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On the initial value problem for a class of nonlinear biharmonic equation with time-fractional derivative

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In this study, we investigate the intial value problem (IVP) for a time-fractional fourth-order equation with nonlinear source terms. More specifically, we consider the time-fractional biharmonic with exponential nonlinearity and the time-fractional Cahn–Hilliard equation. By using the Fourier transform concept, the generalized formula for the mild solution as well as the smoothing effects of resolvent operators are proved. For the IVP associated with the first one, by using the Orlicz space with the function $\Xi(z) = \mathrm{e}^{|z|^p} - 1$ and some embeddings between it and the usual Lebesgue spaces, we prove that the solution is a global-in-time solution or it shall blow up in a finite time if the initial value is regular. In the case of singular initial data, the local-in-time/global-in-time existence and uniqueness are derived. Also, the regularity of the mild solution is investigated. For the IVP associated with the second one, some modifications to the generalized formula are made to deal with the nonlinear term. We also establish some important estimates for the derivatives of resolvent operators, they are the basis for using the Picard sequence to prove the local-in-time existence of the solution.

Keywords: Biharmonic equations; Cahn–Hilliard equations; exponential nonlinearity; fourth order; global existence; local existence; time-fractional; well-posedness

2020 Mathematics subject classification Primary: 26A33; 33E12; 35B40; 35K30; 35K58

1. Introduction

Our objective in this paper is to study the following initial value problem associated with the time-fractional derivative with biharmonic operator

$$\begin{cases} \partial^{\alpha}_{0|t} u(t,x) + \Delta^2 u(t,x) = G(t,x,u), & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \\ u(0,x) = u_0(x), & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{P}$$

where $0 < \alpha < 1$ and, for an absolutely continuous in time function w, the definition the Caputo time-fractional derivative operator $\partial^{\alpha}_{0|t}$ is introduced in [12] as follows:

$$\partial^{\alpha}_{0|t} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\mathrm{d}w}{\mathrm{d}s} \, \mathrm{d}s, \tag{1.1}$$

here, we assume that the integration makes sense Γ is the Gamma function. The main equation in (P) contains the biharmonic operator Δ^2 which is often called higher-order parabolic equations and began to receive widespread attention for their surprising and unexpected properties. More importantly, the higher-order parabolic equations can be used to model many problems in applications, namely, the study of weak interactions of dispersive waves, the theory of combustion, the phase transition, the higher-order diffusion. The most common higher-order parabolic equation is probably the polyharmonic heat equation, especially, the fourth-order heat equation or also called the biharmonic heat equation. The current paper only considers the source function G to be in two nonlinear cases: the exponential nonlinear type and the Cahn–Hilliard equation form.

1.1. Fractional partial differential equations

Over the last few decades, the theory of fractional derivatives has been well developed. As a result, the fractional partial differential equations (FPDEs) have also been studied more and more widely. These new kinds of PDEs have many unexpected properties and numerous applications in many applied and theoretical fields of science and engineering. This is why many researchers have shown a big interest in the study of FPDEs. Recently, there have been many interesting studies concerning diffusion equations with non-local time derivatives and time-fractional derivatives. We can refer the reader to some interesting papers on FPDEs, for example, Caraballo *et al.* [42, 43], Zacher [44, 45], Vespri *et al.* [17], Nane *et al.* [5, 37], Carvalho and Planas [13], Dong and Kim [20, 21], Tuan *et al.* [38, 39] and references therein. We especially consider the interesting studies [18, 27, 40] because our views in approaching the problem are somewhat similar to theirs. Indeed, the works [18, 27] study time-fractional problems with second-order differential operators through the fundamental solution. Although in [18], Dipierro and co-authors establish existence and uniqueness for the solution in an appropriate functional space, and in [27], Zacher *et al.* consider decay estimates of the solution. The study [40] investigates the same problem as [27] but the results are provided in a bounded domain of $\mathbb{R}^N$ and applied to investigate some specific examples corresponding to their kernel k. When considered these studies, we found some important remarks about appropriate functional spaces to study the fundamental solution (remark 2.1).

Motivated by those, we made some detailed observations for ours. On the contrary, our main contributions are the results for models with the biharmonic operators whose properties are somewhat different from the second-order ones. Furthermore, we focus on studying specific effects of different types of nonlinearity to our mild solution. To provide a clearer view, we will present specific discussions below.

1.2. Discussion on problem (P) with exponential nonlinearity

The source function G is the exponential nonlinearity satisfying $G(0) = 0$ and

$$|G(u) - G(v)| \leqslant L|u - v| \left(|u|^{m-1} \mathrm{e}^{\kappa|u|^p} + |v|^{m-1} \mathrm{e}^{\kappa|v|^p}\right), \tag{1.2}$$

for every $u, v \in \mathbb{R}, m > 2$ or $m = 1, p > 1$ and L is a positive constant independent of u, v. In the following, we will discuss in more detail why we chose this function G as in (1.2).

- *In terms of mathematical theory*: It was common knowledge that when we consider the IVP for the classical Schrödinger equations with the polynomial nonlinearity $u|u|^{p-1}$, $p \in (1, \infty)$ and an initial data function in $H^s(\mathbb{R}^N)$, $s \in [0, N/2)$, the value $\bar{\mathbf{c}} = 1 + 4(N - 2s)^{-1}$ is called the critical exponent. Then, the power case p of the nonlinear function is equal to (respectively less than) $\bar{\mathbf{c}}$ is called the critical case (respectively subcritical case). However, when considering the functional space $H^{\frac{N}{2}}(\mathbb{R}^N)$, the critical value $\bar{\mathbf{c}}$ will be larger than any power exponent of the polynomial nonlinearity. Hence, the nonlinear functions of exponential type grow higher than any kind of a power nonlinearity at infinity and also vanishes like a power at zero, which can be seen as the critical nonlinearity of this case. This is also one of the reasons why exponential nonlinearity has been studied by many mathematicians not only for both the Schrödinger equation and some other types of PDEs. To provide an overview of this kind of nonlinearity, let us recall some related studies. The framework introduced above is based on results in [**33**]. In this study, Nakamura and Ozawa study the small data global $H^{N/2}(\mathbb{R}^N)$-solution of the IVP for the Schrödinger equations with the exponential nonlinearity. The IVP for heat equations with this type of nonlinearity was considered in [**24**]. In [**24**], under the smallness assumption on the initial data in the Orlicz space, Ioku has shown the existence of a global-in-time solution of the semilinear heat equations. Under the smallness condition of initial data, decay estimates and the asymptotic behaviour for global-in-time solutions of a semilinear heat equation with the nonlinearity given by $f(u) = |u|^{4/N} u\, \mathrm{e}^{u^2}$ was investigated in [**22**] by Furioli *et al.* For more results about the exponential nonlinearity, we refer the reader to [**6**, **16**, **25**, **34**] and the references therein.

- *In terms of application*: These exponential nonlinearities as in (1.2) are not only investigated for the nonlinear Schrödinger equations but also for other types of PDEs because of many applications in phenomena modelling. Let us mention two well-known applications in combustion theory as follows. The first one is the IVP for the equation $u_t - \Delta u = k\, \mathrm{e}^u$, it can be used to model the ignition solid fuel. The second one is the description of the small fuel loss steady-state

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model by the IVP associated with equation $-\Delta u = k\, \mathrm{e}^{\frac{u}{(1+\varepsilon u)}}$. More applications and details can be found in [7, 26] and references therein.

- *Contributions, challenges and novelties*: To the best of our knowledge, FPDEs with nonlinearities of exponential type have not been studied yet. Our study can be seen as one of the first results in this topic. Due to the nonlinearity of exponential type, it is not possible to apply $L^p - L^q$ estimates of some previous studies [3, 4] to our current problem. In contrast to the case $s > N/2$, the embedding $L^\infty(\mathbb{R}^N) \hookleftarrow H^{N/2}$ is not true, and in view of Trudinger–Moser's inequality, we obtain the embeddings $H^{N/2}(\mathbb{R}^N) \hookrightarrow L^\Xi(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for any $p \leqslant q < \infty$, where $L^\Xi(\mathbb{R}^N)$ is the Orlicz space with the function $\Xi(z) = \mathrm{e}^{|z|^p} - 1$ (see definition 2.5). To deal with the exponential type, we shall use the Orlicz space and the embeddings between it and the usual Lebesgue spaces. However, since our problem is considered in the whole space $\mathbb{R}^N$, the embedding $L^q(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ does not hold anymore when $q > p$. For Orlicz spaces in classical derivatives, the use of the standard smoothing effect of the exponential resolution and Taylor expansion operators play important roles. However, they are not available for problems with time-fractional derivatives. The appearance of the Mittag-Leffler function and the Gamma function also caused a lot of difficulties in setting up some needed estimates related to the Orlicz space for problem (P). Fortunately, thanks to the results shown in [32], the standard smoothing effect can be achieved with a presentation via the M-Wright function of the Mittag-Leffler function. We can also overcome some of the difficulties caused by the Gamma function with some special inequalities. Our main results in this section are briefly described as follows:
  - In the regular case of $u_0 (u_0 \in L^p(\mathbb{R}^N) \cap C_0(\mathbb{R}^N))$, we can derive that our mild solution blows up at a finite time or the maximal time that ensures the unique existence of the solution is infinity.
  - With the assumption of the initial value in the space $L^\Xi(\mathbb{R}^N)$, the local-in-time existence of mild solution can be obtained by a fixed point argument without any smallness assumptions on the initial function. Furthermore, under the stronger assumption that $u_0 \in L_0^\Xi(\mathbb{R}^N)$, the global well-posed results for the solution will be established. To achieve this goal, we have to use the techniques introduced by Chen *et al.* [14], and weighted spaces to deal with the singular term of the mild solution.

1.3. Discussion on problem (P) with Cahn–Hilliard source term

In this subsection, we introduce and discuss problem (P) with another source $G(u) \equiv \Delta F(u)$, here F denotes the derivative of double-well potential; in general, we consider a cubic polynomial like $F(u) = u^3 - u$. For the case $\alpha = 1$, problem (P) is reduced to the standard Cahn–Hilliard equation. The Cahn–Hilliard equation was proposed for the first time by Cahn and Hilliard [11], and is one of the most often studied problems of mathematical physics, which describes the process of phase separation of a binary alloy below the critical temperature. More recently, it has appeared in nano-technology, in models for stellar dynamics, as well as in the theory of galaxy formation as a model for the evolution of two components of intergalactic material (see [**8**]). Let us mention some previous studies on the standard Cahn–Hilliard equations with derivatives of integer order. In [**1**], the authors considered a Cahn–Hilliard equation which is the conserved gradient flow of a nonlocal total free energy functional. Bosch and Stoll [**9**] proposed a fractional inpainting model based on a fractional-order vector-valued Cahn–Hilliard equation [**10**]. We can list many classical papers related to the study of Cahn–Hilliard equation, see, e.g.Dlotko [**19**], Temam [**36**], Akagi [**2**], Zelik [**15**, **28**] and the references therein.

- *Contributions, challenges and novelties*: As we mentioned above, when we consider the FPDEs in $\mathbb{R}^N$, the relationship between two spaces $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$ with $p \neq q$ is not fulfilled. Specifically, one of the greatest challenges, when we investigate the Cahn–Hilliard equations, is to deal with a nonlinearity of the form $G(u) = \Delta F(u)$. Due to the appearance of the Laplacian, when handling the existence and the uniqueness of the mild solution by the successive approximations method and Young's convolution inequality, the property of $\mathrm{d}/\mathrm{d}x\,(f(x) * g(x))$ needs to be applied to make the second-order derivative of the solution representation operator appear. This novelty in this case is setting up the key estimates as in lemma 2.3. It is worth noting that even when we have the main tools available, it is still not an easy task to prove our desired existence results. Besides, when finding the regularity results, we have to estimate the higher-order derivative of the solution representation operators. By learning about the results and techniques of [**29**, **30**], we found a way to obtain the local-in-time existence and uniqueness result for problem (P) with the Cahn–Hilliard source. The global-in-time existence of the solution is a difficult topic and will probably be studied in a forthcoming study. The section about the time-fractional Cahn–Hilliard equation in this study includes:

- to find the solution representation by the Fourier transform and some related properties;

- to establish some useful linear estimates;

- to prove the existence, uniqueness and regularity of local-in-time solution by using the Picard sequence method and the smallness assumption for the initial data function.

1.4. The outline

In § 2, we demonstrate an approach to present the formula of the mild solution and, based on it, we establish some important linear estimates. Also in this section, we introduce some notations and definitions related to the so-called Orlicz space, a generalization of Lebesgue spaces and some useful embeddings between them. In § 3, we investigate problem (P) with the exponential nonlinearities under two separate assumptions on the initial datum function. In particular, for the first assumption, we show that the mild solution exists on $[0, \infty)$ or blows up in a finite time. The local existence and the global-in-time well-posedness of the solution will be stated under the second assumption of the initial function. The main results on the problem with the second type of nonlinearity source function will be analysed in § 4. In general, by using the smallness assumption on the initial function, we derive the local-in-time existence and uniqueness of the mild solution for the IVP associated with the time-fractional Cahn–Hilliard equation. Furthermore, the regularity result will also be proved.

