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Flow around a wedge of arbitrary angle in a Hele-Shaw cell

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In this note we modify and extend the work of Howison & King [12] to describe the situation of flow around a wedge of arbitrary angle in a Hele-Shaw cell. An ingenious complex-variable method due to Polubarinova-Kochina is used to construct an explicit solution to the zero-surface tension problem.

1 Introduction

In a paper published in 1989, Howison & King [12] presented explicit solutions to a selection of free boundary problems arising in fluid flow and diffusion. These problems involved viscous-film-coating of 360°, 270° and 90° corners (problems (P1), (P2) and (P3) in their notation); dopant diffusion in semiconductors ((P4) and (P5)), and etching of semiconductors (P6). The solution method used (which we shall call the P-K method) was devised by Polubarinova-Kochina [16] to solve the rectangular dam problem, but it is remarkably versatile in its applications. In this note we use the method to solve for Hele-Shaw flow around a wedge of given (arbitrary) angle $\alpha\pi$.

In the conclusions of Howison & King [12], the authors state that their solutions are related to solutions of corresponding Hele-Shaw flows, and outline how the correspondence works for the semiconductor etching problem (P6). In this particular problem, which is described in detail in Kuiken [13], a pit is etched into a solid surface by covering part of the surface with a mask, and immersing in some liquid etchant which 'eats away' at the solid not covered by the mask, and partially penetrates beneath the mask. In industrial applications it is important to know the shape of the pit so produced. In the 'thin mask' models of Howison & King [12] and Kuiken [13], this problem is related to that of Hele-Shaw flow around a thin wall, or zero-angle wedge. Thus the solutions we present in this paper, in addition to describing a Hele-Shaw flow, also describe the free boundary for etching problems involving wedge-shaped masks, via the correspondence outlined in Howison & King [12]. Geometries of this kind are sketched by Kuiken [13] under the heading 'Opportunities for further research'. With reference to Figure 1, the wedge corresponds to the mask, the fluid region to the etchant, and the remainder corresponds to the solid surface being etched.

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The wedge geometry in the Hele-Shaw problem is also of relevance for problems in injection-moulding, where fluid is injected into the narrow space between two (usually curved) plates, which form a mould. The fluid used is typically a molten plastic[1], and the mould may contain solid obstacles to form a finished product of the desired shape (e.g. with 'holes' in particular places). Our solutions describe the local situation as the fluid passes round a corner of a given obstacle within the mould. It is important to understand the motion of the free boundary around such obstacles, since obviously one wants to ensure that the mould is filled entirely, without any air-bubbles being trapped behind obstacles.

After this paper was initially submitted, other similar work by Craster *et al.* [4, 5] and by Hoang *et al.* [9] was drawn to the author's attention. In Craster *et al.* [4], the viscous film coating solutions (P1)–(P3) of Howison & King [12] are extended to arbitrary angles; and in Craster *et al.* [5] and Hoang *et al.* [9], quasi-steady solidification problems in wedge-shaped domains are treated. While Craster *et al.* [4] especially is related to this work (in that the geometry, like ours, involves flow exterior to a wedge of given angle), the other two papers [5, 9] both involve solidification *inside* a wedge. We observe that with the quasi-steady approximation[2] adopted in Craster *et al.* [5] and Hoang *et al.* [9], the solidification problem is identical to the Hele-Shaw problem after a trivial rescaling of time with the Stefan number (carried out in Craster *et al.* [5] but not in Hoang *et al.* [9]). Thus we have another interpretation of our solutions: if the fluid domain Ω is identified with the frozen region, the 'air' exterior to Ω with the unfrozen liquid, and the pressure p with the temperature, we have the quasi-steady approximation to a freezing problem exterior to a wedge.

In constructing our solutions we shall try to indicate how the P-K method works, but we do not attempt to give a comprehensive description. Those unfamiliar with the method are referred elsewhere [16, 3] for details, and for useful overviews/background reading [12, 4, 5].

2 Problem formulation

A Hele-Shaw cell consists of two rigid parallel plates separated by some small distance, between which is sandwiched a layer of viscous fluid. The flow is effectively two-dimensional (assumed to take place in the (x, y)-plane) and irrotational, the pressure p acting as a velocity potential. If $\Omega(t)$ is the region occupied by fluid at time t then the problem satisfied by p is, in dimensionless variables:

$$\nabla^2 p = 0 \quad \text{in } \Omega(t);$$
$$\frac{\partial p}{\partial n} = 0 \quad \text{on fixed boundaries } \Gamma\,;$$

[1] Of course, molten plastics of the type common in injection moulding are generally non-Newtonian, and require a different Hele-Shaw model (e.g based on a power-law fluid) leading to a much more difficult problem. However, the Newtonian case provides an important starting-point for a study of injection-moulding. Additional complications due to a curved mould will not be important on this local lengthscale.

[2] This assumes that the rate of motion of the free boundary is much slower than the rate at which heat is conducted.

