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The Picard groups of inclusions of C^*-algebras induced by equivalence bimodules

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Abstract. For two σ-unital C^*-algebras, we consider two equivalence bimodules over them, respectively. Then, by taking the crossed products by the equivalence bimodules, we get two inclusions of C^*-algebras. Furthermore, we suppose that one of the inclusions of C^*-algebras is irreducible, that is, the relative commutant of one of the σ-unital C^*-algebras in the multiplier C^*-algebra of the crossed product is trivial. We will give a sufficient and necessary condition that the two inclusions are strongly Morita equivalent. Applying this result, we will compute the Picard group of a unital inclusion of unital C^*-algebras induced by an equivalence bimodule over the unital C^*-algebra under the assumption that the unital inclusion of unital C^*-algebras is irreducible.

1 Introduction

In the previous paper [7], we discussed strong Morita equivalence for unital inclusions of unital C^*-algebras induced by involutive equivalence bimodules. That is, let A and B be unital C^*-algebras and X and Y an involutive $A-A$-equivalence bimodule and an involutive $B-B$-equivalence bimodule, respectively. Let C_X and C_Y be unital C^*-algebras induced by X and Y, respectively which are defined in [9]. Then, we get the unital inclusions of unital C^*-algebras $A \subset C_X$ and $B \subset C_Y$, respectively. We suppose that $A' \cap C_X = \mathbf{C}1$. In the paper [7], we showed that $A \subset C_X$ and $B \subset C_Y$ are strongly Morita equivalent if and only if there is an $A-B$-equivalence bimodule M such that $Y \cong \widetilde{M} \otimes_A X \otimes_A M$ as $B-B$-equivalence bimodules. In the present paper, we will show the same result as above in the case of inclusions of C^*-algebras induced by σ-unital C^*-algebra equivalence bimodules.

Let A and B be σ-unital C^*-algebras and X and Y an $A-A$-equivalence bimodule and a $B-B$-equivalence bimodule, respectively. Let $A \rtimes_X \mathbf{Z}$ and $B \rtimes_Y \mathbf{Z}$ be the crossed products of A and B by X and Y, respectively, which are defined in [1]. Then, we get inclusions of C^*-algebras $A \subset A \rtimes_X \mathbf{Z}$ and $B \subset B \rtimes_Y \mathbf{Z}$ with $\overline{A(A \rtimes_X \mathbf{Z})} = A \rtimes_X \mathbf{Z}$ and $\overline{B(B \rtimes_Y \mathbf{Z})} = B \rtimes_Y \mathbf{Z}$. We suppose that $A' \cap M(A \rtimes_X \mathbf{Z}) = \mathbf{C}1$. We will show that $A \subset A \rtimes_X \mathbf{Z}$ and $B \subset B \rtimes_Y \mathbf{Z}$ are strongly Morita equivalent if and only if there is an $A-B$-equivalence bimodule M such that $Y \cong \widetilde{M} \otimes_A X \otimes_A M$ or $\widetilde{Y} \cong \widetilde{M} \otimes_A X \otimes_A M$ as

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$B-B$-equivalence bimodules, where $\widetilde{M}$ and $\widetilde{Y}$ are the dual $B-A$-equivalence bimodule and the dual $B-B$-equivalence bimodule of M and Y, respectively. This is our main result (Theorem 3.6).

In Section 3, we will prove it in the following way. First, we assume that there is an $A-B$- equivalence bimodule M satisfying the above condition. Then, modifying the proof of [1, Theorem 4.2], we can show that $A \subset A \rtimes_X \mathbf{Z}$ and $B \subset B \rtimes_Y \mathbf{Z}$ are strongly Morita equivalent. We note that in this case, we do not need the assumption that $A' \cap M(A \rtimes_X \mathbf{Z}) = \mathbf{C}1$.

Next, we assume that $A \subset A \rtimes_X \mathbf{Z}$ and $B \subset B \rtimes_Y \mathbf{Z}$ are strongly Morita equivalent. Then, there are automorphisms α and β such that $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z}$ and $B \otimes \mathbf{K} \subset (B \otimes \mathbf{K}) \rtimes_{Y \otimes \mathbf{K}} \mathbf{Z}$ are isomorphic to $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z}$ and $B \otimes \mathbf{K} \subset (B \otimes \mathbf{K}) \rtimes_\beta \mathbf{Z}$ as inclusions of C^*-algebras, respectively, in the sense of Definition 2.1 below. Applying [8, Theorem 5.5] to α and β, we can obtain the desired conclusion. When we do it, we need the assumption that $A' \cap M(A \rtimes_X \mathbf{Z}) = \mathbf{C}1$. As mentioned in [8], the condition $A' \cap M(A \rtimes_X \mathbf{Z}) = \mathbf{C}1$ holds if and only if the action of α on $A \otimes \mathbf{K}$ is free. This freeness of α plays an important role in proving [8, Theorem 5.5]. We refer to [8, Section 4] and the references therein for more details about the notion of free action on a C^*-algebra. Furthermore, we remark that the same result as [8, Theorem 5.5] in the case of unital inclusions of unital C^*-algebras induced by coactions of a finite-dimensional C^*-Hopf algebras is obtained in [11].

In Section 4, we will give an application (Theorem 4.9) of the above result, that is, we will compute the Picard group of the inclusion of C^*-algebras $A \subset A \rtimes_X \mathbf{Z}$ under the assumption that $A' \cap M(A \rtimes_X \mathbf{Z}) = \mathbf{C}1$.

2 Preliminaries

Let $\mathbf{K}$ be the C^*-algebra of all compact operators on a countably infinite-dimensional Hilbert space and $\{e_{ij}\}_{i,j \in \mathbf{N}}$ its system of matrix units.

For each C^*-algebra A, we denote by $M(A)$ the multiplier C^*-algebra of A. Let π be an isomorphism of A onto a C^*-algebra B. Then, there is the unique strictly continuous isomorphism of $M(A)$ onto $M(B)$ extending π by Jensen and Thomsen [5, Corollary 1.1.15]. We denote it by $\underline{\pi}$.

For an algebra A, we denote by id_A the identity map on A. If A is unital, we denote by 1_A the unit element of A. If no confusion arises, we denote them by id and 1, respectively.

Let A and B be C^*-algebras and X an $A-B$-bimodule. We denote its left A-action and right B-action on X by $a \cdot x$ and $x \cdot b$ for any $a \in A$, $b \in B$, $x \in X$, respectively. We denote by $\widetilde{X}$ the dual $B-A$-bimodule of X and let $\widetilde{x}$ denote the element in $\widetilde{X}$ associated to an element $x \in X$. Furthermore, we regard X as a Hilbert $M(A)-M(B)$-bimodule in the sense of [4] in the same way as described before [8, Definition 2.4].

Let $A \subset C$ and $B \subset D$ be inclusions of C^*-algebras. We give some definitions.

Definition 2.1 We say that $A \subset C$ and $B \subset D$ are *isomorphic as inclusions of C^*-algebras* if there is an isomorphism π of C onto D such that the restriction of π to A, $\pi|_A$ is an isomorphism of A onto B.

Definition 2.2 [10, Definition 2.1] Let $A \subset C$ and $B \subset D$ be inclusions of C^*-algebras with $\overline{AC} = C$ and $\overline{BD} = D$. Then, the inclusions $A \subset C$ and $B \subset D$ are *strongly Morita equivalent* with respect to a $C-D$-equivalence bimodule Y and its closed subspace X if there are a $C-D$-equivalence bimodule Y and its closed subspace X satisfying the following conditions:

(1) $a \cdot x \in X$, ${}_C\langle x, y\rangle \in A$ for any $a \in A$, $x, y \in X$, and $\overline{{}_C\langle X, X\rangle} = A$, $\overline{{}_C\langle Y, X\rangle} = C$,
(2) $x \cdot b \in X$, $\langle x, y\rangle_D \in B$ for any $b \in B$, $x, y \in X$, and $\overline{\langle X, X\rangle_D} = B$, $\overline{\langle Y, X\rangle_D} = D$.

We note that X can be regarded as an $A-B$-equivalence bimodule. Furthermore, we give the following definition.

