r$ such that

$$\sup_{n\ge1} \left\| \big(T_\xi^c x^c \big)_n \right\|_{L_{1,\infty}(\mathcal{M})} + \sup_{n\ge1} \left\| (T_\xi^r x^r)_n \right\|_{L_{1,\infty}(\mathcal{M})} \le C \sup_{n\ge1} \|x_n\|_1. \tag{1.7}$$

Under the assumption that the filtration $(\mathcal{M}_n)_{n\ge1}$ is regular, this inequality substantially extends (1.4). We also refer the reader to [28, 43] for other results about noncommutative martingale transforms.

Our first main result improves (1.7) and also (1.6) in the case of the regular filtration. The result is read as follows (any unexplained terminologies and symbols can be found in Section 2).

Theorem 1.2 *Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal faithful normalized trace τ, and $(\mathcal{M}_n)_{n\ge1}$ be a regular filtration. If $x = (x_n)_{n\ge1} \in L_1(\mathcal{M}, \tau)$, then there exists a decomposition $x = x^c + x^r$ such that*

$$\left\| \big((T_\xi^c x^c)_n \big)_{n\ge1} \right\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty)} + \left\| \big((T_\xi^r x^r)_n \big)_{n\ge1} \right\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty)} \le C \sup_{n\ge1} \|x_n\|_1. \tag{1.8}$$

Since the left-hand side of (1.7) does not exceed the left-hand side of (1.8) (see Remark 3.4(i)), our result does improve (1.7). The proof depends on the noncommutative Gundy decomposition introduced by Parcet and Randrianantoanina [31] and the Doob maximal inequality (1.3). The proof of Theorem 1.2 is contained in Section 3.

The second main objective of this paper is to get asymmetric maximal inequalities associated with martingale transforms. Recently, using the algebraic atomic decomposition introduced in [22], Hong et al. [12] studied various asymmetric Doob maximal inequalities (see also [13] for the continuous case). In order to explain their main results, we recall the following definition introduced by Defant and Junge [8, p. 328].

Definition 1.3 Let $1 \le p < \infty$ and $0 \le \theta \le 1$. The space $L_p(\mathcal{M}, \ell_\infty^\theta)$ consists of all sequences $(x_n)_{n\ge1} \subset L_p(\mathcal{M})$ with finite

$$\|(x_n)_{n\ge1}\|_{L_p(\mathcal{M},\ell_\infty^\theta)} = \inf\Big\{ \|a\|_{\frac{p}{1-\theta}} \Big(\sup_{n\ge1} \|w_n\|_\infty \Big) \|b\|_{\frac{p}{\theta}} : x_n = aw_n b, \forall n \ge 1 \Big\},$$

where the infimum is taken over all possible factorizations of $(x_n)_{n\ge1}$ in the form $x_n = aw_n b$ with $(a,b) \in L_{\frac{p}{1-\theta}}(\mathcal{M}) \times L_{\frac{p}{\theta}}(\mathcal{M})$ and $(w_n)_{n\ge1}$ uniformly bounded in $\mathcal{M}$.

If $\theta = \frac{1}{2}$, then $L_p(\mathcal{M}, \ell_\infty^\theta)$ is the usual $L_p(\mathcal{M}, \ell_\infty)$ introduced in [33] and [21, p. 173]. We denote $L_p(\mathcal{M}, \ell_\infty^\theta)$ by $L_p(\mathcal{M}, \ell_\infty^c)$ (resp. $L_p(\mathcal{M}, \ell_\infty^r)$) if $\theta = 1$ (resp. $\theta = 0$). The first form of asymmetric Doob's inequality can be found in [21, Corollary 4.6] where Junge established that if $p > 2\max\{\theta, 1-\theta\}$ for $0 \le \theta \le 1$, then there is a constant $C_{p,\theta}$ such that

$$\|(x_n)_{n\ge1}\|_{L_p(\mathcal{M},\ell_\infty^\theta)} \le C_{p,\theta}\|x\|_p, \quad x \in L_p(\mathcal{M}). \tag{1.9}$$

However, estimate (1.9) fails when $p < 2\max\{\theta, 1-\theta\}$ ([8, Example 4.4]). As mentioned in [12, p. 997], in the noncommutative setting, "it seems that the inequality

$$\|(\mathcal{E}_n(x))_{n\ge1}\|_{L_p(\mathcal{M},\ell_\infty^c)} \le C_p\|x\|_{\mathcal{H}_p^c}, \quad 1 \le p \le 2 \tag{1.10}$$

is too good to be true". As a substitute, Hong et al. [12, Theorem A] established weak type forms of (1.10) and also strong forms after arbitrary small perturbations of the asymmetries. In [12], another substitute was suggested. That substitute was stated by using a new version of noncommutative Hardy spaces $\mathcal{H}_{pw}^c(\mathcal{M})$ and $\mathcal{H}_{pw}^r(\mathcal{M})$ (see Definition 2.3) where p and w are scalars such that $1 < p < 2$ and $w \ge 2$. Using this instrument, they proved the following estimates.

Theorem 1.4 ([12, Theorem B]) *Let* $1 < p < 2$. *If* $w > 2$, *then*

$$\Big\|(\mathcal{E}_n(x))_{n\ge1}\Big\|_{L_p(\mathcal{M},\ell_\infty^c)} \le C_{p,w}\|x\|_{\mathcal{H}_{pw}^c}, \qquad x \in \mathcal{H}_{pw}^c(\mathcal{M}),$$

and

$$\Big\|(\mathcal{E}_n(x))_{n\ge1}\Big\|_{L_p(\mathcal{M},\ell_\infty^r)} \le C_{p,w}\|x\|_{\mathcal{H}_{pw}^r}, \qquad x \in \mathcal{H}_{pw}^r(\mathcal{M}).$$

Inspired by Theorem 1.4, we find that similar results hold true for martingale transforms T_ξ. We now state our second main result, which extends Theorem 1.4 and complements Junge's asymmetric Doob inequalitiy [21, Corollary 4.6].

Theorem 1.5 *Let* $1 < p < 2$, *and let* ξ_{n-1} *commute with* $\mathcal{M}_n$ *for every* $n \ge 1$ *and* $\sup_n \|\xi_n\|_\infty \le 1$. *If* $w > 2$, *then*

$$\Big\|((T_\xi x)_n)_{n\ge1}\Big\|_{L_p(\mathcal{M},\ell_\infty^c)} \le C_{p,w}\|x\|_{\mathcal{H}_{pw}^c}, \qquad x \in \mathcal{H}_{pw}^c(\mathcal{M})$$

and

$$\Big\|((T_\xi x)_n)_{n\ge1}\Big\|_{L_p(\mathcal{M},\ell_\infty^r)} \le C_{p,w}\|x\|_{\mathcal{H}_{pw}^r}, \qquad x \in \mathcal{H}_{pw}^r(\mathcal{M}).$$

Finally, we explain the third main objective of this paper. It consists in showing that in fact the estimates (1.10) holds in the special case when the martingale transform is given by the noncommutative fractional integral operator I^α $(0 < \alpha < 1)$. Recall that

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the operator I^α $(0 < \alpha < 1)$, a special kind of martingale transform, was studied by Randrianantoanina and Wu [39], where I^α $(0 < \alpha < 1)$ is defined by setting, for a finite martingale $x = (x_k)_{1\le k\le n}$ (for convenience, $x_0 = 0$),

$$I^\alpha x = \sum_{k=1}^{n} \zeta_k^\alpha d_k x$$

for an appropriate scalar sequence $(\zeta_k)_{k\ge1}$ (see Section 2 for details). Now we state our third main result as follows.

Theorem 1.6 *Let $\mathcal{M}$ be a hyperfinite and finite von Neumann algebra. Let $1 \le p < 2$ and $0 < \alpha < \frac{1}{p}$, and let $\frac{1}{q} = \frac{1}{p} - \alpha$. Then*

$$\big\| \big((I^\alpha x)_n\big)_{n\ge1} \big\|_{L_q(\mathcal{M},\ell_\infty^c)} \le C_{\alpha,p} \|x\|_{\mathcal{H}_p^c}, \quad x \in \mathcal{H}_p^c(\mathcal{M}),$$

and

$$\big\| \big((I^\alpha x)_n\big)_{n\ge1} \big\|_{L_q(\mathcal{M},\ell_\infty^r)} \le C_{\alpha,p} \|x\|_{\mathcal{H}_p^r}, \quad x \in \mathcal{H}_p^r(\mathcal{M}).$$

The paper is organized as follows. In next section, we collect definitions, notation, and lemmas from noncommutative martingale theory. In particular, we will give a concrete example of hyperfinite finite von Neumann algebra $\mathcal{R}$ (hyperfinite factor II_1) with increasing regular filtration $(\mathcal{R})_{n\ge1}$. We present the proofs of Theorems 1.2 and 1.5 in Section 3. Section 4 is devoted to the fractional integrals I^α. More precisely, Theorem 1.6 is proved in Section 4.1, and in Section 4.2 (see Theorems 4.3 and 4.4), we show some new noncommutative maximal inequalities for "fractional Doob operator".

Throughout the paper, the symbol C_p is a constant that only depends on p and can vary from line to line; we denote by p' the conjugate index of p; for two Banach spaces $(X_1, \|\cdot\|_{X_1})$ and $(X_2, \|\cdot\|_{X_2})$, the notation $X_1 \simeq X_2$ means that X_1 and X_2 are isomorphic.

2 Preliminaries

This sections contains definitions, notation, and technical results that are used throughout the text.

2.1 Noncommutative Martingales and Spaces

Throughout, let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal faithful normalized trace τ $(\tau(1) = 1)$. Denote by $L_0(\mathcal{M}, \tau)$ the space of τ-measurable operators. For $0 < p \le \infty$, let $L_p(\mathcal{M}, \tau)$ (simply $L_p(\mathcal{M})$) be the associated noncommutative L_p-space (see [35, p. 1463]). When $p = \infty$, $L_\infty(\mathcal{M})$ is just $\mathcal{M}$ with the usual operator norm. We refer the reader to [35] for more information about the noncommutative Lebesgue spaces. Let $(\mathcal{M}_n)_{n\ge1}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that $\cup_{n\ge1}\mathcal{M}_n$ is weak*-dense in $\mathcal{M}$. Let $\mathcal{E}_n$ be the conditional expectation (the existence of $\mathcal{E}_n$ is referred to [44, Proposition 2.36]) of $\mathcal{M}$ onto $\mathcal{M}_n$. A sequence $x = (x_n)_{n\ge1}$ in $L_1(\mathcal{M})$ is called a *noncommutative martingale* with respect to $(\mathcal{M}_n)_{n\geq 1}$ if

$$\mathcal{E}_n(x_{n+1}) = x_n, \qquad \forall n \geq 1.$$

If in addition, all the x_n's are in $L_p(\mathcal{M})$ for some $1 \leq p \leq \infty$, x is called an L_p-*martingale*. In this case, we set

$$\|x\|_p = \sup_{n\geq 1} \|x_n\|_p.$$

If $\|x\|_p < \infty$, x is called an L_p-*bounded martingale*. For $1 < p < \infty$, x_n converges to an element x_∞ in $L_p(\mathcal{M})$, and $x_n = \mathcal{E}_n(x_\infty)$ for every $n \geq 1$ (see [23, p. 961, Remark] or [?, Proposition 3.1.9]). As usual, we often identify a martingale with its final value, whenever the latter exists.

Recall that $(\mathcal{M}_n)_{n\geq 1}$ is a regular filtration if there exists a positive number $R_0 \geq 1$ such that

$$\mathcal{E}_n(x) \leq R_0 \mathcal{E}_{n-1}(x) \tag{2.1}$$

for each positive x. R_0 is usually called the regularity constant. The martingale difference sequence $(d_k x)_{k\geq 1}$ is defined by (with convenience $d_1 x = x_1$)

$$d_k x = x_k - x_{k-1}, \qquad \forall k \geq 2.$$

For every $x \in L_0(\mathcal{M})$, $|x| = (x^* x)^{1/2}$. If $x \in L_0(\mathcal{M})$ and $x = \int_{\mathbb{R}} s de_s^x$ is its spectral decomposition, then for any Borel subset $B \subseteq \mathbb{R}$, we denote by $\chi_B(x)$ the corresponding spectral projection $\int_{\mathbb{R}} \chi_B(s) de_s^x$, where χ_B is the characteristic function of B.