2. Preliminaries

2.1. Generalized mild solution

It is well known that for the following IVP involving a classical homogeneous biharmonic equation

$$\begin{cases} \partial_t \varphi(t,x) + \Delta^2 \varphi(t,x) = 0, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \\ \varphi(t,x) = \varphi_0(x), & \text{in} \quad \{0\} \times \mathbb{R}^N, \end{cases}$$

the solution is given by

$$\varphi(t,x) = \left[\mathcal{F}^{-1}\left(\mathrm{e}^{-t|\xi|^4}\right) * \varphi_0\right](x) = \left[(2\pi)^{\frac{-N}{2}} \int_{\mathbb{R}^N} \mathrm{e}^{i<\xi,x>-t|\xi|^4}\, \mathrm{d}\xi\right] * \varphi_0(x),$$

where the Fourier transform and its inverse are denoted by $\mathcal{F}, \mathcal{F}^{-1}$, respectively, and $< \xi, x >= \sum_{j=1}^N \xi_j x_j, (\xi, x) \in \mathbb{R}^{2N}$. We recall the following lemma for the kernel $\mathscr{K}(t,x) = \mathcal{F}^{-1}(\mathrm{e}^{-t|\xi|^4})$.

Lemma 2.1. *Suppose that $p \geqslant 1$. Then, for any $t > 0$ we have*

$$\|\mathscr{K}(t)\|_{L^p} \leqslant c_p t^{-\frac{N}{4})(1-\frac{1}{p})}, \quad \|D^m \mathscr{K}(t)\|_{L^p} \leqslant c_{p,m} t^{-\frac{N}{4})(1-\frac{1}{p})-\frac{m}{4}}.$$

In view of the above approach, to find the representation for the mild solution to problem (P), we consider the IVP for a homogeneous time-fractional biharmonic equation as follows:

$$\begin{cases} \partial^\alpha_{0|t} \varphi(t,x) + \Delta^2 \varphi(t,x) = 0, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \\ \varphi(0,x) = \varphi_0(x), & \text{in} \quad \mathbb{R}^N. \end{cases} \tag{2.1}$$

Applying the Laplace transform, $\mathscr{L}$, with respect to the time variable to the first equation of (2.1), we have

$$z^\alpha \mathscr{L}\{\varphi\}(z,x) - z^{\alpha-1}\varphi_0(x) + \Delta^2 \mathscr{L}\{\varphi\}(z,x) = 0.$$

Then, by assuming that φ_0 belongs to some appropriate spaces and using the Fourier transform with respect to the spatial variable, the following equation holds

$$z^\alpha \mathcal{F}(\mathscr{L}\{\varphi\})(z,\xi) - z^{\alpha-1}\mathcal{F}(\varphi_0)(\xi) + |\xi|^4 \mathcal{F}(\mathscr{L}\{\varphi\})(z,\xi) = 0.$$

By some simple calculations, one has

$$\mathcal{F}(\mathscr{L}\{\varphi\})(z,\xi) = \frac{z^{\alpha-1}}{z^\alpha - |\xi|^4}.$$

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We now use the inverse Laplace transform to obtain

$$\mathcal{F}(\varphi)(t,\xi) = \widehat{\varphi}(t,\xi) = E_{\alpha,1}\left(-t^{\alpha}|\xi|^4\right)\widehat{\varphi_0}(\tau).$$

Thanks to the Duhamel principle, the Fourier transform of the solution to problem (P) is given by

$$\widehat{u}(t,\xi) = E_{\alpha,1}(-t^{\alpha}|\xi|^4)\widehat{u_0}(\xi) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-\tau)^{\alpha}|\xi|^4)\widehat{G(u)}(\tau,\xi)\,\mathrm{d}\tau.$$

Where $E_{\alpha,1}, E_{\alpha,\alpha}$ are Mittag-Leffler functions. Using the inverse Fourier transform, we obtain

$$\begin{aligned} u(t,x) &= \left[\mathcal{F}^{-1}\Big(E_{\alpha,1}(-t^{\alpha}|\xi|^4)\Big) * u_0\right](x) \\ &\quad + \int_0^t \mathcal{F}^{-1}\Big((t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}|\xi|^4)\Big)(x) * G(u(\tau,x))\,\mathrm{d}\tau, \end{aligned}$$

where we have used the fact that $\widehat{f*g}(\tau) = \widehat{f}(\tau)\widehat{g}(\tau)$. For convenience, we denote

$$\mathbb{K}_{1,\alpha}(t,x) = \mathcal{F}^{-1}\Big(E_{\alpha,1}(-t^{\alpha}|\xi|^4)\Big)(x), \quad \mathbb{K}_{2,\alpha}(t,x) = \mathcal{F}^{-1}\Big(t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}|\xi|^4)\Big)(x), \tag{2.2}$$

and set operators $\mathbb{Z}_{i,\alpha}(i=1,2)$ as follows:

$$\mathbb{Z}_{i,\alpha}(t,x)v(t,x) = \mathbb{K}_{i,\alpha}(t,x) * v(t,x) = \int_{\mathbb{R}^N} \mathbb{K}_{i,\alpha}(t,x-y)v(t,y)\,\mathrm{d}y, \quad i=1,2.$$

Then, we rewrite our solution formula in a concise form

$$u(t,x) = \mathbb{Z}_{1,\alpha}(t,x)u_0(x) + \int_0^t \mathbb{Z}_{2,\alpha}(t-\tau,x)G(u(\tau,x))\,\mathrm{d}\tau. \tag{2.3}$$

Remark 2.1. It is worth noting that the above approach is similar to the common one used to construct the fundamental solution for time-fractional problems with second-order differential operators. Let us provide some remarks from interesting studies about functional spaces in which the fundamental kernels are considered.

(i) The work [**18**] studies an evolution problem with the Caputo derivative of order $\alpha \in (0,1)$ and the Dirac delta distribution centred at $x=0$ (the initial data function). The solution formula of this problem is given by

$$u(\xi,t) = \mathcal{F}^{-1}\Big(E_{\alpha,1}\left[(a-4\pi^2 b|\xi|^2)t^{\alpha}\right]\Big), \quad a \geqslant 0,\ b>0.$$

Then, based on it, the authors have made very interesting comments about functional spaces to which the Fourier transform of the solution belongs. More precisely, they showed that $\mathcal{F}u(\cdot,t)$ is in $L^p(\mathbb{R}^N)$ if and only if $p \in (N/2,\infty)$. This result implies some different case for functional spaces depending on the dimension N.

(ii) In [27, § 3] the authors proved optimal decay estimates for solutions to a time-fractional diffusion equation. Their solution is as follows:

$$u(x,t) = \int_{\mathbb{R}^N} Z(x-y,t)u_0(y)\,\mathrm{d}y, \quad \text{where } \mathcal{F}\{Z\}(\xi,t) = E_{\alpha,1}\big(-|\xi|^2 t^\alpha\big).$$

From the above, they deduced as a conclusion that $Z(t)$ fails to belong to $L^p(\mathbb{R}^N)$ for $N \geqslant 4$ and $p \geqslant \frac{N}{(N-2)}$.

These facts are important when using the fundamental solution to establish well-posed results. In the spirit of the above studies, we also present some similar comments on the estimates for kernels $\mathbb{K}_{1,\alpha}, \mathbb{K}_{2,\alpha}$ in remark 2.3.

In order to achieve the standard smoothing effect of $\mathbb{Z}_{i,\alpha}(i=1,2)$, we also present the mild solution in another form. To this end, we recall the definition of Mittag-Leffler function via the M-Wright type function as follows:

$$E_{\alpha,1}(-z) = \int_0^\infty \mathcal{M}_\alpha(\zeta)\,\mathrm{e}^{-z\zeta}\,\mathrm{d}\zeta, \quad E_{\alpha,\alpha}(-z) = \int_0^\infty \alpha\zeta\mathcal{M}_\alpha(\zeta)\,\mathrm{e}^{-z\zeta}\,\mathrm{d}\zeta, \quad z \in \mathbb{C}. \tag{2.4}$$

Then, we have the second type representation of the solution of problem (P1)

$$\begin{aligned} u(t,x) &= \int_0^\infty \mathcal{M}_\alpha(\zeta)\,[\mathscr{K}(\zeta t^\alpha, x) * u_0(x)]\,\mathrm{d}\zeta \\ &\quad + \int_0^t \int_0^\infty (t-\tau)^{\alpha-1}\alpha\zeta\mathcal{M}_\alpha(\zeta)\,[\mathscr{K}(\zeta(t-\tau)^\alpha, x) * G(u(\tau,x))]\,\mathrm{d}\zeta\,\mathrm{d}\tau. \end{aligned} \tag{2.5}$$

Due to the great impact of the operator $\mathbb{K}_{1,\alpha}, \mathbb{K}_{2,\alpha}$ to our results for mild solutions, we present the following Theorem which can be seen as the combination of theorem 3.1, theorem 3.2 and remark 1.6 of [41].

Theorem 2.2. *Let $X = L^p(\mathbb{R}^N)(1 \leqslant p < \infty)$ or $X = C_0(\mathbb{R}^N)$. Then, $\mathbb{Z}_{1,\alpha}(t)$ and $t^{1-\alpha}\mathbb{Z}_{2,\alpha}(t)$ are bounded linear operators on X. In addition, for $w \in X$, $t \to \mathbb{Z}_{1,\alpha}(t)$, $t \to t^{1-\alpha}\mathbb{Z}_{2,\alpha}(t)$ are continuous functions from $\mathbb{R}^+$ to X.*

Remark 2.2. In fact, although theorems of [41] can be applied for other spaces in which Δ^2 generates a strongly continuous semigroup, in this study, we only focus on the spaces $L^p(\mathbb{R}^N)(1 \leqslant p < \infty)$ and $C_0(\mathbb{R}^N)$.

We continue the study by introducing some useful L^p-estimates for the kernel $\mathbb{K}_{1,\alpha}, \mathbb{K}_{2,\alpha}$ by the following lemma.

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Lemma 2.3. *Let $p \geqslant 1$ and $k \in \mathbb{N}$ be constants such that $k < 4 - N\left(1 - \frac{1}{p}\right)$. Then, there exist two constants $\mathscr{C}_{k,p}, \overline{\mathscr{C}_{k,p}}$ which depend only on α and N, such that*

$$\left\| D^k \mathbb{K}_{1,\alpha}(t) \right\|_{L^p} \leqslant \mathscr{C}_{k,p}(\alpha, N) t^{-\frac{\alpha N}{4} - \frac{\alpha k}{4} + \frac{\alpha N}{4p}} \tag{2.6}$$

and

$$\left\| D^k \mathbb{K}_{2,\alpha}(t) \right\|_{L^p} \leqslant \overline{\mathscr{C}_{k,p}}(\alpha, N) t^{\alpha - \frac{\alpha N}{4} - 1 + \frac{\alpha N}{4p} - \frac{\alpha k}{4}}. \tag{2.7}$$

Proof. ***Step 1. To verify the first inequality***

In this step, we deal with the term $\mathbb{K}_{1,\alpha}(t,x)$. In fact, the representation of $\mathbb{K}_{1,\alpha}(t,x)$ and (2.4) together with Fubini's theorem allow us to deduce

$$\begin{aligned}
\mathbb{K}_{1,\alpha}(t,x) &= \mathcal{F}^{-1}\Big(E_{\alpha,1}(-t^\alpha|\xi|^4)\Big)(x) = (2\pi)^{-N} \int_{\mathbb{R}^N} \mathrm{e}^{i<\xi,x>} \Big(E_{\alpha,1}(-t^\alpha|\xi|^4)\Big)\,\mathrm{d}\xi \\
&= (2\pi)^{-N} \int_{\mathbb{R}^N} \mathrm{e}^{i<\xi,x>} \Big(\int_0^\infty \mathcal{M}_\alpha(\zeta)\,\mathrm{e}^{-t^\alpha\zeta|\xi|^4}\,\mathrm{d}\zeta\Big)\,\mathrm{d}\xi \\
&= (2\pi)^{-N} \int_0^\infty \int_{\mathbb{R}^N} \mathcal{M}_\alpha(\zeta)\,\mathrm{e}^{i<\xi,x>}\,\mathrm{e}^{-t^\alpha\zeta|\xi|^4}\,\mathrm{d}\xi\,\mathrm{d}\zeta.
\end{aligned}$$

By setting $\xi = \vartheta(t^\alpha\zeta)^{-\frac{1}{4}}$, it is straightforward that $\mathrm{d}\xi = (t^\alpha\zeta)^{-\frac{N}{4}}\,\mathrm{d}\vartheta$ and $|\xi|^4 = (t^\alpha\zeta)^{-1}|\vartheta|^4$. Let us denote by

$$\overline{\mathscr{B}}_k(y) = \int_{\mathbb{R}^N} |\vartheta|^k\,\mathrm{e}^{i<y,\vartheta>}\,\mathrm{e}^{-|\vartheta|^4}\,\mathrm{d}\vartheta, \quad k \geqslant 0. \tag{2.8}$$