Definition 2.3 Let α and β be actions of a discrete group G on A and B, respectively. We say that α and β are *strongly Morita equivalent with respect to* (X, λ) if there are an $A-B$-equivalence bimodule X and a linear automorphism action λ on X satisfying the following:

(1) $\alpha_t({}_A\langle x, y\rangle) = {}_A\langle \lambda_t(x), \lambda_t(y)\rangle$,
(2) $\beta_t(\langle x, y\rangle_B) = \langle \lambda_t(x), \lambda_t(y)\rangle_B$, for any $x, y \in X$ and $t \in G$.

Then, we have the following:

$$\lambda_t(a \cdot x) = \alpha_t(a) \cdot \lambda_t(x), \quad \lambda_t(x \cdot b) = \lambda_t(x) \cdot \beta_t(b),$$

for any $a \in A$, $b \in B$, $x \in X$, and $t \in G$.

Let A and B be C^*-algebras and π an isomorphism of B onto A. We construct an $A-B$-equivalence bimodule X_π as follows: Let $X_\pi = A$ as a $\mathbf{C}$-vector space. For any $a \in A$, $b \in B$, and $x, y \in X_\pi$,

$$a \cdot x = ax, \quad x \cdot b = x\pi(b),$$
$${}_A\langle x, y\rangle = xy^*, \quad \langle x, y\rangle_B = \pi^{-1}(x^* y).$$

By easy computations, we can see that X_π is an $A-B$-equivalence bimodule. We call X_π an $A-B$-equivalence bimodule induced by π. Let α be an automorphism of A. Then, in the same way as above, we construct X_α, an $A-A$-equivalence bimodule. Let u_α be a unitary element in $M(A \rtimes_\alpha \mathbf{Z})$ implementing α. Hence, $\alpha = \mathrm{Ad}(u_\alpha)$. We regard Au_α as an $A-A$-equivalence bimodule as follows:

$$a \cdot xu_\alpha = axu_\alpha, \quad xu_\alpha \cdot a = x\alpha(a),$$
$${}_A\langle xu_\alpha, yu_\alpha\rangle = xy^*, \quad \langle xu_\alpha, yu_\alpha\rangle_A = \alpha^{-1}(x^* y),$$

for any $a, x, y \in A$.

Lemma 2.1 *With the above notation, $X_\alpha \cong Au_\alpha$ as $A-A$-equivalence bimodules.*

Proof This is immediate by easy computations. ∎

Let A be a C^*-algebra and X an $A-A$-equivalence bimodule. Let $A \rtimes_X \mathbf{Z}$ be the crossed product of A by X defined in [1]. We regard the C^*-algebra $\mathbf{K}$ as the trivial $\mathbf{K}-\mathbf{K}$-equivalence bimodule. Then, we obtain an $A \otimes \mathbf{K} - A \otimes \mathbf{K}$-equivalence bimodule

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$X \otimes \mathbf{K}$, and we can also consider the crossed product

$$(A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z}$$

of $A \otimes \mathbf{K}$ by $X \otimes \mathbf{K}$. Hence, we have the following inclusions of C^*-algebras:

$$A \subset A \rtimes_X \mathbf{Z}, \quad A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z}.$$

Because there is an isomorphism π of $(A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z}$ onto $(A \rtimes_X \mathbf{Z}) \otimes \mathbf{K}$ such that $\pi|_{A \rtimes \mathbf{K}} = \mathrm{id}$ on $A \otimes \mathbf{K}$, we identify $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z}$ with $A \otimes \mathbf{K} \subset (A \rtimes_X \mathbf{Z}) \otimes \mathbf{K}$. Thus, $A \subset A \rtimes_X \mathbf{Z}$ and $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z}$ are strongly Morita equivalent.

Let H_A be the $A \otimes \mathbf{K} - A$-equivalence bimodule defined as follows: Let $H_A = (A \otimes \mathbf{K})(1_{M(A)} \otimes e_{11})$ as a $\mathbf{C}$-vector space. For any $a \in A$, $k \in \mathbf{K}$, and $x, y \in A \otimes \mathbf{K}$,

$$\begin{aligned}
&(a \otimes k) \cdot x(1 \otimes e_{11}) = (a \otimes k)x(1 \otimes e_{11}), \\
&x(1 \otimes e_{11}) \cdot a = x(a \otimes e_{11}), \\
&{}_{A \otimes \mathbf{K}}\langle x(1 \otimes e_{11}), y(1 \otimes e_{11})\rangle = x(1 \otimes e_{11})y^*, \\
&\langle x(1 \otimes e_{11}), y(1 \otimes e_{11})\rangle_A = (1 \otimes e_{11})x^* y(1 \otimes e_{11}),
\end{aligned}$$

where we identify A with $A \otimes e_{11}$. Let B be a C^*-algebra. Let H_B be as above.

Lemma 2.2 *With the above notation, let M be an $A - B$- equivalence bimodule. Then,*

$$(1 \otimes e_{11}) \cdot (M \otimes \mathbf{K}) \cdot (1 \otimes e_{11}) \cong M$$

as $A - B$-equivalence bimodules, where we regard $M \otimes \mathbf{K}$ as an $A \otimes \mathbf{K} - B \otimes \mathbf{K}$-equivalence bimodule.

Proof Because the linear span of the set

$$\{x \otimes e_{ij} \mid x \in M, \ i, j \in \mathbf{N}\}$$

is dense in $M \otimes \mathbf{K}$, $(1 \otimes e_{11}) \cdot (M \otimes \mathbf{K}) \cdot (1 \otimes e_{11}) \cong M \otimes e_{11}$ as $A - B$-equivalence bimodules. Hence,

$$(1 \otimes e_{11}) \cdot (M \otimes \mathbf{K}) \cdot (1 \otimes e_{11}) \cong M$$

as $A - B$-equivalence bimodules. ∎

Lemma 2.3 *With the above notation, let M be an $A - B$-equivalence bimodule. Then,*

$$\widetilde{H_A} \otimes_{A \otimes \mathbf{K}} (M \otimes \mathbf{K}) \otimes_{B \otimes \mathbf{K}} H_B \cong M$$

as $A - B$-equivalence bimodules.

Proof Let π be the map from $\widetilde{H_A} \otimes_{A \otimes \mathbf{K}} (M \otimes \mathbf{K}) \otimes_{B \otimes \mathbf{K}} H_B$ to $(1 \otimes e_{11}) \cdot (M \otimes \mathbf{K}) \cdot (1 \otimes e_{11})$ defined by

$$\pi([a(1 \otimes e_{11})]\widetilde{\ } \otimes x \otimes b(1 \otimes e_{11})) = (1 \otimes e_{11}) \cdot (a^* \cdot x \cdot b) \cdot (1 \otimes e_{11}),$$

for any $a \in A \otimes \mathbf{K}$, $b \in B \otimes \mathbf{K}$, and $x \in M \otimes \mathbf{K}$. Then, by easy computations, π is an $A - B$-equivalence bimodule isomorphism of $\widetilde{H_A} \otimes_{A \otimes \mathbf{K}} (M \otimes \mathbf{K}) \otimes_{B \otimes \mathbf{K}} H_B$

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onto $(1 \otimes e_{11}) \cdot (M \otimes \mathbf{K}) \cdot (1 \otimes e_{11})$. Thus, by Lemma 2.2,

$$\widetilde{H_A} \otimes_{A \otimes \mathbf{K}} (M \otimes \mathbf{K}) \otimes_{B \otimes \mathbf{K}} H_B \cong M$$

as $A - B$-equivalence bimodules. ∎

We prepare the following lemma which is applied in the next section.