The noncommutative weak L_p-space, denoted by $L_{p,\infty}(\mathcal{M})$, is defined as the collection of all $x \in L_0(\mathcal{M})$ for which the quasi-norm (we also refer the reader to [26, p. 187] for more details)

$$\|x\|_{L_{p,\infty}} := \sup_{\lambda > 0} \lambda \tau(\chi_{(\lambda,\infty)}(|x|))$$

is finite.

We recall the definition of noncommutative martingale Hardy spaces introduced in [34].

Definition 2.1 For $1 \leq p < \infty$, we define the Hardy space $\mathcal{H}_p^c(\mathcal{M})$ (resp. $\mathcal{H}_p^r(\mathcal{M})$) as the collection of all martingales $x = (x_n)_{n\geq 1}$ in $L_p(\mathcal{M})$ with finite norm

$$\|x\|_{\mathcal{H}_p^c} = \Big\| \Big(\sum_{n=1}^{\infty} |d_n x|^2 \Big)^{\frac{1}{2}} \Big\|_p \quad \Big(\text{resp. } \|x\|_{\mathcal{H}_p^r} = \Big\| \Big(\sum_{n=1}^{\infty} |(d_n x)^*|^2 \Big)^{\frac{1}{2}} \Big\|_p \Big).$$

Let $(X_1, \|\cdot\|_{X_1})$ and $(X_2, \|\cdot\|_{X_2})$ be two Banach spaces such that they are embedded into a Hausdorff topological vector space. Denote by $X_1 + X_2$ (resp. $X_1 \cap X_2$) the space of their sum (resp. intersection) equipped with the usual norm (see [4, p. 25]). The noncommutative Burkholder–Gundy inequalities can be stated as follows (see [34, Theorem 2.1]):

$$L_p(\mathcal{M}) \simeq \mathcal{H}_p^c + \mathcal{H}_p^r =: \mathcal{H}_p, \quad 1 < p < 2 \tag{2.2}$$

and

$$L_p(\mathcal{M}) \simeq \mathcal{H}_p^c \cap \mathcal{H}_p^r =: \mathcal{H}_p, \quad 2 \leq p < \infty. \tag{2.3}$$

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We also recall the conditional version of noncommutative martingale Hardy spaces developed by Junge and Xu in [23].

Definition 2.2 For $1 \le p < \infty$, we define the conditional Hardy space $h_p^c(\mathcal{M})$ (resp. $h_p^r(\mathcal{M})$) as the collection of all martingales $x = (x_n)_{n\ge1}$ in $L_p(\mathcal{M})$ with finite norm

$$\|x\|_{h_p^c} = \Big\| \Big(\sum_{n=1}^{\infty} \mathcal{E}_{n-1}|d_n x|^2 \Big)^{\frac{1}{2}} \Big\|_p \left(\text{resp. } \|x\|_{h_p^r} = \Big\| \Big(\sum_{n=1}^{\infty} \mathcal{E}_{n-1}|(d_n x)^*|^2 \Big)^{\frac{1}{2}} \Big\|_p \right).$$

The algebraic atomic decomposition of the conditional Hardy spaces $h_p^c(\mathcal{M}) \simeq h_{p2}^c(\mathcal{M})$ was introduced in [22], where $h_{p2}^c(\mathcal{M})$ is the so-called algebraic atomic Hardy space introduced below with $w = 2$. Here, inspired by [12, p. 1008], we suggest the following definition of (generalized algebraic) atomic Hardy spaces.

Definition 2.3 Let $1 \le p \le 2$ and $w, s \ge 2$ such that $1/p = 1/w + 1/s$. Define

$$h_{pw}^c(\mathcal{M}) = \big\{ x \in L_0(\mathcal{M}, \tau) : \|x\|_{h_{pw}^c} < \infty \big\},$$
$$h_{pw}^{1_c}(\mathcal{M}) = \big\{ x \in L_0(\mathcal{M}, \tau) : \|x\|_{h_{pw}^{1_c}} < \infty \big\},$$

where

$$\|x\|_{h_{pw}^c} = \inf_{\substack{x=\sum_n a_n b_n \\ \mathcal{E}_n(a_n)=0,\, b_n\in L_s(\mathcal{M}_n)}} \Big\| \sum_{n=1}^{\infty} a_n \otimes e_{1,n} \Big\|_w \Big\| \sum_{n=1}^{\infty} b_n \otimes e_{n,1} \Big\|_s,$$

$$\|x\|_{h_{pw}^{1_c}} = \inf_{\substack{x=\sum_n d_n(a_n b_n) \\ a_n\in L_w(\mathcal{M}),\, b_n\in L_s(\mathcal{M})}} \Big\| \sum_{n=1}^{\infty} a_n \otimes e_{1,n} \Big\|_w \Big\| \sum_{n=1}^{\infty} b_n \otimes e_{n,1} \Big\|_s.$$

The analogous families of row Hardy spaces h_{pw}^r and $h_{pw}^{1_r}$ are defined by taking adjoint as usual. Given $1 \le p < 2$ and $w \ge 2$, we define

$$\mathcal{H}_{pw}^c(\mathcal{M}) = h_{pw}^c(\mathcal{M}) + h_{pw}^{1_c}(\mathcal{M}) \quad \text{and} \quad \mathcal{H}_{pw}^r(\mathcal{M}) = h_{pw}^r(\mathcal{M}) + h_{pw}^{1_r}(\mathcal{M}).$$

From [12, Lemma 3.3], we know that $\|\cdot\|_{h_{pw}^c}$ is a norm if $w \ge 2$. Recall that $h_p^d(\mathcal{M})$ $(1 \le p < \infty)$ is the space of all martingales $x = (x_n)_{n\ge1} \in L_p(\mathcal{M})$ such that the norm

$$\|x\|_{h_p^d} = \Big(\sum_{n=1}^{\infty} \|d_n x\|_p^p \Big)^{\frac{1}{p}}$$

is finite. As proved in [22, Remark 5.8], we know that $h_{p2}^{1_c}(\mathcal{M}) \subset h_p^d(\mathcal{M})$ for $1 \le p < 2$. And we can see from [22, Theorems 5.1 and 5.7] that the space $h_{p2}^{1_c}(\mathcal{M})$ plays similar role as $h_p^d(\mathcal{M})$.

By [12, Remark 3.2 and Theorem 2.1], we have the following algebraic Davis decomposition.

Theorem 2.4 *Let* $1 \le p < 2$. *We have*

$$h_p^c(\mathcal{M}) \simeq h_{p2}^c(\mathcal{M})$$

and

$$\mathcal{H}_p^c(\mathcal{M}) \simeq \mathcal{H}_{p2}^c(\mathcal{M}) = h_{p2}^c(\mathcal{M}) + h_{p2}^{1_c}(\mathcal{M}).$$

Similar results hold for the row spaces.

Remark 2.5 Let $1 \le p < 2$. Then $h_{p2}^{1_c}(\mathcal{M}) = h_{p2}^{1_r}(\mathcal{M})$. By [12, Lemma 3.9], for $1 < p < 2$ and $w \ge 2$, we have

$$\mathcal{H}_{pw}^c(\mathcal{M}) \subset \mathcal{H}_p^c(\mathcal{M}), \quad \mathcal{H}_{pw}^r(\mathcal{M}) \subset \mathcal{H}_p^c(\mathcal{M}).$$

Moreover, by [12, Theorem Bi], we have

$$L_p(\mathcal{M}) \simeq \mathcal{H}_{pw}^c(\mathcal{M}) + \mathcal{H}_{pw}^r(\mathcal{M}), \quad 1 < p < 2, w \ge 2.$$

2.2 Noncommutative Fractional Integrals

In this subsection, we recall fractional integrals for noncommutative martingales introduced in [39, Section 2]. Assume that $\mathcal{M}$ is a hyperfinite and finite von Neumann algebra and the filtration $(\mathcal{M}_k)_{k\ge1}$ consists of finite dimensional von Neumann subalgebras of $\mathcal{M}$.

Since $\dim(\mathcal{M}_k) < \infty$, the $L_p(\mathcal{M}_k)$'s are finite dimensional subspaces of $L_p(\mathcal{M})$ for all $1 \le p \le \infty$. Moreover, for $p \ne q$, the two spaces $L_p(\mathcal{M}_k)$ and $L_q(\mathcal{M}_k)$ coincide as sets. In particular, the formal identity $\iota_k \colon L_\infty(\mathcal{M}_k) \to L_2(\mathcal{M}_k)$ forms a natural isomorphism between the two spaces.

For $k \ge 1$, set

$$\zeta_k := 1/\|\iota_k^{-1}\|^2. \tag{2.4}$$

Clearly, $0 < \zeta_k \le 1$ for all $k \ge 1$ and $\lim_{k\to\infty} \zeta_k = 0$. Moreover, for every $x \in L_2(\mathcal{M}_k)$, we have

$$\|x\|_\infty \le \zeta_k^{-1/2}\|x\|_2. \tag{2.5}$$

Furthermore, one can easily verify that for every $x \in L_1(\mathcal{M}_k)$, $(\iota_k^{-1})^*(x) = x \in L_2(\mathcal{M}_k)$ such that

$$\|x\|_2 \le \zeta_k^{-1/2}\|x\|_1. \tag{2.6}$$

Our primary example of a hyperfinite and finite von Neumann algebra the hyperfinite type II_1 factor $\mathcal{R}$ and $(m(k))_{k=1}^\infty \subset \mathbb{N}$ such that $m(k) \ge 2$ for every $k \ge 1$. Set $M_n = \prod_{k=1}^n m(k)$. Denote by $\mathbb{M}_n$ the space of $n \times n$ complex valued matrices with usual normalised trace tr_n satisfying $\mathrm{tr}_n(1_n) = 1$, where 1_n is the $n \times n$ identity matrix. We identify $\mathcal{R}$ with the relative infinite tensor product

$$(\mathcal{R}, \tau) = \bigotimes_{k=1}^\infty (\mathbb{M}_{m(k)}, \mathrm{tr}_{m(k)}).$$

Note that such τ is a faithful normal trace on $\mathcal{R}$. Consider the von Neumann subalgebras of $\mathcal{R}$ defined by setting

$$(\mathcal{R}_n, \tau_n) = \bigotimes_{k=1}^n (\mathbb{M}_{m(k)}, \mathrm{tr}_{m(k)}).$$

In fact, we view $\mathcal{R}_n$ as a von Neumann subalgebra of $\mathcal{R}_{n+1}$ (resp. $\mathcal{R}$) via the inclusion

$$x \in \mathcal{R}_n \longmapsto x \otimes 1_{m(n+1)} \in \mathcal{R}_{n+1} \quad \Big(\text{resp. } x \otimes \Big(\bigotimes_{k=n+1}^{\infty} 1_{m(k)} \Big) \in \mathcal{R} \Big).$$

The conditional expectation $\mathcal{E}_n : \mathcal{R} \to \mathcal{R}_n$ is given by

$$\mathcal{E}_n = \Big(\bigotimes_{k=1}^{n} 1_{m(k)} \Big) \otimes \Big(\bigotimes_{k=n+1}^{\infty} \mathrm{tr}_{m(k)} \Big).$$

Then we can see that the filtration $(\mathcal{R}_n)_{n=1}^{\infty}$ is increasing and $\bigcup_n \mathcal{R}_n$ is weak-$*$ dense in $\mathcal{R}$ according to the definition of infinite tensor product.

Now the sequence $(\zeta_n)_{n=1}^{\infty}$ with respect to the filtration $(\mathcal{R}_n)_{n=1}^{\infty}$ is $(M_n^{-1})_{n=1}^{\infty}$, that is,

$$\zeta_n = \frac{1}{\prod_{k=1}^{n} m(k)}, \quad n \geq 1. \tag{2.7}$$

In particular, if $m(k) = 2$ for each $k \in \mathbb{N}$, then $\zeta_n = 2^{-n}$, $n \geq 1$. In this case, the filtration $(\mathcal{R}_n)_{n=1}^{\infty}$ is regular (see [45] or [20]), and martingales corresponding to $(\mathcal{R}_n)_{n=1}^{\infty}$ are called *noncommutative dyadic martingales.* By the way, the martingale transform of noncommutative dyadic martingales was proved to be strong type (p,p) $(1 < p < \infty)$ by Ferleger and Sukochev [43]. On the other hand, it is also shown in [41, Lemma 3.3] that $(\mathcal{R}_n)_{n=1}^{\infty}$ is regular for general $(m(k))_{n=1}^{\infty}$ with $\sup_k m(k) < \infty$ (the regularity constan R_0 does not exceed $(\sup_k m(k) + 1)!$).

