

A DICHOTOMY FOR SAMPLING BARRIER-CROSSING EVENTS OF RANDOM WALKS WITH REGULARLY VARYING TAILS

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Abstract

We study how to sample paths of a random walk up to the first time it crosses a fixed barrier, in the setting where the step sizes are independent and identically distributed with negative mean and have a regularly varying right tail. We introduce a desirable property for a change of measure to be suitable for exact simulation. We study whether the change of measure of Blanchet and Glynn (2008) satisfies this property and show that it does so if and only if the tail index α of the right tail lies in the interval $(1, 3/2)$.

Keywords: Conditional sampling; perfect sampling; efficiency; random walk; heavy tail; regularly varying tail; change of measure; likelihood ratio

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1. Introduction

Barrier-crossing events of random walks appear in numerous engineering and science models. Examples range from stationary waiting times in queues to ruin events in insurance risk processes [4], [13], [20]. Random walks with regularly varying step-size distributions are of particular interest, and their special analytic structure facilitates an increasingly complete understanding of associated rare events.

In this paper we consider the problem of sampling a path of a random walk until it crosses a given fixed barrier in the setting of heavy-tailed step sizes with negative mean. The higher the barrier, the lower the likelihood of reaching it. This poses challenges for conditional sampling, since naive Monte Carlo sampling devotes much computational time to paths that never cross the barrier and must therefore be ultimately discarded.

The ability to sample up to the first barrier-crossing time plays a central role in several related problems, such as for sampling paths up to their maximum [10] or for sampling only the maximum itself [14]. In turn, these have applications to perfect sampling from stationary distributions [7], [11] and to approximately solving stochastic differential equations [12].

Main contributions. The central question in this paper is: can the change of measure proposed by Blanchet and Glynn [8] be used for *exact*, that is, unbiased, conditional sampling of heavy-tailed random walks given a barrier-crossing rare event? The Blanchet–Glynn measure is designed to approximate such a conditional distribution, so one might therefore expect an answer to this question in the affirmative. Surprisingly, we answer this question, by and large,

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in the negative: this measure cannot be used for conditional sampling. Our results are a consequence of a delicate second-order analysis of tail probabilities of a sum of heavy-tailed random variables that are related to the *residual life tail distribution* of the random walk increments. The asymptotic decomposition and analysis we put forth may be of interest in itself.

We introduce a ‘desirable’ property for a candidate change of measure to be suitable for our exact sampling problem and reveal an intriguing dichotomy on the suitability of the Blanchet–Glynn measure: this measure satisfies this property if and only if the tail index is below the threshold $3/2$. It is worthwhile to stress two immediate consequences. First, our result roughly implies that only the heaviest tails stand a chance to be efficient in this setting, since the desirable property we introduce is intuitively a proxy for efficiency of the proposal measure. This is counterintuitive, since heavier tails typically make problems more difficult. Second, the threshold is not directly connected to the existence of integer moments for the step-size distribution.

The threshold $3/2$ also arises in the simulation literature involving barrier-crossing events with regularly varying step sizes [9], [19]. The nature of the threshold we obtain here is, however, different from these works for three reasons. First, these papers focus on estimating the rare event probability of exceeding a barrier; in contrast, our work focuses on sampling barrier-crossing paths. Second, these papers obtain that the heaviest tails are inefficient in their framework, while we obtain a form of inefficiency for lighter tails. Finally, and perhaps most importantly, the threshold $3/2$ in the existing literature is a direct consequence of requiring second-moment conditions of the estimator, while a direct relation with moments is absent for conditional sampling problems.

A by-product of our work is a counterexample for the statement of Proposition 4 of Blanchet and Glynn [8]. This proposition states that, for a broad class of heavy-tailed step sizes, the expected hitting time of the barrier grows linearly in the barrier level under the Blanchet–Glynn change of measure. We show, though, that this result does not always hold. This proposition is not central to the framework introduced in [8], and the issue we expose here can also be deduced from Corollary 1 of [9], but our result reopens the question of when the measure of Blanchet and Glynn induces a linear hitting time expectation.

Related literature. The primary means for exact or unbiased sampling from heavy-tailed random walks is based on the change of measure technique. Simply put, this procedure consists in sampling from a distribution different from the desired one and determining (or computing) the output using the likelihood ratio. The essential idea is that the changed or proposal distribution should emphasize characteristics of barrier-crossing paths.

The literature of *exact simulation* of barrier-crossing paths is closely related to the one of *estimating the probability* of exceeding the barrier. In the heavy-tailed setting, the latter problem has already been studied for two decades. In contrast, the exact path-sampling problem has only recently received attention, mostly driven by applications of *dominated coupling from the past* when in the presence of heavy tails; see [11].

For the probability estimating problem under heavy tails, early approaches can be found in [2], [5], and [17]. An important contribution for the current paper is [8], which was later followed by [9] and [19]. A recent new technique can be found in [16], in which *Markov Chain Monte Carlo* was used to estimate the multiplicative inverse of the probability of crossing the barrier.

The problem of exact sampling of paths with heavy tails, on the other hand, has only recently been tackled; see [11]. In this paper the authors modified the measure of [19], which focused on the probability estimation problem, and built on the scheme for *exact sampling* of paths introduced in [10, Section 4]. The approach studied in this paper is based on the

Blanchet–Glynn change of measure, which is conceptually simpler than the approach proposed in [11]. The search for a simpler algorithm provided the motivation for this paper.

Outline. This paper is organized as follows. In Section 2 we discuss the general preliminaries for our conditional sampling problem: a change of measure technique and the criterion we propose as a desirable property for efficiency for exact conditional sampling. In Section 3 we state our main result of efficiency for conditional sampling when using the Blanchet–Glynn change of measure [8] and with regularly varying step sizes. In Section 4 we compare our threshold result of Section 3 with similar ones in the literature of rare event sampling. In Section 5 we state a proof of the main result of Section 4.

Notation. We denote by $\{S_n\}$ the infinite length paths of the random walk. Given a probability measure \mathbb{Q} over $\{S_n\}$, we denote the expectation with respect to measure \mathbb{Q} as $\mathbb{E}^{\mathbb{Q}}$. We write $\mathbb{E}_y^{\mathbb{Q}}[\cdot] := \mathbb{E}^{\mathbb{Q}}[\cdot \mid S_0 = y]$ and omit y when $y = 0$, as customary in the literature. Given two probability measures P and Q over the same space, we denote *absolute continuity* of P with respect to Q as $P \ll Q$, meaning that, for all measurable B , $Q(B) = 0$ implies that $P(B) = 0$. For x, y real, we denote $x^+ := \max\{x, 0\}$, $x^- := -\min\{x, 0\}$, $x \wedge y := \min\{x, y\}$, and $x \vee y := \max\{x, y\}$. Also, for two functions f and g , we write $f(t) \sim g(t)$ when $\lim_{t \rightarrow \infty} f(t)/g(t) = 1$; we write $f(t) = O(g(t))$ when $\limsup_{t \rightarrow \infty} |f(t)/g(t)| < \infty$, and $f(t) = o(g(t))$ when $\lim_{t \rightarrow \infty} |f(t)/g(t)| = 0$.

2. Preliminaries

In this section we provide the background necessary for the exposition of our main result. In Section 2.1 we describe techniques for *exact*, or *unbiased*, conditional sampling using the change of measure technique. In Section 2.2 we give the criterion we propose as a desirable property for efficiency for this problem. In Section 2.3 we briefly introduce the Blanchet–Glynn [8] change of measure.

General setting. We consider a random walk $S_n := \sum_{i=1}^n X_i$, where X_i are independent and identically distributed, $\mathbb{E}|X_i| < \infty$, and $S_0 = 0$ unless explicitly stated otherwise. We assume that $\{S_n\}$ has *negative drift*, meaning that $\mathbb{E}X_i < 0$. We also assume that X_i has unbounded right support; that is, $\mathbb{P}(X_i > t) > 0$ for all $t \in \mathbb{R}$.

Given a *barrier* $b \geq 0$, let $\tau_b := \inf\{n \geq 0: S_n > b\}$ be the first barrier-crossing time. Since the random walk has negative drift, we have $S_n \rightarrow -\infty$ almost surely (a.s.) as $n \rightarrow \infty$, and also $\mathbb{P}(\tau_b = \infty) > 0$.

Our main goal is to study the suitability, with efficiency in mind, of using a change of measure to sample exactly paths (S_1, \dots, S_{τ_b}) conditional on $\{\tau_b < \infty\}$.

We remark that, for the sake of clarity of exposition, we will abuse notation and write ‘ (S_0, \dots, S_{τ_b}) follows the distribution $\mathbb{P}(\cdot \mid \tau_b < \infty)$ ’ to mean that, for all finite $n \in \mathbb{N}$, the random vector (S_0, \dots, S_{τ_b}) with $\tau_b = n$ has the distribution $\mathbb{P}(\cdot \mid \tau_b = n)$.

2.1. Exact conditional sampling via change of measure

We tackle the problem of *exact* or *unbiased* conditional sampling using the acceptance–rejection algorithm, which uses the change of measure technique. Here we give a brief exposition of these two methods.