Lemma 2.4 *Let A and B be C^*-algebras and X and Y an $A - A$-equivalence bimodule and a $B - B$-equivalence bimodule, respectively. Let $A \subset A \rtimes_X \mathbf{Z}$ and $B \subset B \rtimes_Y \mathbf{Z}$ be inclusions of C^* algebras induced by X and Y, respectively. We suppose that there is an $A - B$-equivalence bimodule M such that $Y \cong \widetilde{M} \otimes_A X \otimes_A M$ or $\widetilde{Y} \cong \widetilde{M} \otimes_A X \otimes_A M$ as $B - B$-equivalence bimodules. Then, there is an $A \rtimes_X \mathbf{Z} - B \rtimes_Y \mathbf{Z}$-equivalence bimodule N satisfying the following:*

(1) *M is included in N as a closed subspace,*

(2) *$A \subset A \rtimes_X \mathbf{Z}$ and $B \subset B \rtimes_Y \mathbf{Z}$ are strongly Morita equivalent with respect to N and its closed subspace M.*

Proof Modifying the proof of [1, Theorem 4.2], we prove this lemma. We suppose that $Y \cong \widetilde{M} \otimes_A X \otimes_A M$ as $B - B$-equivalence bimodules. Let L_M be the linking C^*-algebra for M defined by

$$L_M = \begin{bmatrix} A & M \\ \widetilde{M} & B \end{bmatrix}.$$

Furthermore, let W be the $L_M - L_M$- equivalence bimodule defined in the proof of [1, Theorem 4.2], which is defined by

$$W = \begin{bmatrix} X & X \otimes_A M \\ Y \otimes_B \widetilde{M} & Y \end{bmatrix}.$$

Let $L_M \rtimes_W \mathbf{Z}$ be the crossed product of L_M by W, and let

$$p = \begin{bmatrix} 1_{M(A)} & 0 \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 & 0 \\ 0 & 1_{M(B)} \end{bmatrix}.$$

Furthermore, let $N = p(L_M \rtimes_W \mathbf{Z})q$. Then, because $M = pL_M q$, M is a closed subspace of N. Hence, by the proof of [1, Theorem 4.2], $A \subset A \rtimes_X \mathbf{Z}$ and $B \subset B \rtimes_Y \mathbf{Z}$ are strongly Morita equivalent with respect N and its closed subspace M.

Next, we suppose that $\widetilde{Y} \cong \widetilde{M} \otimes_A X \otimes_A M$ as $B - B$-equivalence bimodules. Let

$$W_0 = \begin{bmatrix} X & X \otimes_A M \\ \widetilde{Y} \otimes_B \widetilde{M} & \widetilde{Y} \end{bmatrix}.$$

Then, W_0 is an $L_M - L_M$-equivalence bimodule. Let $N_0 = p(L_M \rtimes_{W_0} \mathbf{Z})q$. By the above discussions, $A \subset A \rtimes_X \mathbf{Z}$ and $B \subset B \rtimes_{\widetilde{Y}} \mathbf{Z}$ are strongly Morita equivalent with respect to N_0 and its closed subspace M. On the other hand, there is an isomorphism π of $B \rtimes_Y \mathbf{Z}$ onto $B \rtimes_{\widetilde{Y}} \mathbf{Z}$ such that $\pi|_B = \mathrm{id}$ on B. Let X_π be the $B \rtimes_{\widetilde{Y}} \mathbf{Z} - B \rtimes_Y \mathbf{Z}$ -equivalence bimodule induced by π. Then, B is a closed subspace of X_π, and we regard B as the trivial $B - B$-equivalence bimodule, because $\pi|_B = \mathrm{id}$ on B. Thus, $A \subset A \rtimes_X \mathbf{Z}$

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and $B \subset B \rtimes_Y \mathbf{Z}$ are strongly Morita equivalent with respect to $N_0 \otimes_{B \rtimes_{\widetilde{Y}} \mathbf{Z}} X_\pi$ and its closed subspace $M \otimes_B B(\cong M)$. Therefore, we obtain the conclusion. ∎

Lemma 2.5 *With the above notation, we suppose that A is a σ-unital C^*-algebra. Then, there is an automorphism α of $A \otimes \mathbf{K}$ such that $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z}$ is isomorphic to $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z}$ as inclusions of C^*-algebras.*

Proof Because A is σ-unital, by [3, Corollary 3.5], there is an automorphism α of $A \otimes \mathbf{K}$ such that $X \otimes \mathbf{K} \cong X_\alpha$ as $A \otimes \mathbf{K} - A \otimes \mathbf{K}$-equivalence bimodules, where X_α is the $A \otimes \mathbf{K} - A \otimes \mathbf{K}$-equivalence bimodule induced by α. Let u_α be a unitary element in $M((A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z})$ implementing α. We regard $(A \otimes \mathbf{K})u_\alpha$ as an $A \otimes \mathbf{K} - A \otimes \mathbf{K}$-equivalence bimodule as above. Then, by Lemma 2.1, $X_\alpha \cong (A \otimes \mathbf{K})u_\alpha$ as $A \otimes \mathbf{K} - A \otimes \mathbf{K}$-equivalence bimodules. Let $(A \otimes \mathbf{K}) \rtimes_{(A \otimes \mathbf{K})u_\alpha} \mathbf{Z}$ be the crossed product of $A \otimes \mathbf{K}$ by $(A \otimes \mathbf{K})u_\alpha$. Then, by the definition of the crossed product of a C^*-algebra by an equivalence bimodule, we can see that

$$(A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z} \cong (A \otimes \mathbf{K}) \rtimes_{(A \otimes \mathbf{K})u_\alpha} \mathbf{Z}$$

as C^*-algebras. Because $X_\alpha \cong X \otimes \mathbf{K}$ as $A \otimes \mathbf{K} - A \otimes \mathbf{K}$-equivalence bimodules, we obtain that

$$(A \otimes \mathbf{K}) \rtimes_{(A \otimes \mathbf{K})u_\alpha} \mathbf{Z} \cong (A \otimes \mathbf{K}) \rtimes_{X_\alpha} \mathbf{Z} \cong (A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z}$$

as C^*-algebras. Because the above isomorphisms leave any element in $A \otimes \mathbf{K}$ invariant, we can see that $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z}$ is isomorphic to $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z}$ as inclusions of C^*-algebras. ∎

3 Strong Morita equivalence

Let A and B be σ-unital C^*-algebras and X and Y an $A - A$-equivalence bimodule and a $B - B$-equivalence bimodule, respectively. Let $A \subset A \rtimes_X \mathbf{Z}$ and $B \subset B \rtimes_Y \mathbf{Z}$ be the inclusions of C^*-algebras induced by X and Y, respectively. We suppose that $A \subset A \rtimes_X \mathbf{Z}$ and $B \subset B \rtimes_Y \mathbf{Z}$ are strongly Morita equivalent with respect to an $A \rtimes_X \mathbf{Z} - B \rtimes_Y \mathbf{Z}$-equivalence bimodule N and its closed subspace M. We suppose that $A' \cap M(A \rtimes_X \mathbf{Z}) = \mathbf{C}1$. Then, because the inclusion $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z}$ is isomorphic to the inclusion $A \otimes \mathbf{K} \subset (A \rtimes_X \mathbf{Z}) \otimes \mathbf{K}$ as inclusions of C^*-algebras, by [8, Lemma 3.1],

$$(A \otimes \mathbf{K})' \cap ((A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z}) = \mathbf{C}1.$$

Furthermore, by the above assumptions, the inclusion $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z}$ is strongly Morita equivalent to the inclusion $B \otimes \mathbf{K} \subset (B \otimes \mathbf{K}) \rtimes_{Y \otimes \mathbf{K}} \mathbf{Z}$ with respect to the $(A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z} - (B \otimes \mathbf{K}) \rtimes_{Y \otimes \mathbf{K}} \mathbf{Z}$-equivalence bimodule $N \otimes \mathbf{K}$ and its closed subspace $M \otimes \mathbf{K}$. By Lemma 2.5, there are an automorphism α of $A \otimes \mathbf{K}$ and an automorphism β of $B \otimes \mathbf{K}$ such that $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z}$ and $B \otimes \mathbf{K} \subset (B \otimes \mathbf{K}) \rtimes_{Y \otimes \mathbf{K}} \mathbf{Z}$ are isomorphic to $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z}$ and $B \otimes \mathbf{K} \subset (B \otimes \mathbf{K}) \rtimes_\beta \mathbf{Z}$ as inclusions of C^*-algebras, respectively. Hence, we can assume that $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z}$ and $B \otimes \mathbf{K} \subset (B \otimes \mathbf{K}) \rtimes_\beta \mathbf{Z}$ are strongly Morita equivalent with respect to an $(A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z} - (B \otimes \mathbf{K}) \rtimes_\beta \mathbf{Z}$-equivalence bimodule $N \otimes \mathbf{K}$ and its closed subspace $M \otimes \mathbf{K}$. Because A and B are σ-unital, in the same way as in the proof of [6, Proposition

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3.5] or [3, Proposition 3.1], there is an isomorphism θ of $(B\otimes\mathbf{K})\rtimes_\beta\mathbf{Z}$ onto $(A\otimes\mathbf{K})\rtimes_\alpha\mathbf{Z}$ satisfying the following:

(1) $\theta|_{B\otimes\mathbf{K}}$ is an isomorphism of $B\otimes\mathbf{K}$ onto $A\otimes\mathbf{K}$,

(2) There is an $(A\otimes\mathbf{K})\rtimes_\alpha\mathbf{Z}-(B\otimes\mathbf{K})\rtimes_\beta\mathbf{Z}$- equivalence bimodule isomorphism Φ of $N\otimes\mathbf{K}$ onto Y_θ such that $\Phi|_{M\otimes\mathbf{K}}$ is an $A\otimes\mathbf{K}-B\otimes\mathbf{K}$-equivalence bimodule isomorphism of $M\otimes\mathbf{K}$ onto X_θ, where Y_θ is the $(A\otimes\mathbf{K})\rtimes_\alpha\mathbf{Z}-(B\otimes\mathbf{K})\rtimes_\beta\mathbf{Z}$-equivalence bimodule induced by θ and X_θ is the $A\otimes\mathbf{K}-B\otimes\mathbf{K}$-equivalence bimodule induced by $\theta|_{B\otimes\mathbf{K}}$.