By some simple transformations, we find the following equality

$$\begin{aligned}
\mathbb{K}_{1,\alpha}(t,x) &= \int_0^\infty \int_{\mathbb{R}^N} (t^\alpha\zeta)^{-\frac{N}{4}} \mathcal{M}_\alpha(\zeta)\,\mathrm{e}^{i<\vartheta,x>t^{-\frac{\alpha}{4}}\zeta^{-\frac{1}{4}}}\,\mathrm{e}^{-|\vartheta|^4}\,\mathrm{d}\vartheta\,\mathrm{d}\zeta \\
&= t^{-\frac{\alpha N}{4}} \left(\int_0^\infty \zeta^{-\frac{N}{4}} \mathcal{M}_\alpha(\zeta) \int_{\mathbb{R}^N} \mathrm{e}^{i<\vartheta,x>t^{-\frac{\alpha}{4}}\zeta^{-\frac{1}{4}}}\,\mathrm{e}^{-|\vartheta|^4}\,\mathrm{d}\vartheta\,\mathrm{d}\zeta \right) \\
&= t^{-\frac{\alpha N}{4}} \int_0^\infty \zeta^{-\frac{N}{4}} \mathcal{M}_\alpha(\zeta) \overline{\mathscr{B}}_0(x(t^\alpha\zeta)^{-\frac{1}{4}})\,\mathrm{d}\zeta.
\end{aligned}$$

If we set $x(t^\alpha\zeta)^{-\frac{1}{4}} = z$, then it follows immediately that $\mathrm{d}x = (t^\alpha\zeta)^{\frac{N}{4}}\,\mathrm{d}z$. Applying Minkowski's inequality in integral form, we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} \left| \int_0^\infty \zeta^{-\frac{N}{4}} \mathcal{M}_\alpha(\zeta) \overline{\mathscr{B}}_0(x(t^\alpha\zeta)^{-\frac{1}{4}})\,\mathrm{d}\zeta \right|^p \mathrm{d}x \right)^{\frac{1}{q}} \\
&\quad \leqslant \int_0^\infty \left(\int_{\mathbb{R}^N} \left| \zeta^{-\frac{N}{4}} \mathcal{M}_\alpha(\zeta) \overline{\mathscr{B}}_0(x(t^\alpha\zeta)^{-\frac{1}{4}}) \right|^p \mathrm{d}x \right)^{\frac{1}{q}} \mathrm{d}\zeta.
\end{aligned}$$

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It follows that

$$\Big\|\mathbb{K}_{1,\alpha}(t,x)\Big\|_{L^p} \leqslant t^{-\frac{\alpha N}{4}} \int_0^\infty \left(\int_{\mathbb{R}^N} \left|\zeta^{-\frac{N}{4}} \mathcal{M}_\alpha(\zeta) \overline{\mathscr{B}}_0 \left(x(t^\alpha \zeta)^{-\frac{1}{4}}\right)\right|^p (t^\alpha\zeta)^{\frac{N}{4}} \,\mathrm{d}z\right)^{\frac{1}{p}} \mathrm{d}\zeta$$
$$= t^{-\frac{\alpha N}{4}+\frac{\alpha N}{4p}} \left(\int_0^\infty \zeta^{\frac{N}{4p}-\frac{N}{4}} \mathcal{M}_\alpha(\zeta)\,\mathrm{d}\zeta\right) \left(\int_{\mathbb{R}^N} \left|\overline{\mathscr{B}}_0(z)\right|^p \mathrm{d}z\right)^{\frac{1}{p}}.$$

By setting $\Theta_{p,N} = (\int_{\mathbb{R}^N} \left|\overline{\mathscr{B}}_0(z)\right|^p \mathrm{d}z)^{\frac{1}{p}}$ and using lemma A.2, we obtain the following bound

$$\Big\|\mathbb{K}_{1,\alpha}(t)\Big\|_{L^p} \leqslant \frac{\Theta_{p,N}\Gamma(\frac{N}{4p}-\frac{N}{4}+1)}{\Gamma(\frac{\alpha N}{4p}-\frac{\alpha N}{4}+1)} t^{-\frac{\alpha N}{4}+\frac{\alpha N}{4p}}. \tag{2.9}$$

Next, let us consider the derivative of $\mathbb{K}_{1,\alpha}(t,x)$. It is easy to see that

$$\Big\|D\mathbb{K}_{1,\alpha}(t)\Big\|_{L^p} = \left(\int_{\mathbb{R}^N} \left|D\mathbb{K}_{1,\alpha}(t,x)\right|^p \mathrm{d}x\right)^{1/p}.$$

We have a view on the modus of the boundness for the term $D^k\mathbb{K}_{1,\alpha}(t,x)$ as follows:

$$\left|D^k\mathbb{K}_{1,\alpha}(t,x)\right| \leqslant t^{-\frac{\alpha N}{4}} \int_0^\infty \zeta^{-\frac{N}{4}} \mathcal{M}_\alpha(\zeta) \left|D^k\overline{\mathscr{B}}_0(x(t^\alpha\zeta)^{-\frac{1}{4}})\right| \mathrm{d}\zeta \tag{2.10}$$

It follows from $|\vartheta|^k \leqslant N^{k-1}\sum_{j=1}^N |\vartheta_j|^k$ that

$$\left|D^k\overline{\mathscr{B}}_0\big(x(t^\alpha\zeta)^{-\frac{1}{4}}\big)\right| \leqslant N^{k-1} \sum_{j=1}^N \left|\int_{\mathbb{R}^N} \left(i\vartheta_j t^{-\frac{\alpha}{4}}\zeta^{-\frac{1}{4}}\right)^k \mathrm{e}^{i<\vartheta,x>t^{-\frac{\alpha}{4}}\zeta^{-\frac{1}{4}}} \,\mathrm{e}^{-|\vartheta|^4}\,\mathrm{d}\vartheta\right|$$
$$\leqslant N^{k-1} t^{-\frac{\alpha k}{4}} \zeta^{-\frac{k}{4}} \int_{\mathbb{R}^N} |\vartheta|^k \left|\mathrm{e}^{i<\vartheta,x>t^{-\frac{\alpha}{4}}\zeta^{-\frac{1}{4}}} \,\mathrm{e}^{-|\vartheta|^4}\right| \mathrm{d}\vartheta. \tag{2.11}$$

Then, by using a new variable $x(t^\alpha\zeta)^{-\frac{1}{4}} = z$ and some simple calculations, we find that

$$\int_{\mathbb{R}^N} |\vartheta|^k \left|\mathrm{e}^{i<\vartheta,x>t^{-\frac{\alpha}{4}}\zeta^{-\frac{1}{4}}} \,\mathrm{e}^{-|\vartheta|^4}\right| \mathrm{d}\vartheta = \overline{\mathscr{B}}_k(z). \tag{2.12}$$

Combining (2.10)–(2.12),

$$\Big\|D^k\mathbb{K}_{1,\alpha}(t,x)\Big\|_{L^p} = \left(\int_{\mathbb{R}^N} \left|D^k\mathbb{K}_{1,\alpha}(t,x)\right|^p \mathrm{d}x\right)^{\frac{1}{p}}$$
$$\leqslant N^{k-1}\left(\int_{\mathbb{R}^N} \left|t^{-\frac{\alpha N}{4}-\frac{\alpha k}{4}} \int_0^\infty \mathcal{M}_\alpha(\zeta)\zeta^{-\frac{k}{4}-\frac{N}{4}} \overline{\mathscr{B}}_k(z)\,\mathrm{d}\zeta\right|^p \mathrm{d}x\right)^{\frac{1}{p}}$$
$$\leqslant N^{k-1} t^{-\frac{\alpha N}{4}-\frac{\alpha k}{4}} \left(\int_{\mathbb{R}^N} \left|\int_0^\infty \mathcal{M}_\alpha(\zeta)\zeta^{-\frac{k}{4}-\frac{N}{4}} \overline{\mathscr{B}}_k(z)\,\mathrm{d}\zeta\right|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

Applying Minkowski's inequality in integral form and noting that $\mathrm{d}x = (t^\alpha \zeta)^{\frac{N}{4}}\,\mathrm{d}z$,

$$
\begin{aligned}
&\left\| D^k \mathbb{K}_{1,\alpha}(t,x) \right\|_{L^p} \\
&\quad \leqslant N^{k-1} t^{-\frac{\alpha N}{4}-\frac{\alpha k}{4}} \left(\int_{\mathbb{R}^N} \left| \int_0^\infty \mathcal{M}_\alpha(\zeta) \zeta^{-\frac{k}{4}-\frac{N}{4}} \overline{\mathscr{B}}_k(z)\,\mathrm{d}\zeta \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \\
&\quad \leqslant N^{k-1} t^{-\frac{\alpha N}{4}-\frac{\alpha k}{4}} \int_0^\infty \left(\int_{\mathbb{R}^N} \left| \mathcal{M}_\alpha(\zeta) \zeta^{-\frac{k}{4}-\frac{N}{4}} \overline{\mathscr{B}}_k(z) \right|^p (t^\alpha \zeta)^{\frac{N}{4}}\,\mathrm{d}z \right)^{\frac{1}{p}} \mathrm{d}\zeta \\
&\quad \leqslant N^{k-1} t^{-\frac{\alpha N}{4}-\frac{\alpha k}{4}+\frac{\alpha N}{4p}} \int_0^\infty \zeta^{-\frac{k}{4}-\frac{N}{4}+\frac{N}{4p}} \mathcal{M}_\alpha(\zeta) \left(\int_{\mathbb{R}^N} \left| \overline{\mathscr{B}}_k(z) \right|^p \mathrm{d}z \right)^{\frac{1}{p}} \mathrm{d}\zeta. \qquad (2.13)
\end{aligned}
$$

Due to the condition $1 - \frac{k}{4} - \frac{N}{4} + \frac{N}{4p} > 0$ and lemma A.2, we obtain

$$
\int_0^\infty \zeta^{-\frac{k}{4}-\frac{N}{4}+\frac{N}{4p}} \mathcal{M}_\alpha(\zeta)\,\mathrm{d}\zeta = \frac{\Gamma\left(1 - \frac{k}{4} - \frac{N}{4} + \frac{N}{4p}\right)}{\Gamma\left(\frac{-\alpha k}{4} - \frac{\alpha N}{4} + \frac{\alpha N}{4p} + 1\right)}
$$

and, together with (2.13), allow us to deduce the following boundness result

$$
\left\| D^k \mathbb{K}_{1,\alpha}(t,x) \right\|_{L^p} \leqslant \mathscr{C}_{k,p}(\alpha,N) t^{-\frac{\alpha N}{4}-\frac{\alpha k}{4}+\frac{\alpha N}{4p}},
$$

where we denote

$$
\mathscr{C}_{k,p}(\alpha,N) = \frac{N^{k-1}\Gamma\left(1 - \frac{k}{4} - \frac{N}{4} + \frac{N}{4p}\right)}{\Gamma\left(\frac{-\alpha k}{4} - \frac{\alpha N}{4} + \frac{\alpha N}{4p} + 1\right)} \left(\int_{\mathbb{R}^N} \left| \overline{\mathscr{B}}_k(z) \right|^p \mathrm{d}z \right)^{\frac{1}{p}}.
$$

Step 2. Verify the second inequality

The representation of $\mathbb{K}_{2,\alpha}(t,x)$ and (2.4) together with Fubini's theorem imply

$$
\begin{aligned}
\mathbb{K}_{2,\alpha}(t,x) &= \mathcal{F}^{-1}\left(t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha |\xi|^4)\right) = t^{\alpha-1} \int_{\mathbb{R}^N} \mathrm{e}^{-i<\xi,x>} \left(E_{\alpha,\alpha}(-t^\alpha|\xi|^4)\right) \mathrm{d}\xi \\
&= t^{\alpha-1} (2\pi)^{-N} \int_{\mathbb{R}^N} \mathrm{e}^{i<\xi,x>} \left(\int_0^\infty \alpha\zeta \mathcal{M}_\alpha(\zeta)\, \mathrm{e}^{-t^\alpha \zeta |\xi|^4}\,\mathrm{d}\zeta \right) \mathrm{d}\xi \\
&= t^{\alpha-1} (2\pi)^{-N} \int_0^\infty \alpha\zeta \mathcal{M}_\alpha(\zeta) \left(\int_{\mathbb{R}^N} \mathrm{e}^{i<\xi,x>}\, \mathrm{e}^{-t^\alpha \zeta |\xi|^4}\,\mathrm{d}\xi \right) \mathrm{d}\zeta. \qquad (2.14)
\end{aligned}
$$

By setting $\xi = \vartheta (t^\alpha \zeta)^{-\frac{1}{4}}$, we deduce $\mathrm{d}\xi = (t^\alpha\zeta)^{-\frac{N}{4}}\,\mathrm{d}\vartheta$ and $|\xi|^4 = (t^\alpha\zeta)^{-1}|\vartheta|^4$. Using (2.14),

$$
\mathbb{K}_{2,\alpha}(t,x) = \alpha t^{\alpha-1} \int_0^\infty \zeta (t^\alpha\zeta)^{-\frac{N}{4}} \mathcal{M}_\alpha(\zeta) \int_{\mathbb{R}^N} \mathrm{e}^{i<\vartheta,x>t^{-\frac{\alpha}{4}}\eta^{-\frac{1}{4}}}\, \mathrm{e}^{-|\vartheta|^4} d\vartheta\,\mathrm{d}\zeta.
$$