Now we give the definition of fractional integrals.

Definition 2.6 ([39, Definition 2.1]) Assume that $\mathcal{M}$ is a hyperfinite and finite von Neumann algebra. For a given noncommutative martingale $x = (x_n)_{n\geq 1}$ and $0 < \alpha < 1$, we define the fractional integral of order α of x to be the sequence $I^{\alpha}x = \{(I^{\alpha}x)_n\}_{n\geq 1}$ where for every $n \geq 1$,

$$(I^{\alpha}x)_n = \sum_{k=1}^{n} \zeta_k^{\alpha} d_k x$$

with the sequence of scalars $(\zeta_k)_{k\geq 1}$ from (2.4).

We need several lemmas for the proofs of Theorem 1.6 and other results in Section 4.

Lemma 2.7 *Let $0 < \alpha, \alpha_0 < 1$. Assume that $\mathcal{M}$ is a hyperfinite and finite von Neumann algebra. If $x \in L_1(\mathcal{M}_k)$, then*

$$\zeta_k^{\alpha_0} \|x\|_q \leq \|x\|_1, \quad 1 - \frac{1}{q} = \alpha_0.$$

Moreover, if $x \in L_p(\mathcal{M}_k)$ for $1 < p < \infty$, then

$$\zeta_k^{\alpha} \|x\|_q \leq \|x\|_p, \quad \frac{1}{p} - \frac{1}{q} = \alpha.$$

Proof The proof is simple. Let $x \in L_1(\mathcal{M}_k)$. By (2.5) and (2.6), we have

$$\zeta_k^{\alpha_0}\|x\|_q = \zeta_k^{\alpha_0}\tau(|x|^q)^{\frac{1}{q}} = \zeta_k^{\alpha_0}\tau(|x|^{q-1}|x|)^{\frac{1}{q}} \leq \zeta_k^{\alpha_0}\|x\|_\infty^{\frac{q-1}{q}}\|x\|_1^{\frac{1}{q}} \leq \zeta_k^{\alpha_0}\zeta_k^{-\alpha_0}\|x\|_1,$$

which finishes the proof of first inequality.

Now take $x \in L_p(\mathcal{M}_k)$. Observing that $q > p$ and using the proved inequality above, we get

$$\zeta_k^{\alpha}\|x\|_q = \zeta_k^{\alpha}\||x|^p\|_{q/p}^{1/p} \leq \zeta_k^{\alpha}\Big(\zeta_k^{-1+p/q}\||x|^p\|_1\Big)^{1/p} = \zeta_k^{\alpha}\zeta_k^{-1/p+1/q}\|x\|_p = \|x\|_p.$$

The proof is complete. ∎

Lemma 2.8 ([39, Theorem 2.9]) *Let* $1 < p < q < \infty$ *and* $\alpha = 1/p - 1/q$. *Assume that* $\mathcal{M}$ *is a hyperfinite and finite von Neumann algebra. Then there exists a constant* C_α *such that for every* $x \in L_p(\mathcal{M})$,

$$\Big\|\Big((I^\alpha x)_n\Big)_{n\geq 1}\Big\|_{L_q(\mathcal{M},\ell_\infty^{1/2})} \leq C_{\alpha,p}\|x\|_p.$$

Lemma 2.9 ([39, Theorem 2.11]) *Let* $0 < \alpha < 1$. *Assume that* $\mathcal{M}$ *is a hyperfinite and finite von Neumann algebra. Then there exists a constant* C_α *such that for every* $x \in \mathcal{H}_1^c(\mathcal{M})$,

$$\Big\|\Big((I^\alpha x)_n\Big)_{n\geq 1}\Big\|_{\mathcal{H}^c_{1/(1-\alpha)}} \leq C_\alpha\Big\|x\Big\|_{\mathcal{H}_1^c}.$$

3 Martingale Transforms

In this section, we prove our main theorems of martingale transforms including Theorems 1.2 and 1.5. The main tool in the proof of Theorem 1.2 is noncommutative Gundy's decomposition from [31]. We begin with the so-called Cuculescu projections that are now well known in this field.

Lemma 3.1 ([7] or [31, Proposition 1.4]) *If* $x = (x_n)_{n\geq 1}$ *is a positive* L_1*-bounded martingale and* $\lambda > 0$, *then there exists a sequence of decreasing projections* $(q_n^{(\lambda)})_{n\geq 1}$ *in* $\mathcal{M}$ *satisfying the following properties:*

(i) *for every* $n \geq 1$, $q_n^{(\lambda)} \in \mathcal{M}_n$;
(ii) *for every* $n \geq 1$, $q_n^{(\lambda)}$ *commutes with* $q_{n-1}^{(\lambda)}x_n q_{n-1}^{(\lambda)}$;
(iii) *for every* $n \geq 1$, $q_n^{(\lambda)}x_n q_n^{(\lambda)} \leq \lambda q_n^{(\lambda)}$;
(iv) *if we set* $q^\lambda = \wedge_{n=1}^\infty q_n^{(\lambda)}$, *then* $\tau(1 - q^{(\lambda)}) \leq \|x\|_1/\lambda$.

In what follows, for a fixed $\lambda > 0$, we will simply write $(q_n)_{n\geq 1}$ for the sequence of Cuculescu's projections $(q_n^{(\lambda)})_{n\geq 1}$ associated with the martingale $x = (x_n)_{n\geq 1} \in L_1^+(\mathcal{M})$ and q for the corresponding q^λ. Set $p_n = q_{n-1} - q_n$ for $n \geq 1$. For $\lambda > 0$ and positive martingale $x = (x_n)_{n\geq 1} \in L_1(\mathcal{M})$, the projections $(q_n)_{n\geq 1}$ are usually defined by induction: $q_0 = 1$,

$$q_n = q_{n-1}\chi_{[0,\lambda]}(q_{n-1}x_n q_{n-1}), \quad n \geq 1.$$

Observe that it follows immediately from the definition that $q_n \leq q_{n-1}$ for all $n > 1$.

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We recall the noncommutative Gundy decomposition.

Lemma 3.2 ([31, Theorem 2.4]) *Let $x = (x_n)_{n\geq 1} \in L_1(\mathcal{M})$ be a positive L_1-bounded martingale and let $\lambda > 0$. Then there exist four bounded L_1-martingales y, z, v, and w satisfying the following properties:*

(i) $x = y + z + v + w$ *with*

$$d_n y = q_n d_n x q_n - \mathcal{E}_{n-1}(q_n d_n x q_n), \quad d_n z = q_{n-1} d_n x q_{n-1} - d_n y,$$
$$d_n v = q_{n-1} d_n x (1 - q_{n-1}), \qquad d_n w = (1 - q_{n-1}) d_n x,$$

where (q_n) are the projections associated with x and λ given by Lemma 3.1;

(ii) $\|y\|_1 \leq 8\|x\|_1$ *and* $\|y\|_2^2 \leq 6\lambda\|x\|_1$;

(iii) $\sum_{n\geq 1} \|d_n z\|_1 \leq 6\|x\|_1$;

(iv) $\max\{\tau(\bigvee_n \operatorname{supp}(d_n v)), \tau(\bigvee_n \operatorname{supp}(d_n w))\} \leq \lambda^{-1}\|x\|_1$.

Lemma 3.3 *Use the same assumption as Lemma 3.2. In addition, if the filtration $(\mathcal{M}_n)_{n\geq 1}$ is regular, then*

$$\sup_{n\geq 1} \|d_n z\|_\infty \leq (6 + 2R_0)\lambda,$$

where z and λ are as in Lemma 3.2, R_0 is the regularity constant mentioned in Section 2. Moreover, we have

$$\|z\|_2^2 \leq 6(6 + 2R_0)\lambda\|x\|_1.$$

Proof By Lemma 3.2, we have $d_n z = q_{n-1} d_n x q_{n-1} - d_n y$. Set

$$\alpha_n = q_{n-1} x_n q_{n-1} - q_n x_n q_n \quad \text{and} \quad \beta_n = q_{n-1} x_{n-1} q_{n-1} - q_n x_{n-1} q_n.$$

Noting that

$$\mathcal{E}_{n-1}(q_n x_n q_n) = \mathcal{E}_{n-1}(q_n x_n q_n - q_{n-1} d_n x q_{n-1}),$$

we have

$$d_n z = \alpha_n - \beta_n - \mathcal{E}_{n-1}(\alpha_n - \beta_n).$$

By Lemma 3.1(iii) and regularity, for each $n \geq 1$, we obtain

$$\begin{aligned}
\|d_n z\|_\infty &\leq 2\|\alpha_n - \beta_n\|_\infty \\
&\leq 4\lambda + 2\|q_{n-1} x_n q_{n-1}\|_\infty + 2\|q_n x_{n-1} q_n\|_\infty \\
&\overset{(2.1)}{\leq} 4\lambda + 2R_0\|q_{n-1} x_{n-1} q_{n-1}\|_\infty + 2\|q_n q_{n-1} x_{n-1} q_{n-1} q_n\|_\infty \\
&\leq 6\lambda + 2R_0\lambda,
\end{aligned}$$

where R_0 is the regularity constant appearing in (2.1) and the property $q_n \leq q_{n-1}$ is used. Finally, by Lemma 3.2(iii),

$$\|z\|_2^2 = \sum_{n\geq 1}^{\infty} \|d_n z\|_2^2 \leq \sum_{n\geq 1}^{\infty} \|d_n z\|_\infty \|d_n z\|_1 \leq 6(6 + 2R_0)\lambda\|x\|_1. \qquad \blacksquare$$

Recall an important property: if the sequence $(x_n)_{n\geq 1} \subset L_p(\mathcal{M})$ is positive, then we have (see [?, p. 111] or [24, p. 392])

$$\|(x_n)_{n\geq 1}\|_{L_p(\mathcal{M}, \ell_\infty^{1/2})} = \inf\{\|a\|_{L_p(\mathcal{M})} : x_n \leq a, \forall n \geq 1\}. \tag{3.1}$$

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2 We use Cuculescu projections introduced in Lemma 3.1. For $j \geq 1$ and fixed $k \geq 1$, define

$$\pi_{0,k} = \bigwedge_{i\geq 0} q_k^{(2^i)} \qquad \text{and} \qquad \pi_{j,k} = \bigwedge_{i\geq j} q_k^{(2^i)} - \bigwedge_{i\geq j-1} q_k^{(2^i)}.$$

Note that $\sum_{j\geq 0} \pi_{j,k} = 1$ in the sense of strong operator topology for every fixed $k \geq 1$ (see [37, Proposition 1.4]). We deduce that for every fixed $k \geq 1$,

$$d_k x = \sum_{i<j} \pi_{i,k-1} d_k x \pi_{j,k-1} + \sum_{i\geq j} \pi_{i,k-1} d_k x \pi_{j,k-1} =: \Delta^c_{k-1}(d_k x) + \Delta^r_{k-1}(d_k x).$$

It is obvious that $(\pi_{j,k})_j$ are mutually orthogonal projections for every fixed k. Then the operators Δ^c_{k-1} and Δ^r_{k-1} are actually triangular truncations studied in [11]. It follows from Lemma 3.1(i) that $q_{k-1}^{(2^i)} \in \mathcal{M}_{k-1}$ for every i, and consequently for each j and k, $\pi_{j,k-1} \in \mathcal{M}_{k-1}$, we know that $\Delta^c_{k-1}(d_k x)$ and $\Delta^r_{k-1}(d_k x)$ are still martingale differences. Set $x = x^c + x^r$ with

$$x^c = \sum_{k\geq 1} \Delta^c_{k-1}(d_k x) \quad \text{and} \quad x^r = \sum_{k\geq 1} \Delta^r_{k-1}(d_k x).$$

Fix $\lambda = 2^\ell$ for some $\ell \in \mathbb{Z}$. By Lemma 3.2, for fixed $\lambda > 0$, we get the Gundy decomposition $x = y + z + v + w$. Then

$$\begin{aligned} x^c &= \sum_{k\geq 1} \Delta^c_{k-1}(d_k y) + \sum_{k\geq 1} \Delta^c_{k-1}(d_k z) + \sum_{k\geq 1} \Delta^c_{k-1}(d_k v) + \sum_{k\geq 1} \Delta^c_{k-1}(d_k w) \\ &=: y^c + z^c + v^c + w^c. \end{aligned}$$

Taking into the account that the arguments for row and columns are totally similar, in order to finish the proof, we only need to prove that there is a constant $K > 0$ such that

$$\big\| \big((T^c_\xi x^c)_n \big)_{n\geq 1} \big\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty)} \leq K \|x\|_1.$$

To this end, according to the definition of $\Lambda_{1,\infty}(\mathcal{M}, \ell_\infty)$ (see (1.2)), it suffices to show that for every fixed $\lambda > 0$, we have

(I) $\inf_{e\in\mathcal{P}(\mathcal{M})}\{\lambda\tau(1-e) : \|e\mathcal{E}_n(T^c_\xi y^c)e\|_\infty \leq \lambda, \quad \forall n \geq 1\} \leq C\|x\|_1$;
(II) $\inf_{e\in\mathcal{P}(\mathcal{M})}\{\lambda\tau(1-e) : \|e\mathcal{E}_n(T^c_\xi z^c)e\|_\infty \leq \lambda, \quad \forall n \geq 1\} \leq C\|x\|_1$;
(III) $\inf_{e\in\mathcal{P}(\mathcal{M})}\{\lambda\tau(1-e) : \|e\mathcal{E}_n(T^c_\xi v^c)e\|_\infty \leq \lambda, \quad \forall n \geq 1\} \leq C\|x\|_1$;
(IV) $\inf_{e\in\mathcal{P}(\mathcal{M})}\{\lambda\tau(1-e) : \|e\mathcal{E}_n(T^c_\xi w^c)e\|_\infty \leq \lambda, \quad \forall n \geq 1\} \leq C\|x\|_1$.

