# Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 December 2022.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

**106.E** (Stan Dolan and Kieren MacMillan)

Extend the definition $T_k = \frac{1}{2}k(k+1)$ of a triangular number to all integers $k$. Given that $a + b + c = d + e + f = 0$, find integers $x$, $y$, $z$ such that

$$T_x + T_y + T_z = (T_a + T_b + T_c)(T_d + T_e + T_f).$$

**106.F** (George Stoica)

If the polynomial $P(z) = a_0 + a_1 z + \dots + a_n z^n$ satisfies $|P(z)| \leqslant 1$ for all complex numbers $z$ with $|z| \leqslant 1$, then show that $|a_i| \leqslant 1$ for $i = 0, 1, \dots, n$.

**106.G** (Seán Stewart)

If $C_n$ is the $n$th Catalan number defined by the recurrence relation $C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$ with $C_0 = 1$, prove that:

(a) $$\sum_{n=0}^{\infty} \frac{C_n}{4^n(2n+3)} = 2 - \frac{\pi}{2},$$

(b) $$\sum_{n=0}^{\infty} \frac{C_n}{4^n(2n+3)^2} = 2 - \frac{\pi}{4}(1 + \ln 4),$$

(c) $$\sum_{n=0}^{\infty} \frac{C_n}{4^n(2n+3)^3} = 2 - \frac{\pi}{16}(2 + \ln 16 + \ln^2 4) - \frac{\pi^3}{48}.$$

**106.H** (Isaac Sofair)

A bag contains $p$ white marbles and $q$ black marbles. Marbles are randomly drawn from the bag one at a time without replacement until $n$ white marbles are obtained, where $n \leqslant p$. Prove that the mean number of marbles that need to be drawn is $n\left(1 + \frac{q}{n+1}\right)$.

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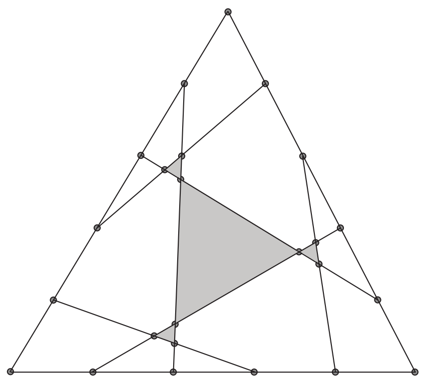
Solutions and comments on **105.I**, **105.J**, **105.K**, **105.L** (November 2021).

**105.I** (Arsalan Wares)

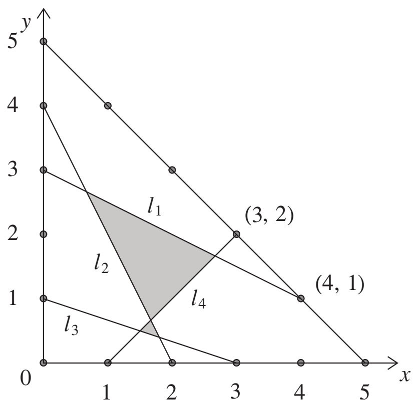
The figure shows a triangle with four points on each side that split the sides into five equal parts. Pairs of these points are joined by six line segments as shown, forming the four small shaded triangles. Exactly what proportion of the original triangle is shaded?



*Answer*: $\dfrac{16}{125}$

There was a variety of coordinate geometry and vector methods used by solvers of this problem, often using an area-ratio preserving affine transformation to map the given triangle to one of a more convenient shape. M. G. Elliott, Graham Howlett, James Mundie and Didier Pinchon's use of such a reduction caught my eye.

First. map to an equilateral triangle to see (by rotational symmetry) that the 3 smallest shaded triangles have equal area. Then map the triangle to the isosceles right-angled triangle shown in the Figure (where some lines have deliberately been omitted).

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The lines $l_1, l_2, l_3, l_4$ have equations $x + 2y = 6$, $2x + y = 4$, $x + 3y = 3, x - y = 1$ with intersection points

| $l_1 \cap l_2$ | $l_1 \cap l_4$ | $l_2 \cap l_4$ | $l_2 \cap l_3$ | $l_3 \cap l_4$ |
|---|---|---|---|---|
| $\left(\frac{2}{3}, \frac{8}{3}\right)$ | $\left(\frac{8}{3}, \frac{5}{3}\right)$ | $\left(\frac{5}{3}, \frac{2}{3}\right)$ | $\left(\frac{9}{5}, \frac{2}{5}\right)$ | $\left(\frac{3}{2}, \frac{1}{2}\right)$ |

The larger shaded triangle has area

$$\frac{1}{2}\begin{vmatrix} \frac{2}{3} & \frac{8}{3} & 1 \\ \frac{5}{3} & \frac{2}{3} & 1 \\ \frac{8}{3} & \frac{5}{3} & 1 \end{vmatrix} = \frac{3}{2}$$

and the smaller shaded triangle has area

$$\frac{1}{2}\begin{vmatrix} \frac{5}{3} & \frac{2}{3} & 1 \\ \frac{3}{2} & \frac{1}{2} & 1 \\ \frac{9}{5} & \frac{2}{5} & 1 \end{vmatrix} = \frac{1}{30}.$$

Since the whole triangle has area $\dfrac{25}{2}$, the required ratio is $\dfrac{\frac{3}{2} + 3 \times \frac{1}{30}}{\frac{25}{2}} = \dfrac{16}{125}$.