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By a similar argument as in step 1,

$$\begin{aligned}
&\left\|\mathbb{K}_{2,\alpha}(t,x)\right\|_{L^p} \\
&\quad \leqslant \alpha t^{\alpha-\frac{\alpha N}{4}-1}\int_0^\infty \left(\int_{\mathbb{R}^N}\left|\zeta^{1-\frac{N}{4}}\mathcal{M}_\alpha(\zeta)\overline{\mathscr{B}}_0(x(t^\alpha\zeta)^{-\frac{1}{4}})\right|^p \mathrm{d}x\right)^{\frac{1}{p}}\mathrm{d}\zeta \\
&\quad = \alpha t^{\alpha-\frac{\alpha N}{4}-1}\int_0^\infty \left(\int_{\mathbb{R}^N}\left|\zeta^{1-\frac{N}{4}}\mathcal{M}_\alpha(\zeta)\overline{\mathscr{B}}_0\left(x(t^\alpha\zeta)^{-\frac{1}{4}}\right)\right|^p (t^\alpha\zeta)^{\frac{N}{4}}\,\mathrm{d}z\right)^{\frac{1}{p}}\mathrm{d}\zeta \\
&\quad = \alpha t^{\alpha-\frac{\alpha N}{4}-1+\frac{\alpha N}{4p}}\left(\int_{\mathbb{R}^N}\left|\overline{\mathscr{B}}_0(z)\right|^p \mathrm{d}z\right)^{\frac{1}{p}}\int_0^\infty \zeta^{1+\frac{N}{4p}-\frac{N}{4}}\mathcal{M}_\alpha(\zeta)\,\mathrm{d}\zeta \\
&\quad = \frac{\alpha\Theta_{p,N}\Gamma(\frac{N}{4p}-\frac{N}{4}+2)}{\Gamma(\frac{\alpha N}{4p}-\frac{\alpha N}{4}+1+\alpha)}t^{\alpha-\frac{\alpha N}{4}-1+\frac{\alpha N}{4p}}.
\end{aligned}$$

Now, we estimate the derivative of the quantity $\mathbb{K}_{2,\alpha}$. Let us recall the following formula

$$\mathbb{K}_{2,\alpha}(t,x) = t^{\alpha-1-\frac{\alpha N}{4}}\int_0^\infty \alpha\zeta^{1-\frac{N}{4}}\mathcal{M}_\alpha(\zeta)\overline{\mathscr{B}}_0\big(xt^{-\frac{\alpha}{4}}\zeta^{-\frac{1}{4}}\big)\,\mathrm{d}\zeta.$$

In view of the boundedness of $D\mathbb{K}_{1,\alpha}(t,x)$, we have

$$\left|D^k\mathbb{K}_{2,\alpha}(t,x)\right| \leqslant t^{\alpha-1-\frac{\alpha N}{4}}\int_0^\infty \alpha\zeta^{1-\frac{N}{4}}\mathcal{M}_\alpha(\zeta)\left|D^k\overline{\mathscr{B}}_0(x(t^\alpha\zeta)^{-\frac{1}{4}})\right|\mathrm{d}\zeta. \tag{2.15}$$

It follows from $|\vartheta|^k \leqslant N^{k-1}\sum_{j=1}^N|\vartheta_j|^k$ that

$$\begin{aligned}
\left|D^k\overline{\mathscr{B}}_0\big(xt^{-\frac{1}{4}}\eta^{-\frac{1}{4}}\big)\right| &\leqslant N^{k-1}\sum_{j=1}^N\left|\int_{\mathbb{R}^N}\left(i\vartheta_j t^{-\frac{\alpha}{4}}\zeta^{-\frac{1}{4}}\right)^k \mathrm{e}^{i<\vartheta,x>t^{-\frac{\alpha}{4}}\zeta^{-\frac{1}{4}}}\,\mathrm{e}^{-|\vartheta|^4}\mathrm{d}\vartheta\right| \\
&\leqslant N^{k-1}t^{-\frac{\alpha k}{4}}\zeta^{-\frac{k}{4}}\int_{\mathbb{R}^N}|\vartheta|^k\left|\mathrm{e}^{i<\vartheta,x>t^{-\frac{\alpha}{4}}\zeta^{-\frac{1}{4}}}\,\mathrm{e}^{-|\vartheta|^4}\right|\mathrm{d}\vartheta. \qquad (2.16)
\end{aligned}$$

By using substitution $x(t^\alpha\zeta)^{-\frac{1}{4}} = z$, the second derivative with respect to x of $\mathbb{K}_{2,\alpha}(t,x)$ is estimated by

$$\begin{aligned}
\left\|D^k\mathbb{K}_{2,\alpha}(t,x)\right\|_{L^p} &= \left(\int_{\mathbb{R}^N}\left|D^k\mathbb{K}_{2,\alpha}(t,x)\right|^p\mathrm{d}x\right)^{\frac{1}{p}} \\
&\leqslant N^{k-1}\Bigg(\int_{\mathbb{R}^N}\Bigg|t^{\alpha-1-\frac{\alpha N}{4}-\frac{\alpha k}{4}}\int_0^\infty \alpha\zeta^{1-\frac{N}{4}-\frac{k}{4}}\mathcal{M}_\alpha(\zeta)\int_{\mathbb{R}^N}|\vartheta|^k\Big| \\
&\qquad \times\exp\left(i<\vartheta,x>t^{-\frac{1}{4}}\zeta^{-\frac{1}{4}}\right)\mathrm{e}^{-|\vartheta|^4}\Big|\mathrm{d}\vartheta\,\mathrm{d}\zeta\Bigg|^p\mathrm{d}x\Bigg)^{\frac{1}{p}}
\end{aligned}$$

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$$= N^{k-1}\left(\int_{\mathbb{R}^N}\left|t^{\alpha-1-\frac{\alpha N}{4}-\frac{\alpha k}{4}}\int_0^\infty \alpha\mathcal{M}_\alpha(\zeta)\zeta^{1-\frac{N}{4}-\frac{k}{4}}\overline{\mathscr{B}}_k(z)\,\mathrm{d}\zeta\right|^p \mathrm{d}x\right)^{\frac{1}{p}}$$

$$\leqslant N^{k-1}t^{\alpha-1-\frac{\alpha N}{4}-\frac{\alpha k}{4}}\left(\int_{\mathbb{R}^N}\left|\int_0^\infty \alpha\mathcal{M}_\alpha(\zeta)\zeta^{1-\frac{N}{4}-\frac{k}{4}}\overline{\mathscr{B}}_k(z)\,\mathrm{d}\zeta\right|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

Applying Minkowski's inequality in integral form, we find that

$$\begin{aligned}
&\left\|D^k\mathbb{K}_{2,\alpha}(t,x)\right\|_{L^p}\\
&\leqslant N^{k-1}t^{\alpha-1-\frac{\alpha N}{4}-\frac{\alpha k}{4}}\left(\int_{\mathbb{R}^N}\left|\int_0^\infty \alpha\mathcal{M}_\alpha(\zeta)\zeta^{1-\frac{N}{4}-\frac{k}{4}}\overline{\mathscr{B}}_k(z)\,\mathrm{d}\zeta\right|^p \mathrm{d}x\right)^{\frac{1}{p}}\\
&\leqslant N^{k-1}\alpha t^{\alpha-1-\frac{\alpha N}{4}-\frac{\alpha k}{4}}\int_0^\infty\left(\int_{\mathbb{R}^N}\left|\mathcal{M}_\alpha(\zeta)\zeta^{1-\frac{N}{4}-\frac{k}{4}}\overline{\mathscr{B}}_k(z)\right|^p (t^\alpha\zeta)^{\frac{N}{4}}\,\mathrm{d}z\right)^{\frac{1}{p}}\mathrm{d}\zeta\\
&\leqslant N^{k-1}\alpha t^{\alpha-\frac{\alpha N}{4}-1+\frac{\alpha N}{4p}-\frac{\alpha k}{4}}\int_0^\infty \zeta^{1-\frac{N}{4}+\frac{N}{4p}-\frac{k}{4}}\mathcal{M}_\alpha(\zeta)\left(\int_{\mathbb{R}^N}\left|\overline{\mathscr{B}}_k(z)\right|^p \mathrm{d}z\right)^{\frac{1}{p}}\mathrm{d}\zeta\\
&= N^{k-1}\alpha\left(\int_{\mathbb{R}^N}\left|\overline{\mathscr{B}}_k(z)\right|^p \mathrm{d}z\right)^{\frac{1}{p}} t^{\alpha-\frac{\alpha N}{4}-1+\frac{\alpha N}{4p}-\frac{\alpha k}{4}}\int_0^\infty \zeta^{1-\frac{N}{4}+\frac{N}{4p}-\frac{k}{4}}\mathcal{M}_\alpha(\zeta)\,\mathrm{d}\zeta.
\end{aligned} \tag{2.17}$$

Let us continue by computing the integral term on the right-hand side (RHS) of (2.4). Indeed, using lemma (5.2) and noting that $2-\frac{N}{4}+\frac{N}{4p}-\frac{k}{4}>0$, we immediately derive

$$\int_0^\infty \zeta^{1-\frac{N}{4}+\frac{N}{4p}-\frac{k}{4}}\mathcal{M}_\alpha(\zeta)\,\mathrm{d}\zeta = \frac{\Gamma\left(2-\frac{N}{4}+\frac{N}{4p}-\frac{k}{4}\right)}{\Gamma\left(1+\alpha-\frac{\alpha N}{4}+\frac{\alpha N}{4p}-\frac{\alpha k}{4}\right)}. \tag{2.18}$$

Combining (2.17) and (2.18), we find that there exists $\overline{\mathscr{C}_{k,p}}(\alpha,N)$ such that

$$\begin{aligned}
\left\|D^k\mathbb{K}_{2,\alpha}(t,x)\right\|_{L^p} &\leqslant N^{k-1}\left(\int_{\mathbb{R}^N}\left|\overline{\mathscr{B}}_k(z)\right|^p \mathrm{d}z\right)^{\frac{1}{p}}\\
&\quad\times\frac{\alpha\Gamma\left(2-\frac{N}{4}+\frac{N}{4p}-\frac{k}{4}\right)}{\Gamma\left(1+\alpha-\frac{\alpha N}{4}+\frac{\alpha N}{4p}-\frac{\alpha k}{4}\right)}t^{\alpha-\frac{\alpha N}{4}-1+\frac{\alpha N}{4p}-\frac{\alpha k}{4}}.
\end{aligned}$$

We end the proof here. □

REMARK 2.3. Let us state some comments on the assumptions of the above lemma as follows.

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(i) When $p = 1$ the assumption $k < 4 - N(1 - \frac{1}{p})$ implies that we can take k from the set $\{0, 1, 2, 3\}$. In addition, when $p = 1$ and $k = 0$, from the facts that $\Theta_{1,N} = 1$, $\frac{\alpha}{\Gamma(1+\alpha)} < \Gamma(2) = 1$, we can bound $\mathscr{C}_{k,p}$ and $\overline{\mathscr{C}_{k,p}}$ by 1.

(ii) When $k = 0$, the assumption becomes $\frac{N}{4}(1 - \frac{1}{p}) < 1$. This assumption is always satisfied whenever $N \leqslant 4$. On the other hand, when $N \geqslant 5$, we need to consider the condition that $p < \frac{N}{N-4}$ further, if we want to apply this lemma.

(iii) When $k \geqslant 1$, we have a certain restriction on the hypothesis for p. For example, when $N = 4$ the hypothesis $1 \leqslant p < \frac{N}{N-k}$ implies that $p \in \{1, 2, 3\}$. In short, when using lemma 2.3, the larger k and dimension N, the more restricted on the amount of p.

Remark 2.4. From [**23**, proposition 2.1], we can bound $\Theta_{p,N}$ by a constant Θ_N independent of p. This fact will be needed when we set up some linear estimates.

2.2. Space setting

Definition 2.4. Assume that a function $\Xi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ is increasing convex, right continuous at 0 and

$$\lim_{z\to\infty} \Xi(z) = \infty.$$

Then, we define the Orlicz space $L^{\Xi}(\mathbb{R}^N)$ in the following fashion

$$L^{\Xi}(\mathbb{R}^N) = \left\{\varphi \in L^1_{\rm loc}(\mathbb{R}^N);\int_{\mathbb{R}^N} \Xi\left(\frac{|\varphi(x)|}{\kappa}\right) \mathrm{d}x < \infty,\ \text{for some } \kappa > 0\right\}.$$

Remark 2.5. The Orlicz space $L^{\Xi}(\mathbb{R}^N)$ mentioned above is a Banach space, endowed with the Luxemburg norm

$$\|\varphi\|_{\Xi} = \inf\left\{\kappa > 0; \int_{\mathbb{R}^N} \Xi\left(\frac{|\varphi(x)|}{\kappa}\right) \mathrm{d}x \leqslant 1\right\}.$$

Remark 2.6. Let $1 < p < \infty$, by choosing $\Xi(z) = z^p$, we can identify the space $L^{\Xi}(\mathbb{R}^N)$ with the usual Lebesgue space $L^p(\mathbb{R}^N)$. For the sake of brevity, we set

$$\|\cdot\|_{L^p+L^q} := \|\cdot\|_{L^p(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)}, \quad p,\ q \in [1; \infty].$$

Definition 2.5. Let $1 \leqslant p < \infty$, in the rest of this study, we use the symbol $L^{\Xi}(\mathbb{R}^N)$ to indicate the Orlicz spaces with $\Xi(z) = \mathrm{e}^{|z|^p} - 1$. We also denote

$$\|\cdot\|_{L^q+\Xi} := \|\cdot\|_{L^q(\mathbb{R}^N)\cap L^{\Xi}(\mathbb{R}^N)}, \quad q \in [1; \infty].$$

Definition 2.6. Let $1 \leqslant p < \infty$. We define the following subspace of $L^{\Xi}(\mathbb{R}^N)$

$$L^{\Xi}_0(\mathbb{R}^N) = \left\{\varphi \in L^1_{\rm loc}(\mathbb{R}^N);\int_{\mathbb{R}^N} \Xi\left(\frac{|\varphi(x)|}{\kappa}\right) \mathrm{d}x < \infty,\ \text{for every } \kappa > 0\right\}.$$

Remark 2.7. It can be shown from [**25**] that $L^{\Xi}_0(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N)}^{L^{\Xi}(\mathbb{R}^N)}$.