Let

$$\gamma=\theta|_{B\otimes\mathbf{K}}\circ\beta\circ\theta|_{B\otimes\mathbf{K}}^{-1},$$

and let λ be the linear automorphism of X_θ defined by $\lambda(x)=\gamma(x)$ for any $x\in X_\theta(=A\otimes\mathbf{K})$.

Lemma 3.1 *With the above notation, γ and β are strongly Morita equivalent with respect to (X_θ,λ).*

Proof For any $x,y\in X_\theta$,

$$\begin{aligned}&{}_{A\otimes\mathbf{K}}\langle\lambda(x),\lambda(y)\rangle=\gamma(xy^*)=\gamma({}_{A\otimes\mathbf{K}}\langle x,y\rangle),\\&\langle\lambda(x),\lambda(y)\rangle_{B\otimes\mathbf{K}}=\theta|_{B\otimes\mathbf{K}}^{-1}(\gamma(x^*y))=\beta(\theta|_{B\otimes\mathbf{K}}^{-1}(x^*y))=\beta(\langle x\ y\rangle_{B\otimes\mathbf{K}}).\end{aligned}$$

Hence, γ and β are strongly Morita equivalent with respect to (X_θ,λ). ■

By the proof of [8, Theorem 5.5], there is an automorphism ϕ of $\mathbf{Z}$ satisfying that γ^ϕ and α are exterior equivalent, that is, there is a unitary element $z\in M(A\otimes\mathbf{K})$ such that

$$\gamma^\phi=\mathrm{Ad}(z)\circ\alpha,\quad \underline{\alpha}(z)=z,$$

where γ^ϕ is the automorphism of $A\otimes\mathbf{K}$ induced by γ and ϕ, that is, γ^ϕ is defined by $\gamma^\phi=\gamma^{\phi(1)}$. We note that $\gamma^\phi=\gamma$ or $\gamma^\phi=\gamma^{-1}$. We regard $A\otimes\mathbf{K}$ as the trivial $A\otimes\mathbf{K}-A\otimes\mathbf{K}$-equivalence bimodule. Let μ be the linear automorphism of $A\otimes\mathbf{K}$ defined by

$$\mu(x)=\alpha(x)z^*,$$

for any $x\in A\otimes\mathbf{K}$.

Lemma 3.2 *With the above notation, α and γ^ϕ are strongly Morita equivalent with respect to $(A\otimes\mathbf{K},\mu)$.*

Proof For any $x,y\in A\otimes\mathbf{K}$,

$$\begin{aligned}&{}_{A\otimes\mathbf{K}}\langle\mu(x),\mu(y)\rangle={}_{A\otimes\mathbf{K}}\langle\alpha(x)z^*,\alpha(y)z^*\rangle=\alpha(xy^*)=\alpha({}_{A\otimes\mathbf{K}}\langle x,y\rangle),\\&\langle\mu(x),\mu(y)\rangle_{A\otimes\mathbf{K}}=z\alpha(x^*y)z^*=\gamma^\phi(x^*y)=\gamma^\phi(\langle x,y\rangle_{A\otimes\mathbf{K}}).\end{aligned}$$

Therefore, we obtain the conclusion. ■

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Let v be the linear automorphism of X_θ defined by

$$v(x) = \gamma^\phi(z^* x),$$

for any $x \in X_\theta(= A \otimes \mathbf{K})$.

Lemma 3.3 *With the above notation, α and β^ϕ are strongly Morita equivalent with respect to (X_θ, v), where β^ϕ is the automorphism of $B \otimes \mathbf{K}$ induced by β and ϕ, that is, β^ϕ is defined by $\beta^\phi = \beta^{\phi(1)}$.*

Proof For any $x, y \in X_\theta$,

$$\begin{aligned}
{}_{A\otimes\mathbf{K}}\langle v(x)\,,\, v(y)\rangle &= {}_{A\otimes\mathbf{K}}\langle \gamma^\phi(z^*x)\,,\, \gamma^\phi(z^*y)\rangle = \gamma^\phi(z^*xy^*z) = z\alpha(z^*xy^*z)z^* \\
&= \alpha(xy^*) = \alpha({}_{A\otimes\mathbf{K}}\langle x\,,\, y\rangle), \\
\langle v(x)\,,\, v(y)\rangle_{B\otimes\mathbf{K}} &= \langle \gamma^\phi(z^*x)\,,\, \gamma^\phi(z^*y)\rangle_{B\otimes\mathbf{K}} = \theta|_{B\otimes\mathbf{K}}^{-1}(\gamma^\phi(x^*y)) = \beta^\phi(\theta|_{B\otimes\mathbf{K}}^{-1}(x^*y)) \\
&= \beta^\phi(\langle x\,,\, y\rangle_{B\otimes\mathbf{K}}).
\end{aligned}$$

Therefore, we obtain the conclusion. ∎

Because $\beta^\phi = \beta$ or $\beta^\phi = \beta^{-1}$, by Lemma 3.3, α is strongly Morita equivalent to β or β^{-1}.

(I) We suppose that α is strongly Morita equivalent to β. Then, by Lemma 3.3, there is the linear automorphism v of X_θ satisfying the following:

(1) $v(a \cdot x) = \alpha(a) \cdot v(x)$,
(2) $v(x \cdot b) = v(x) \cdot \beta(b)$,
(3) ${}_{A\otimes\mathbf{K}}\langle v(x)\,,\, v(y)\rangle = \alpha({}_{A\otimes\mathbf{K}}\langle x\,,\, y\rangle)$,
(4) $\langle v(x)\,,\, v(y)\rangle_{B\otimes\mathbf{K}} = \beta(\langle x\,,\, y\rangle_{B\otimes\mathbf{K}})$, for any $a \in A \otimes \mathbf{K}$, $b \in B \otimes \mathbf{K}$, and $x, y \in X_\theta$.

Lemma 3.4 *With the above notation and assumptions, let X_α and X_β be the $A \otimes \mathbf{K} - A \otimes \mathbf{K}$-equivalence bimodule and the $B \otimes \mathbf{K} - B \otimes \mathbf{K}$-equivalence bimodule induced by α and β, respectively. Then,*

$$X_\beta \cong \widetilde{X_\theta} \otimes_{A\otimes\mathbf{K}} X_\alpha \otimes_{A\otimes\mathbf{K}} X_\theta$$

as $B \otimes \mathbf{K} - B \otimes \mathbf{K}$-equivalence bimodules.