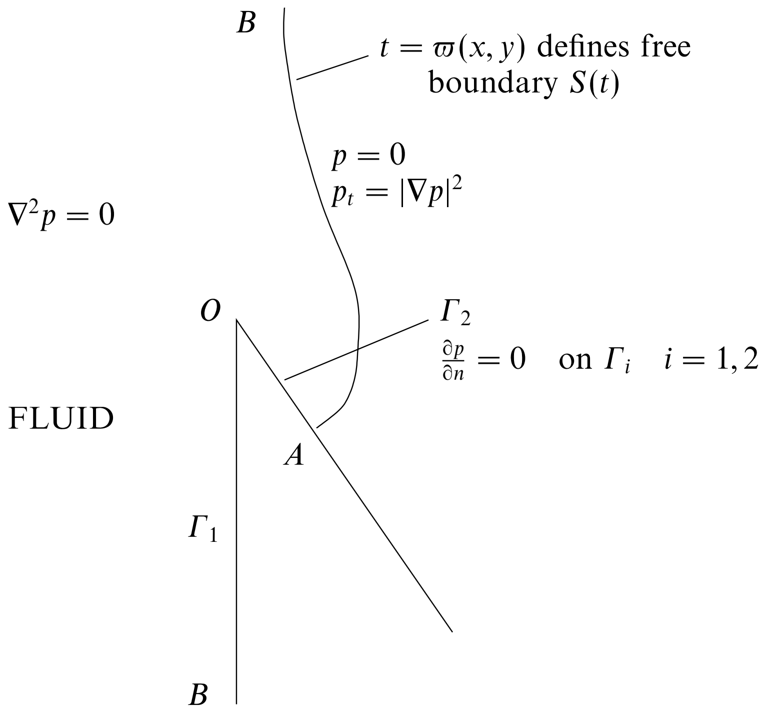

FIGURE 1. The geometry of the problem.

$$p = 0, \quad \frac{\partial p}{\partial t} + |\nabla p|^2 = 0 \quad \text{on free boundaries } S(t);$$

see, for example, Crank [3] and Elliott & Ockendon [6]: the total fluid boundary $\partial\Omega(t)$ is the union of Γ and $S(t)$. The geometry we consider is sketched in Figure 1; we assume that at time $t = 0$ the fluid occupies the left half-space $x \leqslant 0$, so that it is in contact with and parallel to the nearside of the wedge, which lies along the negative y-axis, the wedge apex being at the origin. If $\omega(z, t)$ is the complex potential for the pressure ($p = -\Re(\omega)$ and $z = x + iy$) then the solution to the outer problem, in which $|z| \gg 1$ and the free boundary is assumed to be asymptotic to the y-axis for y large and positive, is easily written down as

$$\omega = f(t)\sqrt{2iz},$$

where $f(t)$ is any function of time, and the factor of 2 is just for convenience later. This clearly has vanishing real part on the positive y-axis, and its derivative has vanishing real part on the negative y-axis: it corresponds to the asymptotic pressure

$$p \sim f(t)[(x^2 + y^2)^{\frac{1}{2}} - y]^{\frac{1}{2}}; \tag{2.1}$$

cf. problem P6 in Howison & King [12]. The question we now address is, what will be the free boundary shape for $t > 0$?

The problem and boundary conditions are not, as they stand, amenable to application of the P-K method (see elsewhere [12, 16, 3] for the general suitable problem formulation). The first step is to eliminate time from the problem except as a parameter by applying the well-known Baiocchi transformation to the pressure [3, 6, 14]. For $t \geqslant 0$ this is defined in the two regions $x \geqslant 0$, $x < 0$ as follows:

$$\text{in } x \geqslant 0: \qquad u(x, y, t) = \int_{\varpi(x,y)}^{t} p(x, y, \tau)\, d\tau,$$

where $t = \varpi(x, y)$ is a parametrisation of the free boundary $S(t)$; and

$$\text{in } x < 0: \qquad u(x, y, t) = u_0(x, y) + \int_0^t p(x, y, \tau)\, d\tau,$$

where u_0 is the solution to the Cauchy problem

$$\nabla^2 u_0 = 1 \quad \text{in } \Omega(0),$$
$$u_0 = 0 = \frac{\partial u_0}{\partial n} \quad \text{on } S(0), \qquad \frac{\partial u_0}{\partial n} = 0 \quad \text{on } \Gamma,$$

so here $u_0 \equiv x^2/2$. The problem satisfied by u is easily seen to be

$$\nabla^2 u = 1 \quad \text{in } \Omega(t),$$
$$u = 0 = \frac{\partial u}{\partial n} \quad \text{on } S(t), \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma,$$
$$u \sim \frac{x^2}{2} + [(x^2 + y^2)^{\frac{1}{2}} - y]^{\frac{1}{2}} \int_0^t f(\tau)\, d\tau, \qquad \text{as } x^2 + y^2 \to \infty.$$

We can scale time t out of the problem altogether if we introduce similarity variables via

$$(X, Y) \int_0^t f(\tau)\, d\tau = (x, y), \qquad U \left(\int_0^t f(\tau)\, d\tau \right)^2 = u;$$

U then satisfies the problem

$$\nabla^2 U = 1 \quad \text{in } \hat{\Omega},$$
$$U = 0 = \frac{\partial U}{\partial n} \quad \text{on } \hat{S}, \qquad \frac{\partial U}{\partial n} = 0 \quad \text{on } \hat{\Gamma},$$
$$U \sim \frac{X^2}{2} + [(X^2 + Y^2)^{\frac{1}{2}} - Y]^{\frac{1}{2}}, \qquad \text{as } X^2 + Y^2 \to \infty. \tag{2.2}$$

We have used hats to distinguish between domains/boundaries in the two problems (though in fact $\hat{\Gamma}$ and Γ are identical, both being time-independent). We denote the unknown point $\hat{A}$ where the free boundary $\hat{S}$ meets the fixed boundary $\hat{\Gamma}$ by (X_0, Y_0). Taking the complex variable $Z = X + iY$, with the point $\hat{A}$ described by $Z_0 = X_0 + iY_0 = -iR_0 e^{i\alpha\pi}$, we define the complex-valued function W by

$$W = U_X - iU_Y - X. \tag{2.3}$$

Then clearly $\partial W / \partial \bar{Z} = 0$, hence W is an analytic function of Z in $\hat{\Omega}$. Our problem is now in a form suitable for the P-K technique. Corresponding to $\hat{\Omega}$ is a region $\overline{\Omega}$ in the W-plane. We conformally map both $\hat{\Omega}$ and $\overline{\Omega}$ onto the upper half-plane $\Re(\zeta) > 0$ in a third complex plane ($\zeta = \xi + i\eta$), via the mapping functions

$$Z = \mathscr{Z}(\zeta), \qquad W = \mathscr{W}(\zeta);$$

when these maps are found the solution is completed by elimination of the auxiliary variable ζ. The boundary $\partial\hat{\Omega}$ consists of the three distinct segments $\hat{\Gamma}_1$, $\hat{\Gamma}_2$ and $\hat{S}$ (cf.