Change of measure technique. Let $P(y, dz)$ be the *transition kernel* of the random walk, that is, $P(y, dz) = \mathbb{P}(S_1 \in dz \mid S_0 = y)$. We consider a ‘changed’ or ‘proposal’ transition kernel $Q(y, dz)$, which may be chosen *state dependent*, meaning that $Q(y_1, y_1 + \cdot)$ and $Q(y_2, y_2 + \cdot)$ may be different measures for $y_1 \neq y_2$. We assume that $P(y, \cdot) \ll Q(y, \cdot)$ for all y , which implies that the *likelihood ratio* function $dP/dQ(y, \cdot)$ exists. Letting \mathbb{Q} be the

distribution of $\{S_n\}$ induced by the proposal kernel Q , we slightly abuse notation and denote by $d\mathbb{P}/d\mathbb{Q}(S_n : 0 \leq n \leq T)$ the likelihood ratio of a finite path (S_0, \dots, S_T) . More precisely, for T finite $d\mathbb{P}/d\mathbb{Q}(S_n : 0 \leq n \leq T) := L_T$, where L_T is the nonnegative random variable satisfying $\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_B L_T] = \mathbb{E}^{\mathbb{P}}[\mathbf{1}_B]$ for all B in the σ -algebra $\sigma(S_n : 0 \leq n \leq T)$, where $\mathbf{1}_B$ is the indicator on event B . With this, it holds that

$$\frac{d\mathbb{P}}{d\mathbb{Q}}(S_n : 0 \leq n \leq T) = \frac{dP}{dQ}(S_0, S_1) \cdots \frac{dP}{dQ}(S_{T-1}, S_T)$$

for all T finite or \mathbb{Q} -a.s. finite stopping time. See [1, Section XIII.3] for further details.

Acceptance–rejection algorithm for exact conditional sampling. This procedure considers the situation of a distribution that is ‘difficult’ to sample from, and another distribution that is ‘easy’ to sample from; the aim is to simulate from the difficult distribution. The acceptance–rejection algorithm allows one to sample from the difficult distribution by repeatedly sampling from the easy, ‘proposal’, distribution. Here we show a known specialization of this technique to the problem of sampling paths from the conditional distribution $\mathbb{P}(\cdot \mid \tau_b < \infty)$; see [10].

Let P be the transition kernel of the random walk, and consider a ‘proposal’ kernel Q , possibly state dependent, such that $P(y, \cdot) \ll Q(y, \cdot)$ for all y . Assume that, for some computable constant $C > 0$, we have

$$\frac{d\mathbb{P}}{d\mathbb{Q}}(S_n : 0 \leq n \leq \tau_b) \mathbf{1}_{\{\tau_b < \infty\}} \leq C, \quad \mathbb{Q}\text{-a.s.}$$

If U is uniformly distributed on $[0, 1]$ under \mathbb{Q} and drawn independently from $\{S_n\}$, then it can be verified that

$$\mathbb{Q}\left(U \leq \frac{\mathbf{1}_{\{\tau_b < \infty\}}}{C} \frac{d\mathbb{P}}{d\mathbb{Q}}(S_n : 0 \leq n \leq \tau_b)\right) = \frac{\mathbb{P}(\tau_b < \infty)}{C}, \tag{2.1}$$

$$\mathbb{Q}\left(\{S_n\} \in \cdot \mid U \leq \frac{\mathbf{1}_{\{\tau_b < \infty\}}}{C} \frac{d\mathbb{P}}{d\mathbb{Q}}(S_n : 0 \leq n \leq \tau_b)\right) = \mathbb{P}(\{S_n\} \in \cdot \mid \tau_b < \infty), \tag{2.2}$$

over events $B \in \mathcal{F}_{\tau_b}$ such that $B \subseteq \{\tau_b < \infty\}$.

The acceptance–rejection procedure consists of iterating the steps:

- (i) sample jointly $(U, (S_0, \dots, S_{\tau_b}))$ from \mathbb{Q} , and
- (ii) check whether

$$U \leq \frac{\mathbf{1}_{\{\tau_b < \infty\}}}{C} \frac{d\mathbb{P}}{d\mathbb{Q}}(S_n : 0 \leq n \leq \tau_b) \tag{2.3}$$

holds.

The algorithm stops, ‘accepts’, the first time (2.3) is satisfied, and then outputs the path (S_0, \dots, S_{τ_b}) . Equation (2.1) states that a sample is ‘accepted’ with probability $\mathbb{P}(\tau_b < \infty)/C$; and (2.2) assures that *the simulation is exact*, that is, the distribution of the output is $\mathbb{P}(\cdot \mid \tau_b < \infty)$.

2.2. A desirable property for conditional sampling

We now propose a criterion for when a ‘changed’ or ‘proposal’ transition kernel Q is useful in sampling paths up to τ_b from the conditional distribution $\mathbb{P}(\cdot \mid \tau_b < \infty)$. Simply put, our proposed criterion states that all crossing events occur with higher probability under the proposal measure than under the original. Intuitively thus, this criterion is an efficiency condition for the exact conditional sampling problem.

Definition 2.1. (Direct proposal for exact conditional sampling.) Let $Q(y, dz)$ be a transition kernel such that $P(y, \cdot) \ll Q(y, \cdot)$ for all y . Let \mathbb{Q} be the distribution of $\{S_n\}$ on $\mathbb{R}^{\mathbb{N}}$ induced by Q . We say that \mathbb{Q} is a direct proposal for exact conditional sampling from $\mathbb{P}(\cdot \mid \tau_b < \infty)$ if and only if

$$\mathbb{Q}(\{S_n\} \in B) \geq \mathbb{P}(\{S_n\} \in B)$$

for all events $B \in \mathcal{F}_{\tau_b}$ such that $B \subseteq \{\tau_b < \infty\}$, and the inequality is strict for some such B . Here \mathcal{F}_{τ_b} is the usual σ -algebra associated to the stopping time τ_b .

We remark that the previous notion does not require $\mathbb{Q}(\tau_b < \infty) = 1$, although that is true for the Blanchet–Glynn change of measure, as we will see in Proposition 3.2. We also remark that, by the definition of the likelihood ratio, we have, for all $B \subseteq \{\tau_b < \infty\}$, it holds that $\mathbb{P}(\{S_n\} \in B) = \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_B \mathbf{1}_{\{\tau_b < \infty\}} d\mathbb{P}/d\mathbb{Q}(S_n : 0 \leq n \leq \tau_b)]$. Together with Definition 2.1, this identity yields the following equivalent condition for a proposal measure being direct for exact conditional sampling.

Corollary 2.1. The following statements are equivalent:

- (i) \mathbb{Q} is a direct proposal for exact conditional sampling from $\mathbb{P}(\cdot \mid \tau_b < \infty)$;
- (ii) $\mathbf{1}_{\{\tau_b < \infty\}} d\mathbb{P}/d\mathbb{Q}(S_n : 0 \leq n \leq \tau_b) \leq 1$ holds \mathbb{Q} -a.s.

An algorithmic equivalence and motivation. We now show yet another equivalent condition for a measure to be a direct proposal for exact conditional sampling. Informally speaking, this is an ‘algorithmic’ characterization, since it states the property of being a direct proposal as a *correctness* property of a simulation algorithm. This algorithm has been a keystone of several recent exact simulation works, see, e.g. [7], [10], [11], [12], and [18], since, in particular, it samples a Bernoulli random variable with parameter $\mathbb{P}(\tau_b < \infty)$, without the need to know the actual value of $\mathbb{P}(\tau_b < \infty)$. This algorithmic property initially motivated the research presented in the current paper, and also motivates the terminology for calling a proposal *direct* for exact conditional sampling.

Consider the following procedure: sample a path (S_0, \dots, S_{τ_b}) from \mathbb{Q} ; set $I := 1$ if $d\mathbb{P}/d\mathbb{Q}(S_n : 0 \leq n \leq \tau_b) \leq 1$, and $I := 0$ otherwise; output $(I, (S_0, \dots, S_{\tau_b}))$. It holds that if $\mathbb{Q}(\tau_b < \infty) = 1$ then parts (i) and (ii) of Corollary 2.1 are actually equivalent to the following statement: I is distributed as a Bernoulli random variable with parameter $\mathbb{P}(\tau_b < \infty)$ and if $I = 1$ then the sample path (S_0, \dots, S_{τ_b}) follows the distribution $\mathbb{P}(\cdot \mid \tau_b < \infty)$. Indeed, this is direct from (2.1) and (2.2) using $C = 1$, by Corollary 2.1(ii).

2.3. The Blanchet–Glynn change of measure

We now present the essential ideas of the Blanchet–Glynn change of measure [8]. This measure proved efficient for estimating the probability $\mathbb{P}(\tau_b < \infty)$ as $b \rightarrow \infty$. In the current paper we are interested in its use for the exact conditional sampling problem.