Proof Let Ψ be the map from $\widetilde{X_\theta} \otimes_{A\otimes\mathbf{K}} X_\alpha \otimes_{A\otimes\mathbf{K}} X_\theta$ to X_β defined by

$$\Psi(\widetilde{x} \otimes a \otimes y) = \langle x\,,\, a \cdot v(y)\rangle_{B\otimes\mathbf{K}},$$

for any $x, y \in X_\theta$ and $a \in X_\alpha$. Then, for any $x, x_1, y, y_1 \in X_\theta$ and $a, a_1 \in X_\alpha$,

$$\begin{aligned}
{}_{B\otimes\mathbf{K}}\langle \widetilde{x} \otimes a \otimes y\,,\, \widetilde{x_1} \otimes a_1 \otimes y_1\rangle &= {}_{B\otimes\mathbf{K}}\langle \widetilde{x} \cdot {}_{A\otimes\mathbf{K}}\langle a \otimes y\,,\, a_1 \otimes y_1\rangle\,,\, \widetilde{x_1}\rangle \\
&= \langle\, {}_{A\otimes\mathbf{K}}\langle a_1 \otimes y_1\,,\, a \otimes y\rangle \cdot x\,,\, x_1\rangle_{B\otimes\mathbf{K}} \\
&= \langle\, {}_{A\otimes\mathbf{K}}\langle a_1 \cdot {}_{A\otimes\mathbf{K}}\langle y_1\,,\, y\rangle\,,\, a\rangle \cdot x\,,\, x_1\rangle_{B\otimes\mathbf{K}} \\
&= \langle\, {}_{A\otimes\mathbf{K}}\langle a_1\alpha({}_{A\otimes\mathbf{K}}\langle y_1\,,\, y\rangle)\,,\, a\rangle \cdot x\,,\, x_1\rangle_{B\otimes\mathbf{K}} \\
&= \langle a_1\alpha({}_{A\otimes\mathbf{K}}\langle y_1\,,\, y\rangle)a^* \cdot x\,,\, x_1\rangle_{B\otimes\mathbf{K}}.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
{}_{B\otimes\mathbf{K}}\langle \Psi(\widetilde{x}\otimes a\otimes y),\, \Psi(\widetilde{x_1}\otimes a_1\otimes y_1)\rangle &= {}_{B\otimes\mathbf{K}}\langle\langle x,\, a\cdot v(y)\rangle_{B\otimes\mathbf{K}},\, \langle x_1,\, a_1\cdot v(y_1)\rangle_{B\otimes\mathbf{K}}\rangle \\
&= \langle x,\, a\cdot v(y)\rangle_{B\otimes\mathbf{K}}\, \langle a_1\cdot v(y_1),\, x_1\rangle_{B\otimes\mathbf{K}} \\
&= \langle x,\, a\cdot v(y)\cdot\langle a_1\cdot v(y_1),\, x_1\rangle_{B\otimes\mathbf{K}}\,\rangle_{B\otimes\mathbf{K}} \\
&= \langle x,\, {}_{A\otimes\mathbf{K}}\langle a\cdot v(y),\, a_1\cdot v(y_1)\rangle\cdot x_1\rangle_{B\otimes\mathbf{K}} \\
&= \langle x,\, a\, {}_{A\otimes\mathbf{K}}\langle v(y),\, v(y_1)\rangle a_1^*\cdot x_1\rangle_{B\otimes\mathbf{K}} \\
&= \langle a_1\, {}_{A\otimes\mathbf{K}}\langle v(y_1),\, v(y)\rangle a^*\cdot x,\, x_1\rangle_{B\otimes\mathbf{K}} \\
&= \langle a_1\alpha({}_{A\otimes\mathbf{K}}\langle y_1,\, y\rangle)a^*\cdot x,\, x_1\rangle_{B\otimes\mathbf{K}}.
\end{aligned}
$$

Hence, Ψ preserves the left $B\otimes\mathbf{K}$-valued inner products. Furthermore,

$$
\begin{aligned}
\langle \widetilde{x}\otimes a\otimes y,\, \widetilde{x_1}\otimes a_1\otimes y_1\rangle_{B\otimes\mathbf{K}} &= \langle y,\, \langle \widetilde{x}\otimes a,\, \widetilde{x_1}\otimes a_1\rangle_{A\otimes\mathbf{K}}\cdot y_1\rangle_{B\otimes\mathbf{K}} \\
&= \langle y,\, \langle a,\, {}_{A\otimes\mathbf{K}}\langle x,\, x_1\rangle\cdot a_1\rangle_{A\otimes\mathbf{K}}\cdot y_1\rangle_{B\otimes\mathbf{K}} \\
&= \langle y,\, \langle a,\, {}_{A\otimes\mathbf{K}}\langle x,\, x_1\rangle a_1\rangle_{A\otimes\mathbf{K}}\cdot y_1\rangle_{B\otimes\mathbf{K}} \\
&= \langle y,\, \alpha^{-1}(a^*\, {}_{A\otimes\mathbf{K}}\langle x,\, x_1\rangle a_1)\cdot y_1\rangle_{B\otimes\mathbf{K}}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\langle \Psi(\widetilde{x}\otimes a\otimes y),\, \Psi(\widetilde{x_1}\otimes a_1\otimes y_1)\rangle_{B\otimes\mathbf{K}} &= \langle\langle x,\, a\cdot v(y)\rangle_{B\otimes\mathbf{K}},\, \langle x_1,\, a_1\cdot v(y_1)\rangle_{B\otimes\mathbf{K}}\rangle_{B\otimes\mathbf{K}} \\
&= \beta^{-1}(\langle a\cdot v(y),\, x\rangle_{B\otimes\mathbf{K}}\, \langle x_1,\, a_1\cdot v(y_1)\rangle_{B\otimes\mathbf{K}}) \\
&= \beta^{-1}(\langle a\cdot v(y),\, x\cdot\langle x_1,\, a_1\cdot v(y_1)\rangle_{B\otimes\mathbf{K}}\,\rangle_{B\otimes\mathbf{K}}) \\
&= \beta^{-1}(\langle a\cdot v(y),\, {}_{A\otimes\mathbf{K}}\langle x,\, x_1\rangle a_1\cdot v(y_1)\rangle_{B\otimes\mathbf{K}}) \\
&= \beta^{-1}(\langle v(y),\, a^*\, {}_{A\otimes\mathbf{K}}\langle x,\, x_1\rangle a_1\cdot v(y_1)\rangle_{B\otimes\mathbf{K}}) \\
&= \langle y,\, \alpha^{-1}(a^*\, {}_{A\otimes\mathbf{K}}\langle x,,\, x_1\rangle a_1)\cdot y_1\rangle_{B\otimes\mathbf{K}}.
\end{aligned}
$$

Hence, Ψ preserves the right $B\otimes\mathbf{K}$-valued inner products. Therefore, we obtain the conclusion. ■

(II) We suppose that α is strongly Morita equivalent to β^{-1}. Then, by Lemma 3.4,

$$X_{\beta^{-1}} \cong \widetilde{X_\theta}\otimes_{A\otimes\mathbf{K}} X_\alpha \otimes_{A\otimes\mathbf{K}} X_\theta$$

as $B\otimes\mathbf{K}-B\otimes\mathbf{K}$-equivalence bimodules. Thus, we obtain the following lemma.

Lemma 3.5 *With the above notation and assumptions,*

$$\widetilde{X_\beta}\cong X_{\beta^{-1}} \cong \widetilde{X_\theta}\otimes_{A\otimes\mathbf{K}} X_\alpha \otimes_{A\otimes\mathbf{K}} X_\theta$$

as $B\otimes\mathbf{K}-B\otimes\mathbf{K}$-equivalence bimodules.

We recall that there is an $(A\otimes\mathbf{K})\rtimes_\alpha\mathbf{Z}-(B\otimes\mathbf{K})\rtimes_\beta\mathbf{Z}$- equivalence bimodule isomorphism Φ of $N\otimes\mathbf{K}$ onto Y_θ such that $\Phi|_{M\otimes\mathbf{K}}$ is an $A\otimes\mathbf{K}-B\otimes\mathbf{K}$-equivalence bimodule isomorphism of $M\otimes\mathbf{K}$ onto X_θ. We identify $M\otimes\mathbf{K}$ with X_θ by $\Phi|_{M\otimes\mathbf{K}}$. Then, by Lemmas 3.4 and 3.5,

$$X_\beta\cong(\widetilde{M\otimes\mathbf{K}})\otimes_{A\otimes\mathbf{K}} X_\alpha\otimes_{A\otimes\mathbf{K}}(M\otimes\mathbf{K})$$

or

$$\widetilde{X_\beta} \cong (\widetilde{M \otimes \mathbf{K}}) \otimes_{A\otimes\mathbf{K}} X_\alpha \otimes_{A\otimes\mathbf{K}} (M \otimes \mathbf{K})$$