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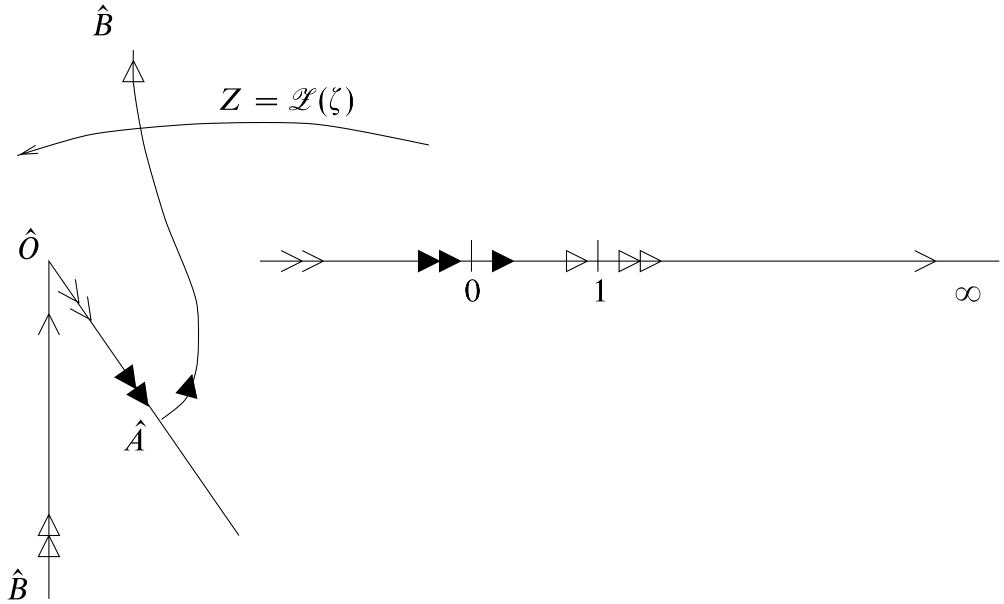

FIGURE 2. The mapping from the ζ-plane to the Z-plane. The arrows indicate the correspondence and sense of the mapping from the real ζ-axis to the fluid boundary.

Figure 1); the points $\hat{B}$ (at infinity), $\hat{O}$ and $\hat{A}$ separating these three segments are singular points of the flow. By Riemann's mapping theorem we can prescribe the position of the three points $\hat{A}$, $\hat{B}$ and $\hat{O}$ in the ζ-plane; for convenience we place them at 0, 1 and ∞. The point is that we are able to determine the singular behaviour of the functions $\mathscr{Z}$ and $\mathscr{W}$ at these points, and this in turn is sufficient to determine fully $\mathscr{Z}$ and $\mathscr{W}$, which are analytic except at the three singular points. By performing a local analysis on our problem at each of the singular points we can find, at each point $\xi_i = 0, 1, \infty$, two *exponents* β_i and γ_i such that the local behaviour of $\mathscr{Z}$ and $\mathscr{W}$ satisfies:

$$\mathscr{Z} \sim L_1(\zeta - \xi_i)^{\beta_i} F_i(\zeta) + L_2(\zeta - \xi_i)^{\gamma_i} G_i(\zeta),$$
$$\mathscr{W} \sim M_1(\zeta - \xi_i)^{\beta_i} F_i(\zeta) + M_2(\zeta - \xi_i)^{\gamma_i} G_i(\zeta),$$

for functions F_i and G_i analytic in some neighbourhood of ξ_i. (In the general theory it is sometimes necessary to have more complicated local behaviour than the above, involving logarithms, but such complications do not arise in our calculation.) The only singularities of $\mathscr{Z}$ and $\mathscr{W}$ are then algebraic (or logarithmic) branch-points, hence $\mathscr{Z}$ and $\mathscr{W}$ must be branches of the *Riemann P-function*

$$P \left\{ \begin{matrix} 0 & 1 & \infty & \\ \beta_0 & \beta_1 & \beta_2 & \zeta \\ \gamma_0 & \gamma_1 & \gamma_2 & \end{matrix} \right\},$$

which has the desired singular behaviour at the given points (see elsewhere [1, 2, 8] for a discussion of Riemann's P-equation and Riemann P-functions). In this paper, we

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content ourselves with the above outline of the solution procedure (which §3 should clarify) without attempting to justify the statements. For a more convincing argument, see Crank [3] or Polubarinova-Kochina [16].

3 Solution to the problem

We need to find the local behaviour of $\mathscr{Z}(\zeta)$ and $\mathscr{W}(\zeta)$ at each of the singular points $\zeta = 0, 1, \infty$. To do this we must first solve the local problem for $U(X, Y)$ at the points $\hat{A}$ $(Z = Z_0)$, $\hat{B}$ $(Z = \infty)$, $\hat{O}$ $(Z = 0)$.