We first prove (I). The condition $\sup_n \|\xi_n\|_\infty \leq 1$ implies that the martingale transforms T^c_ξ and T^r_ξ are both bounded in $L_2(\mathcal{M})$. Combining the fact that triangular truncations are contractive in $L_2(\mathcal{M})$ (see [10] or [11]) and Lemma 3.2(ii), we deduce that

$$\begin{aligned} \|T^c_\xi y^c\|_2^2 \leq \|y^c\|_2^2 &= \sum_{k\geq 1} \|\Delta^c_{k-1}(d_k y)\|_2^2 \leq C \sum_{k\geq 1} \|d_k y\|_2^2 \\ &= C\|y\|_2^2 \leq 6C\lambda\|x\|_1. \end{aligned}$$

We now decompose $T^c_\xi y^c$ into the combination of four positive elements:

$$T^c_\xi y^c = h_1 - h_2 + ih_3 - ih_4$$

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such that

$$\|h_j\|_2 \le \|T_\xi^c y^c\|_2, \quad j \in \{1,2,3,4\}.$$

Actually, we can take h_1 (resp. h_3) as the positive part of $\mathrm{Re}(T_\xi^c y^c)$ (resp. $\mathrm{Im}(T_\xi^c y^c)$), and take h_2 (resp. h_4) as the positive part of $\mathrm{Re}(T_\xi^c y^c)$ (resp. $\mathrm{Im}(T_\xi^c y^c)$), where

$$\mathrm{Re}(T_\xi^c y^c) = \frac{T_\xi^c y^c + (T_\xi^c y^c)^*}{2} \quad \text{and} \quad \mathrm{Im}(T_\xi^c y^c) = \frac{T_\xi^c y^c - (T_\xi^c y^c)^*}{2i}.$$

By the Doob inequality (1.3), there is a constant $C > 0$ such that

$$\left\| \left(\mathcal{E}_n(h_j) \right)_{n\ge1} \right\|_{L_2(\mathcal{M},\ell_\infty^{1/2})} \le C\|h_j\|_2$$

for every $j \in \{1,2,3,4\}$. Then, by (3.1), there exist positive elements a_j satisfying that for every $n \ge 1$,

$$\mathcal{E}_n(h_j) \le a_j, \quad \|a_j\|_2 \le C\|h_j\|_2 \le C\|T_\xi^c y^c\|_2.$$

Set

$$e_j = \chi_{(0,\frac{\lambda}{4})}(a_j) \quad \text{and} \quad e_\lambda = \bigwedge_{j=1}^{4} e_j.$$

Then, for every $j \in \{1,2,3,4\}$,

$$e_j \mathcal{E}_n(h_j) e_j \le e_j a_j e_j = a_j \chi_{(0,\frac{\lambda}{4})}(a_j) \le \frac{\lambda}{4},$$

which implies that

$$\|e_\lambda \mathcal{E}_n(T_\xi^c y^c) e_\lambda\|_\infty \le \sum_{j=1}^{4} \|e_j \mathcal{E}_n(h_j) e_j\|_\infty \le \lambda.$$

Now, for the projection e_λ, by the Chebyshev inequality, we have

$$\begin{aligned}\lambda\tau(1-e_\lambda) &\le \sum_{j=1}^{4} \lambda\tau(1-e_j) = \sum_{j=1}^{4} \lambda\tau(\chi_{[\frac{\lambda}{4},\infty)}(a_j)) \\ &\le \sum_{j=1}^{4} \frac{16}{\lambda}\|a_j\|_2^2 \le \frac{64C}{\lambda}\|T_\xi^c y^c\|_2^2 \le 384C\|x\|_1.\end{aligned}$$

This finishes the proof of (I).

Comparing Lemma 3.3 and Lemma 3.2(ii), we find that the "z-part" plays a similar role to that of the "y-part". Then the proof of (II) can be finished by applying Lemma 3.3 and similar argument used in the proof of (I). This is the only place where we have used the regularity assumption (by referring to Lemma 3.3) in the whole proof of the theorem.

Now we turn to the proofs of (III) and (IV). Set

$$\widehat{q} = \bigwedge_{s\ge\ell} \bigwedge_{k\ge1} q_k^{(2^s)},$$

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recalling that $\ell \in \mathbb{Z}$ is given by the equality $\lambda = 2^\ell$. By Lemma 3.1(iv),

$$\lambda\tau(1-\widehat{q}) \le \lambda \sum_{s\ge\ell} \tau\Big(1 - \bigwedge_{k\ge1} q_k^{(2^s)}\Big) \le \lambda \sum_{s\ge\ell}\sum_{k\ge1} \tau(1-q_k^{(2^s)})$$
$$\le \lambda \sum_{s\ge\ell} \frac{1}{2^s}\|x\|_1 = \|x\|_1.$$

Then, by the definition of $\Lambda_{1,\infty}(\mathcal{M},\ell_\infty)$, it remains to show

$$\|\widehat{q}\mathcal{E}_n(T_\xi^c v^c)\widehat{q}\|_\infty \le \lambda, \qquad \forall n \ge 1$$

and

$$\|\widehat{q}\mathcal{E}_n(T_\xi^c w^c)\widehat{q}\|_\infty \le \lambda, \qquad \forall n \ge 1.$$

Actually, we will prove below that $\mathcal{E}_n(T_\xi^c v^c)\widehat{q} = 0$ and $\mathcal{E}_n(T_\xi^c w^c)\widehat{q} = 0$. Note that

$$\mathcal{E}_n(T_\xi^c v^c)\widehat{q} = \sum_{k=1}^n \Delta_{k-1}^c(d_k v)\widehat{q} = \sum_{k=1}^n \Delta_{k-1}^c(d_k v)\widehat{q}_{k-1}\widehat{q},$$

where $\widehat{q}_{k-1} = \bigwedge_{s\ge\ell} q_{k-1}^{(2^s)} \ge \widehat{q}$. It suffices to show

$$\Delta_{k-1}^c(d_k v)\widehat{q}_{k-1} = 0 \quad \text{and} \quad \Delta_{k-1}^c(d_k w)\widehat{q}_{k-1} = 0. \tag{3.2}$$

Note that $\bigwedge_{s\ge i} q_{k-1}^{(2^s)} \ge \widehat{q}_{k-1}$ for $i > \ell$. From the definition of $\pi_{i,k-1}$, we have

$$\pi_{i,k-1} = \bigwedge_{s\ge i} q_{k-1}^{(2^s)} - \bigwedge_{s\ge i-1} q_{k-1}^{(2^s)}.$$

Hence,

$$\pi_{i,k-1}\widehat{q}_{k-1} = \widehat{q}_{k-1}\pi_{i,k-1} = 0, \qquad \text{for } i > \ell,\ k \ge 1,$$

which further implies

$$\Delta_{k-1}^c(d_k v)\widehat{q}_{k-1} = \sum_{i<j\le\ell} \pi_{i,k-1} d_k v \pi_{j,k-1}$$

and

$$\Delta_{k-1}^c(d_k w)\widehat{q}_{k-1} = \sum_{i<j\le\ell} \pi_{i,k-1} d_k w \pi_{j,k-1}.$$

Since $d_k v = q_{k-1}^{(2^\ell)} d_k x (1 - q_{k-1}^{(2^\ell)})$ (by Lemma 3.2(i)) and $\pi_{i,k-1} \le q_{k-1}^{(2^\ell)}$ for $i \le \ell$, it follows that

$$\Delta_{k-1}^c(d_k v)\widehat{q}_{k-1} = \sum_{i<j\le\ell} \pi_{i,k-1} q_{k-1}^{(2^\ell)} d_k x (1 - q_{k-1}^{(2^\ell)})\pi_{j,k-1} = 0.$$

Similarly, $\Delta_{k-1}^c(d_k w)\widehat{q}_{k-1} = 0$. Then (3.2) is proved, and the proof is complete. ∎

Remark 3.4 This remark contains two points.

(i) Our Theorem 1.2 improves (1.7). To explain this, it suffices to show the embedding property $\Lambda_{1,\infty}(\mathcal{M},\ell_\infty) \subset L_{1,\infty}(\mathcal{M})$, which implies that the left-hand side of (1.8) is greater than the left-hand side of (1.7). Assume $\mathcal{M}$ is acting on a Hilbert space H. Take a martingale $x = (x_n)_{n\ge1} \in \Lambda_{1,\infty}(\mathcal{M},\ell_\infty)$, $n_0 \in \mathbb{N}$ and $\lambda > 0$. Set $e_{n_0} = \chi_{[0,\lambda]}(|x_{n_0}|)$. Let e be a projection in $\mathcal{M}$ such that $e|x_n|^2 e \le \lambda^2$ for every $n \in \mathbb{N}$.

We claim that $e_{n_0}^{\perp} \wedge e = 0$ where $e_{n_0}^{\perp} = 1 - e_{n_0}$. Suppose that $e_{n_0}^{\perp} \wedge e$ is not zero. Then $e_{n_0}^{\perp} \wedge e(H) \neq \{0\}$. For every nonzero element $\xi \in e_{n_0}^{\perp} \wedge e(H)$, we have

$$\begin{aligned}\langle e_{n_0}^{\perp} \wedge e|x_{n_0}|^2 e_{n_0}^{\perp} \wedge e\xi, \xi\rangle &= \langle e_{n_0}^{\perp} \wedge e(e_{n_0}^{\perp}|x_{n_0}|^2 e_{n_0}^{\perp})e_{n_0}^{\perp} \wedge e\xi, \xi\rangle \\ &> \lambda^2\langle e_{n_0}^{\perp} \wedge e\xi, \xi\rangle = \lambda^2\|\xi\|_H^2.\end{aligned}$$

This is a contradiction to the fact for each $n \in \mathbb{N}$, $e_{n_0}^{\perp} \wedge e|x_n|^2 e e_{n_0}^{\perp} \wedge e = e_{n_0}^{\perp} \wedge e(e|x_n|^2 e)e_{n_0}^{\perp} \wedge e \leq \lambda^2$. Hence, $e_{n_0}^{\perp} \wedge e = 0$. Then $e_{n_0} \vee e^{\perp} = 1$. By [44, p. 292, Proposition 1.6], we arrive at

$$e = e_{n_0} \vee e^{\perp} - e^{\perp} \sim e_{n_0} - e_{n_0} \wedge e^{\perp} \leq e_{n_0},$$

which further implies

$$\sup\left\{\tau(e) : e \in \mathcal{P}(\mathcal{M}), \|e|x_n|^2 e\|_\infty \leq \lambda^2, \forall n \geq 1\right\} \leq \tau(e_{n_0})$$

and

$$\tau(1 - e_{n_0}) \leq \inf\{\tau(1 - e) : e \in \mathcal{P}(\mathcal{M}), \|e|x_n|^2 e\|_\infty^{1/2} \leq \lambda, \forall n \geq 1\}.$$

Note that $\|ex_n e\|_\infty = \|ex_n^* ex_n e\|_\infty^{1/2} \leq \|e|x_n|^2 e\|_\infty^{1/2}$ for each n. Hence,

$$\inf\{\tau(1 - e) : \|e|x_n|^2 e\|_\infty^{1/2} \leq \lambda, \forall n \geq 1\} \leq \inf\{\tau(1 - e) : \|ex_n e\|_\infty \leq \lambda, \forall n \geq 1\}.$$

Thus, for every $\lambda > 0$, we have

$$\lambda\tau(1 - e_{n_0}) \leq \inf\left\{\lambda\tau(1 - e) : e \in \mathcal{P}(\mathcal{M}), \|ex_n e\|_\infty \leq \lambda, \forall n \geq 1\right\}.$$

Since n_0 is arbitrary, the above inequality means $\|x\|_{L_{1,\infty}(\mathcal{M})} \leq \|x\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty)}$ and

$$\Lambda_{1,\infty}(\mathcal{M}, \ell_\infty) \subset L_{1,\infty}(\mathcal{M}).$$

(ii) For the proof of (II) ("z-part") in the proof of Theorem 1.2, if we want to use Lemma 3.2(iii), we have to show $\sum_{k\geq 1}\|\Delta_{k-1}^c(d_k z)\|_1 \leq C\sum_{k\geq 1}\|d_k z\|_1$. However, the triangular truncation Δ_{k-1}^c is just of weak type $(1,1)$ (see [11]). Here, we have employed Lemma 3.3 to avoid such difficulty. We still do not know how to prove Theorem 1.2 without the condition "$(\mathcal{M}_n)_{n\geq 1}$ is regular" (see also [14, Remark 4.1]).