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From the previous definitions, we can note that the Orlicz space is a generalization of the usual Lebesgue space. Let us introduce some of the useful embeddings between Orlicz spaces and Lebesgue spaces that we will need in our main results section.

LEMMA 2.7. *For every constants p, q satisfying $1 \leqslant p \leqslant q < \infty$, the embedding $L^{\Xi}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ holds. In addition,*

$$\|\varphi\|_{L^q} \leqslant \left[\Gamma\left(\frac{q}{p}+1\right)\right]^{\frac{1}{q}} \|\varphi\|_{\Xi}. \tag{2.19}$$

LEMMA 2.8. *Given $1 \leqslant q \leqslant p$, we have $L^q(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \hookrightarrow L_0^{\Xi}(\mathbb{R}^N) \subsetneq L^{\Xi}(\mathbb{R}^N)$. In particular, for any $\varphi \in L^q(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ the following bound holds*

$$\|\varphi\|_{\Xi} \leqslant (\log 2)^{\frac{-1}{p}} \left[\|\varphi\|_{L^q} + \|\varphi\|_{L^\infty}\right].$$

LEMMA 2.9. *Let $p \geqslant 1$ and $\alpha \in (0,1)$. Then, we can find constants $\mathscr{C}_{1,h}, \mathscr{C}_{2,h}, \mathscr{C}_{\Xi}$ such that the following results hold.*

(i) Suppose that $h \in [1,p]$ satisfies $h > N/4$, for any $\varphi \in L^h(\mathbb{R}^N)$, we have

$$\|\mathbb{Z}_{1,\alpha}(t)\varphi\|_{\Xi} \leqslant \mathscr{C}_{1,h} t^{\frac{-\alpha N}{4h}} \left[\log\left(1+t^{\frac{-\alpha N}{4}}\right)\right]^{\frac{-1}{p}} \|\varphi\|_{L^h},$$
$$\|\mathbb{Z}_{2,\alpha}(t)\varphi\|_{\Xi} \leqslant \mathscr{C}_{2,h} t^{\frac{-\alpha N}{4q}} \left[\log\left(1+t^{\frac{-\alpha N}{4}}\right)\right]^{\frac{-1}{p}} \|\varphi\|_{L^h}.$$

(ii) For any $\varphi \in L^{\Xi}(\mathbb{R}^N)$, we have

$$\|\mathbb{Z}_{1,\alpha}(t)\varphi\|_{\Xi} \leqslant \|\varphi\|_{\Xi}, \quad \|\mathbb{Z}_{2,\alpha}(t)\varphi\|_{\Xi} \leqslant t^{\alpha-1} \|\varphi\|_{\Xi}.$$

Proof. Firstly, by using Young's convolution inequality, there exists a constant $q \in [1,\infty]$ such that

$$\|\mathbb{Z}_{i,\alpha}(t)\varphi\|_{L^p} \leqslant \|\mathbb{K}_{i,\alpha}(t)\|_{L^q} \|\varphi\|_{L^h}. \tag{2.20}$$

Then, thanks to lemma 2.3, we have

$$\|\mathbb{Z}_{1,\alpha}(t)\varphi\|_{L^p} \leqslant \frac{\Theta_N \Gamma\left(1-\frac{N}{4}\left(\frac{1}{h}-\frac{1}{p}\right)\right)}{\Gamma\left(1-\frac{\alpha N}{4}\left(\frac{1}{h}-\frac{1}{p}\right)\right)} t^{\frac{-\alpha N}{4}\left(\frac{1}{h}-\frac{1}{p}\right)} \|\varphi\|_{L^h},$$

$$\begin{aligned}\left\|t^{1-\alpha}\mathbb{Z}_{2,\alpha}(t)\varphi\right\|_{L^p} &= t^{1-\alpha}\|\mathbb{Z}_{2,\alpha}(t)\varphi\|_{L^p}\\ &\leqslant \frac{\alpha\Theta_N \Gamma\left(2-\frac{N}{4}\left(\frac{1}{h}-\frac{1}{p}\right)\right)}{\Gamma\left(2-\frac{\alpha N}{4}\left(\frac{1}{h}-\frac{1}{p}\right)\right)} t^{\frac{-\alpha N}{4}\left(\frac{1}{h}-\frac{1}{p}\right)} \|\varphi\|_{L^h}.\end{aligned}$$

We can show that the constants on the RHS of the above estimates can be bounded by two constants $\mathscr{C}_{1,h}, \mathscr{C}_{2,h}$ that are independent of p, respectively. In fact, by properties of the Gamma function when $0 < \frac{N}{4}\left(\frac{1}{h}-\frac{1}{p}\right) < 1 < \frac{29}{20}$, we obtain

$$\|\mathbb{Z}_{1,\alpha}(t)\varphi\|_{L^p} \leqslant \Theta_N \Gamma\left(1-\frac{N}{4h}\right) t^{\frac{-\alpha N}{4}\left(\frac{1}{h}-\frac{1}{p}\right)} \|\varphi\|_{L^h}.$$

On the other hand, the Gautschi inequality implies

$$\frac{\Gamma\left(2-\frac{N}{4}\left(\frac{1}{h}-\frac{1}{p}\right)\right)}{\Gamma\left(2-\frac{\alpha N}{4}\left(\frac{1}{h}-\frac{1}{p}\right)\right)} \leqslant \left[1-\frac{\alpha N}{4}\left(\frac{1}{h}-\frac{1}{p}\right)\right]^{\frac{(\alpha-1)N}{4}\left(\frac{1}{h}-\frac{1}{p}\right)} \leqslant \left(1-\frac{\alpha N}{4h}\right)^{\frac{(\alpha-1)N}{4h}}.$$

It follows that

$$\left\|t^{1-\alpha}\mathbb{Z}_{2,\alpha}(t)\varphi\right\|_{L^p} \leqslant \alpha\Theta_N \left(1-\frac{\alpha N}{4h}\right)^{\frac{(\alpha-1)N}{4h}} t^{\frac{-\alpha N}{4}\left(\frac{1}{h}-\frac{1}{p}\right)} \|\varphi\|_{L^h}.$$

We are now ready to verify our main statements. Because the techniques are the same, we will present only the proof for the second one, $\mathbb{Z}_{2,\alpha}(t)\varphi$. We note that for $j \geqslant 1$

$$\left\|t^{1-\alpha}\mathbb{Z}_{2,\alpha}(t)\varphi\right\|_{L^{pj}}^{pj} \leqslant \mathscr{C}_{2,h}^{pj} t^{\frac{-\alpha N pj}{4}\left(\frac{1}{h}-\frac{1}{pj}\right)} \|\varphi\|_{L^h}^{pj} \leqslant t^{\frac{\alpha N}{4}} \left[\mathscr{C}_{2,h} t^{\frac{-\alpha N}{4h}} \|\varphi\|_{L^h}\right]^{pj}.$$

Then, the Taylor expansion of the exponential leads us to

$$\begin{aligned}\int_{\mathbb{R}^N}\left[\exp\left(\frac{\left|t^{1-\alpha}\mathbb{Z}_{2,\alpha}(t)\varphi(x)\right|^p}{\kappa^p}\right)-1\right]\mathrm{d}x &= \sum_{j=1}^{\infty}\frac{\left\|t^{1-\alpha}\mathbb{Z}_{2,\alpha}(t)\varphi\right\|_{L^{pj}}^{pj}}{j!\kappa^{pj}} \\ &\leqslant t^{\frac{\alpha N}{4}}\sum_{j=1}^{\infty}\frac{\left[\mathscr{C}_{2,h}t^{\frac{-\alpha N}{4h}}\|\varphi\|_{L^h}\right]^{pj}}{j!\kappa^{pj}} \\ &= t^{\frac{\alpha N}{4}}\left[\exp\left(\frac{\mathscr{C}_{2,h}t^{\frac{-\alpha N}{4h}}\|\varphi\|_{L^h}}{\kappa}\right)^p-1\right].\end{aligned} \tag{2.21}$$

Next, assume that the RHS of the above estimate is less than or equal 1. Then, we can easily find that

$$\mathscr{C}_{2,h} t^{\frac{-\alpha N}{4h}} \left[\log\left(1+t^{\frac{-\alpha N}{4}}\right)\right]^{\frac{-1}{p}} \|\varphi\|_{L^h} \leqslant \kappa.$$

In view of (2.21), if we set

$$A := \left\{\kappa > 0; \int_{\mathbb{R}^N}\left[\exp\left(\frac{|\mathbb{Z}_{2,\alpha}(t)\varphi(x)|^p}{\kappa^p}\right)-1\right]\mathrm{d}x \leqslant 1\right\},$$

$$B := \left\{\kappa > 0; \mathscr{C}_{2,h} t^{\frac{-\alpha N}{4h}} \left[\log\left(1+t^{\frac{-\alpha N}{4}}\right)\right]^{\frac{-1}{p}} \|\varphi\|_{L^h} \leqslant \kappa\right\},$$

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the cover result $B \subset A$ holds. This implies that

$$\inf A \leqslant \inf B = \mathscr{C}_{2,h} t^{\frac{-\alpha N}{4h}} \left[\log\left(1 + t^{\frac{-\alpha N}{4}}\right)\right]^{\frac{-1}{p}} \|\varphi\|_{L^h}.$$

We obtain the first results of this lemma. To prove the remaining result, we only need to modify slightly inequality (2.21) in the following way

$$\int_{\mathbb{R}^N} \left[\exp\left(\frac{\left|t^{1-\alpha}\mathbb{Z}_{2,\alpha}(t)\varphi(x)\right|^p}{\kappa^p}\right) - 1\right] \mathrm{d}x = \sum_{j=1}^{\infty} \frac{\left\|t^{1-\alpha}\mathbb{Z}_{2,\alpha}(t)\varphi\right\|^{pj}_{L^{pj}(\mathbb{R}^N)}}{j!\kappa^{pj}}$$
$$\leqslant \sum_{j=1}^{\infty} \frac{\|\varphi\|^{pj}_{L^{pj}(\mathbb{R}^N)}}{j!\kappa^{pj}} = \exp\left(\frac{\|\varphi\|_{L^p}}{\kappa}\right)^p - 1.$$

Then, our statements follow. □

PROPOSITION 2.1. *Assume that $\varphi \in L^{\Xi}_0(\mathbb{R}^N)$. Then, we have*

$$\mathbb{Z}_{1,\alpha}(t)\varphi \in C\left([0,T]; L^{\Xi}_0(\mathbb{R}^N)\right).$$

Proof. Since $\varphi \in L^{\Xi}_0(\mathbb{R}^N)$, there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C^{\infty}_0(\mathbb{R}^N)$ such that φ_n converges to φ with respect to $L^{\Xi}(\mathbb{R}^N)$ norm. This implies that, for any $t > 0$, $\mathbb{Z}_{1,\alpha}(t)\varphi_n$ will converge to $\mathbb{Z}_{1,\alpha}(t)\varphi$. Indeed, by applying lemma 2.9, we have

$$\|\mathbb{Z}_{1,\alpha}(t)\varphi_n - \mathbb{Z}_{1,\alpha}(t)\varphi\|_{\Xi} \leqslant \|\varphi_n - \varphi\|_{\Xi} \xrightarrow{n\to\infty} 0.$$

By taking two number $t_1, t_2 > 0$, the triangle inequality implies

$$\begin{aligned} &\|\mathbb{Z}_{1,\alpha}(t_2)\varphi - \mathbb{Z}_{1,\alpha}(t_1)\varphi\|_{\Xi} \\ &\quad \leqslant \|\mathbb{Z}_{1,\alpha}(t_2)\varphi_n - \mathbb{Z}_{1,\alpha}(t_2)\varphi\|_{\Xi} + \|\mathbb{Z}_{1,\alpha}(t_1)\varphi_n - \mathbb{Z}_{1,\alpha}(t_1)\varphi\|_{\Xi} \\ &\qquad + \|\mathbb{Z}_{1,\alpha}(t_2)\varphi_n - \mathbb{Z}_{1,\alpha}(t_1)\varphi_n\|_{\Xi}. \end{aligned}$$

Combining lemma 2.9, the definition of $L^{\Xi}_0(\mathbb{R}^N)$ and the application of theorem 2.2 for $L^p(\mathbb{R}^N)$ and $C_0(\mathbb{R}^N)$

$$\begin{cases} \lim_{t_2\to t_1} \|\mathbb{Z}_{1,\alpha}(t_2)\varphi_n - \mathbb{Z}_{1,\alpha}(t_1)\varphi_n\|_{L^p} = 0, & \varphi_n \in C^{\infty}_0(\mathbb{R}^N), \\ \lim_{t_2\to t_1} \|\mathbb{Z}_{1,\alpha}(t_2)\varphi_n - \mathbb{Z}_{1,\alpha}(t_1)\varphi_n\|_{L^\infty} = 0, & \varphi_n \in C^{\infty}_0(\mathbb{R}^N), \end{cases}$$

and

$$\begin{cases} \lim_{n\to\infty} \|\mathbb{Z}_{1,\alpha}(t_1)\varphi_n - \mathbb{Z}_{1,\alpha}(t_1)\varphi\|_{\Xi} = 0, \\ \lim_{n\to\infty} \|\mathbb{Z}_{1,\alpha}(t_2)\varphi_n - \mathbb{Z}_{1,\alpha}(t_2)\varphi\|_{\Xi} = 0. \end{cases}$$

Consequently, by an appropriate choice of n, the desired conclusion of this proposition can be drawn easily. □

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3. Time-fractional biharmonic equation with exponential nonlinearity

In this section, we investigate the IVP for the time-fractional biharmonic equation with exponential nonlinearity

$$\begin{cases} \partial^{\alpha}_{0|t} u(t,x) + \Delta^2 u(t,x) = G(u(t,x)), & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \\ u(0,x) = u_0(x), & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{P1}$$

with the following assumptions of the initial function:

Assumption 1. The initial function u_0 belongs to $L^p(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$.