The free boundary $\hat{S}$ must meet the fixed boundary $\hat{\Gamma}$ at right angles at $\hat{A}$. On $\hat{S}$ both U and its normal derivative must vanish, while on $\hat{\Gamma}$ only the normal derivative vanishes. Hence, if $(\hat{X}, \hat{Y})$ are co-ordinates along and perpendicular to the side of the wedge, with origin at $\hat{A}$, the local solution is

$$U \sim \frac{\hat{X}^2}{2} \equiv \frac{1}{2}((X - X_0)\sin\alpha\pi - (Y - Y_0)\cos\alpha\pi)^2.$$

In terms of Z, then, the local behaviour of W is

$$W \sim -\tfrac{1}{2}(Z + \bar{Z}_0) - \tfrac{1}{2}e^{-2i\alpha\pi}(Z - Z_0), \tag{3.1}$$

near $Z = Z_0$. At large distances (infinity, or the point $\hat{B}$) W satisfies

$$W \sim \pm\left(\frac{i}{2Z}\right)^{\frac{1}{2}} \qquad \text{as } |Z| \to \infty, \tag{3.2}$$

as may be seen from (2.2) and (2.3); the branch of the square-root must be chosen consistently later. Near the wedge tip $\hat{O}$ the leading-order behaviour of U is

$$U \sim \lambda R^{1/(2-\alpha)}\cos\left(\frac{3\pi/2 - \theta}{2 - \alpha}\right) + \frac{R^2}{4},$$

where $R^2 = X^2 + Y^2$, $\tan\theta = Y/X$ and λ is a (real) constant; hence the behaviour of W is

$$W \sim kZ^{-(1-\alpha)/(2-\alpha)} - \frac{Z}{2}, \tag{3.3}$$

where the complex constant k has the form $k = \tilde{\lambda}\exp(-3\pi i/(2(2-\alpha)))$, $\tilde{\lambda} \in \mathbb{R}$. If we now write down the behaviour of the mapping function $\mathscr{Z}(\zeta)$ at each of the singular points, the above allows us to deduce the local behaviour of $\mathscr{W}(\zeta)$ also, and hence the values of the exponents in the Riemann P-function.

The unknown point $\hat{A} = (X_0, Y_0)$ is the image of $\zeta = 0$. From the geometry, simple considerations of conformal mapping show that near $\zeta = 0$ we require

$$\mathscr{Z}(\zeta) \sim Z_0 + \mu e^{i\alpha\pi}\zeta^{\frac{1}{2}} \qquad \mu \in \mathbb{R}^+, \tag{3.4}$$

so from (3.1) we find

$$\mathscr{W}(\zeta) \sim -R_0\sin\alpha\pi - \mu\zeta^{\frac{1}{2}}\cos\alpha\pi. \tag{3.5}$$

Thus the values of the exponents at $\zeta = 0$ are $\beta_0 = 0$, $\gamma_0 = \frac{1}{2}$.

The point at infinity $\hat{B}$ is the image of $\zeta = 1$; near this point we have

$$\mathscr{Z}(\zeta) \sim \frac{iv}{1-\zeta} \qquad v \in \mathbb{R}^+; \tag{3.6}$$

(3.2) then gives

$$\mathscr{W}(\zeta) \sim \pm \left(\frac{1-\zeta}{2v}\right)^{\frac{1}{2}}, \tag{3.7}$$

so the exponents at $\zeta = 1$ are $\beta_1 = -1$, $\gamma_1 = \frac{1}{2}$.

Finally, near the wedge-tip $\hat{O}$, $\zeta \to \infty$ and we have

$$\mathscr{Z}(\zeta) \sim -i\sigma\zeta^{-(2-\alpha)} \qquad \sigma \in \mathbb{R}^+, \tag{3.8}$$

so that by (3.3)

$$\mathscr{W}(\zeta) \sim \tilde{\sigma} \exp\left(-\frac{i\pi(2+\alpha)}{2(2-\alpha)}\right)\zeta^{1-\alpha} \qquad \tilde{\sigma} \in \mathbb{R}. \tag{3.9}$$

Thus the exponents at $\zeta = \infty$ are $\beta_2 = 2-\alpha$, $\gamma_2 = -1+\alpha$. It then follows from the P-K theory that the functions $\mathscr{Z}(\zeta)$ and $\mathscr{W}(\zeta)$ are branches of the Riemann P-function

$$P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & -1 & 2-\alpha & \zeta \\ \frac{1}{2} & \frac{1}{2} & -1+\alpha & \end{array}\right\} \equiv (1-\zeta)^{-1} P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & 1-\alpha & \zeta \\ \frac{1}{2} & \frac{3}{2} & -2+\alpha & \end{array}\right\};$$

the equality is due to a transformation given in Carrier *et al.* [2]. This is equivalent to being able to express $\mathscr{Z}$ and $\mathscr{W}$ in terms of branches of a hypergeometric function, since the branches of the P-function on the right-hand side are in fact branches of the hypergeometric function $F(1-\alpha, -2+\alpha; \frac{1}{2}; \zeta)$ [2].