The main idea motivating the Blanchet–Glynn change of measure is to approximate the transition kernel of the conditional distribution. Indeed, it is well known that the one-step transition kernel of $\mathbb{P}(\cdot \mid \tau_b < \infty)$, say $Q^{b,\infty}$, satisfies

$$Q^{b,\infty}(y, dz) = P(y, dz) \frac{\mathbb{P}_z(\tau_b < \infty)}{\mathbb{P}_y(\tau_b < \infty)}, \tag{2.4}$$

where P is the original transition kernel of $\{S_n\}$; see [3, Section VI.7]. Here, the term $\mathbb{P}_y(\tau_b < \infty)$ in the denominator of (2.4) can be interpreted as a normalizing term, since

$\int P(y, dz)\mathbb{P}_z(\tau_b < \infty) = \mathbb{P}_y(\tau_b < \infty)$ for all y . It is nevertheless impractical to simulate from this kernel because usually the values of $\mathbb{P}_z(\tau_b < \infty)$ for $z < b$ are not readily known. To be consistent with the notation of Blanchet and Glynn [8], we denote $u^*(x) := \mathbb{P}_x(\tau_0 < \infty)$ for all $x \in \mathbb{R}$; in particular, the definition of $Q^{b,\infty}$ in (2.4) takes the form

$$Q^{b,\infty}(y, dz) = P(y, dz) \frac{u^*(z - b)}{u^*(y - b)}.$$

The idea put forth by Blanchet and Glynn is to approximate u^* using the asymptotic approximation given by the Pakes–Veraverbeke theorem; see [15, Chapter 5]. This result states that

$$u^*(x) = \mathbb{P}_x(\tau_0 < \infty) \sim \frac{1}{|\mathbb{E}X|} \int_{-x}^\infty \mathbb{P}(X > s) ds \quad \text{as } x \rightarrow -\infty \tag{2.5}$$

for random walks with negative drift and step sizes X which are (right) *strongly subexponential*. Inspired by this fact, the Blanchet–Glynn change of measure uses the following transition kernel:

$$Q^{b,c}(y, dz) := P(y, dz) \frac{v(z - b - c)}{w(y - b - c)}, \tag{2.6}$$

where

$$v(x) := \min \left\{ 1, \frac{1}{|\mathbb{E}X|} \int_{-x}^\infty \mathbb{P}(X > s) ds \right\}$$

and $w(y - b - c) := \int P(y, dz)w(z - b - c) = \int P(y - b, dz)v(z - b - c)$ is a normalizing term. The constant $c \in \mathbb{R}$ is a translation parameter, which in [8] and in our work, we will see, is eventually chosen sufficiently large. Nonetheless, a heuristic but ultimately fallacious argument for choosing large c is that if we could choose ‘ $c = \infty$ ’ then by the Pakes–Veraverbeke asymptotic result (2.5), we would have ‘ $Q^{b,c} = Q^{b,\infty}$ ’, that is, the Blanchet–Glynn measure $Q^{b,c}$ matches the conditional one-step transition kernel $Q^{b,\infty}$.

In proving our results for this transition kernel we heavily rely on the fact that the functions v and w are closely related to the *residual life tail distribution* of X . That is, a random variable Z with distribution given by

$$\mathbb{P}(Z > t) := \min \left\{ 1, \frac{1}{|\mathbb{E}X|} \int_t^\infty \mathbb{P}(X > s) ds \right\} \quad \text{for all } t. \tag{2.7}$$

We thus have $v(x) = \mathbb{P}(Z > -x)$ and $w(x) = \mathbb{P}(X + Z > -x)$ for all x , and, in particular, $Q^{b,c}(y, \cdot) = Q^{b,c}(S_1 - S_0 \in \cdot \mid S_0 = y) = \mathbb{P}(X \in \cdot \mid X + Z > c + b - y)$, where X and Z are independent. For further details, we refer the reader to [3, Section VI.7] and [8].

3. Main result: a threshold for being a direct proposal

In this section we present our main result on whether the Blanchet–Glynn [8] change of measure is a direct proposal for exact conditional sampling of random walks with regularly varying step sizes. Our main result is Theorem 3.1, in which we establish a dichotomy for the tail index of the step-size distribution. We state two results from which our main result easily follows. The first is a characterization for when the Blanchet–Glynn change of measure is a direct proposal for exact conditional sampling, and the second explores this characterization in the case of regularly varying step sizes. Finally, we study the time at which the barrier is hit under the Blanchet–Glynn measure.

Main result. We now describe the main result of this paper. We work under the following assumptions on the distribution of the step sizes, in addition to the assumption of negative drift, that is, $\mathbb{E}X < 0$.

Assumption 3.1. *We have the following assumptions.*

- (A1) *The right tail $\mathbb{P}(X^+ > \cdot)$ is regularly varying with tail index $\alpha > 1$; that is, for all $u > 0$ we have $\mathbb{P}(X > ut) \sim u^{-\alpha}\mathbb{P}(X > t)$ as $t \rightarrow \infty$.*
- (A2) *The left tail $\mathbb{P}(X^- > \cdot)$ decays fast enough so that there exists a function $h(t) = o(t)$ such that $h(t) \rightarrow \infty$ and $\int_{h(t)}^\infty \mathbb{P}(X^- > s) ds = o(t \cdot \mathbb{P}(X^+ > t))$ as $t \rightarrow \infty$.*
- (A3) *The step size distribution has a continuous density which is regularly varying with tail index $\alpha + 1$.*

We note that the more natural condition $\mathbb{P}(X^- > t) = o(\mathbb{P}(X^+ > t))$ as $t \rightarrow \infty$ does not necessarily imply assumption (A2); although it does imply that $\int_{h(t)}^\infty \mathbb{P}(X^- > s) ds = o(h(t)\mathbb{P}(X^+ > h(t)))$ for all h such that $h(t) \rightarrow \infty$. Nonetheless, assumption (A2) is not overly restrictive. Indeed, a stronger condition is that there exists $\delta > 0$ such that $t^\delta \mathbb{P}(X^- > t) = O(\mathbb{P}(X^+ > t))$ as $t \rightarrow \infty$; the latter holds, for instance, when $\mathbb{P}(X^- > \cdot)$ is light-tailed, or when $\mathbb{P}(X^- > \cdot)$ is regularly varying with tail index β satisfying $\beta > \alpha$. We also note that assumption (A3) can be replaced by the less restrictive assumption that the step-size distribution be ultimately absolutely continuous with respect to Lebesgue measure, with continuous and regularly varying density. More precisely, it can be replaced by the assumption that there exists some t_0 such that on $[t_0, \infty)$, the step-size distribution has a continuous density $f(\cdot)$ which is regularly varying with tail index $\alpha + 1$.

The main result of this paper follows.

Theorem 3.1. (Direct proposal with regularly varying right tails.) *Let $\mathbb{Q}^{b,c}$ be the distribution of $\{S_n\}$ induced by the transition kernel $Q^{b,c}$ defined in (2.6). Under assumptions (A1)–(A3) the following hold.*

- (i) *If $\alpha \in (1, 3/2)$ then there exists some sufficiently large c so that $\mathbb{Q}^{b,c}$ is a direct proposal for exact conditional sampling from $\mathbb{P}(\cdot \mid \tau_b < \infty)$ for all $b \geq 0$.*
- (ii) *If $\alpha \in (3/2, 2)$ then, for all $c \in \mathbb{R}$ and all $b \geq 0$, it holds that $\mathbb{Q}^{b,c}$ is not a direct proposal for exact conditional sampling from $\mathbb{P}(\cdot \mid \tau_b < \infty)$.*

It is noteworthy that the change of measure is direct for exact conditional sampling only for step sizes with very heavy tails. Indeed, recall that the tail index α is an indicator of how heavy a tail is; c.f. $\mathbb{E}[(X^+)^p] < \infty$ for $p \in (1, \alpha)$ and $\mathbb{E}[(X^+)^p] = \infty$ for $p > \alpha$.

Proof elements. We show here the main elements of the proof of Theorem 3.1, and start by investigating in Proposition 3.1 how the following statements are related. The proof is deferred to Appendix A.

- (S1_b^c) *The distribution $\mathbb{Q}^{b,c}$ induced by the Blanchet–Glynn kernel (2.6) is direct for exact conditional sampling from $\mathbb{P}(\cdot \mid \tau_b < \infty)$.*
- (S2) *We have $\mathbb{P}(X + Z > t) \leq \mathbb{P}(Z > t)$ for all sufficiently large t , where Z has the residual life distribution (2.7) and is independent of X .*

Proposition 3.1. (i) *If (S2) holds then there exists some sufficiently large c so that (S1_b^c) holds for all $b \geq 0$.*

(ii) *Suppose that $\mathbb{P}(|X| \leq \delta) > 0$ for all $\delta > 0$. If $(S1_b^c)$ holds for some $b \geq 0$ and some $c \in \mathbb{R}$ then $(S2)$ also holds.*

We remark that part (i) says that the same parameter c , chosen sufficiently large, works for all barriers $b \geq 0$; that is, b is independent of c in this case. We also remark that in the case of (ii), applying (i), we find that $(S1_b^c)$ actually holds for all $b \geq 0$, possibly after changing the constant c .

It is shown in [8] that $\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t) = o(\mathbb{P}(X > t))$ as $t \rightarrow \infty$ for the family of strongly subexponential distributions, which includes regularly varying tails. Hence, the previous proposition shows that for a measure to be direct for exact conditional sampling it is not enough to know that the difference decays faster than $\mathbb{P}(X > t)$, we actually need the sign of the difference as $t \rightarrow \infty$.

The following result shows that, in the case of step sizes satisfying assumptions (A1) and (A2), the sign of $\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t)$ when $t \rightarrow \infty$ is fully determined by the tail index α of the right tail distribution. The proof is given in Section 5.

Theorem 3.2. *Suppose that assumptions (A1)–(A3) hold. Let Z be a random variable independent of X with the residual life distribution (2.7). Then the following statements hold.*

- (i) *If $\alpha \in (1, 3/2)$ then $\mathbb{P}(X + Z > t) \leq \mathbb{P}(Z > t)$ for all $t > 0$ sufficiently large.*
- (ii) *If $\alpha \in (3/2, 2)$ then $\mathbb{P}(X + Z > t) \geq \mathbb{P}(Z > t)$ for all $t > 0$ sufficiently large.*

With this, Theorem 3.1 is a corollary of Proposition 3.1 and Theorem 3.2.