Now we show the proof of Theorem 1.5.

Proof of Theorem 1.5 We only prove the result for column space since the row analog can be proved similarly.

By Theorem 1.4, $\|((T_\xi x)_n)_{n\geq 1}\|_{L_p(\mathcal{M},\ell_\infty^c)} \leq C_{p,w}\|T_\xi x\|_{\mathcal{H}_{pw}^c}$. It suffices to show that the following inequality holds true for every $x \in \mathcal{H}_{pw}^c$,

$$\|T_\xi x\|_{\mathcal{H}_{pw}^c} \leq C\|x\|_{\mathcal{H}_{pw}^c}, \quad 1 < p < 2, \quad w > 2.$$

According to Definition 2.3, we have to prove

$$\|T_\xi x\|_{h_{pw}^c} \leq C\|x\|_{h_{pw}^c} \quad \text{and} \quad \|T_\xi x\|_{h_{pw}^{1_c}} \leq C\|x\|_{h_{pw}^{1_c}}.$$

First, we deal with the left estimate. Given $x \in h^c_{pw}(\mathcal{M})$ with $1/p = 1/w + 1/s$, there exists a decomposition $x = \sum_n a_n b_n$ satisfying

$$\Big\| \sum_{n=1}^{\infty} a_n \otimes e_{1,n} \Big\|_w \Big\| \sum_{n=1}^{\infty} b_n \otimes e_{n,1} \Big\|_s \le (1+\delta) \|x\|_{h^c_{pw}},$$

where $a_n \in L_w(\mathcal{M})$, $\mathcal{E}_n(a_n) = 0$, $b_n \in L_s(\mathcal{M}_n)$ and δ is as small as we wish. Observe that for every $k \ge 1$,

$$d_k x = \mathcal{E}_k\Big(\sum_{n=1}^{k-1} a_n b_n \Big) - \mathcal{E}_{k-1}\Big(\sum_{n=1}^{k-1} a_n b_n \Big) = \sum_{n=1}^{k-1} (d_k a_n) b_n. \tag{3.3}$$

Then

$$T_\xi x = \sum_{k=1}^{\infty} \xi_{k-1} d_k x = \sum_{k=1}^{\infty} \xi_{k-1} \sum_{n=1}^{k-1} (d_k a_n) b_n = \sum_{n=1}^{\infty} \Big(\underbrace{\sum_{k=n+1}^{\infty} \xi_{k-1} d_k a_n}_{\widehat{a}_n} \Big) b_n.$$

It follows from (1.5) that $\|\widehat{a}_n\|_w = \|T_\xi a_n\|_w \le c_w \|a_n\|_w$. And it is easy to check that $\mathcal{E}_n(\widehat{a}_n) = 0$. We have

$$\begin{aligned} \Big\| \sum_{n=1}^{\infty} \widehat{a}_n \otimes e_{1,n} \Big\|_w &= \Big\| \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} (\xi_{k-1} \otimes 1)(d_k a_n \otimes e_{1,n}) \Big\|_w \\ &= \Big\| \sum_{k=1}^{\infty} (\xi_{k-1} \otimes 1) \sum_{n=1}^{k-1} (d_k a_n \otimes e_{1,n}) \Big\|_w \\ &= \Big\| \sum_{k=1}^{\infty} (\xi_{k-1} \otimes 1) \Big[\widehat{\mathcal{E}}_k\Big(\sum_{n=1}^{\infty} a_n \otimes e_{1,n} \Big) - \widehat{\mathcal{E}}_{k-1}\Big(\sum_{n=1}^{\infty} a_n \otimes e_{1,n} \Big) \Big] \Big\|_w \\ &= \Big\| T_{\xi \otimes 1}\Big(\sum_{n=1}^{\infty} a_n \otimes e_{1,n} \Big) \Big\|_w \le C \Big\| \sum_{n=1}^{\infty} a_n \otimes e_{1,n} \Big\|_w, \end{aligned}$$

where $\widehat{\mathcal{E}}_k = \mathcal{E}_k \otimes \mathrm{id}_{B(\ell_2)}$, $\xi \otimes 1 = (\xi_k \otimes 1)_{k \ge 1}$, and the last inequality is due (1.5), since $\xi_{k-1} \otimes 1$ commutes with $\mathcal{M}_k \otimes B(\ell_2)$ for every k. Consequently,

$$\|T_\xi x\|_{h^c_{pw}} \le C \|x\|_{h^c_{pw}}.$$

Now we turn to prove that T_ξ is bounded on $h^{1_c}_{pw}(\mathcal{M})$. For $x \in h^{1_c}_{pw}(\mathcal{M})$, there exists a decomposition $x = \sum_n d_n(a_n b_n)$ satisfying

$$\Big\| \sum_{n=1}^{\infty} a_n \otimes e_{1,n} \Big\|_w \Big\| \sum_{n=1}^{\infty} b_n \otimes e_{n,1} \Big\|_s \le (1+\delta) \|x\|_{h^{1_c}_{pw}},$$

where $a_n \in L_w(\mathcal{M})$, $b_n \in L_s(\mathcal{M})$ and δ is small enough. Then

$$T_\xi x = \sum_{k=1}^{\infty} \xi_{k-1} d_k(a_k b_k) = \sum_{k=1}^{\infty} d_k(\underbrace{\xi_{k-1} a_k}_{\widetilde{a}_k} b_k).$$

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Observe that $\sup_n \|\xi_n\|_\infty \le 1$ and $\widetilde{a}_k \in L_w(\mathcal{M})$. Notice that ξ_{k-1} commutes with $\mathcal{M}_k$ for each $k \ge 1$; then

$$\Big\|\sum_{n=1}^{\infty} \widetilde{a}_n \otimes e_{1,n}\Big\|_w = \Big\|\Big(\sum_{n=1}^{\infty} \xi_{n-1} a_n a_n^* \xi_{n-1}^*\Big)^{1/2}\Big\|_w = \Big\|\Big(\sum_{n=1}^{\infty} a_n \xi_{n-1} \xi_{n-1}^* a_k^*\Big)^{\frac{1}{2}}\Big\|_w$$
$$\le \Big\|\Big(\sum_{n=1}^{\infty} a_n a_n^*\Big)^{\frac{1}{2}}\Big\|_w = \Big\|\sum_{n=1}^{\infty} a_n \otimes e_{1,n}\Big\|_w,$$

which implies that $\|T_\xi x\|_{h_{pw}^{1_c}} \le \|x\|_{h_{pw}^{1_c}}$. The proof is complete. ∎

4 Fractional Integrals

In this section, we show several results related to fractional integrals in noncommutative martingale setting. In classical harmonic analysis, fractional integrals (or Riesz potentials, see *e.g.*, [42]) plays an important role in the proof of Sobelov inequality; see *e.g.*, [42, p. 124, Theorem 2]. In the same spirit, we can naturally expect that the outcome of this study would be useful in the operator-valued harmonic analysis ([27]).

All results in the section are obtained for hyperfinite and finite von Neumann algebras.

4.1 Fractional Integral

In this subsection, we prove Theorem 1.6.

Before going further, we recall some results from [21]. The following lemma is taken from the proof of [21, Proposition 2.8].

Lemma 4.1 *For $n \ge 1$, let $\mathcal{M}_n$ be a subalgebra of $\mathcal{M}$ with conditional expectation $\mathcal{E}_n$. There exists an isometric right $\mathcal{M}_n$-module map $u_n : \mathcal{M} \to \mathcal{M}_n \otimes B(\ell_2)$ whose image is the space of columns with entries in $\mathcal{M}_n$ satisfying*

$$\mathcal{E}_n(x^* y) = u_n(x)^* u_n(y).$$

In the same paper, Junge also proved the following dual Doob inequality.

Lemma 4.2 ([21, Theorem 0.1]) *Let $1 \le p < \infty$ and let $(x_n)_{n\ge1}$ be a sequence of positive elements in $L_p(\mathcal{M})$. Then*

$$\Big\|\sum_{n\ge1} \mathcal{E}_n(x_n)\Big\|_p \le C_p \Big\|\sum_{n\ge1} x_n\Big\|_p.$$

Now we are in a position to prove Theorem 1.6.

Proof of Theorem 1.6 We only prove the result for column spaces, since the row analog can be proved similarly.

Case 1: $0 < \alpha < 1/2$.

Note that $\|\cdot\|_{L_q(\mathcal{M},\ell_\infty^c)}$ is a quasi-norm (see [8, Theorem 3.2]). According to Theorem 2.4, we have $\mathcal{H}_p^c(\mathcal{M}) \simeq h_{p2}^c(\mathcal{M}) + h_{p2}^{1_c}(\mathcal{M})$. Then it suffices to show

(4.1) $$\big\|\big((I^\alpha x)_n\big)_{n\ge1}\big\|_{L_q(\mathcal{M},\ell_\infty^c)} \le C_{\alpha,p}\|x\|_{h_{p2}^c}$$

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and

(4.2) $$\big\| \big(I^\alpha x)_n \big)_{n\geq 1} \big\|_{L_q(\mathcal{M},\ell_\infty^c)} \leq C_{\alpha,p} \|x\|_{h_{p2}^{1c}}$$

We first prove (4.1). Let $x \in h_{p2}^c(\mathcal{M})$ with $1/p = 1/2 + 1/s$. Then there exists a decomposition $x = \sum_n a_n b_n$ satisfying

$$\Big\| \sum_{n=1}^{\infty} a_n \otimes e_{1,n} \Big\|_2 \Big\| \sum_{n=1}^{\infty} b_n \otimes e_{n,1} \Big\|_s \leq (1+\delta) \|x\|_{h_{p2}^c},$$

where $\mathcal{E}_n(a_n) = 0$, $b_n \in L_s(\mathcal{M}_n)$ and $\delta > 0$ is as small as we wish. By (3.3), for every $N \geq 1$, we have

$$\begin{aligned}(I^\alpha x)_N &= \sum_{k=1}^{N} \zeta_k^\alpha d_k x = \sum_{k=1}^{N} \sum_{n=1}^{k-1} \zeta_k^\alpha (d_k a_n) b_n = \sum_{n=1}^{N} \Big(\sum_{k=n+1}^{N} \zeta_k^\alpha d_k a_n \Big) b_n \\ &= \sum_{n=1}^{N} (I^\alpha a_n)_N b_n = \Big(\sum_{n=1}^{N} (I^\alpha a_n)_N \otimes e_{1,n} \Big) \Big(\sum_{n=1}^{\infty} b_n \otimes e_{n,1} \Big).\end{aligned}$$

Set $A_1 = \sum_{n=1}^{\infty} I^\alpha a_n \otimes e_{1,n}$ and $B_1 = \sum_{n=1}^{\infty} b_n \otimes e_{n,1}$. It is not hard to check that for every $N \geq 1$,

$$(I^\alpha x)_N = \widehat{\mathcal{E}}_N(A_1) B_1,$$

where $\widehat{\mathcal{E}}_N = \mathcal{E}_N \otimes \mathrm{id}_{B(\ell_2)}$. In fact, notice that $\mathcal{E}_n(a_n) = 0$ for each $n \geq 1$, we have

$$\begin{aligned}\widehat{\mathcal{E}}_N(A_1) &= \sum_{n=1}^{N} \mathcal{E}_N(I^\alpha a_n) \otimes e_{1,n} + \sum_{n=N+1}^{\infty} \mathcal{E}_N(I^\alpha a_n) \otimes e_{1,n} \\ &= \sum_{n=1}^{N} (I^\alpha a_n)_N \otimes e_{1,n}.\end{aligned}$$