For convenience of notation, we write $a = 1-\alpha$, then $\mathscr{Z}(\zeta)$ and $\mathscr{W}(\zeta)$ are linear combinations of independent branches of $(1-\zeta)^{-1}F(a, -1-a; \frac{1}{2}; \zeta)$. Abramowitz & Stegun [1] and Gradshteyn & Ryzhik [8] give closed-form expressions for one branch of each of the hypergeometric functions $F(a, -a; \frac{1}{2}; \zeta)$ and $F(a, 1-a; \frac{1}{2}; \zeta)$; these functions are simply related to the function we seek by one of Gauss' relations for contiguous hypergeometric functions ((15.2.11) in Abramowitz & Stegun [1]; (9.137.3) in Gradshteyn & Ryzhik [8]). For our purposes, this relation takes the form:

$$\begin{aligned}(a+\tfrac{1}{2})F(a, -1-a; \tfrac{1}{2}; \zeta) - (\tfrac{1}{2} + 2a(1-\zeta))F(a, -a; \tfrac{1}{2}; \zeta)& \\ +a(1-\zeta)F(a, 1-a; \tfrac{1}{2}; \zeta) &= 0,\end{aligned}$$

where

$$F(a, -a; \tfrac{1}{2}; \zeta) = \cos(2a \sin^{-1}\sqrt{\zeta}),$$
$$F(a, 1-a; \tfrac{1}{2}; \zeta) = \tfrac{1}{\sqrt{1-\zeta}}\cos((2a-1)\sin^{-1}\sqrt{\zeta}),$$

(formulae (15.1.17), (15.1.18) in Abramowitz & Stegun [1] and (9.121.31), (9.121.32) in Gradshteyn & Ryzhik [8]). Hence one branch of the required hypergeometric function is

$$\begin{aligned}(a+\tfrac{1}{2})F(a, -1-a; \tfrac{1}{2}; \zeta) = (\tfrac{1}{2} + 2a(1-\zeta))\cos(2a \sin^{-1}\sqrt{\zeta})& \\ -a\sqrt{1-\zeta}\cos((2a-1)\sin^{-1}\sqrt{\zeta}),&\end{aligned} \tag{3.10}$$

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which is readily verified to be the branch analytic at $\zeta = 0$ (corresponding to the zero index at $\zeta = 0$). Another, independent branch is given [1, 2, 8] by $\sqrt{\zeta}F(a+\frac{1}{2}, -(a+\frac{1}{2}); \frac{3}{2}; \zeta)$; we again use properties of contiguous hypergeometric functions and closed-form expressions for the related functions (given this time by (15.1.15), (15.1.16) in Abramowitz & Stegun [1] and (9.121.29), (9.121.30) in Gradshteyn & Ryzhik [8]) to find

$$2a(a+1)\sqrt{\zeta}F(a+\tfrac{1}{2}, -(a+\tfrac{1}{2}); \tfrac{3}{2}; \zeta) = (\tfrac{1}{2} + 2a(1-\zeta))\sin(2a\sin^{-1}\sqrt{\zeta}) \\ -a\sqrt{1-\zeta}\sin((2a-1)\sin^{-1}\sqrt{\zeta}). \tag{3.11}$$

To summarise, then, if we denote by $B_1(\zeta)$ and $B_2(\zeta)$ the two independent branches we require for our solutions, we have (from (3.10) and (3.11), respectively)

$$B_1(\zeta) = (\tfrac{1}{2} + 2a(1-\zeta))\cos(2a\sin^{-1}\sqrt{\zeta}) - a\sqrt{1-\zeta}\cos((2a-1)\sin^{-1}\sqrt{\zeta}), \tag{3.12}$$

$$B_2(\zeta) = (\tfrac{1}{2} + 2a(1-\zeta))\sin(2a\sin^{-1}\sqrt{\zeta}) - a\sqrt{1-\zeta}\sin((2a-1)\sin^{-1}\sqrt{\zeta}), \tag{3.13}$$

where $a = 1 - \alpha$, and the solutions $\mathscr{Z}(\zeta)$, $\mathscr{W}(\zeta)$ are

$$\mathscr{Z}(\zeta) = \frac{L_1 B_1(\zeta)}{1-\zeta} + \frac{L_2 B_2(\zeta)}{1-\zeta}, \tag{3.14}$$

$$\mathscr{W}(\zeta) = \frac{M_1 B_1(\zeta)}{1-\zeta} + \frac{M_2 B_2(\zeta)}{1-\zeta}, \tag{3.15}$$

for some constants L_i, $M_i \in \mathbb{C}$ to be determined.

To find the values of L_i and M_i we must determine the local behaviour of $B_i(\zeta)$ at each of the singular points so that we can match the behaviour of the solutions as written above to the known behaviour given in (3.4)–(3.9). When evaluating the behaviour as $\zeta \to \infty$ it seems we need to make some assumption about the parameter α, since this determines the relative sizes of the terms in the large-ζ expansion. If we assume an acute-angled wedge $0 < \alpha < \frac{1}{2}$ ($\frac{1}{2} < a < 1$), then analysis of the functions $B_i(\zeta)/(1-\zeta)$ allows us to calculate the terms of orders ζ^a, ζ^{a-1}, ζ^{-a}, ζ^{a-2}, ζ^{-a-1}, which will be of descending size. We find, for $|\zeta| \gg 1$,

$$\frac{B_1(\zeta)}{1-\zeta} = (-4)^a \left[a\zeta^a + \left(-\frac{a^2}{2} + \frac{a}{4} - \frac{1}{4} \right)\zeta^{a-1} + 0 \cdot \zeta^{-a} + \frac{1}{16}(2a^3 - 5a^2 + 5a - 4)\zeta^{a-2} \right. \\ \left. - \frac{(a+1)}{4 \cdot 16^a}\zeta^{-a-1} + \left(\frac{a^4}{48} + \frac{a^3}{16} - \frac{49a^2}{192} + \frac{21a}{64} - \frac{1}{4} \right)\zeta^{a-3} + O(\zeta^{-a-2}) \right] \tag{3.16}$$

and

$$\frac{B_2(\zeta)}{1-\zeta} = i(-4)^a \left[a\zeta^a + \left(-\frac{a^2}{2} + \frac{a}{4} - \frac{1}{4} \right)\zeta^{a-1} + 0 \cdot \zeta^{-a} + \frac{1}{16}(2a^3 - 5a^2 + 5a - 4)\zeta^{a-2} \right. \\ \left. + \frac{(a+1)}{4 \cdot 16^a}\zeta^{-a-1} + \left(\frac{a^4}{48} + \frac{a^3}{16} - \frac{49a^2}{192} + \frac{21a}{64} - \frac{1}{4} \right)\zeta^{a-3} + O(\zeta^{-a-2}) \right]. \tag{3.17}$$