Hitting time analysis. We now investigate the finiteness and mean value of the hitting time τ_b under the Blanchet–Glynn change of measure $\mathbb{Q}^{b,c}$. The motivation is that τ_b gives a rough estimate of the computational effort of sampling a barrier-crossing path using the measure $\mathbb{Q}^{b,c}$.

The following result explores the hitting time in the regularly varying right tails setting of assumptions (A1) and (A2). Its proof is deferred to Appendix B.

Proposition 3.2. (Hitting time under $\mathbb{Q}^{b,c}$.) *Let \mathbb{Q} be the distribution of $\{S_n\}$ induced by the transition kernel $Q^{b,c}$ defined in (2.6). Consider the setting of assumptions (A1)–(A3). For any sufficiently large c , the following hold for all $b \geq 0$.*

- (i) *If $\alpha > 1$ then $\mathbb{Q}^{b,c}(\tau_b < \infty) = 1$.*
- (ii) *If $\alpha \in (1, 3/2)$ then $\mathbb{E}^{\mathbb{Q}^{b,c}} \tau_b = \infty$ for all $b \geq 0$.*
- (iii) *If $\alpha > 2$ then $\mathbb{E}^{\mathbb{Q}^{b,c}} \tau_b = O(b)$ as $b \rightarrow \infty$.*

We remark that Proposition 3.1(ii), although a negative result, is actually independent of the Blanchet–Glynn measure $\mathbb{Q}^{b,c}$ and holds essentially because we have $\mathbb{E}^{\mathbb{P}}[\tau_b \mid \tau_b < \infty] = \infty$ when $\alpha \in (1, 2)$; see, e.g. [4, Theorem 1.1]. In other words, if $\alpha \in (1, 2)$ no algorithm or change of measure—direct or not—sampling paths (S_0, \dots, S_{τ_b}) from $\mathbb{P}(\cdot \mid \tau_b < \infty)$ can produce paths of finite expected length.

We also remark that Proposition 3.2(ii) is a counterexample for Proposition 4 of [8]. Indeed, the latter result claims that we have $\mathbb{E}^{\mathbb{Q}^{b,c}} \tau_b = O(b)$ as $b \rightarrow \infty$ when the step sizes satisfy $\mathbb{E}^{\mathbb{P}}[X^p; X > 0] < \infty$ for some $p > 1$. Proposition 3.2(ii) shows that the latter condition is not enough in general. Alternatively, this issue with Proposition 4 of [8] can also be derived from Corollary 1 of [9]. The latter result shows that if $\alpha \in (1, 3/2)$ no change of measure can be at the same time *strongly efficient for importance sampling* and have linear expected hitting time; in contrast, Proposition 4 of [8] states that, for any $\alpha > 1$, the Blanchet–Glynn measure

is both *strongly efficient for importance sampling* and has linear expected hitting time. Clearly both results are contradictory.

4. Threshold 3/2: a comparison

In this section we compare the threshold result of Theorem 3.1 with previous simulation works where, when using regularly varying step sizes, some form of efficiency of the method has given rise to the same threshold 3/2 for the tail index. We argue that the threshold arises in this existing literature for reasons unrelated to our work.

Review. Previous works in which the 3/2 threshold appears in the context of efficiency are Blanchet and Liu [9] and Murthy *et al.* [19]. In both papers, the authors focus on solving the probability estimation problem via importance sampling; that is, their aim is to estimate the probability $\mathbb{P}(\tau_b < \infty)$ for arbitrarily large barriers b , using Monte Carlo sampling from another measure. Blanchet and Liu proposed a parameterized and state-dependent change of measure, say \mathbb{Q}^{BL} , which in the regularly varying case takes the form of a mixture between a *big-* and a *small-jump* transition kernel. Murthy *et al.* [19] proposed a similar big- and small-jump mixture kernel, say \mathbb{Q}^{MJB} , however their change of measure is state-independent and additionally conditions on the time interval at which the barrier-crossing event occurs.

Both Blanchet and Liu [9] and Murthy *et al.* [19] have two requirements on their proposed measures:

- (i) the linear scaling $\mathbb{E}^{\mathbb{Q}}\tau_b = O(b)$ as $b \rightarrow \infty$ of the hitting time, and
- (ii) *strong efficiency* of the estimation procedure.

In short, the latter means that, under the proposed change of measure \mathbb{Q} , the coefficient of variation of the random variable

$$\frac{d\mathbb{P}}{d\mathbb{Q}}(S_n : 0 \leq n \leq \tau_b) \mathbf{1}_{\{\tau_b < \infty\}} \quad (4.1)$$

stays bounded as $b \rightarrow \infty$; this is a second-moment condition on (4.1). The authors of both papers arrive at the same threshold result: for regularly varying step sizes, the proposed change of measure satisfies the previous two requirements for some combination of tuning parameters if and only if the tail index α is greater than 3/2.

Comparison. Given that the same threshold appears, it is natural to ask if there is a connection between our result in Corollary 3.1 and the results in prior works. We now argue why there is no clear or direct connection between these results.

In [9] and [19], the threshold 3/2 is strictly related to the second-moment condition over the likelihood ratio (4.1) that is imposed by the requirement of efficiency for importance sampling. More precisely, in both these works if a moment condition is imposed on a different moment than the second, then we have a different threshold for the admissible tail indexes. In contrast, our result arises from imposing an almost sure condition on the Blanchet–Glynn change of measure. Indeed, by Corollary 2.1, the condition of a proposal measure being direct for exact conditional sampling is a \mathbb{Q} -almost sure condition on the random variable (4.1). In contrast, and as said before, efficiency for importance sampling is a second-moment condition on (4.1).

5. Proof of Theorem 3.2

In this section we prove Theorem 3.2, which is the main component of our main result of Theorem 3.1. That is, we prove that the tail index α completely determines the sign of the

difference

$$\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t) \tag{5.1}$$

when t is large enough. We work under assumptions (A1)–(A3), which state, roughly speaking, that the step sizes have regularly varying right tails with tail index α , and lighter left tails. In short, in Theorem 3.2 we establish that if $\alpha \in (1, 3/2)$ then the difference (5.1) is negative for large t , and positive if $\alpha \in (3/2, 2)$.

The following is a roadmap for the main steps of the proof. First, in Lemma 5.1 we write the difference (5.1) as a sum of several terms. Second, in Lemma 5.2 we carry out an asymptotic analysis to determine which terms dominate when $t \rightarrow \infty$. It follows that the sign of the difference (5.1) when $t \rightarrow \infty$ can be reduced to the sign of the sum of dominant terms when $t \rightarrow \infty$. Finally, the latter is analyzed in Lemma 5.3, which reveals the dichotomy for α in $(1, 3/2)$ or $(3/2, 2)$.

Before embarking on the proof, some remarks on our notation are in order. Recall that we say that the random variable Z has the *residual life distribution of X* if its distribution is given by

$$\mathbb{P}(Z > t) = \min \left\{ 1, \frac{1}{|\mathbb{E}X|} \int_t^\infty \mathbb{P}(X > s) ds \right\} \text{ for all } t.$$

We write the left-most point of the support of Z as $z_0 := \inf\{t : \mathbb{P}(Z > t) < 1\}$, which is finite since $\mathbb{E}X$ is also finite. Additionally, we use the fact that the density of Z is $\mathbb{P}(X > t)/|\mathbb{E}X|$ for all $t > z_0$ and that $\int_{z_0}^\infty \mathbb{P}(X > s) ds = |\mathbb{E}X|$. We also use the notation $F(t) := \mathbb{P}(X \leq t)$ and $\bar{F}(t) := \mathbb{P}(X > t)$ for all t . Finally, we recall that assumption (A1) establishes that the right tail $\mathbb{P}(X > \cdot)$ is regularly varying with *tail index* $\alpha > 1$.

We start with a general decomposition of the difference (5.1).

Lemma 5.1. *Let X be a random variable with negative mean, and let Z be independent of X with the residual life distribution of X . Consider any function h such that $\max\{z_0, 0\} < h(t) < t/2$ for all $t > \max\{2z_0, 0\}$. Then the following holds for $t > \max\{2z_0, 0\}$:*

$$\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t) = p(t) - q(t) + \varepsilon_1(t) - \varepsilon_2(t),$$

where we define, for $t > \max\{2z_0, 0\}$,

$$\begin{aligned} p(t) &:= \frac{1}{|\mathbb{E}X|} \int_{h(t)}^{t-h(t)} \bar{F}(t-s)\bar{F}(s) ds, \\ q(t) &:= \frac{\bar{F}(t)}{|\mathbb{E}X|} \int_{h(t)}^\infty [2\bar{F}(s) - F(-s)] ds, \\ \varepsilon_1(t) &:= \frac{1}{|\mathbb{E}X|} \left[\left(\int_0^{h(t)} + \int_{z_0}^{h(t)} \right) [\bar{F}(t-s) - \bar{F}(t)]\bar{F}(s) ds \right. \\ &\quad \left. + \int_{-h(t)}^0 [\bar{F}(t) - \bar{F}(t-s)]F(s) ds \right], \\ \varepsilon_2(t) &:= \frac{1}{|\mathbb{E}X|} \int_{-\infty}^{-h(t)} \bar{F}(t-s)F(s) ds. \end{aligned}$$