For $\frac{1}{w} = \frac{1}{2} - \alpha < \frac{1}{2}$, by applying the $L_2 - L_w$ boundedness of the fractional integral I^α (see Lemma 2.8), we find

$$\begin{aligned}\|A_1\|_w &= \Big\| \sum_{n=1}^{\infty} |(I^\alpha a_n)^*|^2 \Big\|_{\frac{w}{2}}^{\frac{1}{2}} \leq \Big(\sum_{n=1}^{\infty} \|I^\alpha a_n\|_w^2 \Big)^{\frac{1}{2}} \\ &\overset{\text{Lem. 2.8}}{\leq} C_\alpha \Big(\sum_{n=1}^{\infty} \|a_n\|_2^2 \Big)^{\frac{1}{2}} = C_\alpha \Big\| \sum_{n=1}^{\infty} a_n \otimes e_{1,n} \Big\|_2.\end{aligned}$$

Since $w > 2$, it follows from the Doob inequality (1.3) and (3.1) that there exists $\eta \in L_{w/2}(\mathcal{M} \otimes B(\ell_2))$ such that

$$\widehat{\mathcal{E}}_n(|A_1|^2) \leq \eta, \quad \|\eta\|_{w/2} \leq C_w \||A_1|^2\|_{w/2} = \|A_1\|_w^2, \quad \forall n \geq 1.$$

Then there exist contractions $y_n \in \mathcal{M} \otimes B(\ell_2)$ such that

$$\widehat{\mathcal{E}}_n(|A_1|^2)^{1/2} = y_n \eta^{1/2} \quad \text{and} \quad \widehat{\mathcal{E}}_n(|A_1|^2) = \eta^{1/2} y_n^* y_n \eta^{1/2}.$$

On the other hand, according to the polar decomposition, there is $v_{\mathrm{row},A_1} \in \mathcal{M} \otimes B(\ell_2)$ such that

$$A_1 = v_{\mathrm{row},A_1} |A_1|.$$

Hence, by Lemma 4.1, we get

$$\widehat{\mathcal{E}}_n(A_1) = \widehat{u}_n (v_{\mathrm{row},A_1}^*)^* \widehat{u}_n(|A_1|),$$

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where the map $\widehat{u}_n$ is corresponding to the conditional expectation $\widehat{\mathcal{E}}_n$ as in Lemma 4.1. Using Lemma 4.1 again, we have

$$\widehat{u}_n\big(|A_1|\big)^*\widehat{u}_n\big(|A_1|\big) = \widehat{\mathcal{E}}_n\big(|A_1|^2\big) = \eta^{1/2}y_n^*y_n\eta^{1/2}. \tag{4.3}$$

Again, applying the polar decomposition twice, we can write

$$\widehat{u}_n(|A_1|) = \gamma_{n,1}\big|\widehat{u}_n\big(|A_1|\big)\big| = \gamma_{n,1}|y_n\eta^{1/2}| = \gamma_{n,1}\gamma_{n,2}^*y_n\eta^{1/2},$$

where the second equality is due to (4.3). We conclude from the above argument that

$$\begin{aligned}(I^\alpha x)_n &= \widehat{\mathcal{E}}_n(A_1)B_1 = \widehat{u}_n(v_{\mathrm{row},A_1}^*)^*\gamma_{n,1}\gamma_{n,2}^*y_n\eta^{1/2}B_1\\ &= \underbrace{\widehat{u}_n(v_{\mathrm{row},A_1}^*)^*\gamma_{n,1}\gamma_{n,2}^*y_n\eta^{1/2}B_1|\eta^{1/2}B_1|^{-1}}_{w_n}|\eta^{1/2}B_1|.\end{aligned}$$

Since $\widehat{u}_n(v_{\mathrm{row},A_1}^*)^*$ is a row matrix and B_1 is a column matrix, it follows that $w_n\in\mathcal{M}$. We still have to show that $|\eta^{1/2}B_1|\in L_q(\mathcal{M})$. Observe that $1/q = 1/w+1/s$. By the Hölder inequality, we obtain

$$\begin{aligned}\big\||\eta^{1/2}B_1|\big\|_{L_q(\mathcal{M})} &= \|\eta^{1/2}B_1\|_{L_q(\mathcal{M}\otimes B(\ell_2))} \le \|\eta^{1/2}\|_w\|B_1\|_s \le \|A_1\|_w\|B_1\|_s\\ &\le C_\alpha\Big\|\sum_{n=1}^\infty a_n\otimes e_{1,n}\Big\|_2\Big\|\sum_{n=1}^\infty b_n\otimes e_{n,1}\Big\|_s \le C_\alpha\|x\|_{h_{p2}^c}.\end{aligned}$$

Now we prove (4.2). Let $x\in h_{p2}^{1_c}(\mathcal{M})$ with $1/p = 1/2+1/s$. From the definition, we can find a decomposition $x=\sum_n d_n(a_nb_n)$ satisfying

$$\Big\|\sum_{n=1}^\infty a_n\otimes e_{1,n}\Big\|_2\Big\|\sum_{n=1}^\infty b_n\otimes e_{n,1}\Big\|_s \le (1+\delta)\|x\|_{h_{p2}^{1_c}},$$

where $a_n\in L_2(\mathcal{M})$, $b_n\in L_s(\mathcal{M})$ and $\delta>0$ is small enough. A simple calculation gives

$$(I^\alpha x)_n = \sum_{k=1}^n\zeta_k^\alpha d_k(a_kb_k) = \sum_{k=1}^n\zeta_k^\alpha\mathcal{E}_k(a_kb_k) - \sum_{k=1}^n\zeta_k^\alpha\mathcal{E}_{k-1}(a_kb_k) =: Y_n - Z_n.$$

We only estimate $\|(Y_n)_{n\ge1}\|_{L_q(\mathcal{M},\ell_\infty^c)}$, since $\|(Z_n)_{n\ge1}\|_{L_q(\mathcal{M},\ell_\infty^c)}$ can be similarly proved. Define the conditional expectation E_n in $B(\ell_2)$ by setting

$$E_n\big((m_{j,k})_{j,k\ge1}\big) = \big((m_{j,k})_{1\le j,k\le n}\big)\oplus(m_{k,k})_{k>n}.$$

Then, from Lemma 4.1, we have

$$\begin{aligned}Y_n &= \sum_{k=1}^n u_k(\zeta_k^\alpha a_k^*)^*u_k(b_k) = \Big(\sum_{k=1}^n u_k(\zeta_k^\alpha a_k^*)^*\otimes e_{1,k}\Big)\Big(\sum_{k=1}^\infty u_k(b_k)\otimes e_{k,1}\Big)\\ &= \mathbb{E}_n(A_2)B_2,\end{aligned}$$

where $\mathbb{E}_n = \mathrm{id}_\mathcal{M}\otimes\mathrm{id}_{B(\ell_2)}\otimes E_n$ and $A_2 = \sum_{k=1}^\infty u_k(\zeta_k^\alpha a_k^*)^*\otimes e_{1,k}$. Note that $|A_2|^2\in L_{w/2}(\mathcal{M}\otimes B(\ell_2))$. In fact, using Lemma 2.7, we have

$$\begin{aligned}\|A_2\|_w^2 &= \Big\|\sum_{k=1}^\infty\zeta_k^{2\alpha}\mathcal{E}_k(a_ka_k^*)\Big\|_{w/2} \le \sum_{k=1}^\infty\|\zeta_k^{2\alpha}\mathcal{E}_k(a_ka_k^*)\|_{w/2}\\ &\le \sum_{k=1}^\infty\|a_ka_k^*\|_1 = \Big\|\sum_{k=1}^\infty a_k\otimes e_{1,k}\Big\|_2^2.\end{aligned}$$

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Observe that $w/2 > 1$ for $\frac{1}{w} = \frac{1}{2} - \alpha$. Similar to the proof of (4.1), we can apply the Doob inequality (1.3) and (3.1) to get the factorization

$$\mathbb{E}_n(|A_2|^2) = \beta z_n^* z_n \beta, \quad n \geq 1$$

satisfying

$$0 \leq \beta \in L_w(\mathcal{M} \otimes B(\ell_2)), \quad \|\beta\|_w \leq C_{w/2}\|A_2\|_w, \quad \text{and} \quad \sup_{n\geq 1}\|z_n\|_\infty \leq 1.$$

On the other hand, combining the polar decomposition and Lemma 4.1, there is a contraction v_{row,A_2} such that

$$\mathbb{E}_n(A_2) = \mathbb{E}_n(v_{\mathrm{row},A_2}|A_2|) = \widetilde{u}_n(v_{\mathrm{row},A_2}^*)^* \widetilde{u}_n(|A_2|),$$

where the operator $\widetilde{u}_n$ is corresponding to the conditional expectation $\mathbb{E}_n$ as in Lemma 4.1. Then

$$\big|\widetilde{u}_n\big(|A_2|\big)\big|^2 = \mathbb{E}_n(|A_2|^2) = |z_n\beta|^2,$$

which further implies that there is a contraction $\rho_n \in \mathcal{M} \otimes B(\ell_2)$ such that $\widetilde{u}_n(|A|) = \rho_n z_n \beta$. Now we get the factorization

$$Y_n = \widetilde{u}_n(v_{\mathrm{row},A_2}^*)^* \rho_n y_n \beta B_2 = \underbrace{\widetilde{u}_n(v_{\mathrm{row},A_2}^*)^* \rho_n z_n \beta B_2 |\beta B_2|^{-1}}_{w_{n,2}} |\beta B_2|.$$

Since v_{row,A_2} is a row matrix, by the definition of $\widetilde{u}_n$, we know that $\widetilde{u}_n(v_{\mathrm{row},A_2}^*)^*$ is a row matrix. From this and B_2 is column matrix, we know that $w_{n,2}$ is affiliated with $\mathcal{M}$. It is obvious that $\|w_{n,2}\|_\infty \leq 1$ and $|\beta B_2|$ is also affiliated with $\mathcal{M}$. Hence, the factorization above is our desired one. Now, to prove $\|(Y_n)_{n\geq 1}\|_{L_q(\mathcal{M},\ell_\infty^c)} \leq C\|x\|_{h_{p2}^{1_c}}$ we only need to show $\|\beta B_2\|_q \leq C\|x\|_{h_{p2}^{1_c}}$. Indeed, by Hölder's inequality for $1/q = 1/w + 1/s$, we have

$$\begin{aligned}\|\beta B_2\|_q &\leq \|\beta\|_w \|B_2\|_s \leq C_{w/2}C_{s/2}\Big\|\sum_{n=1}^\infty a_n \otimes e_{1,n}\Big\|_2 \Big\|\sum_{n=1}^\infty b_n \otimes e_{n,1}\Big\|_s \\ &\leq (1+\delta)\|x\|_{h_{p2}^{1_c}},\end{aligned}$$

where

$$\|B_2\|_s = \Big\|\sum_{k=1}^\infty \mathcal{E}_k|b_k|^2\Big\|_{s/2}^{1/2} \leq C_{s/2}\Big\|\sum_{k=1}^\infty |b_k|^2\Big\|_{s/2}^{1/2} = C_{s/2}\Big\|\sum_{k=1}^\infty b_k \otimes e_{k,1}\Big\|_s$$

is due to $s \geq 2$ and the dual Doob inequality (Lemma 4.2).

Case 2: $1/2 \leq \alpha < 1/p$.