Matching the behaviour of $\mathscr{Z}(\zeta)$, $\mathscr{W}(\zeta)$ as given by (3.14), (3.15) with that in (3.8), (3.9) yields the following relations between the coefficients and the unknown constants σ, $\tilde{\sigma}$:

$$L_1 = -iL_2, \qquad L_2 = \frac{2\sigma 4^a e^{i\alpha\pi}}{a+1}, \tag{3.18}$$

$$M_1 - iM_2 = \frac{i\tilde{\sigma}}{a4^a}\exp\left(\frac{i\alpha\pi(1-\alpha)}{2-\alpha}\right). \tag{3.19}$$

Due to the way the coefficients appear in the expressions (3.16) and (3.17), it is clear that the same matching follows through also for the case $a = 1$, and for the obtuse angled wedge $\frac{1}{2} < \alpha < 1$ and the right-angled wedge $\alpha = \frac{1}{2}$. The case $a \to 0$ is a singular limit in our solution, as we would expect: as the wedge angle approaches π in the configuration of Figure 1, the distance OA must increase very rapidly from zero as the free boundary shoots up the side of the wedge. When the wedge angle is actually equal to π the solution is trivial, since the initial 'free boundary' is everywhere in contact with the wedge, and the fluid must just sit there.

The behaviour of $B_1(\zeta)$ and $B_2(\zeta)$ near $\zeta = 0$ ($Z = Z_0$, the point $\hat{A}$) in (3.14), (3.15) gives

$$\mathscr{Z}(\zeta) = L_1(a+\tfrac{1}{2}) + 2aL_2(a+1)\sqrt{\zeta} - L_1\zeta(2a^2+2a-1) + O(\zeta^{\frac{3}{2}}),$$

$$\mathscr{W}(\zeta) = M_1(a+\tfrac{1}{2}) + 2aM_2(a+1)\sqrt{\zeta} - M_1\zeta(2a^2+2a-1) + O(\zeta^{\frac{3}{2}});$$

comparing these with (3.4) and (3.5) we find the further relations

$$L_1 = -\frac{iR_0 e^{i\alpha\pi}}{a+\frac{1}{2}}, \qquad L_2 = \frac{R_0 e^{i\alpha\pi}}{a+\frac{1}{2}}, \qquad \mu = \frac{2a(a+1)R_0}{a+\frac{1}{2}}; \tag{3.20}$$

$$M_1 = -\frac{R_0\sin\alpha\pi}{a+\frac{1}{2}}, \qquad M_2 = -\frac{\mu\cos\alpha\pi}{2a(a+1)} = -\frac{R_0\cos\alpha\pi}{a+\frac{1}{2}}, \tag{3.21}$$

(R_0 is still unknown here, being the modulus of Z_0). Finally, we examine the neighbourhood of $\zeta = 1$, where (3.14) and (3.15) give

$$\mathscr{Z}(\zeta) = (-L_1\cos\alpha\pi + L_2\sin\alpha\pi)\left(\frac{1}{2(1-\zeta)} + a(a+1)\right)$$
$$+(L_1\sin\alpha\pi + L_2\cos\alpha\pi)\frac{2a}{3}(2a+1)(a+1)(1-\zeta)^{\frac{1}{2}} + O(1-\zeta),$$

$$\mathscr{W}(\zeta) = (-M_1\cos\alpha\pi + M_2\sin\alpha\pi)\left(\frac{1}{2(1-\zeta)} + a(a+1)\right)$$
$$+(M_1\sin\alpha\pi + M_2\cos\alpha\pi)\frac{2a}{3}(2a+1)(a+1)(1-\zeta)^{\frac{1}{2}} + O(1-\zeta);$$

comparison with (3.6) and (3.7) gives

$$-L_1\cos\alpha\pi + L_2\sin\alpha\pi = 2iv, \tag{3.22}$$

$$-M_1\cos\alpha\pi + M_2\sin\alpha\pi = 0, \tag{3.23}$$

$$\frac{2a}{3}(2a+1)(a+1)(M_1\sin\alpha\pi + M_2\cos\alpha\pi) = \pm\frac{1}{\sqrt{2v}}. \tag{3.24}$$

Putting all this information together we find that, provided we take the negative branch of the square-root in (3.2), giving minus signs in (3.7) and (3.24), the conditions (3.18)–(3.24) are all compatible with the restrictions that the constants σ, μ, v be real and positive, and