Proof. First note that

$$\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t) = \mathbb{P}(X + Z > t, Z \leq t) - \mathbb{P}(X + Z \leq t, Z > t).$$

For t satisfying $\max\{z_0, 0\} < h(t) < t/2$, we decompose the first term on the right-hand side as follows:

$$\begin{aligned} \mathbb{P}(X + Z > t, Z \leq t) &= \int_{z_0}^t \bar{F}(t-s) \frac{1}{|\mathbb{E}X|} \bar{F}(s) \, ds \\ &= \frac{1}{|\mathbb{E}X|} \left[\int_{z_0}^{h(t)} \bar{F}(t) \bar{F}(s) \, ds + \int_{z_0}^{h(t)} [\bar{F}(t-s) - \bar{F}(t)] \bar{F}(s) \, ds \right. \\ &\quad + \int_{h(t)}^{t-h(t)} \bar{F}(t-s) \bar{F}(s) \, ds + \int_0^{h(t)} \bar{F}(t) \bar{F}(s) \, ds \\ &\quad \left. + \int_0^{h(t)} [\bar{F}(t-s) - \bar{F}(t)] \bar{F}(s) \, ds \right]. \end{aligned}$$

A similar decomposition follows for the second term:

$$\begin{aligned} \mathbb{P}(X + Z \leq t, Z > t) &= \int_t^\infty F(t-s) \frac{1}{|\mathbb{E}X|} \bar{F}(s) \, ds \\ &= \frac{1}{|\mathbb{E}X|} \left[\int_{-\infty}^{-h(t)} F(s) \bar{F}(t-s) \, ds + \int_{-h(t)}^0 F(s) \bar{F}(t) \, ds \right. \\ &\quad \left. + \int_{-h(t)}^0 F(s) [\bar{F}(t-s) - \bar{F}(t)] \, ds \right]. \end{aligned}$$

Subtracting both terms, we obtain

$$\begin{aligned} &|\mathbb{E}X| [\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t)] \\ &= \int_{h(t)}^{t-h(t)} \bar{F}(t-s) \bar{F}(s) \, ds \\ &\quad - \bar{F}(t) \left[\int_{-h(t)}^0 F(s) \, ds - \int_{z_0}^{h(t)} \bar{F}(s) \, ds - \int_0^{h(t)} \bar{F}(s) \, ds \right] \\ &\quad + \left[\left(\int_{z_0}^{h(t)} + \int_0^{h(t)} \right) [\bar{F}(t-s) - \bar{F}(t)] \bar{F}(s) \, ds \right. \\ &\quad \left. - \int_{-h(t)}^0 [\bar{F}(t-s) - \bar{F}(t)] F(s) \, ds \right] - \int_{-\infty}^{-h(t)} F(s) \bar{F}(t-s) \, ds \\ &= |\mathbb{E}X| [p(t) - q(t) + \varepsilon_1(t) - \varepsilon_2(t)]. \end{aligned}$$

The last equality comes from using the definition of p, q, ε_1 , and ε_2 , and noting that $|\mathbb{E}X| = \int_{z_0}^\infty \bar{F}(s) \, ds$ and $\mathbb{E}X < 0$, so we have

$$\begin{aligned} &\int_{-h(t)}^0 F(s) \, ds - \int_{z_0}^{h(t)} \bar{F}(s) \, ds - \int_0^{h(t)} \bar{F}(s) \, ds \\ &= \mathbb{E}X^- - |\mathbb{E}X| - \mathbb{E}X^+ + 2 \int_{h(t)}^\infty \bar{F}(s) \, ds - \int_{-\infty}^{-h(t)} F(s) \, ds \\ &= \int_{h(t)}^\infty [2\bar{F}(s) - F(-s)] \, ds. \end{aligned}$$

This concludes the proof. □

The next step consists in determining which terms dominate when $t \rightarrow \infty$; this is carried out in the following result.

Lemma 5.2. *Let X be a random variable with negative mean satisfying assumptions (A1)–(A3) for some index of regular variation $\alpha \in (1, 2)$. Let Z be independent of X with the residual life distribution of X . In the definition of p, q, ε_1 , and ε_2 , consider a function h satisfying assumption (A2); in particular, h satisfies $\max\{z_0, 0\} < h(t) < t/2$ for all $t > \max\{2z_0, 0\}$, and it holds that $h(t) \rightarrow \infty$ and $h(t) = o(t)$ as $t \rightarrow \infty$. Then the following hold as $t \rightarrow \infty$:*

- (i) $p(t) - q(t) \sim \frac{t\bar{F}(t)^2}{|\mathbb{E}X|} \left(2 \int_0^{1/2} [(1-u)^{-\alpha} - 1]u^{-\alpha} du - \frac{2^\alpha}{\alpha - 1} \right)$,
- (ii) $\varepsilon_1(t) = o(t\bar{F}(t)^2)$ and $\varepsilon_2(t) = o(t\bar{F}(t)^2)$.

We remark that Lemmas 5.1 and 5.2 together establish that if X satisfies assumptions (A1)–(A3) and $\alpha \in (1, 2)$ then

$$\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t) \sim K_\alpha \mathbb{P}(Z > t)\mathbb{P}(X > t) \quad \text{as } t \rightarrow \infty, \tag{5.2}$$

where

$$K_\alpha := (\alpha - 1) \int_0^1 ((1-u)^{-\alpha} - 1)(u^{-\alpha} - 1) du - (\alpha + 1).$$

This comes from $\mathbb{P}(Z > t) \sim t\bar{F}(t)/((\alpha - 1)|\mathbb{E}X|)$ by Karamata’s theorem [6, Theorem 1.6.1]. We note that, in contrast, Proposition 3 of [8] shows that $\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t) = o(\mathbb{P}(X > t))$, so result (5.2) is much finer.

Proof of Lemma 5.2. (i) First rewrite $|\mathbb{E}X|(p(t) - q(t))/t\bar{F}(t)^2$ as

$$2 \int_{h(t)}^{t/2} \frac{\bar{F}(t-s) - \bar{F}(t)}{\bar{F}(t)} \frac{\bar{F}(s)}{t\bar{F}(t)} ds - \frac{2 \int_{t/2}^\infty \bar{F}(s) ds}{t\bar{F}(t)} + \frac{\int_{h(t)}^\infty \bar{F}(s) ds}{t\bar{F}(t)}.$$

The third term goes to 0 by assumption (A2), so we can ignore it for the proof of the statement. For the second term, note that since $\alpha > 1$, Karamata’s theorem [6, Theorem 1.6.1] yields

$$2 \int_{t/2}^\infty \bar{F}(s) ds \sim \frac{t\bar{F}(t/2)}{\alpha - 1} \sim \frac{2^\alpha}{\alpha - 1} t\bar{F}(t).$$

It remains to investigate the first term. To this end, we first rewrite the integral as

$$\int_{h(t)/t}^{1/2} \left[\frac{\bar{F}(t(1-u))}{\bar{F}(t)} - 1 \right] \frac{\bar{F}(tu)}{\bar{F}(t)} du; \tag{5.3}$$

we need to show that as $t \rightarrow \infty$ this integral converges to $\int_0^{1/2} [(1-u)^{-\alpha} - 1]u^{-\alpha} du$. To this end, consider $\delta \in (0, \frac{1}{2})$ and note that, since $h(t) = o(t)$, we can write (5.3), for all sufficiently large t , as

$$\int_{h(t)/t}^\delta \left[\frac{\bar{F}(t(1-u))}{\bar{F}(t)} - 1 \right] \frac{\bar{F}(tu)}{\bar{F}(t)} du + \int_\delta^{1/2} \left[\frac{\bar{F}(t(1-u))}{\bar{F}(t)} - 1 \right] \frac{\bar{F}(tu)}{\bar{F}(t)} du. \tag{5.4}$$

We start by analyzing the second term in (5.4). Since \bar{F} is regularly varying with tail index α , by the uniform convergence theorem [6, Theorem 1.2.1], we obtain

$$\sup_{u \in [\delta, 1/2]} \left| \left[\frac{\bar{F}(t(1-u))}{\bar{F}(t)} - 1 \right] \frac{\bar{F}(tu)}{\bar{F}(t)} - [(1-u)^{-\alpha} - 1]u^{-\alpha} \right| \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

so

$$\int_{\delta}^{1/2} \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1 \right] \frac{\overline{F}(tu)}{\overline{F}(t)} du \rightarrow \int_{\delta}^{1/2} [(1-u)^{-\alpha} - 1]u^{-\alpha} du \quad \text{as } t \rightarrow \infty. \quad (5.5)$$