as $B \otimes \mathbf{K} - B \otimes \mathbf{K}$-equivalence bimodules. Furthermore, we recall that $X_\alpha \cong X \otimes \mathbf{K}$ as $A \otimes \mathbf{K} - A \otimes \mathbf{K}$-equivalence bimodules and that $X_\beta \cong Y \otimes \mathbf{K}$ as $B \otimes \mathbf{K} - B \otimes \mathbf{K}$-equivalence bimodules. Thus,

$$Y \otimes \mathbf{K} \cong (\widetilde{M \otimes \mathbf{K}}) \otimes_{A\otimes\mathbf{K}} (X \otimes \mathbf{K}) \otimes_{A\otimes\mathbf{K}} (M \otimes \mathbf{K})$$

or

$$\widetilde{Y \otimes \mathbf{K}} \cong (\widetilde{M \otimes \mathbf{K}}) \otimes_{A\otimes\mathbf{K}} (X \otimes \mathbf{K}) \otimes_{A\otimes\mathbf{K}} (M \otimes \mathbf{K})$$

as $B \otimes \mathbf{K} - B \otimes \mathbf{K}$-equivalence bimodules. Furthermore, by Lemma 2.3,

$$X \otimes \mathbf{K} \cong H_A \otimes_A X \otimes_A \widetilde{H_A}$$

as $A \otimes \mathbf{K} - A \otimes \mathbf{K}$-equivalence bimodules and

$$Y \otimes \mathbf{K} \cong H_B \otimes_B Y \otimes_B \widetilde{H_B}$$

as $B \otimes \mathbf{K} - B \otimes \mathbf{K}$-equivalence bimodules. Thus,

$$\begin{aligned} Y &\cong \widetilde{H_B} \otimes_{B\otimes\mathbf{K}} (Y \otimes \mathbf{K}) \otimes_{B\otimes\mathbf{K}} H_B \\ &\cong \widetilde{H_B} \otimes_{B\otimes\mathbf{K}} (\widetilde{M \otimes \mathbf{K}}) \otimes_{A\otimes\mathbf{K}} (X \otimes \mathbf{K}) \otimes_{A\otimes\mathbf{K}} (M \otimes \mathbf{K}) \otimes_{B\otimes\mathbf{K}} H_B \\ &\cong \widetilde{H_B} \otimes_{B\otimes\mathbf{K}} (\widetilde{M \otimes \mathbf{K}}) \otimes_{A\otimes\mathbf{K}} H_A \otimes_A X \otimes_A \widetilde{H_A} \otimes_{A\otimes\mathbf{K}} (M \otimes \mathbf{K}) \otimes_{B\otimes\mathbf{K}} H_B \end{aligned}$$

or similarly

$$\widetilde{Y} \cong \widetilde{H_B} \otimes_{B\otimes\mathbf{K}} (\widetilde{M \otimes \mathbf{K}}) \otimes_{A\otimes\mathbf{K}} H_A \otimes_A X \otimes_A \widetilde{H_A} \otimes_{A\otimes\mathbf{K}} (M \otimes \mathbf{K}) \otimes_{B\otimes\mathbf{K}} H_B$$

as $B \otimes \mathbf{K} - B \otimes \mathbf{K}$-equivalence bimodules. Furthermore, by Lemma 2.3,

$$\widetilde{H_A} \otimes_{A\otimes\mathbf{K}} (M \otimes \mathbf{K}) \otimes_{B\otimes\mathbf{K}} H_B \cong M$$

as $A - B$-equivalence bimodules. Hence,

$$Y \cong \widetilde{M} \otimes_A X \otimes_A M \quad \text{or} \quad \widetilde{Y} \cong \widetilde{M} \otimes_A X \otimes_A M$$

as $B - B$-equivalence bimodules. Therefore, we obtain the following theorem.

Theorem 3.6 *Let A and B be σ-unital C^*-algebras and X and Y an $A - A$-equivalence bimodule and a $B - B$-equivalence bimodule, respectively. Let $A \subset A \rtimes_X \mathbf{Z}$ and $B \subset B \rtimes_Y \mathbf{Z}$ be the inclusions of C^*-algebras induced by X and Y, respectively. We suppose that $A' \cap M(A \rtimes_X \mathbf{Z}) \cong \mathbf{C}1$. Then, the following conditions are equivalent:*

(1) *$A \subset A \rtimes_X \mathbf{Z}$ and $B \subset B \rtimes_Y \mathbf{Z}$ are strongly Morita equivalent with respect to an $A \rtimes_X \mathbf{Z} - B \rtimes_Y \mathbf{Z}$-equivalence bimodule N and its closed subspace M,*

(2) *$Y \cong \widetilde{M} \otimes_A X \otimes_A M$ or $\widetilde{Y} \cong \widetilde{M} \otimes_A X \otimes_A M$ as $B - B$-equivalence bimodules.*

Proof (1)⇒(2): This is immediate by the above discussions. (2)⇒(1): This is immediate by Lemma 2.4. ∎

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Remark 3.7 The above theorem says that the inclusions $A \subset A \rtimes_X \mathbf{Z}$ and $B \subset B \rtimes_Y \mathbf{Z}$ are strongly Morita equivalent if and if X and Y are "flip" conjugate as equivalence bimodules. This is natural, because α and β, the corresponding actions on $A \otimes \mathbf{K}$ and $B \otimes \mathbf{K}$ to X and Y, respectively, are "flip" exterior equivalent, that is, α and β (or β^{-1}) are exterior equivalent.

4 The Picard groups

Let A be a unital C^*-algebra and X an $A-A$-equivalence bimodule. Let $A \subset A \rtimes_X \mathbf{Z}$ be the inclusion of unital C^*-algebras induced by X. We suppose that $A' \cap (A \rtimes_X \mathbf{Z}) = \mathbf{C}1$. In this section, we shall compute $\mathrm{Pic}(A, A \rtimes_X \mathbf{Z})$, the Picard group of the inclusion $A \subset A \rtimes_X \mathbf{Z}$ (See [6]).

Let G be the subgroup of $\mathrm{Pic}(A)$ defined by

$$G = \{[M] \in \mathrm{Pic}(A) \mid X \cong \widetilde{M} \otimes_A X \otimes_A M \text{ or } \widetilde{X} \cong \widetilde{M} \otimes_A X \otimes_A M \\ \text{as } A-A\text{-equivalence bimodules}\}.$$

Let f_A be the homomorphism of $\mathrm{Pic}(A, A \rtimes_X \mathbf{Z})$ to $\mathrm{Pic}(A)$ defined by

$$f_A([M, N]) = [M]$$

for any $[M, N] \in \mathrm{Pic}(A, A \rtimes_X \mathbf{Z})$. First, we show $\mathrm{Im} f_A = G$, where $\mathrm{Im} f_A$ is the image of f_A.

Lemma 4.1 *With the above notation,* $\mathrm{Im} f_A = G$.

Proof Let $[M, N] \in \mathrm{Pic}(A, A \rtimes_X \mathbf{Z})$. Then, by the definition of $\mathrm{Pic}(A, A \rtimes_X \mathbf{Z})$, the inclusion $A \subset A \rtimes_X \mathbf{Z}$ is strongly Morita equivalent to itself with respect to an $A \rtimes_X \mathbf{Z} - A \rtimes_X \mathbf{Z}$-equivalence bimodule N and its closed subspace M. Hence, by Theorem 3.6, $X \cong \widetilde{M} \otimes_A X \otimes_A M$ or $\widetilde{X} \cong \widetilde{M} \otimes_A X \otimes_A M$ as $A-A$-equivalence bimodules. Thus, $\mathrm{Im} f_A \subset G$. Next, let $[M] \in G$. Then, by Lemma 2.4, there is an $A \rtimes_X \mathbf{Z} - A \rtimes_X \mathbf{Z}$-equivalence bimodule N satisfying the following:

(1) M is included in N as a closed subspace,
(2) $[M, N] \in \mathrm{Pic}(A, A \rtimes_X \mathbf{Z})$.