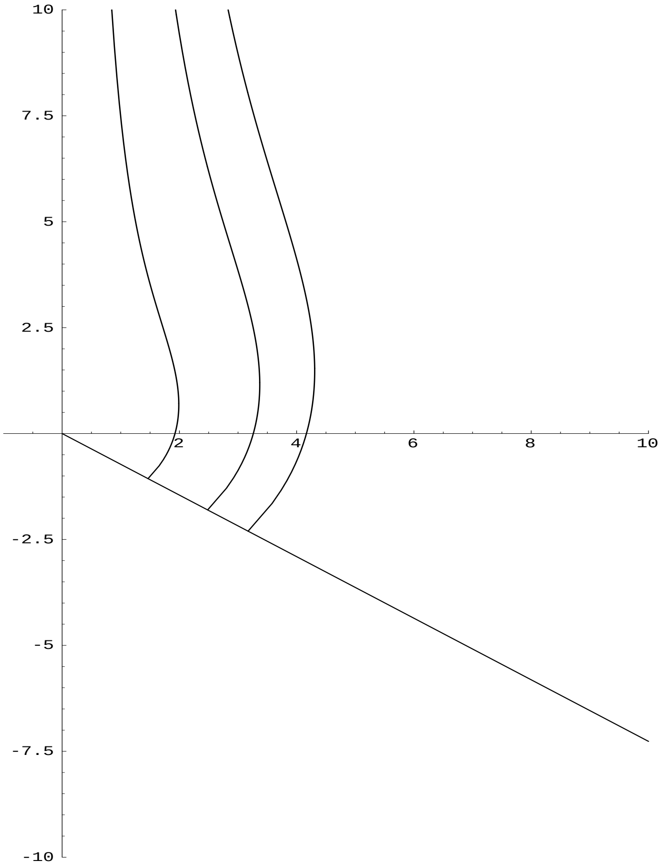

FIGURE 3. Plot showing the free boundary at times $t = 0.5, 1, \pi/2$ with wedge angle $\alpha = 0.3\pi$. The fluid flow is from left to right, and the initial free boundary is along the positive y-axis.

that $\tilde{\sigma}$ be real. In terms of the wedge angle $\alpha\pi$ and $a = 1 - \alpha$ we have

$$L_1 = -i\mathrm{e}^{i\alpha\pi}[(2a/3)(2a+1)(a+1)]^{-\frac{2}{3}}, \quad L_2 = \mathrm{e}^{i\alpha\pi}[(2a/3)(2a+1)(a+1)]^{-\frac{2}{3}};$$
$$M_1 = -\sin\alpha\pi[(2a/3)(2a+1)(a+1)]^{-\frac{2}{3}}, \quad M_2 = -\cos\alpha\pi[(2a/3)(2a+1)(a+1)]^{-\frac{2}{3}};$$
$$R_0 = \left[\frac{9(2a+1)}{32a^2(a+1)^2}\right]^{\frac{1}{3}}.$$

Hence the solution to the problem. In terms of the original variables a parametric representation of the free boundary is given, for $0 \leqslant \xi \leqslant 1$, by

$$z = \mathscr{Z}(\xi)\int_0^t f(\tau)\,d\tau = \frac{L_1\int_0^t f(\tau)\,d\tau}{(1-\xi)}(1 - i\cot\alpha\pi)(B_1(\xi) + iB_2(\xi)),$$
$$= \frac{i\mathrm{e}^{-ia\pi}(i\sqrt{\xi} + \sqrt{1-\xi})^{2a}\int_0^t f(\tau)\,d\tau}{(1-\xi)[(2a/3)(2a+1)(a+1)]^{\frac{2}{3}}}\left\{\frac{1}{2} + 2a(1-\xi) - \frac{a\sqrt{1-\xi}}{i\sqrt{\xi} + \sqrt{1-\xi}}\right\}.$$

The function $f(t)$ is arbitrary, being the dimensionless timescale on which the pressure

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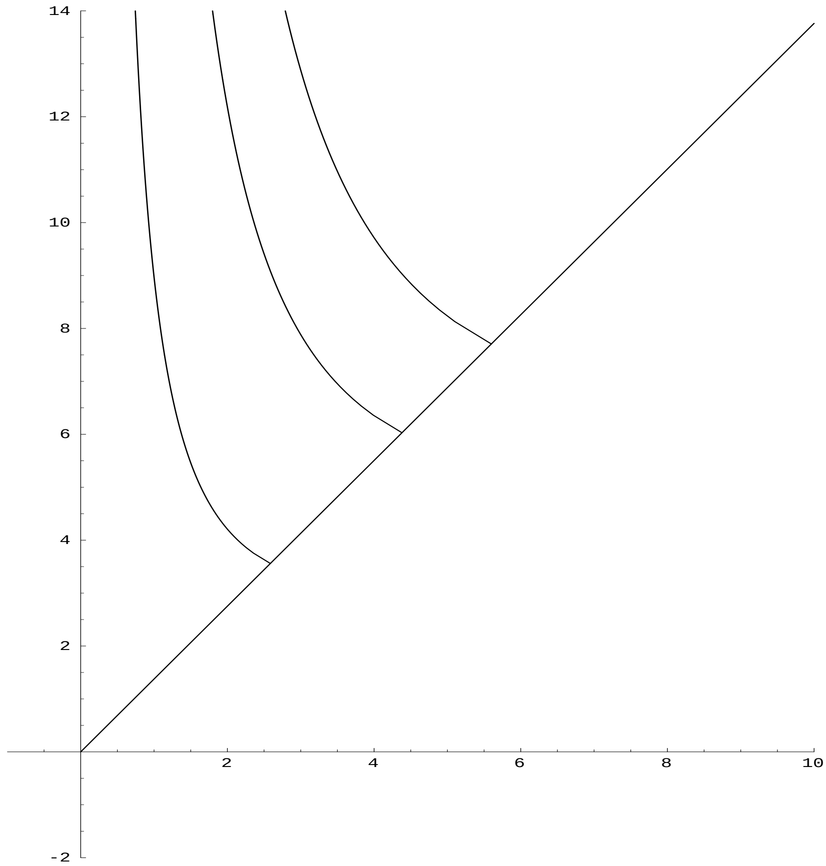

FIGURE 4. Plot showing the free boundary at times $t = 0.5, 1, \pi/2$ with wedge angle $\alpha = 0.8\pi$. The fluid flow is from left to right, and the initial free boundary is along the positive y-axis.

is varying at infinity. However, if we wish the analysis to hold for large times then the free boundary must not move off to $x = +\infty$ as $t \to \infty$, since then the outer solution for the pressure (2.1) would no longer be appropriate. Thus we require $\int_0^\infty f(\tau)\, d\tau < \infty$, which requires $f(t) \sim t^{-\lambda}$ as $t \to \infty$ for some $\lambda > 1$ (though this cannot be the small-time behaviour).