If $1 < p < 2$, we know that $1/q = 1/p - \alpha < 1/2$ implies $q > 2$. Then combining (1.9), Lemma 2.8, and (2.2), we obtain

$$\big\|\big((I^\alpha x)_n\big)_{n\geq 1}\big\|_{L_q(\mathcal{M},\ell_\infty^c)} \overset{(1.9)}{\leq} C_q\|I^\alpha x\|_q \leq C_q C_\alpha \|x\|_p \leq C_q C_\alpha C_p \|x\|_{\mathcal{H}_p^c}.$$

It remains to consider the case $p = 1$. In this case, if $\alpha = 1/2$, then, by Case 1 and Lemma 2.9, we get

$$\begin{aligned}\big\|\big((I^{1/2}x)_n\big)_{n\geq 1}\big\|_{L_2(\mathcal{M},\ell_\infty^c)} &= \big\|\big((I^{1/4}I^{1/4}x)_n\big)_{n\geq 1}\big\|_{L_2(\mathcal{M},\ell_\infty^c)} \\ &\leq C_\alpha\|I^{1/4}x\|_{\mathcal{H}_{4/3}^c} \leq C_\alpha\|x\|_{\mathcal{H}_1^c}.\end{aligned}$$

If $1/2 < \alpha < 1$, then $1/(1-\alpha) > 2$, and it follows from (1.9), Lemma 2.8, and Lemma 2.9 that

$$\begin{aligned}\|((I^\alpha x)_n)_{n\geq 1}\|_{L_{1/(1-\alpha)}(\mathcal{M},\ell_\infty^c)} &\overset{(1.9)}{\leq} C_\alpha \|I^\alpha x\|_{L_{1/(1-\alpha)}} \\ &= C_\alpha \|I^{\alpha-1/2} I^{1/2} x\|_{L_{1/(1-\alpha)}} \leq C_\alpha \|I^{1/2} x\|_{L_2} \\ &= C_\alpha \|I^{1/2} x\|_{\mathcal{H}_2^c} \leq C_\alpha \|x\|_{\mathcal{H}_1^c}.\end{aligned}$$

The proof of the theorem is complete. ■

4.2 Fractional Doob Maximal Inequalities

We find from [6, p. 36] that, for $1 < p < q < \infty$ and classical dyadic martingale $f = (f_n)_{n\geq 1}$, (4.4) holds true with $\mathcal{M} = L_\infty[0,1)$ and $\zeta_n = 2^{-n}$

$$\tag{4.4} \begin{aligned}\Big\| \big(\zeta_n^\alpha f_n \big)_{n\geq 1} \Big\|_{L_q(L_\infty[0,1),\ell_\infty^{1/2})} &= \Big\| \sup_n \zeta_n^\alpha |f_n| \Big\|_{L_q(L_\infty[0,1))} \\ &\leq C_\alpha \|f\|_p, \quad \alpha = \frac{1}{p} - \frac{1}{q}.\end{aligned}$$

Usually, $\sup_n \zeta_n^\alpha |f_n|$ is called a fractional Doob maximal function of a martingale $f = (f_n)_{n\geq 1}$. First, with the help of Lemmas 2.8 and 2.9, we extend (4.4) into noncommutative setting:

Theorem 4.3 *Let $0 < \alpha < 1$. Assume that $\mathcal{M}$ is a hyperfinite and finite von Neumann algebra. If $\sup_{n\geq 1} \frac{\zeta_n}{\zeta_{n-1}} := C_\zeta < 1$, then there exists a constant C_α such that*

$$\tag{4.5} \begin{aligned} &\big\| (\zeta_n^\alpha x_n)_{n\geq 1} \big\|_{L_q(\mathcal{M},\ell_\infty^{1/2})} \leq C_\alpha \|x\|_p, \\ &\qquad x \in L_p(\mathcal{M}), \quad 1 < p < q < \infty, \quad \alpha = \frac{1}{p} - \frac{1}{q}; \\ &\big\| (\zeta_n^\alpha x_n)_{n\geq 1} \big\|_{L_{\frac{1}{1-\alpha}}(\mathcal{M},\ell_\infty^{1/2})} \leq C_\alpha \|x\|_{\mathcal{H}_1}, \quad x \in \mathcal{H}_1(\mathcal{M}).\end{aligned}$$

Furthermore, combining Theorem 4.3 and the Gundy decomposition established in [31], we get the following weak type fractional maximal inequality.

Theorem 4.4 *Let $0 < \alpha < 1$. Assume that $\mathcal{M}$ is a hyperfinite and finite von Neumann algebra. If $x = (x_n)_{n\geq 1} \in L_1(\mathcal{M})$ and $\sup_{n\geq 1} \frac{\zeta_n}{\zeta_{n-1}} := C_\zeta < 1$, then there exists a constant C_α such that*

$$\|(\zeta_n^\alpha x_n)_{n\geq 1}\|_{\Lambda_{\frac{1}{1-\alpha},\infty}(\mathcal{M},\ell_\infty)} \leq C_\alpha \sup_{n\geq 1} \|x_n\|_1.$$

We give an example for the special condition $\sup_{n\geq 1} \frac{\zeta_n}{\zeta_{n-1}} := C_\zeta < 1$ used in the above results.

Remark 4.5 Take $(\zeta_n)_{n\geq 1}$ defined as in (2.4) corresponding to $(\mathcal{R}_n)_{n\geq 1}$ (usually called noncommutative Vilenkin filtration; see (2.7)). We have $\sup_k \zeta_k/\zeta_{k-1} = \sup_k 1/m(k) \leq 1/2$, since $m(k) \geq 2$ for each $k \geq 1$.

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Now we prove Theorem 4.3. It is the origin of the subsection, since many results are based on it.

Proof of Theorem 4.3 Note that for $p \geq 1$, $L_p(\mathcal{M}, \ell_\infty^{1/2})$ is a Banach space (see [24, p. 392] or [?, Proposition 4.1.3]). For positive $x \in L_p(\mathcal{M})$, we have

$$\begin{aligned}
&\left\| \left(\zeta_n^\alpha \mathcal{E}_n(x) \right)_{n\geq 1} \right\|_{L_q(\mathcal{M}, \ell_\infty^{1/2})} \\
&\quad = \left\| \left(\zeta_n^\alpha \mathcal{E}_n(x) - \zeta_n^\alpha \mathcal{E}_{n-1}(x) + \zeta_n^\alpha \mathcal{E}_{n-1}(x) \right)_{n\geq 1} \right\|_{L_q(\mathcal{M}, \ell_\infty^{1/2})} \\
&\quad = \left\| \left(\mathcal{E}_n(I^\alpha x) - \mathcal{E}_{n-1}(I^\alpha x) + \zeta_n^\alpha \mathcal{E}_{n-1}(x) \right)_{n\geq 1} \right\|_{L_q(\mathcal{M}, \ell_\infty^{1/2})} \\
&\quad \leq 2 \left\| \left(\mathcal{E}_n(I^\alpha x) \right)_{n\geq 1} \right\|_{L_q(\mathcal{M}, \ell_\infty^{1/2})} + \sup_{n\geq 1} \frac{\zeta_n^\alpha}{\zeta_{n-1}^\alpha} \left\| \left(\zeta_{n-1}^\alpha \mathcal{E}_{n-1}(x) \right)_{n\geq 1} \right\|_{L_q(\mathcal{M}, \ell_\infty^{1/2})},
\end{aligned}$$

This and Lemma 2.8 imply

$$\begin{aligned}
\left\| (\zeta_n^\alpha \mathcal{E}_n(x))_{n\geq 1} \right\|_{L_q(\mathcal{M}, \ell_\infty^{1/2})} &\leq \frac{2}{1 - C_\zeta^\alpha} \left\| \left(\mathcal{E}_n(I^\alpha x) \right)_{n\geq 1} \right\|_{L_q(\mathcal{M}, \ell_\infty^{1/2})} \\
&\leq \frac{C_\alpha}{1 - C_\zeta^\alpha} \|x\|_p.
\end{aligned}$$

The inequality (4.5) follows, since every $x \in L_p(\mathcal{M})$ can be written into the combination of four positive elements.

Combining the above argument and Doob's inequality (1.3), we get

$$\begin{aligned}
\left\| (\zeta_n^\alpha \mathcal{E}_n(x))_{n\geq 1} \right\|_{L_{\frac{1}{1-\alpha}}(\mathcal{M}, \ell_\infty^{1/2})} &\leq \frac{2}{1 - C_\zeta^\alpha} \left\| \left(\mathcal{E}_n(I^\alpha x) \right)_{n\geq 1} \right\|_{L_{\frac{1}{1-\alpha}}(\mathcal{M}, \ell_\infty^{1/2})} \\
&\leq C_{\frac{1}{1-\alpha}} \frac{2}{1 - C_\zeta^\alpha} \|I^\alpha x\|_{\frac{1}{1-\alpha}}.
\end{aligned}$$

For $0 < \alpha \leq 1/2$, we use Burkholder–Gundy's inequality (2.2) and Lemma 2.9 to deduce

$$\|I^\alpha x\|_{\frac{1}{1-\alpha}} \leq C_{\frac{1}{1-\alpha}} \|I^\alpha x\|_{\mathcal{H}_{\frac{1}{1-\alpha}}} \leq C_\alpha C_{\frac{1}{1-\alpha}} \|x\|_{\mathcal{H}_1}.$$

For $1/2 < \alpha < 1$, applying Lemmas 2.8 and 2.9, we have

$$\|I^\alpha x\|_{\frac{1}{1-\alpha}} = \|I^{\alpha - \frac{1}{2}} I^{\frac{1}{2}} x\|_{\frac{1}{1-\alpha}} \leq C_\alpha \|I^{\frac{1}{2}} x\|_2 \leq C_\alpha \|x\|_{\mathcal{H}_1}.$$

Conbining these esimates, the second inequality follows and the proof is complete. ■

To prove the Doob inequality (1.3), Junge [21] first obtained the dual form of the Doob inequality (Lemma 4.2). In this paper, as an application of this (p,q)-type maximal inequality (4.5), we establish the dual version of (4.5). The following result for $\alpha = 0$ is just Lemma 4.2.

Corollary 4.6 *Let $1 \leq p < \infty$ and $0 \leq \alpha < \frac{1}{p}$. Assume that $\mathcal{M}$ is a hyperfinite and finite von Neumann algebra. If $\sup_{n\geq 1} \frac{\zeta_n}{\zeta_{n-1}} := C_\zeta < 1$ and $(b_n) \subset L_p(\mathcal{M})$ is a sequence of positive elements, then there exists a constant C_α such that*

$$\left\| \sum_n \zeta_n^\alpha \mathcal{E}_n(b_n) \right\|_q \leq C_\alpha \left\| \sum_n b_n \right\|_p, \quad \frac{1}{q} = \frac{1}{p} - \alpha.$$

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Proof The result for $\alpha = 0$ is just Lemma 4.2. Hence, we only consider $0 < \alpha < \frac{1}{p}$. The case $p = 1$ follows from Lemma 2.7. It remains to consider the case $1 < p < \infty$. By duality, we have

$$\Big\| \sum_n \zeta_n^\alpha \mathcal{E}_n(b_n) \Big\|_q = \sup_{g \geq 0, \|g\|_{q'} \leq 1} \tau\Big(\sum_n \zeta_n^\alpha \mathcal{E}_n(b_n) g \Big)$$
$$= \sup_{g \geq 0, \|g\|_{q'} \leq 1} \tau\Big(\sum_n b_n \zeta_n^\alpha \mathcal{E}_n(g) \Big).$$

Note that $1/q' - 1/p' = \alpha$. According to Theorem 4.3 and the definition of $\|\cdot\|_{L_{p'}(\mathcal{M}, \ell_\infty^{1/2})}$, we find suitable a such that

$$0 \leq \zeta_n^\alpha \mathcal{E}_n(g) \leq a, \quad \forall n \geq 1 \quad \text{and} \quad \|a\|_{p'} \leq C_\alpha \|g\|_{q'}.$$

Then, by Hölder's inequality,

$$\Big\| \sum_n \zeta_n^\alpha \mathcal{E}_n(b_n) \Big\|_q \leq \tau\Big(\sum_n b_n a \Big) \leq \Big\| \sum_n b_n \Big\|_p \|a\|_{p'} \leq C_\alpha \Big\| \sum_n b_n \Big\|_p,$$

which finishes the proof. ∎

We also find the following fractional version of the Stein inequality ([34, Theorem 2.3]). We use Corollary 4.6 to prove it. This is new even in the commutative martingale setting.