Assumption 2. The initial function u_0 belongs to $L^{\Xi}(\mathbb{R}^N)$ or $L^{\Xi}_0(\mathbb{R}^N)$.

3.1. Unique existence of mild solution under the first assumption for the initial function

In this section, we investigate problem (P1) with the assumption that u_0 belongs to the space $L^p(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$.

Theorem 3.1. *Assume that $u_0 \in L^{\Xi}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$. Then, there exists a unique solution of problem (P1) that belongs to $C\left((0,T]; L^{\Xi}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)\right)$.*

Proof. The proof is begun by fixing a constant $\Im > 0$ and choosing a small time $T < \sqrt[\alpha]{\alpha \Im_1^{-1}}$, where $\Im_1$ is defined in (3.7). Next, we consider the following space

$$\mathbf{A} := \left\{ u \in C\left((0,T]; L^{\Xi}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)\right); \sup_{t \in (0,T]} \|u(t) - \mathbb{Z}_{1,\alpha}(t) u_0\|_{L^\infty + \Xi} \leqslant \Im \right\},$$

and the operator $\mathscr{F} : \mathbf{A} \to \mathbf{A}$ given by

$$\mathscr{F} u(t) = \mathbb{Z}_{1,\alpha}(t) u_0 + \int_0^t \mathbb{Z}_{2,\alpha}(t-\tau) G(u(\tau)) \, \mathrm{d}\tau. \tag{3.1}$$

By Young's convolution inequality and lemma 2.3, we have

$$\begin{aligned} &\left\| \int_{\mathbb{R}^N} \mathbb{K}_{1,\alpha}(t, \cdot - y) u_0(y) \, \mathrm{d}y \right\|_{L^\infty} \\ &\quad = \left\| \mathbb{K}_{1,\alpha}(t) * u_0 \right\|_{L^\infty} \leqslant \|\mathbb{K}_{1,\alpha}(t)\|_{L^1(\mathbb{R}^N)} \|u_0\|_{L^\infty} \leqslant \|u_0\|_{L^\infty}. \end{aligned}$$

Then, for every $u \in \mathbf{A}$, the following estimate holds

$$\begin{aligned} \|u(t)\|_{L^\infty + \Xi} &\leqslant \|\mathbb{Z}_{1,\alpha}(t) u_0\|_{L^\infty + \Xi} + \|u(t) - \mathbb{Z}_{1,\alpha}(t) u_0\|_{L^\infty + \Xi} \\ &\leqslant 2(\log 2)^{\frac{-1}{p}} \|u_0\|_{L^\infty + \Xi} + \Im =: \Im_0. \end{aligned} \tag{3.2}$$

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Our main goal is to prove that the integral equation (3.1) has a unique solution by the fixed point argument. To this end, we present the following two steps.

Step 1. From (1.2), for any $u \in \mathbb{A}$ and $t > 0$, we deduce

$$\begin{cases} \|G(u(t))\|_{L^\infty} \leqslant L\,\|u(t)\|_{L^\infty}^m \, \mathrm{e}^{\kappa\|u(t)\|_{L^\infty}^p} \leqslant L\Im_0^m \, \mathrm{e}^{\kappa\Im_0^p}, \\ \|G(u(t))\|_{L^p} \leqslant L\,\|u(t)\|_{L^{mp}}^m \, \mathrm{e}^{\kappa\|u(t)\|_{L^\infty}^p} \leqslant L\Im_0^m \, \mathrm{e}^{\kappa\Im_0^p} \, (\Gamma(m+1))^{\frac{1}{p}}, \end{cases} \tag{3.3}$$

where we have used lemma 2.7 to achieve the second inequality. It follows that

$$\begin{aligned} \|G(u(t))\|_{L^\infty+\Xi} &\leqslant (\log 2)^{\frac{-1}{p}} \, \|G(u(t))\|_{L^p+L^\infty} + \|G(u(t))\|_{L^\infty} \\ &\leqslant L\left(2 + (\Gamma(m+1))^{\frac{1}{p}}\right)\Im_0^m (\log 2)^{\frac{-1}{p}} \, \mathrm{e}^{\kappa\Im_0^p}. \end{aligned}$$

In addition, by using lemma 2.8 and applying theorem 2.2 with respect to the spaces $L^p(\mathbb{R}^N), C_0(\mathbb{R}^N)$, for any $\tau, t \in (0, T]$, we deduce

$$t^{1-\alpha}\mathbb{Z}_{2,\alpha}(t)G(u(\tau)) \in C\left((0,T]; L^{\Xi}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)\right). \tag{3.4}$$

Next, by taking a positive number h and $r \in \{t, t+h\}$, for any $t \geqslant \xi$, we obtain

$$\begin{aligned} &(t+h-\tau)^{\alpha-1} \left\|(r-\tau)^{1-\alpha}\mathbb{Z}_{2,\alpha}(r-\tau)G(u(\tau))\right\|_{L^\infty+\Xi} \\ &\quad \leqslant (t-\tau)^{\alpha-1} \, \|G(u(\tau))\|_{L^\infty+\Xi} \leqslant C(t-\tau)^{\alpha-1}. \end{aligned}$$

Then, applying the Lebesgue dominated convergence theorem,

$$\lim_{h\to 0} \left\| \int_0^t (t+h-\xi)^{\alpha-1} \left[\frac{\mathbb{Z}_{2,\alpha}(t+h-\tau)}{(t+h-\tau)^{\alpha-1}} - \frac{\mathbb{Z}_{2,\alpha}(t-\tau)}{(t-\tau)^{\alpha-1}} \right] G(u(\tau))\,\mathrm{d}\tau \right\|_{L^\infty+\Xi} = 0. \tag{3.5}$$

On the other hand, using the fact that $\lim_{h\to 0} |(t+h)^\alpha - h^\alpha - t^\alpha| = 0$ and (3.4), we further find that

$$\lim_{h\to 0} \left\| \int_0^t \left[(t+h-\xi)^{\alpha-1} - (t-\xi)^{\alpha-1}\right] \frac{\mathbb{Z}_{2,\alpha}(t+h-\tau)}{(t+h-\tau)^{\alpha-1}} G(u(\tau))\,\mathrm{d}\tau \right\|_{L^\infty+\Xi} = 0. \tag{3.6}$$

For the purpose of proving that the integral term on the RHS of (3.1) is continuous on $(0, T]$, we also need the upcoming fact

$$\int_t^{t+h} \|\mathbb{Z}_{2,\alpha}(t+h-\tau)G(u(\tau))\,\mathrm{d}\tau\|_{L^\infty+\Xi} \,\mathrm{d}\tau \leqslant Ch^\alpha \xrightarrow{h\to 0} 0.$$

Taking into consideration the above limit results and applying theorem 2.2 to $u_0 \in L^p(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, we can claim that $\mathscr{F}u(t)$ is a continuous mapping on $(0, T]$.

Step 2. By using the Hölder inequality, for a constant $\aleph \in [1, \infty)$ and $u_1, u_2 \in \mathbf{A}$, we have

$$\begin{aligned} &\|G(u_1(\tau)) - G(u_2(\tau))\|_{L^p} \\ &\quad \leqslant L\left[\sum_{j=1,2} \left\| |u_j(\tau)|^{m-1} \, \mathrm{e}^{\kappa|u_j(\tau)|^p} \right\|_{L^{\frac{\aleph p}{\aleph-1}}}\right] \|u_1(\tau) - u_2(\tau)\|_{L^{\aleph p}}. \end{aligned}$$

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Since the embedding $L^{\Xi}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ holds for any $q \in [p, \infty)$, and $u_j(\tau) \in C_0(\mathbb{R}^N)(j = 1, 2)$, the above inequality becomes

$$\|G(u_1(\tau)) - G(u_2(\tau))\|_{L^p} \leqslant 2L\left(\Gamma(\aleph+1)\right)^{\frac{1}{\aleph p}} \left(\Gamma\left(\frac{\aleph(m-1)}{\aleph-1}+1\right)\right)^{\frac{\aleph-1}{\aleph p}} \times \Im_0^{m-1} \mathrm{e}^{\kappa \Im_0^p} \|u_1(\tau) - u_2(\tau)\|_{\Xi}.$$

On the other hand, it is a simple matter to check that

$$\begin{aligned}\|G(u_1(\tau)) - G(u_2(\tau))\|_{L^\infty} &\leqslant L\left[\sum_{j=1,2} \|u_j(\tau)\|_{L^\infty}^{m-1} \mathrm{e}^{\kappa \|u_j(\tau)\|_{L^\infty}^p}\right] \|u_1(\tau) - u_2(\tau)\|_{L^\infty} \\ &\leqslant 2L\Im_0^{m-1} \mathrm{e}^{\kappa \Im_0^p} \|u_1(\tau) - u_2(\tau)\|_{L^\infty}.\end{aligned}$$

The two above results help us to deduce

$$\begin{aligned}&\|G(u_1(\tau)) - G(u_2(\tau))\|_{L^\infty+\Xi} \\ &\quad\leqslant 2L\left[2 + \left(\Gamma(\aleph+1)\right)^{\frac{1}{\aleph p}} \left(\Gamma\left(\frac{\aleph(m-1)}{\aleph-1}+1\right)\right)^{\frac{\aleph-1}{\aleph p}}\right] \\ &\quad\times (\log 2)^{\frac{-1}{p}} \Im_0^{m-1} \mathrm{e}^{\kappa \Im_0^p} \|u_1(\tau) - u_2(\tau)\|_{L^\infty+\Xi} \\ &\quad=: \Im_1 \|u_1(\tau) - u_2(\tau)\|_{L^\infty+\Xi}.\end{aligned} \tag{3.7}$$

Also, lemma 2.3 shows us that

$$\begin{aligned}&\left\|\int_{\mathbb{R}^N} \mathbb{K}_{2,\alpha}(t-\tau, x-y)\left[G(u_1(\tau,y)) - G(u_2(\tau,y))\right] \mathrm{d}y\right\|_{L^\infty} \\ &\quad\leqslant \|\mathbb{K}_{2,\alpha}(t-\tau, x)\|_{L^1(\mathbb{R}^N)} \|G(u_1(\tau)) - G(u_2(\tau))\|_{L^\infty} \\ &\quad\leqslant (t-\tau)^{\alpha-1} \|G(u_1(\tau)) - G(u_2(\tau))\|_{L^\infty}.\end{aligned}$$

From this result and lemma 2.9, we find that

$$\begin{aligned}\|\mathscr{F}u_1(t) - \mathscr{F}u_2(t)\|_{L^\infty+\Xi} &\leqslant \int_0^t \|\mathbb{Z}_{2,\alpha}(t-\tau)\left[G(u_1(\tau)) - G(u_2(\tau))\right]\|_{L^\infty+\Xi} \,\mathrm{d}\tau \\ &\leqslant \int_0^t (t-\tau)^{\alpha-1} \|G(u_1(\tau)) - G(u_2(\tau))\|_{L^\infty+\Xi} \,\mathrm{d}\tau \\ &\leqslant \frac{T^\alpha \Im_1}{\alpha} \sup_{t\in(0,T]} \|u_1(t) - u_2(t)\|_{L^\infty+\Xi}.\end{aligned}$$

By choosing $u_2 \equiv 0$ and $u_1 \in \mathbf{A}$ in step 2, we derive

$$\sup_{t\in(0,T]} \|\mathscr{F}u_1(t) - \mathbb{Z}_{1,\alpha}(t)u_0\|_{L^\infty+\Xi} \leqslant \frac{T^\alpha \Im_1}{\alpha} \sup_{t\in(0,T]} \|u_1(t)\|_{L^\infty+\Xi} \leqslant \Im.$$