We next analyze the first term in (5.4). Using assumption (A3), we apply the mean value theorem on the interval $(0, u)$ to the function $s \mapsto \overline{F}(t(1-s))$ and establish that $\overline{F}(t(1-u))/\overline{F}(t) - 1 = f(t(1-\xi))tu/\overline{F}(t)$ for some $\xi = \xi(t, u) \in (0, u)$, where f is the density of X . Additionally, for all sufficiently large t , it holds that $f(t(1-\xi))/\overline{F}(t) \leq 2(1+2^{\alpha+1})\alpha$ for all $\xi \in (0, \delta)$. Indeed, since f is regularly varying with tail index $\alpha + 1$ then by Karamata's theorem [6, Theorem 1.6.1], we obtain $tf(t)/\overline{F}(t) \rightarrow \alpha$; additionally, by the uniform convergence theorem [6, Theorem 1.2.1], we have, for any large enough t ,

$$\sup_{\xi \in (0, \delta)} \left| \frac{f(t(1-\xi))}{f(t)} - \frac{1}{(1-\xi)^{\alpha+1}} \right| \leq 1,$$

so $f(t(1-\xi))/f(t) \leq 1 + 1/(1-\xi)^{\alpha+1} \leq 1 + 2^{\alpha+1}$ for all sufficiently large t and for all $\xi \in (0, \delta) \subset (0, \frac{1}{2})$. We conclude that $\overline{F}(t(1-u))/\overline{F}(t) - 1 \leq 2(1+2^{\alpha+1})\alpha u$ for all large enough t . We use this inequality to bound the term in the brackets of the first term of (5.4), obtaining, for all sufficiently large t ,

$$\int_{h(t)/t}^{\delta} \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1 \right] \frac{\overline{F}(tu)}{\overline{F}(t)} du \leq 2(1+2^{\alpha+1})\alpha \int_{h(t)/t}^{\delta} \frac{tu\overline{F}(tu)}{t\overline{F}(t)} du.$$

We now argue that $\int_{h(t)/t}^{\delta} (tu\overline{F}(tu))/(t\overline{F}(t)) du \leq 2\delta^{2-\alpha}/(2-\alpha)$ for all sufficiently large t . Indeed,

$$\int_{h(t)/t}^{\delta} \frac{tu\overline{F}(tu)}{t\overline{F}(t)} du = \left(\int_0^{\delta t} - \int_0^{h(t)} \right) \frac{s\overline{F}(s)}{t^2\overline{F}(t)} ds,$$

so using the fact that $t\overline{F}(t)$ is regularly varying with tail index $\alpha - 1 \in (0, 1)$, we can apply Karamata's theorem [6, Theorem 1.6.1] and the fact that $h(t) \rightarrow \infty$ to obtain

$$\int_{h(t)/t}^{\delta} \frac{tu\overline{F}(tu)}{t\overline{F}(t)} du \sim \frac{1}{2-\alpha} \left(\delta^2 \frac{\overline{F}(\delta t)}{\overline{F}(t)} - \left(\frac{h(t)}{t} \right)^2 \frac{\overline{F}(h(t))}{\overline{F}(t)} \right) \sim \frac{\delta^{2-\alpha}}{2-\alpha} \quad \text{as } t \rightarrow \infty.$$

Indeed, note that since $h(t) = o(t)$ and $s \mapsto s^2\overline{F}(s)$ is regularly varying with tail index $\alpha - 2 \in (-1, 0)$ then $(h(t)/t)^2(\overline{F}(h(t))/\overline{F}(t)) \rightarrow 0$. All in all, we find that the first term of (5.4) satisfies, for all large enough t ,

$$\int_{h(t)/t}^{\delta} \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1 \right] \frac{\overline{F}(tu)}{\overline{F}(t)} du \leq \frac{4(1+2^{\alpha+1})\alpha}{2-\alpha} \delta^{2-\alpha}.$$

Finally, note that $\delta \in (0, \frac{1}{2})$ is arbitrary, so letting δ decrease to 0 in the latter inequality leads to

$$\lim_{\delta \searrow 0} \limsup_{t \rightarrow \infty} \int_{h(t)/t}^{\delta} \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1 \right] \frac{\overline{F}(tu)}{\overline{F}(t)} du = 0. \quad (5.6)$$

Similarly, letting δ decrease to 0 in (5.5), we obtain

$$\lim_{\delta \searrow 0} \lim_{t \rightarrow \infty} \int_{\delta}^{1/2} \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1 \right] \frac{\overline{F}(tu)}{\overline{F}(t)} du = \int_0^{1/2} [(1-u)^{-\alpha} - 1]u^{-\alpha} du. \quad (5.7)$$

From (5.6), (5.7), and (5.4) (the decomposition of (5.3)), we obtain the desired result.

(ii) First, we show that $\varepsilon_1(t) = o(t\bar{F}(t)^2)$. To this end, it is sufficient to prove $\int_0^{h(t)} [\bar{F}(t-s) - \bar{F}(t)]\bar{F}(s) ds = o(t\bar{F}(t)^2)$. Indeed, if $z_0 < 0$ it holds that $\int_{z_0}^{h(t)} [\bar{F}(t-s) - \bar{F}(t)]\bar{F}(s) ds = o(t\bar{F}(t)^2)$ as $t \rightarrow \infty$, since

$$\left| \int_{z_0}^0 [\bar{F}(t-s) - \bar{F}(t)]\bar{F}(s) ds \right| \leq \int_0^{|z_0|} [\bar{F}(t) - \bar{F}(t+s)] ds \leq |z_0| \sup_{s \in [t, t+|z_0|]} f(s) \sim |z_0|f(t)$$

by the uniform convergence theorem [15, Chapter 2] and because f is, in particular, long-tailed by assumption (A3); and since $\alpha \in (1, 2)$ then $f(t) = o(t\bar{F}(t)^2)$. It is sufficient then to prove that the expression

$$\int_0^{h(t)} \frac{\bar{F}(t-s) - \bar{F}(t)}{\bar{F}(t)} \frac{\bar{F}(s)}{t\bar{F}(t)} ds = \int_0^{h(t)/t} \left[\frac{\bar{F}(t(1-u))}{\bar{F}(t)} - 1 \right] \frac{\bar{F}(tu)}{\bar{F}(t)} du$$

goes to 0 as $t \rightarrow \infty$. We proceed by using the same line of reasoning used to prove (5.6), which is delineated in the following. First, apply the mean value theorem on the interval $(0, u)$ to the function $s \mapsto \bar{F}(t(1-s))$, and then use the uniform convergence theorem [6, Theorem 1.2.1] to find that, for all sufficiently large t ,

$$\int_0^{h(t)/t} \left[\frac{\bar{F}(t(1-u))}{\bar{F}(t)} - 1 \right] \frac{\bar{F}(tu)}{\bar{F}(t)} du \leq 2(1 + 2^{\alpha+1})\alpha \int_0^{h(t)/t} \frac{tu\bar{F}(tu)}{t\bar{F}(t)} du. \tag{5.8}$$

Second, apply Karamata’s theorem [6, Theorem 1.6.1] to obtain

$$\int_0^{h(t)/t} \frac{tu\bar{F}(tu)}{t\bar{F}(t)} du = \int_0^{h(t)} \frac{s\bar{F}(s)}{t^2\bar{F}(t)} ds \sim \frac{1}{2-\alpha} \left(\frac{h(t)}{t} \right)^2 \frac{\bar{F}(h(t))}{\bar{F}(t)}, \tag{5.9}$$

since $h(t) \rightarrow \infty$. Third, since the function $s \mapsto s^2\bar{F}(s)$ is regularly varying with $2-\alpha \in (0, 1)$ and $h(t) = o(t)$ then $(h(t)/t)^2\bar{F}(h(t))/\bar{F}(t) \rightarrow 0$ when $t \rightarrow \infty$. The latter fact, together with (5.8) and (5.9), allows us to conclude the desired result.

Finally, $\varepsilon_2(t) = o(t\bar{F}(t)^2)$ also holds because

$$\int_{-\infty}^{-h(t)} \frac{\bar{F}(t-s)}{\bar{F}(t)} \frac{F(s)}{t\bar{F}(t)} ds = \int_{h(t)}^{\infty} \frac{\bar{F}(t+s)}{\bar{F}(t)} \frac{F(-s)}{t\bar{F}(t)} ds \leq \int_{h(t)}^{\infty} \frac{F(-s)}{t\bar{F}(t)} ds,$$

with the last term going to 0 as $t \rightarrow \infty$ by assumption (A2). □

From the previous result we see that when $t \rightarrow \infty$, the sign of the difference $\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t)$ reduces to the sign of the term $p(t) - q(t)$. We now show that the latter is fully determined by the tail index α being either in $(1, 3/2)$ or in $(3/2, 2)$.

Lemma 5.3. *The quantity*

$$\int_0^{1/2} [(1-u)^{-\alpha} - 1]u^{-\alpha} du - \frac{2^{\alpha-1}}{\alpha-1} \tag{5.10}$$

is negative for $\alpha \in (1, 3/2)$ and positive for $\alpha \in (3/2, 2)$.

Proof. First note that, for fixed $u \in (0, \frac{1}{2})$, the function $[(1-u)^{-\alpha} - 1]u^{-\alpha}$ is strictly increasing in $\alpha > 0$, so $\int_0^{1/2} [(1-u)^{-\alpha} - 1]u^{-\alpha} du$ is also strictly increasing. Also, since $2^\beta/\beta$ is strictly decreasing for $\beta \in (0, 1)$ then $-2^{\alpha-1}/(\alpha-1)$ is strictly increasing in α when $\alpha \in (1, 2)$. Thus, (5.10) is strictly increasing in α for $\alpha \in (1, 2)$. Finally, it is easy to verify that if $\alpha = 3/2$ then (5.10) is equal to 0. □

With the results of the previous lemmas, the proof of Theorem 3.2 is straightforward.