Proposition 4.7 *Let $\mathcal{M}$ be a hyperfinite and finite von Neumann algebra. Let $0 \leq \alpha < 1/p$ with $1 < p < \infty$, and let $1/q = 1/p - \alpha$. If $\sup_{n \geq 1} \frac{\zeta_n}{\zeta_{n-1}} := C_\zeta < 1$ and $(b_n) \subset L_p(\mathcal{M})$, then*

$$\Big\| \Big(\sum_n |\zeta_n^\alpha \mathcal{E}_n(b_n)|^2 \Big)^{\frac{1}{2}} \Big\|_q \leq C_\alpha \Big\| \Big(\sum_n |b_n|^2 \Big)^{\frac{1}{2}} \Big\|_p. \tag{4.6}$$

Proof Since for $\alpha = 0$, (4.6) is just [34, Theorem 2.3], we only consider $0 < \alpha < 1/p$. If $2 \leq p < \infty$, by the property $\mathcal{E}_n(a)^* \mathcal{E}_n(a) \leq \mathcal{E}_n(|a|^2)$ for $a \in L_s(\mathcal{M})$ $(s \geq 2)$ and Corollary 4.6, we have

$$\Big\| \Big(\sum_n |\zeta_n^\alpha \mathcal{E}_n(b_n)|^2 \Big)^{\frac{1}{2}} \Big\|_q = \Big\| \sum_{n=1}^\infty |\mathcal{E}_n(b_n)|^2 \zeta_n^{2\alpha} \Big\|_{q/2}^{\frac{1}{2}}$$
$$\leq \Big\| \sum_{n=1}^\infty \mathcal{E}_n(|b_n|^2) \zeta_n^{2\alpha} \Big\|_{q/2}^{\frac{1}{2}}$$
$$\leq C_{2\alpha} \Big\| \sum_{n=1}^\infty |b_n|^2 \Big\|_{p/2}^{\frac{1}{2}} = C_{2\alpha} \Big\| \Big(\sum_n |b_n|^2 \Big)^{\frac{1}{2}} \Big\|_p.$$

Now, it suffices to consider the case $1 < p < 2$. For $1 < q \leq 2$, by the duality $(L_r(\mathcal{M}, \ell_2^c))^* = L_{r'}(\mathcal{M}, \ell_2^c)$ (see [34]) and the proven result of the case $2 \leq p < \infty$, we

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then have

$$\begin{aligned}\Big\|\Big(\sum_n|\zeta_n^\alpha\mathcal{E}_n(b_n)|^2\Big)^{\frac12}\Big\|_q &= \sup\Big\{|\tau\Big(\sum_n\zeta_n^\alpha\mathcal{E}_n(b_n)g_n^*\Big)|:\Big\|\Big(\sum_n|g_n|^2\Big)^{\frac12}\Big\|_{q'}\le 1\Big\}\\ &= \sup\Big\{|\tau\Big(\sum_n b_n\zeta_n^\alpha\mathcal{E}_n(g_n^*)\Big)|:\Big\|\Big(\sum_n|g_n|^2\Big)^{\frac12}\Big\|_{q'}\le 1\Big\}\\ &\le \Big\|\Big(\sum_n|b_n|^2\Big)^{\frac12}\Big\|_p\Big\|\Big(\sum_n|\zeta_n^\alpha\mathcal{E}_n(g_n)|^2\Big)^{\frac12}\Big)\Big\|_{p'}\\ &\le C_\alpha\Big\|\Big(\sum_n|b_n|^2\Big)^{\frac12}\Big\|_p,\end{aligned}$$

where the last "≤" is due to (4.6) (note that $1/q'-1/p'=\alpha$ and $q'\ge 2$). If $2<q<\infty$, then take $0<\alpha_1<1$ and $0<\alpha_2<1$ such that $\alpha=\alpha_1+\alpha_2$,

$$\frac12-\frac1q=\alpha_1 \quad\text{and}\quad \frac1p-\frac12=\alpha_2.$$

Applying (4.6) for $p=2$, we arrive at

$$\begin{aligned}\Big\|\Big(\sum_n|\zeta_n^\alpha\mathcal{E}_n(b_n)|^2\Big)^{\frac12}\Big\|_q &\le C_{\alpha_1}\Big\|\Big(\sum_n|\zeta_n^{\alpha_2}\mathcal{E}_n(b_n)|^2\Big)^{\frac12}\Big\|_2\\ &\le C_{\alpha_1}C_{\alpha_2}\Big\|\Big(\sum_n|b_n|^2\Big)^{\frac12}\Big\|_p,\end{aligned}$$

where we have used (4.6) again with $1<p<2$ and $q=2$ to get the last "≤". The proof is complete. ■

Since $\|\cdot\|_{\Lambda_{1/1-\alpha,\infty}(\mathcal{M},\ell_\infty)}$ is a quasi-norm, we cannot apply the idea used in the proof of Theorem 4.3 to prove the weak type inequality in Theorem 4.4. We apply Gundy's decomposition to overcome this problem.

Observe that

$$\|x\|_{\mathcal{H}_1}\le\Big\|\Big(\sum_{k=1}^\infty|d_kx|^2\Big)^{1/2}\Big\|_1\le\sum_{k=1}^\infty\||d_kx|^2\|_{1/2}^{1/2}=\sum_{k=1}^\infty\|d_kx\|_1=\|x\|_{h_1^d}.$$

Combining this and Theorem 4.3, we get the result below.

Lemma 4.8 *Let* $0<\alpha<1$. *Assume that* $\mathcal{M}$ *is a hyperfinite and finite von Neumann algebra. If* $\sup_{n\ge1}\frac{\zeta_n}{\zeta_{n-1}}:=C_1<1$, *then there exists a constant* C_α *such that*

$$\|(\zeta_n^\alpha x_n)_{n\ge1}\|_{L_{\frac{1}{1-\alpha}}(\mathcal{M},\ell_\infty^{1/2})}\le C_\alpha\|x\|_{h_1^d},\quad x\in h_1^d(\mathcal{M}). \tag{4.7}$$

Now we are ready to prove Theorem 4.4. The idea of the proof is similar to the one of Theorem 1.2. We need to do more calculations, so we still give the proof.

Proof of Theorem 4.4 By linearity and homogeneity, we can assume without loss of generality that $\|x\|_1=1$. Set $\lambda=s^{\frac{1}{1-\alpha}}$ for every fixed $s>0$. Applying Lemma 3.2, for $\lambda>0$, we get the Gundy decomposition

$$x=y+z+v+w.$$

By the quasi-triangle inequality of $\Lambda_{\frac{1}{1-\alpha},\infty}(\mathcal{M},\ell_\infty)$, it suffices to show

$$\|(\zeta_n^\alpha y_n)_{n\geq 1}\|_{\Lambda_{\frac{1}{1-\alpha},\infty}(\mathcal{M},\ell_\infty)} \leq C_\alpha \|x\|_1, \tag{4.8}$$

$$\|(\zeta_n^\alpha z_n)_{n\geq 1}\|_{\Lambda_{\frac{1}{1-\alpha},\infty}(\mathcal{M},\ell_\infty)} \leq C_\alpha \|x\|_1, \tag{4.9}$$

and

$$\|(\zeta_n^\alpha (v+w)_n)_{n\geq 1}\|_{\Lambda_{\frac{1}{1-\alpha},\infty}(\mathcal{M},\ell_\infty)} \leq C_\alpha \|x\|_1. \tag{4.10}$$

To show (4.8), take $1 < p < \min\{2, \frac{1}{\alpha}\}$ and $1/q = 1/p - \alpha$. Similar to the proof of Theorem 1.2, we decompose y into the combination of four positive elements

$$y = h_1 - h_2 + ih_3 - ih_4$$

such that

$$\|h_j\|_p \leq \|y\|_p, \quad j \in \{1,2,3,4\}.$$

By Lemma 2.7, we know that $\|\zeta_n^\alpha \mathcal{E}_n(h_j)\|_q \leq \|\mathcal{E}_n(h_j)\|_p \leq \|h_j\|_p$ for every $n \geq 1$ and j. Applying Theorem 4.3 and the definition of $L_q(\mathcal{M}, \ell_\infty^{1/2})$, for every $j \in \{1,2,3,4\}$, there exist positive elements a_j satisfying for every $n \geq 1$,

$$\zeta_n^\alpha \mathcal{E}_n(h_j) \leq a_j, \quad \|a_j\|_q \leq C_\alpha \|h_j\|_p \leq C_\alpha \|y\|_p.$$

Set

$$e_j = \chi_{(0,\frac{s}{4})}(a_j) \quad \text{and} \quad e_s = \bigwedge_{j=1}^4 e_j.$$

Then, for each j,

$$e_j \zeta_n^\alpha \mathcal{E}_n(h_j) e_j \leq e_j a_j e_j \leq \frac{s}{4},$$

which implies that

$$\|e_s \zeta_n^\alpha \mathcal{E}_n(y) e_s\|_\infty \leq \sum_{j=1}^4 \|e_j \zeta_n^\alpha \mathcal{E}_n(h_j) e_j\|_\infty \leq s.$$

Note that $\|y\|_1 \leq 8$ and $\|y\|_2^2 \leq 6\lambda$ (by Lemma 3.2(ii)) where $\lambda = s^{\frac{1}{1-\alpha}}$. The Chebyshev inequality gives

$$\begin{aligned} s\tau(1-e_s)^{1-\alpha} &\leq \sum_{j=1}^4 s\tau(1-e_j)^{1-\alpha} = \sum_{j=1}^4 s\tau(\chi_{(\frac{s}{4},\infty)}(a_j))^{1-\alpha} \\ &\leq \sum_{j=1}^4 4^q s^{1-q(1-\alpha)} \|a_j\|_q^{q(1-\alpha)} \\ &\leq C_\alpha^{q(1-\alpha)} 4^{q+1} s^{1-q(1-\alpha)} \|y\|_p^{q(1-\alpha)}. \end{aligned}$$

Since $1 < p < \min\{2, \frac{1}{\alpha}\}$, it follows from the Hölder inequality that

$$\begin{aligned} \|y\|_p^{q(1-\alpha)} &= \tau(|y|^{2-p}|y|^{2p-2})^{\frac{q(1-\alpha)}{p}} \\ &\leq \Big(\||y|^{2-p}\|_{\frac{1}{2-p}} \||y|^{2p-2}\|_{\frac{1}{p-1}} \Big)^{\frac{q(1-\alpha)}{p}} \\ &\leq 8^{\frac{(2-p)q(1-\alpha)}{p}} 6^{\frac{(p-1)q(1-\alpha)}{p}} \lambda^{\frac{(p-1)q(1-\alpha)}{p}}, \end{aligned}$$

which, together with the definition of $\lambda = s^{\frac{1}{1-\alpha}}$, further implies

$$s\tau(1-e_s)^{1-\alpha} \le C_\alpha^{q(1-\alpha)} 4^{q+1} 8^{\frac{(2-p)q(1-\alpha)}{p}} 6^{\frac{(p-1)q(1-\alpha)}{p}}.$$

According to the definition of $\Lambda_{\frac{1}{1-\alpha},\infty}(\mathcal{M},\ell_\infty)$, these finish the proof of (4.8).

It follows from (4.7) and Lemma 3.2(iii) that

$$\begin{aligned}\|(\zeta_n^\alpha z_n)_{n\ge1}\|_{\Lambda_{\frac{1}{1-\alpha},\infty}(\mathcal{M},\ell_\infty)} &\le \|(\zeta_n^\alpha z_n)_{n\ge1}\|_{L_{\frac{1}{1-\alpha}}(\mathcal{M},\ell_\infty^{1/2})}\\ &\le C_\alpha \|z\|_{h_1^d} \le 6C_\alpha\|x\|_1,\end{aligned}$$

where the first inequality is due to the definitions of those two spaces. This is just (4.9). Now we turn to show (4.10). Set

$$e_0 = 1 - \Big(\bigvee_n \operatorname{supp}(d_n v)\Big) \vee \Big(\bigvee_n \operatorname{supp}(d_n w)\Big).$$

Then it is easy to check that

$$e_0(v+w)e_0 = 0.$$

Note that $\lambda = s^{\frac{1}{1-\alpha}}$. By the definition of $\Lambda_{\frac{1}{1-\alpha},\infty}(\mathcal{M},\ell_\infty)$ and Lemma 3.2(iv), we have

$$\begin{aligned}\|(\zeta_n^\alpha (v+w)_n)_{n\ge1}\|_{\Lambda_{\frac{1}{1-\alpha},\infty}(\mathcal{M},\ell_\infty)} &\le \sup_{s>0} s\tau(1-e_0)^{1-\alpha}\\ &\le \sup_{s>0} s\Big(\tau\Big(\bigvee_n \operatorname{supp}(d_n v)\Big) + \tau\Big(\bigvee_n \operatorname{supp}(d_n v)\Big)\Big)^{1-\alpha}\\ &\le \sup_{s>0} s(2\lambda^{-1})^{1-\alpha} = 2^{1-\alpha}.\end{aligned}$$

The proof is complete. ∎

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