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This result along with step 1 show that $\mathscr{F}$ is invariant on $\mathbf{A}$. Furthermore, it is obvious that $\mathbf{A}$ is a complete metric space with the metric

$$\mathrm{d}(u,v) := \sup_{t\in(0,T]} \|u(t)-v(t)\|_{L^\infty+\Xi}\,.$$

Therefore, the Banach principle argument can be applied to conclude that $\mathscr{F}$ has a unique fixed point, which means that there exists a unique solution of the problem (P1) belonging to $\mathbf{A}$. □

LEMMA 3.2. *Let $u:(0,T]\to L^\Xi(\mathbb{R}^N)\cap C_0(\mathbb{R}^N)$ be the unique mild solution of problem* (P1). *Then there exists a unique continuous extension u^* of u on $(0,T+h]$, for some $h>0$.*

Proof. The main idea of the proof is to show that there exists a unique solution of (P1) which belongs to the space

$$\mathbf{B} = \left\{ w \in C\left((0,T+h]; L^\Xi(\mathbb{R}^N)\cap C_0(\mathbb{R}^N)\right) \,\middle|\, \begin{array}{c} w(t)=u(t), \forall t\in(0,T] \\ \sup_{t\in[T,T+h]}\|w(t)-u(T)\|_{L^\infty+\Xi} \leqslant \Im \end{array} \right\},$$

where $\Im, h>0$ will be chosen later. To this end, let us consider the function $\mathscr{J}:\mathbf{B}\to\mathbf{B}$, satisfying

$$\mathscr{J}w(t) = \mathbb{Z}_{1,\alpha}(t)u_0 + \int_0^t \mathbb{Z}_{2,\alpha}(t-\tau)G(u(\tau))\,\mathrm{d}\tau.$$

It is easily seen that for any $t\in(0,T]$, $\mathscr{J}(w(t))=\mathscr{J}(u(t))=u(t)$. The continuity result of $\mathscr{J}$ on $(0,T+h]$ can be drawn by the same arguments as in theorem 3.1. Our remaining part of proving the well definition of $\mathscr{J}$ is to find that if $w\in\mathbf{B}$, $\sup_{t\in[T,T+h]}\|\mathscr{J}(w(t))-u(T)\|_{L^\infty+\Xi}\leqslant\Im$. Indeed, for any $t\in[T,T+h]$, we have

$$\begin{aligned}\|\mathscr{J}(w(t))-u(T)\|_{L^\infty+\Xi} \leqslant{}& \underbrace{\|\mathbb{Z}_{1,\alpha}(t)u_0-\mathbb{Z}_{1,\alpha}(T)u_0\|_{L^\infty+\Xi}}_{(I)} \\ &+ \underbrace{\int_T^t \|\mathbb{Z}_{2,\alpha}(t-\tau)G(u(\tau))\|_{L^\infty+\Xi}\,\mathrm{d}\tau}_{(II)} \\ &+ \underbrace{\int_0^T \left\|\left[\mathbb{Z}_{2,\alpha}(t-\tau)-\mathbb{Z}_{2,\alpha}(T-\tau)\right]G(u(\tau))\right\|_{L^\infty+\Xi}\mathrm{d}\tau}_{(III)}.\end{aligned}$$

- Choosing a sufficiently small h_1 and using theorem 2.2, lemma 2.8, for every $t\in[T,T+h_1]$, we can find that $(I)\leqslant\frac{\Im}{3}$.

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- Choosing a sufficiently small h_2, for every $t \in [T, T+h_2]$, we obtain

$$
\begin{aligned}
(II) &\leqslant \frac{L\left(2+(\Gamma(m+1))^{\frac{1}{p}}\right)\left(\|u(T)\|_{L^\infty+\Xi}+\Im\right)^m}{(\log 2)^{\frac{1}{p}}\, \mathrm{e}^{-\kappa\left(\|u(T)\|_{L^\infty+\Xi}+\Im\right)^p}} \int_T^{T+h_2} (t-\tau)^{\alpha-1}\,\mathrm{d}\tau \\
&\leqslant C\left[(t-T)^\alpha-(t-T-h_2)^\alpha\right] \leqslant \frac{\Im}{3}.
\end{aligned}
$$

- Choosing a sufficiently small h_3 and repeating the arguments for (3.5), (3.6), we can also find that

$$
(III) \leqslant \frac{\Im}{3}, \quad \text{for every } t \in [T, T+h_3].
$$

Additionally, $\mathscr{J}$ is a contraction mapping on $\mathbf{B}$ if $h = h_4$ is small enough. In fact, for $u_1, u_2 \in \mathbf{B}$, we have

$$
\begin{aligned}
&\|\mathscr{J}u_1(t)-\mathscr{J}u_2(t)\|_{L^\infty+\Xi} \leqslant \int_T^t (t-\tau)^{\alpha-1}\|G(u_1(\tau))-G(u_2(\tau))\|_{L^\infty+\Xi}\,\mathrm{d}\tau \\
&\leqslant \frac{2L\left[2+(\Gamma(\aleph+1))^{\frac{1}{\aleph p}}\left(\Gamma\left(\frac{\aleph(m-1)}{\aleph-1}+1\right)\right)^{\frac{\aleph-1}{\aleph p}}\right]}{\alpha h_4^{-\alpha}(\log 2)^{\frac{1}{p}}\left(\|u(T)\|+\Im\right)^{1-m}\,\mathrm{e}^{-\kappa(\|u(T)\|+\Im)^p}} \sup_{t\in[T,T+h]}\|u_1(t)-u_2(t)\|_{L^\infty+\Xi} \\
&\leqslant \mathcal{L} \sup_{t\in[T,T+h]}\|u_1(t)-u_2(t)\|_{L^\infty+\Xi}.
\end{aligned}
$$

Thanks to an appropriate choice of h_4, $\mathcal{L}$ can be proved to be less than 1.

By setting $h = \min\{h_1, h_2, h_3, h_4\}$, we can now apply the Banach principle arguments to declare that the solution to problem (P1) has been extended to some larger interval. □

THEOREM 3.3. *Let $T_{\max}$ be the supremum of the set of all $T > 0$ such that problem (P1) has a unique local solution on $(0, T]$. Assume that G satisfies (1.2) and u_0 belongs to $L^p(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$. Then, we can conclude that $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\limsup_{t\to T_{\max}^-}\|u(t)\|_{L^\infty+\Xi} = \infty$.*

Proof. Arguing by contradiction, we assume that $T_{\max} < \infty$ and there exist a positive constant $\Im < \infty$ such that

$$
\|u(t)\|_{L^\infty+\Xi} \leqslant \Im. \tag{3.8}
$$

Let $\{t_j\}_{j\in\mathbb{N}} \subset (0, T_{\max})$ satisfy $t_j \xrightarrow{j\to\infty} T_{\max}$. Our goal is to show that $\{u(t_j)\}_{j\in\mathbb{N}}$ is a Cauchy sequence. To this end, let us take $t_a, t_b \in \{t_j\}_{j\in\mathbb{N}}$, without loss of

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generality, we suppose that $t_a < t_b$. Then, we have

$$\begin{aligned}\|u(t_b)-u(t_a)\|_{L^\infty+\Xi} &\leqslant \|\mathbb{Z}_{1,\alpha}(t_b)u_0-\mathbb{Z}_{1,\alpha}(t_a)u_0\|_{L^\infty+\Xi}\\ &\quad+\int_{t_a}^{t_b}\|\mathbb{Z}_{2,\alpha}(T_{\max}-\tau)G(u(\tau))\|_{L^\infty+\Xi}\,\mathrm{d}\tau\\ &\quad+\int_0^{t_a}\Big\|\left[\mathbb{Z}_{2,\alpha}(T_{\max}-\tau)-\mathbb{Z}_{2,\alpha}(t_a-\tau)\right]G(u(\tau))\Big\|_{L^\infty+\Xi}\,\mathrm{d}\tau\\ &\quad+\int_0^{t_b}\Big\|\left[\mathbb{Z}_{2,\alpha}(t_b-\tau)-\mathbb{Z}_{2,\alpha}(T_{\max}-\tau)\right]G(u(\tau))\Big\|_{L^\infty+\Xi}\,\mathrm{d}\tau.\end{aligned}$$

Given $\varepsilon > 0$, let us consider some large enough constants j_1, j_2, j_3 defined through the following steps.

Step 1. Thanks to the property of $\mathbb{Z}_{1,\alpha}(t)$, we can choose a sufficiently large j_1 such that

$$\|\mathbb{Z}_{1,\alpha}(t_b)u_0-\mathbb{Z}_{1,\alpha}(t_a)u_0\|_{L^\infty+\Xi} < \frac{\varepsilon}{4}, \quad \text{for any } a, b \geqslant j_1.$$

Step 2. For the sake of simplicity, let us set

$$\Im^* = L(\log 2)^{\frac{-1}{p}}\left(2+(\Gamma(m+1))^{\frac{1}{mp}}\right)\Im^m\,\mathrm{e}^{-\kappa\Im^p}.$$

and choose a large j_2 such that

$$|t_j - T_{\max}| < \frac{\varepsilon\alpha}{8\Im^*}, \quad \text{for any } j \geqslant j_2.$$

Then, similar to the proof of theorem 3.1, it is easily seen that

$$\begin{aligned}\int_{t_a}^{t_b}\|\mathbb{Z}_{2,\alpha}(T_{\max}-\tau)G(u(\tau))\|_{L^\infty+\Xi}\,\mathrm{d}\tau &\leqslant \int_{t_a}^{t_b}(T_{\max}-\tau)^{\alpha-1}\|G(u(\tau))\|_{L^\infty+\Xi}\,\mathrm{d}\tau\\ &\leqslant \Im^*\alpha^{-1}\left(|t_b-T_{\max}|^\alpha+|t_a-T_{\max}|^\alpha\right) < \frac{\varepsilon}{4}.\end{aligned}$$

Step 3. The Lebesgue dominated convergence theorem helps us to find a sufficiently large j_3 such that, for any $j \geqslant j_3$,

$$\int_0^{t_j}\|\left[\mathbb{Z}_{2,\alpha}(T_{\max}-\tau)-\mathbb{Z}_{2,\alpha}(t_j-\tau)\right]G(u(\tau))\|_{L^\infty+\Xi}\,\mathrm{d}\tau < \frac{\varepsilon}{4}.$$

Now choosing $j_0 = \max\{j_1, j_2, j_3\}$ yields that

$$\|u(t_a)-u(t_b)\|_{L^\infty+\Xi} \leqslant \varepsilon, \quad \text{for every } a, b \geqslant j_0.$$

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Then, there exists a limit of the Cauchy sequence $\{u(t_j)\}_{j\in\mathbb{N}}$ in $L^{\Xi}(\mathbb{R}^N)\cap C_0(\mathbb{R}^N)$ as j tends to infinity denoted by u^*. Hence, we can check that

$$\begin{aligned}&\left\|\int_0^{t_j}\mathbb{Z}_{2,\alpha}(t_j-\tau)G(u(\tau))\,\mathrm{d}\tau-\int_0^{T_{\max}}\mathbb{Z}_{2,\alpha}(T_{\max}-\tau)G(u(\tau))\,\mathrm{d}\tau\right\|_{L^\infty+\Xi}\\&\quad\leqslant\int_0^{t_j}\left\|\left[\mathbb{Z}_{2,\alpha}(t_j-\tau)-\mathbb{Z}_{2,\alpha}(T_{\max}-\xi)\right]G(u(\tau))\right\|_{L^\infty+\Xi}\,\mathrm{d}\tau\\&\qquad+\frac{\Im^*(T_{\max}-t_j)^\alpha}{\alpha}\xrightarrow{j\to\infty}0.\end{aligned}$$

It implies that

$$\begin{aligned}u^*:&=\lim_{j\to\infty}\left[\mathbb{Z}_{1,\alpha}(t_j)u_0+\int_0^{t_j}\mathbb{Z}_{2,\alpha}(t_j-\tau)G(u(\tau))\,\mathrm{d}\tau\right]\\&=\mathbb{Z}_{1,\alpha}(T_{\max})u_0+\int_0^{T_{\max}}\mathbb{Z}_{2,\alpha}(T_{\max}-\tau)G(u(\tau))\,\mathrm{d}\tau.\end{aligned}$$

This result helps us to enlarge the solution u of the problem (P1) over the interval $[0,T_{\max}]$. Therefore, lemma 3.2 is available to extend u to an interval that is larger than $[0,T_{\max}]$. This fact contradicts our definition of $T_{\max}$. □

3.2. Well-posedness results under the second assumption on the initial function

First of all, let us introduce the important nonlinear estimate for problem (P1). To achieve this aim, we need the following lemma about the Gamma function, that can be found in [**22**, lemma 3.3].