Proof of Theorem 3.2. Consider, in the definition of p, q, ε_1 , and ε_2 , a function h satisfying assumption (A2); in particular, it satisfies $\max\{z_0, 0\} < h(t) < t/2$ for all $t > \max\{2z_0, 0\}$, $h(t) \rightarrow \infty$, and $h(t) = o(t)$ as $t \rightarrow \infty$. By Lemma 5.1 and Lemma 5.2, we have, as $t \rightarrow \infty$,

$$\frac{\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t)}{t\overline{F}(t)^2} = \frac{p(t) - q(t)}{t\overline{F}(t)^2} + o(1).$$

We conclude from Lemma 5.3 that as $t \rightarrow \infty$, the right-hand side is negative for $\alpha \in (1, 3/2)$ and positive for $\alpha \in (3/2, 2)$. \square

Appendix A. Proof of Proposition 3.1

Proof of Proposition 3.1. (i) Assume that (S2) holds; that is, we have $\mathbb{P}(X + Z > t) \leq \mathbb{P}(Z > t)$ for all sufficiently large t . Then take sufficiently large c so that $\mathbb{P}(X + Z > c + t) \leq \mathbb{P}(Z > c + t)$ holds for all $t \geq 0$. Thus, we have, by the definition of v and w ,

$$\frac{w(y - b - c)}{v(y - b - c)} \leq 1 \quad \text{for all } y \leq b. \tag{A.1}$$

Then using the definition of $\mathbb{Q}^{b,c}$ from (2.6), the following holds for $S_0 = 0$ and all $b \geq 0$:

$$\begin{aligned} \mathbf{1}_{\{\tau_b < \infty\}} \frac{d\mathbb{P}}{d\mathbb{Q}^{b,c}}(S_n : 0 \leq n \leq \tau_b) &= \mathbf{1}_{\{\tau_b < \infty\}} \frac{w(S_0 - b - c)}{v(S_{\tau_b} - b - c)} \prod_{n=1}^{\tau_b-1} \frac{w(S_n - b - c)}{v(S_n - b - c)} \\ &\leq \mathbf{1}_{\{\tau_b < \infty\}} \frac{w(S_0 - b - c)}{v(S_{\tau_b} - b - c)}. \end{aligned}$$

Conditional on $\tau_b < \infty$, it follows that $w(S_0 - b - c) \leq v(S_0 - b - c) \leq v(S_{\tau_b} - b - c)$ by inequality (A.1) and the monotonicity of v . From this, we obtain $\mathbf{1}_{\{\tau_b < \infty\}} d\mathbb{P}/d\mathbb{Q}^{b,c}(S_n : 0 \leq n \leq \tau_b) \leq 1$. We conclude that statement (S1^{b,c}) holds, by Corollary 2.1.

(ii) Assume that (S1^{b,c}) holds for some $b \geq 0$ and some $c \in \mathbb{R}$; that is, $\mathbb{Q}^{b,c}$ is a direct proposal for conditional sampling from $\mathbb{P}(\cdot \mid \tau_b < \infty)$. We proceed by contradiction and assume that (S2) does not hold, that is, for all t , we have that there exists a $t_0 > t$ such that $\mathbb{P}(X + Z > t_0) > \mathbb{P}(Z > t_0)$. Using the fact that $w(y) = \mathbb{P}(X + Z > -y)$ and $v(y) = \mathbb{P}(Z > -y)$, it follows that the previous hypothesis implies, in particular, that, for all $y \leq b$, there exists $y_0 < y$ such that $w(y_0 - b - c)/v(y_0 - b - c) > 1$ holds.

With this, we will show that necessarily the following holds:

$$\mathbb{Q}^{b,c} \left(\mathbf{1}_{\{\tau_b < \infty\}} \frac{w(S_0 - b - c)}{v(S_{\tau_b} - b - c)} \prod_{n=1}^{\tau_b-1} \frac{w(S_n - b - c)}{v(S_n - b - c)} > 1 \right) > 0, \tag{A.2}$$

that is, $\mathbb{Q}^{b,c}(\mathbf{1}_{\{\tau_b < \infty\}} d\mathbb{P}/d\mathbb{Q}^{b,c}(S_n : 0 \leq n \leq \tau_b) > 1) > 0$. The latter is a contradiction with hypothesis (S1^{b,c}), by Corollary 2.1. Now, to prove (A.2) the main idea is to construct paths $(S_n : 0 \leq n \leq \tau_b)$ under $\mathbb{Q}^{b,c}$ that, before crossing the barrier b , spend a sufficiently large amount of time in the set

$$Y_{>1}^{c,b} := \left\{ y \leq b : \frac{w(y - b - c)}{v(y - b - c)} > 1 \right\}.$$

For that we distinguish two cases: if $S_0 = 0 \in Y_{>1}^{c,b}$ or not. We start with the former case.

Case 1: if c and $b \geq 0$ are such that $S_0 = 0 \in Y_{>1}^{c,b}$. In this case, it holds that, for all $C > 0$, there exists $N > 0$ such that

$$\mathbb{P}\left(\prod_{n=1}^{N-1} \frac{w(S_n - b - c)}{v(S_n - b - c)} > C\right) > 0.$$

Indeed, this comes from the fact that $\mathbb{P}(|X| \leq \delta) > 0$ for all $\delta > 0$ and that the function $w(\cdot - b - c)/v(\cdot - b - c)$ is continuous; hence, the random walk can stay for an arbitrary amount of steps in a small neighborhood of $S_0 = 0$, a subset of $Y_{>1}^{c,b}$. It follows that, for a sufficiently large $N > 0$,

$$\mathbb{P}\left(\tau_b = N; \prod_{n=1}^{N-1} \frac{w(S_n - b - c)}{v(S_n - b - c)} > \frac{v(S_N - b - c)}{w(S_0 - b - c)}\right) > 0,$$

since X has unbounded right support and $v(\cdot - b - c) \leq 1$. Using then the absolute continuity of \mathbb{P} with respect to $\mathbb{Q}^{b,c}$ over paths with finite number of steps, we obtain

$$\mathbb{Q}^{b,c}\left(\tau_b = N; \prod_{n=1}^{N-1} \frac{w(S_n - b - c)}{v(S_n - b - c)} > \frac{v(S_N - b - c)}{w(S_0 - b - c)}\right) > 0. \tag{A.3}$$

We conclude that (A.2) also holds, since the event in (A.3) is a subset of the event in (A.2). This proves (A.2), which is a contradiction with hypothesis (S1^{b,c}).

Case 2: if c and $b \geq 0$ are such that $S_0 = 0 \notin Y_{>1}^{c,b}$. The idea for this case is to reduce it to the previous one, by constructing paths that, first, move to the set $Y_{>1}^{c,b}$, and second, spend a sufficiently large amount of time in $Y_{>1}^{c,b}$. For that, first define $\tau_{>1} := \inf\{n \geq 0 : S_n \in Y_{>1}^{c,b}\}$. Then take a compact set $A \subseteq \mathbb{R}$ and a large enough $M > 0$, so that they satisfy

$$\mathbb{P}(\tau_{>1} = M; S_n - b \in A \text{ for } n = 0, \dots, M) > 0;$$

here, A is chosen to satisfy $S_0 - b \in \text{int}(A)$ and $Y_{>1}^{c,b} \cap \text{int}(A) \neq \emptyset$. Note that, it holds that $\sup\{\prod_{n=1}^M v(y_n - b - c)/w(y_n - b - c) : y_0, \dots, y_M \in A + b\}$ is finite, since A is compact and $v(\cdot - c)/w(\cdot - c)$ is continuous. With this, and using the same arguments of case 1, it follows that there exists a large enough $N > 0$ such that

$$\mathbb{P}\left(\tau_b = N + M; \frac{v(S_0 - b - c)}{w(S_{N+M} - b - c)} \prod_{n=1}^{M+N-1} \frac{w(S_n - b - c)}{v(S_n - b - c)} > 1\right) > 0. \tag{A.4}$$

Indeed, it is sufficient to condition the probability on the left-hand side of (A.4) on the event $\{\tau_{>1} = M; S_0, \dots, S_M \in A + b\}$ and use the strong Markov property. It follows that, by the absolute continuity of \mathbb{P} with respect to $\mathbb{Q}^{b,c}$ over paths with finite number of steps,

$$\mathbb{Q}^{b,c}\left(\tau_b = N + M; \frac{v(S_0 - b - c)}{w(S_{N+M} - b - c)} \prod_{n=1}^{M+N-1} \frac{w(S_n - b - c)}{v(S_n - b - c)} > 1\right) > 0. \tag{A.5}$$

Clearly, (A.2) holds, since the event of the latter inequality contains the event in (A.5). We have arrived to a contradiction with hypothesis (S1^{b,c}). □

Appendix B. Proof of Proposition 3.2

Before showing the proof of Proposition 3.2, we establish the following lemma, which is a direct corollary of [9, Lemma 2]. It will be used to prove part (iii) of the latter result.

Lemma B.1. *Let \mathbb{Q} be a measure over paths of $\{S_n\}$. Assume that we have, for any large enough $b > 0$,*

$$\liminf_{y \rightarrow -\infty} \left\{ \mathbb{E}^{\mathbb{Q}}[S_1 - S_0 \mid S_0 = y] - \int_{b-y}^{\infty} \mathbb{Q}(S_1 - S_0 > u \mid S_0 = y) du \right\} > 0. \tag{B.1}$$

Then $\mathbb{E}^{\mathbb{Q}}\tau_b = O(b)$ as $b \rightarrow \infty$.