Correct solutions were received from: M. V. Channakeshava, N. Curwen, S. Dolan, M. G. Elliott, M. J. Emson, R. Harris, A. P. Harrison, M. Hennings, G. Howlett, P. F. Johnson, Y. Kong, J. A. Mundie, D. Pinchon, S. Riccarelli, V. Schindler, C. Starr and the proposer Arsalan Wares.

**105.J** (Nicolo Sartori di Borgoricco)

For positive integers $k \geqslant 2$, show that the sum of all the multiples of 3 or 5 between 1 and $10^k$ inclusive is 233… 34166… 68, where there are $k - 2$ digits in each block of 3's and 6's.

Solutions to this popular digits problem were all similar to that given below, although with differences in notation and how the final stage of the calculation was completed.

Let $S_n$ denote the sum of multiples of $n$ between 1 and $10^k$. Then

$$S_3 = \sum_{k=1}^{\frac{1}{3}(10^k - 1)} 3k = \frac{1}{6}\left(10^k - 1\right)\left(10^k + 2\right),$$

$$S_5 = \sum_{k=1}^{\frac{1}{5} \times 10^k} 5k = \frac{1}{10}.10^k\left(10^k + 5\right),$$

and $$S_{15} = \sum_{k=1}^{\frac{1}{15}(10^k - 10)} 15k = \frac{1}{30}\left(10^k - 10\right)\left(10^k + 5\right).$$

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By inclusion-exclusion and simplifying the above expressions, the required sum is

$$S_3 + S_5 - S_{15} = \frac{1}{30}\left(7 \times 10^{2k} + 25 \times 10^k + 40\right)$$
$$= \frac{1}{6}\left(10^{2k} + 4 \times 10^{2k-1} + 5 \times 10^k + 8\right).$$

Thus $S_3 + S_5 - S_{15} = \frac{1}{6} \times 1400\ldots 0500\ldots 08$ where the blocks of 0's have $k - 2$ and $k - 1$ digits, respectively.

This simplifies to $233\ldots 34166\ldots 68$, as required, where the blocks of 3's and 6's each have $k - 2$ digits.

Correct solutions were received from: N. Curwen, S. Dolan, M. G. Elliott, R. Harris, A. P. Harrison, M. Hennings, G. Howlett, P. F. Johnson, P. King, Y. Kong, J. A. Mundie, D. Pinchon, S. Riccarelli, C. Starr, R. Summerson, A. Tee and the proposer Nicolo Sartori di Borgoricco.

**105.K** (Isaac Sofair)

Accounts of the famous "problème des ménages" split into those giving a summation for the result and those giving a recurrence relation. Prove that these are equivalent by showing algebraically that

$$A_n = \sum_{j=0}^{n} (-1)^j \frac{2n}{2n-j}\binom{2n-j}{j}(n-j)!$$

satisfies the recurrence relation

$$(n-1)A_{n+1} = \left(n^2 - 1\right)A_n + (n+1)A_{n-1} + 4(-1)^n, \qquad n > 2,$$

with $A_2 = 0$ and $A_3 = 1$.

Although fewer solutions were received for this problem than the other three, they proved to be the most varied. The 'bare hands' solution that follows is based on those of Graham Howlett, James Mundie, Samuele Riccarelli and (especially) Ray Harris. Didier Pinchon utilised formulae for the Chebyshev polynomials of the first kind, and Yong Kong used the Gosper-Zeilberger method to verify the result. James Mundie located a proof of the equivalence in [1] and Mark Hennings noted that Muir's discussion in [2] of the recurrence relation in **105.K** also contains an expression for the generating function of $(A_n)$.