Typical free boundary evolution is shown in Figures 3, 4 and 5; Figure 3 depicts an acute-angled wedge ($\alpha = 0.3$), Figure 4 depicts an obtuse-angled wedge ($\alpha = 0.8$), and Figure 5 shows the right-angled wedge. For convenience the function $f(t)$ is chosen as $f(t) = 5 \arctan t$, since this gives acceptable $t \to \infty$ behaviour.

4 Conclusions

We have constructed an exact time-dependent solution for the flow of fluid in a Hele-Shaw cell driven by the asymptotic pressure field given in (2.1), around a wedge of arbitrary angle $0 \leqslant \alpha\pi < \pi$. The case $\alpha = 0$ (the zero-angle wedge) may also be obtained from the results of problem (P6) in Howison & King [12] (in this context we note that the term $8\zeta^2$

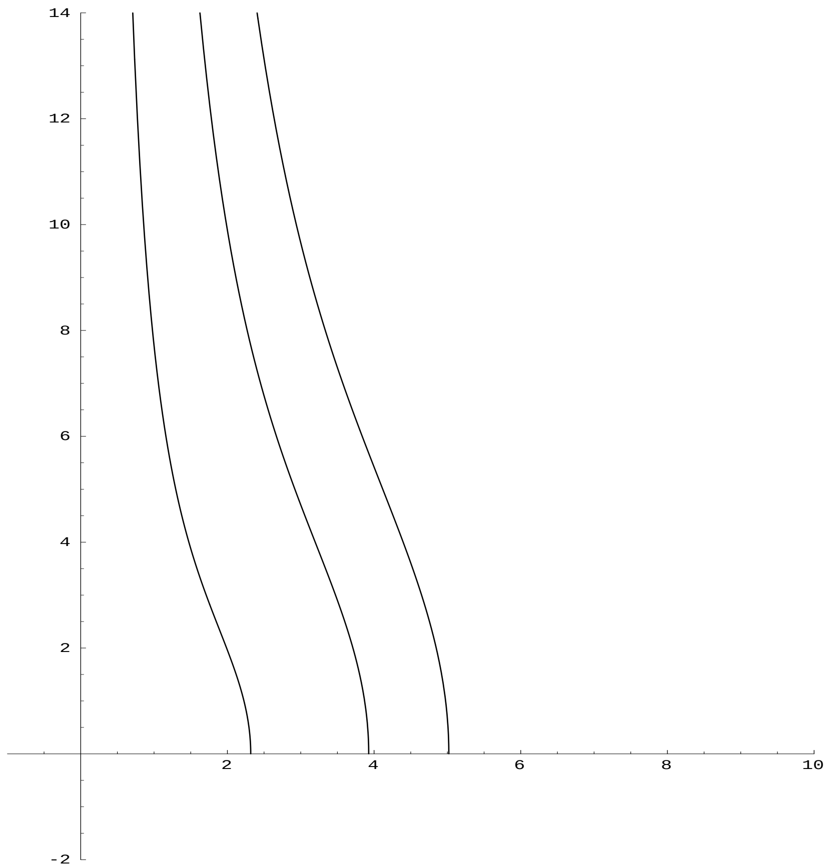

FIGURE 5. Plot showing the free boundary at times $t = 0.5, 1, \pi/2$ with wedge angle $\alpha = \pi/2$. The fluid flow is from left to right, and the initial free boundary is along the positive y-axis.

occurring in the formulae for Z_6 and W_6 should be $8\zeta^2/3$). The case $\alpha = 1$ is, as noted, trivial with our assumed initial geometry, since the 'free boundary' then lies along the wedge for all time. At a given time t, the distance which the free boundary has advanced along the wedge is given by

$$R_0 \int_0^t f(\tau)\, d\tau \equiv \left[\frac{9(2a+1)}{32a^2(a+1)^2}\right]^{\frac{1}{3}} \int_0^t f(\tau)\, d\tau.$$

As mentioned in the Introduction, with a simple rescaling of time the plots also represent the motion of the free boundary as a fluid freezes around a corner of given angle, in the quasi-static approximation.

Obviously there are techniques other than the one we have used for solving zero surface tension Hele-Shaw free boundary problems. Other common procedures include: (i) conformally mapping the unit disk onto the fluid domain and reformulating the boundary conditions on the unit circle (another technique due to Polubarinova-Kochina [15], as well as to Galin [7]), and (ii) making use of a relation between the Schwarz function of the free boundary and the complex potential of the flow (e.g. see Howison [11]). The difficulty in our problem is the fact that we have a free boundary and a rigid boundary on which we have to apply different boundary conditions. No satisfactory conformal mapping or Schwarz function method has yet been formulated to deal with general problems of this type. Certain special geometries can be solved for, using the more conventional techniques, e.g. channel flow [10, 17, 18], or flow problems *inside* a wedge of angle $2\pi/n$ for integers n [10]. In such symmetrical cases the flow problem may be 'reflected' in the walls so that one need only solve for a flow with free boundaries (the boundary condition on the walls is satisfied automatically). However, we do not have such symmetry in our problem, and this approach does not work.

Finally, we note that since the problem we consider is of the well-posed type in which the viscous fluid is advancing, we do not expect the addition of small positive surface tension to greatly affect our solution behaviour.

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