Proof. The proof consists in showing that if (B.1) holds then the function $h(y) := (C + |b - y|)\mathbf{1}_{\{y \leq b\}}$ satisfies the hypothesis of [9, Lemma 2] for some $C > 0$. That is, $\mathbb{E}_y^{\mathbb{Q}^{b,c}}[h(S_1)] - h(y) < -\rho$ for all $y \leq b$ and for some $\rho > 0$. In order to achieve this, note that, for $y < b$,

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}}[h(S_1) - h(y) \mid S_0 = y] \\ &= -\mathbb{E}^{\mathbb{Q}}[S_1 - S_0 \mid S_0 = y] + \int_{b-y}^{\infty} \mathbb{Q}(S_1 - S_0 > u \mid S_0 = y) du - C\mathbb{Q}(S_1 > b \mid S_0 = y). \end{aligned}$$

Therefore, $\limsup_{y \rightarrow -\infty} \mathbb{E}^{\mathbb{Q}}[h(S_1) - h(y) \mid S_0 = y] < 0$ is equivalent to (B.1). Then using [9, Lemma 2], we conclude that $\mathbb{E}^{\mathbb{Q}}\tau_b \leq h(0)/\rho = C/\rho + \rho^{-1}b = O(b)$. \square

Proof of Proposition 3.2. (i) We have to show that $\mathbb{Q}^{b,c}(\tau_b < \infty \mid S_0 = 0) = 1$ holds for all $b \geq 0$. We will actually show that this holds for all $c \in \mathbb{R}$.

For that, first consider any $c \in \mathbb{R}$ and note that, from [8, Lemma 1], we have

$$\lim_{y \rightarrow -\infty} \mathbb{E}^{\mathbb{Q}^{b,c}}[S_1 - S_0 \mid S_0 = y] > 0.$$

This result applies in our case because X is strongly subexponential, since X has regularly varying right tails with tail index $\alpha > 1$. Also, it can be checked that

$$\mathbb{E}^{\mathbb{Q}^{b,c}}[S_1 - S_0 \mid S_0 = y] = \mathbb{E}^{\mathbb{P}}[X \mid X + Z > c + b - y] \tag{B.2}$$

holds, since

$$\mathbb{Q}^{b,c}(S_1 - S_0 \in \cdot \mid S_0 = y) = \mathbb{P}(X \in \cdot \mid X + Z > c + b - y), \tag{B.3}$$

where X and Z are independent and Z has the residual life distribution of X .

Thus, we have that there exist $\varepsilon > 0$ and $y_0 \in \mathbb{R}$ such that for all $y \leq y_0$, and we have $\mathbb{E}^{\mathbb{Q}^{b,c}}[S_1 - S_0 \mid S_0 = y] > \varepsilon$. It follows that $\mathbb{Q}^{b,c}(\tau_{y_0} < \infty \mid S_0 = y) = 1$ holds for all $y \leq y_0$. We distinguish two cases now: if $b \leq y_0$ and if $y_0 < b$. In the former case, it is direct that $\mathbb{Q}^{b,c}(\tau_b < \infty \mid S_0 = 0) \geq \mathbb{Q}^{b,c}(\tau_{y_0} < \infty \mid S_0 = 0) = 1$ holds, since $0 \leq b \leq y_0$. In the latter case, on the other hand, that is, if $y_0 < b$, we can use a standard geometric trials argument to obtain, for all $y \leq y_0$,

$$\mathbb{Q}^{b,c}(\tau_b < \infty \mid S_0 = y) \geq \mathbb{Q}^{b,c}(\text{geom}(\gamma) < \infty) = 1.$$

Here, $\text{geom}(\gamma)$ is an independent geometric random variable with parameter

$$\gamma := \inf_{\bar{y} \in [y_0, b]} \mathbb{Q}^{b,c}(S_1 - S_0 > b - y_0 \mid S_0 = \bar{y}),$$

where $\gamma > 0$ owing to (B.3) and using the fact that the function $\bar{y} \mapsto \mathbb{P}(X > c + b - y \mid X + Z > c + b - \bar{y})$ is continuous and strictly positive. Indeed, it is continuous because both X and Z have absolutely continuous distributions, the former by assumption (A3) and the latter by definition (2.7). In both cases, $b \leq y_0$ and $y_0 < b$, we have shown that $\mathbb{Q}^{b,c}(\tau_b < \infty \mid S_0 = y) = 1$.

(ii) We proceed by contradiction and assume that $\mathbb{E}^{\mathbb{Q}^{b,c}}[\tau_b] < \infty$. By [4, Theorem 1.1], for $\alpha \in (1, 2)$, it holds that $\mathbb{E}^{\mathbb{P}}[\tau_b \mid \tau_b < \infty] = \infty$. But since $\mathbb{Q}^{b,c}$ is a direct proposal for exact conditional sampling when $\alpha \in (1, 3/2)$, by Corollary 2.1(ii), we have

$$\mathbb{E}^{\mathbb{Q}^{b,c}}[\tau_b] = \mathbb{E}^{\mathbb{Q}^{b,c}}\left[\sum_{n \geq 0} n \mathbf{1}_{\{\tau_b = n\}}\right] \geq \mathbb{E}^{\mathbb{P}}\left[\sum_{n \geq 0} n \mathbf{1}_{\{\tau_b = n\}}\right] = \mathbb{E}^{\mathbb{P}}[\tau_b \mathbf{1}_{\{\tau_b < \infty\}}] = \infty,$$

which is a contradiction.

(iii) By Lemma B.1, it is sufficient to show that (B.1) holds. For that, note that by the definition of the Blanchet–Glynn kernel in (2.6) and the fact that

$$v(y) = \mathbb{P}(Z > -y) = \frac{1}{|\mathbb{E}^{\mathbb{P}} X|} \mathbb{E}^{\mathbb{P}}[[X + y]^+] \tag{B.4}$$

and

$$w(y) = \mathbb{P}(X + Z > -y) = \mathbb{E}^{\mathbb{P}}[v(y + X)], \tag{B.5}$$

we have

$$\mathbb{Q}^{b,c}(S_1 - S_0 \in \cdot \mid S_0 = y) = \mathbb{P}(X \in \cdot \mid X + Z > -y + c),$$

where Z is independent of X and has the residual life distribution of X . With this, (B.2), and (B.3), we obtain

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}^{b,c}}[S_1 - S_0 \mid S_0 = y] - \int_{b-y}^{\infty} \mathbb{Q}^{b,c}(S_1 - S_0 > u \mid S_0 = y) du \\ &= \mathbb{E}^{\mathbb{P}}[X \mid X + Z > c + b - y] - \int_{b-y}^{\infty} \mathbb{E}^{\mathbb{P}}[\mathbf{1}_{\{X > u\}} \mid X + Z > c + b - y] du \\ &= \mathbb{E}^{\mathbb{P}}[X \mid X + Z > c + b - y] - \mathbb{E}^{\mathbb{P}}[[X - b + y]^+ \mid X + Z > c + b - y]. \end{aligned} \tag{B.6}$$

For the first term in the right-hand side of (B.6), we obtain

$$\liminf_{y \rightarrow -\infty} \mathbb{E}^{\mathbb{P}}[X \mid X + Z > c + b - y] \geq (\alpha - 1) |\mathbb{E}^{\mathbb{P}} X|$$

by following the same arguments as in the proof of [8, Lemma 1] and using the fact that $t\mathbb{P}(X > t) \sim |\mathbb{E}^{\mathbb{P}} X|(\alpha - 1)\mathbb{P}(Z > t)$ as $t \rightarrow \infty$, which is direct from Karamata’s theorem; see [6, Theorem 1.6.1]. For the second term in the right-hand side of (B.6), it follows that, if $b - y > z_0$ then by (B.4) and (B.5), we have

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}[[X - b + y]^+ \mid X + Z > c + b - y] \\ &= \frac{\mathbb{E}^{\mathbb{P}}[[X - b + y]^+ \mathbf{1}_{\{X + Z > c + b - y\}}]}{\mathbb{P}(X + Z > c + b - y)} \\ &= \frac{v(y - b - c) \int_{b-y}^{\infty} \mathbb{P}(X > u, X + Z > c + b - y) du}{w(y - b - c) \mathbb{P}(Z > c + b - y)} \end{aligned}$$

$$\begin{aligned} &\leq \frac{v(y-b-c)}{w(y-b-c)} \frac{\int_{b-y}^{\infty} \mathbb{P}(X > u) \, du}{\mathbb{P}(Z > c+b-y)} \\ &= \frac{v(y-b-c)}{w(y-b-c)} \frac{\mathbb{P}(Z > b-y)}{\mathbb{P}(Z > c+b-y)} |\mathbb{E}^{\mathbb{P}} X|. \end{aligned}$$

It follows that

$$\limsup_{y \rightarrow -\infty} \mathbb{E}^{\mathbb{P}} [(X - b + y)^+ | X + Z > b - y + c] \leq |\mathbb{E}^{\mathbb{P}} X|,$$

since $v(y-b-c)/w(y-b-c) \rightarrow 1$ as $y \rightarrow -\infty$ by [8, Proposition 3], and since Z is long-tailed because it is regularly varying with tail index $\alpha - 1 > 0$.

We have thus obtained

$$\liminf_{y \rightarrow -\infty} \left\{ \mathbb{E}^{\mathbb{Q}^{b,c}} [S_1 - S_0 | S_0 = y] - \int_{b-y}^{\infty} \mathbb{Q}^{b,c}(S_1 - S_0 > u | S_0 = y) \, du \right\} \geq (\alpha - 2) |\mathbb{E}^{\mathbb{P}} X|,$$

so applying Lemma B.1, we conclude that if $\alpha > 2$ then $\mathbb{E}^{\mathbb{Q}^{b,c}} \tau_b = O(b)$ as $b \rightarrow \infty$. \square

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