Hence, $G \subset \mathrm{Im} f_A$. Therefore, we obtain the conclusion. ∎

Next, we compute $\mathrm{Ker} f_A$, the kernel of f_A. Let $\mathrm{Aut}(A, A \rtimes_X \mathbf{Z})$ be the group of all automorphisms α of $A \rtimes_X \mathbf{Z}$ such that $\alpha|_A$ is an automorphism of A. Let $\mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z})$ be the group of all automorphisms α of $A \rtimes_X \mathbf{Z}$ such that $\alpha|_A = \mathrm{id}$ on A. It is clear that $\mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z})$ is a normal subgroup of $\mathrm{Aut}(A, A \rtimes_X \mathbf{Z})$. Let π be the homomorphism of $\mathrm{Aut}(A, A \rtimes_X \mathbf{Z})$ to $\mathrm{Pic}(A, A \rtimes_X \mathbf{Z})$ defined by

$$\pi(\alpha) = [M_\alpha, N_\alpha],$$

for any $\alpha \in \mathrm{Aut}(A, A \rtimes_X \mathbf{Z})$, where $[M_\alpha, N_\alpha]$ is an element in $\mathrm{Pic}(A, A \rtimes_X \mathbf{Z})$ induced by α (See [6, Section 3]).

Lemma 4.2 *With the above notation,*

$$\mathrm{Ker} f_A = \{[A, N_\beta] \in \mathrm{Pic}(A, A \rtimes_X \mathbf{Z}) \mid \beta \in \mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z})\}.$$

Proof Let $[M, N] \in \mathrm{Ker} f_A$. Then, $[M] = [A]$ in $\mathrm{Pic}(A)$, and by [6, Lemma 7.5], there is a $\beta \in \mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z})$ such that

$$[M, N] = [A, N_\beta]$$

in $\mathrm{Pic}(A, A \rtimes_X \mathbf{Z})$, where N_β is the $A \rtimes_X \mathbf{Z} - A \rtimes_X \mathbf{Z}$-equivalence bimodule induced by β. Therefore, we obtain the conclusion. ∎

Let $\mathrm{Int}(A, A \rtimes_X \mathbf{Z})$ be the group of all $\mathrm{Ad}(u)$ such that u is a unitary element in A. By [6, Lemma 3.4],

$$\mathrm{Ker}\, \pi \cap \mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z}) = \mathrm{Int}(A, A \rtimes_X \mathbf{Z}) \cap \mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z}).$$

Hence,

$$\begin{aligned}
&\mathrm{Ker}\, \pi \cap \mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z}) \\
&\quad = \{\mathrm{Ad}(u) \in \mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z}) \mid u \text{ is a unitary element in } A\} \\
&\quad = \{\mathrm{Ad}(u) \in \mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z}) \mid u \text{ is a unitary element in } A' \cap A\}.
\end{aligned}$$

Because $A' \cap (A \rtimes_X \mathbf{Z}) = \mathbf{C}1$, $A' \cap A = \mathbf{C}1$. Thus,

$$\mathrm{Ker}\, \pi \cap \mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z}) = \{1\}.$$

It follows that we can obtain the following lemma.

Lemma 4.3 *With the above notation,* $\mathrm{Ker} f_A \cong \mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z})$.

Proof Because $\mathrm{Ker}\, \pi \cap \mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z}) = \{1\}$, by Lemma 4.2,

$$\begin{aligned}
\mathrm{Ker} f_A &= \pi(\mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z})) \cong \mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z})/(\mathrm{Ker}\, \pi \cap \mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z})) \\
&= \mathrm{Aut}_0(A, A \rtimes_X \mathbf{Z}).
\end{aligned}$$

Therefore, we obtain the conclusion. ∎

We recall that the inclusions of C^*-algebras $A \otimes \mathbf{K} \subset (A \rtimes_X \mathbf{Z}) \otimes \mathbf{K}$ and $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z}$ are isomorphic as inclusions of C^*-algebras. Furthermore, there is an automorphism α of $A \otimes \mathbf{K}$ such that $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_{X \otimes \mathbf{K}} \mathbf{Z}$ and $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z}$ are isomorphic as inclusions of C^*-algebras.

Lemma 4.4 *With the above notation, the action α of $\mathbf{Z}$ is free, that is, for any $n \in \mathbf{Z} \setminus \{0\}$, α^n satisfies the following: If $x \in M(A \otimes \mathbf{K})$ satisfies that $xa = \alpha^n(a)x$ for any $a \in A \otimes \mathbf{K}$, then $x = 0$.*

Proof Because $A' \cap (A \rtimes_X \mathbf{Z}) = \mathbf{C}1$, by [8, Lemma 3.1], $(A \otimes \mathbf{K})' \cap M((A \rtimes_X \mathbf{Z}) \otimes \mathbf{K}) = \mathbf{C}1$. Hence, because $A \otimes \mathbf{K} \subset (A \rtimes_X \mathbf{Z}) \otimes \mathbf{K}$ is isomorphic to $A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z}$ as inclusions of C^*-algebras,

$$(A \otimes \mathbf{K})' \cap M((A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z}) = \mathbf{C}1.$$

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Thus, by [8, Corollary 4.2], the action α is free. ∎

For any $n \in \mathbf{Z}$, let δ_n be the function on $\mathbf{Z}$ defined by

$$\delta_n(m) = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}.$$

We regard δ_n as an element in $M((A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z})$.

Let $E^{M(A\otimes\mathbf{K})}$ be the canonical faithful conditional expectation from $M(A \otimes \mathbf{K}) \rtimes_{\underline{\alpha}} \mathbf{Z}$ onto $M(A \otimes \mathbf{K})$ defined in [2, Section 3]. Then, we may let $E^{A\otimes\mathbf{K}}$ be the restriction of $E^{M(A\otimes\mathbf{K})}$ to $(A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z}$, that is, $E^{A\otimes\mathbf{K}} = E^{M(A\otimes\mathbf{K})}|_{(A\otimes\mathbf{K})\rtimes_\alpha\mathbf{Z}}$. Let $\{u_i\}_{i\in I}$ be an approximate unit of $A \otimes \mathbf{K}$. We fix the approximate unit $\{u_i\}_{i\in I}$ of $A \otimes \mathbf{K}$. For any $x \in M((A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z})$, we define the Fourier coefficient of x at $n \in \mathbf{Z}$ as in the same way as in [8, Section 2]. We show that $\mathrm{Aut}_0(A \otimes \mathbf{K}, (A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z}) \cong \mathbf{T}$.

Let $\beta \in \mathrm{Aut}_0(A \otimes \mathbf{K}, (A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z})$. For any $a \in A \otimes \mathbf{K}$,

$$\underline{\beta}(\delta_1)a\underline{\beta}(\delta_1^*) = \beta(\delta_1 a \delta_1^*) = \beta(\alpha(a)) = \alpha(a).$$

Hence, $\underline{\beta}(\delta_1)a = \alpha(a)\underline{\beta}(\delta_1)$ for any $a \in A \otimes \mathbf{K}$.

Lemma 4.5 *With the above notation, let a_n be the Fourier coefficient of $\underline{\beta}(\delta_1)$ at $n \in \mathbf{Z}$. Then, for any $a \in A \otimes \mathbf{K}$,*

$$a_n \alpha^{n-1}(a) = a a_n.$$

Proof Let $a \in A \otimes \mathbf{K}$. Then, because $\|au_i - u_i a\| \to 0 (i \to \infty)$, the Fourier coefficient of $\underline{\beta}(\delta_1)a$ at $n \in \mathbf{Z}$ is given by

$$\begin{aligned}
\lim_i E^{A\otimes\mathbf{K}}(\underline{\beta}(\delta_1)au_i\delta_n) &= \lim_i E^{A\otimes\mathbf{K}}(\underline{\beta}(\delta_1)u_i a\delta_n) \\
&= \lim_i E^{A\otimes\mathbf{K}}(\underline{\beta}(\delta_1)u_i\delta_n\alpha^n(a)) \\
&= \lim_i E^{A\otimes\mathbf{K}}(\underline{\beta}(\delta_1)u_i\delta_n)\alpha^n(a) \\
&= a_n\alpha^n(a).
\end{aligned}$$

Furthermore, the Fourier coefficient of $\alpha(a)\underline{\beta}(\delta_1)$ at $n \in \mathbf{Z}$ is given by

$$\lim_i E^{A\otimes\mathbf{K}}(\alpha(a)\underline{\beta}(\delta_1)u_i\delta_n) = \alpha(a)\lim_i E^{A\otimes\mathbf{K}}(\underline{\beta}(\delta_1)u_i\delta_n) = \alpha(a)a_n.$$