Write $e_{n,j} = (-1)^j \frac{2n}{2n-j}\binom{2n-j}{j}(n-j)!$ so that $A_n = \sum_{j=0}^{n} e_{n,j}$. We aim to show that

$$(n-1)A_{n+1} - \left(n^2 - 1\right)A_n - (n+1)A_{n-1} = 4(-1)^n. \qquad (*)$$

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Consider the terms in $A_{n+1}$, $A_n$, $A_{n-1}$ with the following arrangement:

$$\begin{array}{llllllll}
A_{n+1} : & e_{n+1,0} & + e_{n+1,1} & + e_{n+1,2} + \dots & + e_{n+1,j} + \dots & + e_{n+1,n} & + e_{n+1,n+1} \\
A_n : & e_{n,0} & + e_{n,1} & + e_{n,2} + \dots & + e_{n,j} + \dots & + e_{n,n} & \\
A_{n-1} : & & & e_{n-1,0} + \dots & + e_{n-1,j-2} + \dots & + e_{n-1,n-2} & + e_{n-1,n-1}
\end{array}$$

Next, examine each column for its contribution to the left-hand side of $(*)$:

- $(n-1)e_{n+1,0} - (n^2-1)e_{n,0} = (n-1)(n+1)! - (n^2-1)n! = 0$
- $(n-1)e_{n+1,1} - (n^2-1)e_{n,1} = -[(n-1)2(n+1)n! - (n^2-1)2n(n-1)!]$ $= 0$
- $(n-1)e_{n+1,j} - (n^2-1)e_{n,j} - (n+1)e_{n+1,j-2}$

$$\begin{aligned}
&= (-1)^j\left[(n-1)\frac{2n+2}{2n+2-j}\binom{2n+2-j}{j}(n+1-j)!\right.\\
&\qquad \left. -(n^2-1)\frac{2n}{2n-j}\binom{2n-j}{j}(n-j)! - (n+1)\frac{2n-2}{2n-j}\binom{2n-j}{j-2}(n+1-j)!\right]\\
&= (-1)^j 2(n^2-1)(n-j)!\left[\frac{n+1-j}{2n+2-j}\binom{2n+2-j}{j} - \frac{n}{2n-j}\binom{2n-j}{j}\right.\\
&\qquad \left. - \frac{n+1-j}{2n-j}\binom{2n-j}{j-2}\right]\\
&= (-1)^j 2(n^2-1)\frac{(n-j)!(2n-j-1)!}{j!(2n-2j+2)!}(n+1-j)[(2n+1-j)(2n-j)\\
&\qquad -2n(2n-2j+1) - j(j-1)]\\
&= 0
\end{aligned}$$

- $(n-1)e_{n+1,n+1} - (n+1)e_{n-1,n-1} = (n-1)2(-1)^{n+1} - (n+1)2(-1)^{n-1}$ $= 4(-1)^n$, as required.

*References*

1. M. Hall, *Combinatorial theory*, Blaisdell (1967) p. 14.
2. T. Muir, Additional note on a problem of arrangement, *Proc. Roy. Soc. of Edinburgh* **11** (1880) pp. 187-190.

Correct solutions were received from: R. Harris, M. Hennings, G. Howlett, Y. Kong, J. A. Mundie, D. Pinchon, S. Riccarelli and the proposer Isaac Sofair.

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**105.L** (Geoffrey Brown)

For positive integers $n$, evaluate:

(a) $$\lim_{b\to\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{b^{n+k+1}}{n+k+1},$$

(b) $$\lim_{b\to\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{b^{n+k+\frac{1}{2}}}{n+k+\frac{1}{2}}.$$

*Answers*: (a) $\Gamma(n+1) = n!$, (b) $\Gamma\left(n+\frac{1}{2}\right) = \dfrac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2^n}\sqrt{\pi}$.

Most solvers argued along the following lines, only differing in how many properties of the incomplete/complete gamma function they chose to invoke.

From the exponential series we have, for $\alpha \geqslant 0$,

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}x^{\alpha+k} = x^{\alpha}e^{-x}$$

with the convergence uniform on $[0, b]$, $b > 0$. Termwise integration is thus valid to give

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{b^{\alpha+k+1}}{\alpha+k+1} = \int_0^b x^{\alpha}e^{-x}\,dx.$$

Letting $b \to \infty$ and applying the dominated convergence theorem then justifies

$$\lim_{b\to\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{b^{\alpha+k+1}}{\alpha+k+1} = \int_0^{\infty} x^{\alpha}e^{-x}\,dx = \Gamma(\alpha+1).$$

For (a), set $\alpha = n$ to get the answer $\Gamma(n+1) = n!$.
For (b), set $\alpha = n - \frac{1}{2}$ to get the answer

$$\Gamma\left(n+\tfrac{1}{2}\right) = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\ldots\frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2^n}\sqrt{\pi}.$$

Zoltan Retkes observed that the answers $\Gamma(n+1)$ and $\Gamma(n+\frac{1}{2})$ remain valid for all real $n \geqslant 0$.

Correct solutions were received from: N. Curwen, S. Dolan, M. G. Elliott, M. Hennings, G. Howlett, P. F. Johnson, Y. Kong, J. A. Mundie, Z. Retkes, S. Riccarelli, C. Starr, S. Stewart, R. Summerson and the proposer Geoffrey Brown.

Finally, Marian Dincă's solution to **105.G** arrived just too late for acknowledgement in the March 2022 *Gazette*.

N.J.L.

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