Because $\underline{\beta}(\delta_1)a = \alpha(a)\underline{\beta}(\delta_1)$, we get that

$$a_n\alpha^n(a) = \alpha(a)a_n,$$

for any $a \in A \otimes \mathbf{K}$. Because a is an arbitrary element in $A \otimes \mathbf{K}$, replacing a by $\alpha^{-1}(a)$, we obtain the conclusion. ∎

Lemma 4.6 *With the above notation,*

$$\mathrm{Aut}_0(A \otimes \mathbf{K}, (A \otimes \mathbf{K}) \rtimes_\alpha \mathbf{Z}) \cong \mathbf{T}.$$

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Proof Let $\beta \in \mathrm{Aut}_0(A\otimes \mathbf{K}, (A\otimes\mathbf{K})\rtimes_\alpha \mathbf{Z})$, and let a_n be the Fourier coefficient of $\underline{\beta}(\delta_1)$ at $n\in\mathbf{Z}$. Then, by Lemma 4.5, $a_n\alpha^{n-1}(a) = aa_n$ for any $a\in A\otimes\mathbf{K}$. Because the automorphism α^{n-1} is free for any $n\in\mathbf{Z}\smallsetminus\{1\}$ by Lemma 4.4, $a_n = 0$ for any $n\in\mathbf{Z}\smallsetminus\{1\}$. Thus, $\underline{\beta}(\delta_1) = a_1\delta_1$. Because $\underline{\beta}(\delta_1)a\underline{\beta}(\delta_1^*) = \alpha(a)$, for any $a\in A\otimes\mathbf{K}$,

$$a_1\delta_1 a\delta_1^* a_1^* = \alpha(a).$$

Because $\delta_1 a\delta_1^* = \alpha(a)$,

$$a_1\alpha(a)a_1^* = \alpha(a),$$

for any $a\in A\otimes\mathbf{K}$. Because δ_1 and $\underline{\beta}(\delta_1)$ are unitary elements in $M((A\otimes\mathbf{K})\rtimes_\alpha\mathbf{Z})$, a_1 is a unitary element in $M(A\otimes\mathbf{K})$. Thus,

$$a_1\alpha(a) = \alpha(a)a_1,$$

for any $a\in A\otimes\mathbf{K}$. Because $(A\otimes\mathbf{K})'\cap M(A\otimes\mathbf{K}) = \mathbf{C}1$, $a_1\in\mathbf{C}1$. Because a_1 is a unitary element in $M(A\otimes\mathbf{K})$, there is the unique element $c_\beta\in\mathbf{T}$ such that $a_1 = c_\beta 1$. Let ε be the map from $\mathrm{Aut}_0(A\otimes\mathbf{K}, (A\otimes\mathbf{K})\rtimes_\alpha\mathbf{Z})$ onto $\mathbf{T}$ defined by $\varepsilon(\beta) = c_\beta$. By routine computations, we can see that ε is an isomorphism of $\mathrm{Aut}_0(A\otimes\mathbf{K}, (A\otimes\mathbf{K})\rtimes_\alpha\mathbf{Z})$ onto $\mathbf{T}$. ■

Lemma 4.7 *With the above notation,*

$$\mathrm{Aut}_0(A, A\rtimes_X\mathbf{Z}) \cong \mathrm{Aut}_0(A\otimes\mathbf{K}, (A\otimes\mathbf{K})\rtimes_\alpha\mathbf{Z}).$$

Proof Because $A\otimes\mathbf{K}\subset(A\otimes\mathbf{K})\rtimes_\alpha\mathbf{Z}$ and $A\otimes\mathbf{K}\subset(A\rtimes_X\mathbf{Z})\otimes\mathbf{K}$ are isomorphic as inclusions of C^*-algebras, it suffices to show that

$$\mathrm{Aut}_0(A, A\rtimes_X\mathbf{Z}) \cong \mathrm{Aut}_0(A\otimes\mathbf{K}, (A\rtimes_X\mathbf{Z})\otimes\mathbf{K}).$$

Let κ be the homomorphism of $\mathrm{Aut}_0(A, A\rtimes_X\mathbf{Z})$ to $\mathrm{Aut}_0(A\otimes\mathbf{K}, (A\rtimes_X\mathbf{Z})\otimes\mathbf{K})$ defined by

$$\kappa(\beta) = \beta\otimes\mathrm{id}_{\mathbf{K}},$$

for any $\beta\in\mathrm{Aut}_0(A, A\rtimes_X\mathbf{Z})$. Then, it is clear that κ is a monomorphism of $\mathrm{Aut}_0(A, A\rtimes_X\mathbf{Z})$ to $\mathrm{Aut}_0(A\otimes\mathbf{K}, (A\rtimes_X\mathbf{Z})\otimes\mathbf{K})$. We show that κ is surjective. Let $\gamma\in\mathrm{Aut}_0(A\otimes\mathbf{K}, (A\rtimes_X\mathbf{Z})\otimes\mathbf{K})$. Then,

$$\gamma(a\otimes e_{ij}) = a\otimes e_{ij},$$

for any $a\in A$, $i, j\in\mathbf{N}$. Thus,

$$\gamma(x\otimes e_{11}) = (1\otimes e_{11})\gamma(x\otimes e_{11})(1\otimes e_{11}),$$

for any $x\in A\rtimes_X\mathbf{Z}$. Hence, there is an automorphism β of $A\rtimes_X\mathbf{Z}$ such that

$$\gamma(x\otimes e_{11}) = \beta(x)\otimes e_{11},$$

for any $x\in A\rtimes_X\mathbf{Z}$. For any $i, j\in\mathbf{N}$ and $x\in A\rtimes_X\mathbf{Z}$,

$$\begin{aligned}\gamma(x\otimes e_{ij}) &= \gamma((1\otimes e_{i1})(x\otimes e_{11})(1\otimes e_{1j})) = (1\otimes e_{i1})(\beta(x)\otimes e_{11})(1\otimes e_{1j})\\ &= \beta(x)\otimes e_{ij}.\end{aligned}$$

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Especially, if $a \in A$, $\beta(a) \otimes e_{ij} = \gamma(a \otimes e_{ij}) = a \otimes e_{ij}$, for any $i, j \in \mathbf{N}$. Thus, $\beta(a) = a$, for any $a \in A$. Therefore, $\gamma = \beta \otimes \mathrm{id}_{\mathbf{K}}$ and $\beta \in \mathrm{Aut}_0(A, A \rtimes_X \mathbf{K})$. Hence, we have shown that κ is surjective. ∎

Lemma 4.8 *With the above notation,* $\mathrm{Ker} f_A \cong \mathbf{T}$.

Proof This is immediate by Lemmas 4.3, 4.6, and 4.7. ∎

By Lemmas 4.1 and 4.8, we have the following exact sequence:

$$1 \longrightarrow \mathbf{T} \longrightarrow \mathrm{Pic}(A, A \rtimes_X \mathbf{Z}) \longrightarrow G \longrightarrow 1,$$

where

$$G = \{[M] \in \mathrm{Pic}(A) \,|\, X \cong \widetilde{M} \otimes_A X \otimes_A M \text{ or } \widetilde{X} \cong \widetilde{M} \otimes_A X \otimes_A M \text{ as } A - A\text{-equivalence bimodules}\}.$$

Let g be the map from G to $\mathrm{Pic}(A, A \rtimes_X \mathbf{Z})$ defined by

$$g([M]) = [M, N],$$

where N is the $A \rtimes_X \mathbf{Z} - A \rtimes_X \mathbf{Z}$-equivalence bimodule defined in the proof of Lemma 2.4. Then, g is a homomorphism of G to $\mathrm{Pic}(A, A \rtimes_X \mathbf{Z})$ such that

$$f_A \circ g = \mathrm{id}$$

on G. Thus, we obtain the following theorem.

Theorem 4.9 *Let A be a unital C^*-algebra and X an $A - A$-equivalence bimodule. Let $A \subset A \rtimes_X \mathbf{Z}$ be the unital inclusion of unital C^*-algebras induced by X. We suppose that $A' \cap (A \rtimes_X \mathbf{Z}) = \mathbf{C}1$. Then,* $\mathrm{Pic}(A, A \rtimes_X \mathbf{Z})$ *is a semidirect product of G by* $\mathbf{T}$.

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