Lemma 3.4. *For any $p,q\geqslant 1$, there exists a positive constant $\bar{M}$ such that*

$$[\Gamma(pq+1)]^{\frac{1}{q}}\leqslant\bar{M}\Gamma(p+1)q^p.$$

Lemma 3.5. *Let $u,v\in L^{\Xi}(\mathbb{R}^N)$ and $\mathcal{V}=\max\{\|u\|_\Xi,\|v\|_\Xi\}$. Then, for any $h\geqslant p$, the following estimate holds*

$$\|G(u)-G(v)\|_{L^h}\leqslant\mathscr{C}_h\mathcal{V}^{m-1}\sum_{j=0}^{\infty}(3h\kappa\mathcal{V}^p)^j\|u-v\|_\Xi.$$

Proof. From (1.2), by using Taylor expansion, for any $u,v\in L^{\Xi}(\mathbb{R}^N)$, one has

$$\begin{aligned}\|G(u)-G(v)\|_{L^h}&\leqslant L\sum_{j=0}^{\infty}\frac{\kappa^j}{j!}\|u-v\|_{L^{3h}}\\&\quad\times\left(\|u\|_{L^{3hpj}}^{pj}\|u\|_{L^{3h(m-1)}}^{m-1}+\|v\|_{L^{3hpj}}^{pj}\|v\|_{L^{3h(m-1)}}^{m-1}\right),\end{aligned}\tag{3.9}$$

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where we have used the Hölder inequality with $\frac{1}{h} = \frac{1}{3h} + \frac{1}{3h} + \frac{1}{3h}$. By the embedding stated in lemma 2.7, the estimate (3.9) becomes

$$\|G(u) - G(v)\|_{L^h} \leqslant \frac{2L\mathcal{V}^{m-1}(\Gamma(3hp^{-1}+1))^{\frac{1}{3h}}}{(\Gamma(3h(m-1)p^{-1}+1))^{\frac{-1}{3h}}} \|u-v\|_{\Xi} \times \sum_{j=0}^{\infty} \left(\frac{\kappa^j}{j!} \mathcal{V}^{pj} \left[\Gamma(3hj+1)\right]^{\frac{1}{3h}} \right). \tag{3.10}$$

It follows from lemma 3.4 that

$$\left[\Gamma(3hj+1)\right]^{\frac{1}{3h}} \leqslant \bar{M}\Gamma\left(j+1\right)(3h)^j = \bar{M}(3h)^j j!.$$

Based on the above results, the following approximation is satisfied

$$\|G(u) - G(v)\|_{L^h} \leqslant \frac{2L\mathcal{V}^{m-1}(\Gamma(3hp^{-1}+1))^{\frac{1}{3h}}}{(\Gamma(3h(m-1)p^{-1}+1))^{\frac{-1}{3h}}} \|u-v\|_{\Xi} \sum_{j=0}^{\infty} (3h\kappa\mathcal{V}^p)^j. \tag{3.11}$$

□

3.2.1. Local-in-time solution

THEOREM 3.6. *Let u_0 be in $L^{\Xi}(\mathbb{R}^N)$ with sufficiently small data and $h > \max\{p, \frac{N}{4}\}$. Then there exists a locally unique mild solution to problem (4.3), belonging to an open ball centred at the origin with radius $\varepsilon < (3h\kappa)^{\frac{-1}{p}}$.*

Proof. Our proof starts with the observation that

$$\|\mathbb{Z}_{1,\alpha}(t)u_0\|_{\Xi} \leqslant \|u_0\|_{\Xi}, \quad t > 0, \tag{3.12}$$

where we have used lemma 2.9. In addition, for any $u, v \in L^{\infty}\left(0, T; L^{\Xi}(\mathbb{R}^N)\right),\ T > 0$, lemma 3.5 implies for a fixed constant $h > \max\{p, \frac{N}{4}\}$ and $t \in (0, T)$ that

$$\begin{aligned}
&\int_0^t (t-\tau)^{\alpha\left(1-\frac{N}{4h}\right)-1} \|G(u(\tau)) - G(v(\tau))\|_{L^h} \,\mathrm{d}\tau \\
&\leqslant \mathscr{C}_h \sum_{j=0}^{\infty} (3h\kappa)^j \int_0^t (t-\tau)^{\alpha\left(1-\frac{N}{4h}\right)-1} \|u(\tau) - v(\tau)\|_{\Xi} \left[\mathcal{V}(\tau)\right]^{m-1+pj} \mathrm{d}\tau \\
&\leqslant \frac{\mathscr{C}_h T^{\alpha\left(1-\frac{N}{4h}\right)}}{\alpha\left(1-\frac{N}{4h}\right)} \|u-v\|_{L_T^{\infty}\Xi} \|\mathcal{V}\|_{L_T^{\infty}\Xi}^{m-1} \sum_{j=0}^{\infty} (3h\kappa \|\mathcal{V}\|_{L_T^{\infty}\Xi}^{p})^j,
\end{aligned}$$

here we set $\mathcal{V}(t) = \max\{\|u(t)\|_{\Xi}, \|v(t)\|_{\Xi}\}$. Next, for purpose of using the fixed point principle, let us define the following mapping

$$\mathscr{G}u(t) = \mathbb{Z}_{1,\alpha}(t)u_0 + \int_0^t \mathbb{Z}_{2,\alpha}(t-\tau)G(u(\tau))\,\mathrm{d}\tau, \tag{3.13}$$

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which maps a closed ball $B_{L^\infty(0,T;L^\Xi(\mathbb{R}^N))}(0,\varepsilon)$ to itself, provided that $\varepsilon < (3h\kappa)^{\frac{-1}{p}}$ and T satisfies

$$\frac{\max\left\{\overline{\mathscr{C}_{0,\frac{h}{h-1}}}(\alpha,N)\mathscr{C}_h, \overline{\mathscr{C}_{0,1}}(\alpha,N)\mathscr{C}_p\right\}}{\alpha(\log 2)^{\frac{1}{p}}(1-3h\kappa\varepsilon^p)}\left[\left(1-\frac{N}{4h}\right)^{-1} T^{\alpha\left(1-\frac{N}{4h}\right)} + T^\alpha\right]\varepsilon^{m-1} \leqslant \frac{1}{2}.$$

We first prove the invariance of $B_{L^\infty(0,T;L^\Xi(\mathbb{R}^N))}(0,\varepsilon)$ under the action of $\mathscr{G}$. Indeed, the Young convolution inequality and lemma 2.3 imply

$$\begin{aligned}
&\left\|\int_{\mathbb{R}^N} \mathbb{K}_{2,\alpha}(t-\tau,x-y)\left[G(u(\tau,y)) - G(v(\tau,y))\right]\,\mathrm{d}y\right\|_{L^\infty} \\
&\quad\leqslant \|\mathbb{K}_{2,\alpha}(t-\tau,x)\|_{L^{\frac{h}{h-1}}} \|G(u(\tau)) - G(v(\tau))\|_{L^h} \\
&\quad\leqslant \overline{\mathscr{C}_{0,\frac{h}{h-1}}}(\alpha,N)(t-\tau)^{\alpha\left(1-\frac{N}{4h}\right)-1} \|G(u(\tau)) - G(v(\tau))\|_{L^h}. \qquad (3.14)
\end{aligned}$$

This result leads us to

$$\|\mathscr{G}u - \mathscr{G}v\|_{L^\infty_T L^\infty_x} \leqslant \overline{\mathscr{C}_{0,\frac{h}{h-1}}}(\alpha,N)\int_0^t (t-\tau)^{\alpha\left(1-\frac{N}{4h}\right)-1} \|G(u(\tau)) - G(v(\tau))\|_{L^h}\,\mathrm{d}\tau.$$

In (3.14) and the above inequality, if we take $v \equiv 0$, then from the property of geometric series, the following holds

$$\begin{aligned}
&\|\mathscr{G}u - \mathbb{Z}_{1,\alpha}(\cdot)u_0\|_{L^\infty_T L^\infty_x} \\
&\quad\leqslant \overline{\mathscr{C}_{0,\frac{h}{h-1}}}(\alpha,N)\mathscr{C}_h\left[\alpha\left(1-\frac{N}{4h}\right)\right]^{-1} T^{\alpha\left(1-\frac{N}{4h}\right)}\varepsilon^m\left[\sum_{j=0}^\infty (3h\kappa\varepsilon^p)^j\right] \\
&\quad= \overline{\mathscr{C}_{0,\frac{h}{h-1}}}(\alpha,N)\mathscr{C}_h\left[\alpha\left(1-\frac{N}{4h}\right)\right]^{-1} T^{\alpha\left(1-\frac{N}{4h}\right)}\varepsilon^m(1-3h\kappa\varepsilon^p)^{-1}.
\end{aligned}$$

Similarly, we also obtain the following estimate for the $L^p(\mathbb{R}^N)$ norm

$$\begin{aligned}
\|\mathscr{G}u - \mathbb{Z}_{1,\alpha}u_0\|_{L^\infty_T L^p_x} &\leqslant \overline{\mathscr{C}_{0,1}}(\alpha,N)\mathscr{C}_p\alpha^{-1}T^\alpha\varepsilon^m\left[\sum_{j=0}^\infty (3h\kappa\varepsilon^p)^j\right] \\
&= \overline{\mathscr{C}_{0,1}}(\alpha,N)\mathscr{C}_p\alpha^{-1}T^\alpha\varepsilon^m(1-3p\kappa\varepsilon^p)^{-1}.
\end{aligned}$$

It follows from the above results that

$$\begin{aligned}
\|\mathscr{G}u\|_{L^\infty_T\Xi} &\leqslant \|\mathbb{Z}_{1,\alpha}u_0\|_{L^\infty_T\Xi} + (\log 2)^{\frac{-1}{p}} \\
&\qquad\times\left(\|\mathscr{G}u - \mathbb{Z}_{1,\alpha}u_0\|_{L^\infty_T L^\infty_x} + \|\mathscr{G}u - \mathbb{Z}_{1,\alpha}u_0\|_{L^\infty_T L^p_x}\right) \\
&\leqslant \|u_0\|_\Xi + \frac{\max\left\{\overline{\mathscr{C}_{0,\frac{h}{h-1}}}(\alpha,N)\mathscr{C}_h, \overline{\mathscr{C}_{0,1}}(\alpha,N)\mathscr{C}_p\right\}}{\alpha(\log 2)^{\frac{1}{p}}(1-3h\kappa\varepsilon^p)} \\
&\qquad\times\left[\left(1-\frac{N}{4h}\right)^{-1} T^{\alpha\left(1-\frac{N}{4h}\right)} + T^\alpha\right]\varepsilon^m \leqslant \varepsilon,
\end{aligned}$$

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where we have used the smallness assumption for the initial data function that $2\|u_0\|_{\Xi} \leqslant \varepsilon < (3h\kappa)^{\frac{-1}{p}}$. Hence, the invariance property of $\mathscr{G}$ is ensured. Furthermore, by using analogous arguments and the chosen time T, we can easily show that $\mathscr{G}$ is a strict contraction on $B_{L^\infty(0,T;L^\Xi(\mathbb{R}^N))}(0,\varepsilon)$. Then, the Banach principle argument yields that our problem has a unique local-in-time mild solution in $L^\infty\left(0,T;L^\Xi(\mathbb{R}^N)\right)$ and the proof is complete. □

3.2.2. Global-in-time well-posedness results

LEMMA 3.7 (see [**14**, lemma 8)]. *Let $m, n > -1$ such that $m+n > -1$. Then*

$$\sup_{t\in[0,T]} t^h \int_0^1 s^m (1-s)^n \, e^{-\mu t(1-s)} \mathrm{d}s \xrightarrow{\mu\longrightarrow\infty} 0.$$

DEFINITION 3.8. Let X be a Banach space. Then, we denote by $L^\infty_{a,b}\left((0,T];X\right)$ the subspace of $L^\infty(0,T;X)$ such that

$$\sup_{t\in(0,T]} t^a \, \mathrm{e}^{-bt} \|\varphi(t)\|_X < \infty, \quad \varphi \in L^\infty_{a,b}\left((0,T];X\right).$$

for some positive numbers a, b.

THEOREM 3.9. *Assume that the initial function u_0 belongs to $L^\Xi_0(\mathbb{R}^N)$ and $m = 1$. Then, problem (P1) has a unique solution in $C\left((0,T];L^q(\mathbb{R}^N)\right) \cap L^\infty_{a,b}\left((0,T];L^\Xi_0(\mathbb{R}^N)\right)$ for $a \in (0,1)$ and some $b_0 > 0$. Furthermore, if $N < 4p, a < \min\{\frac{1}{2}, \frac{\alpha N}{4p}\}$, we have*

$$\|u(t)\|_{L^p} \leqslant Ct^{-a} \, e^{b_0 t} \|u_0\|_{\Xi}. \tag{3.15}$$

Proof. The proof starts by handling the nonlinear source term on the RHS of the first equation of problem (P1). For every $u \in L^\infty_{a,b}\left((0,T];L^\Xi_0(\mathbb{R}^N)\right)$, we can choose two functions u_1, u_2 such that

$$\begin{cases} u(t) = u_1(t) + u_2(t) & \forall t \in (0,T], \\ u_1(t) \in C_0^\infty(\mathbb{R}^N), & \forall t \in (0,T], \\ \|u_2(t)\|_{\Xi} < (3h\kappa 2^{p-1})^{\frac{-1}{p}}, & \forall t \in (0,T]. \end{cases} \tag{3.16}$$

For brevity, we set

$$\mathscr{C}_T := \sup_{u\in L^\infty_{a,b}\left((0,T];L^\Xi_0(\mathbb{R}^N)\right)} \left\{ \sup_{t\in(0,T]} \left\{ |u_1(t)| : u_1(t) \text{ satisfies } (3.16) \right\} \right\}.$$

Then, for any $h \in [p,\infty)$,

$$\|G(u(t)) - G(v(t))\|_{L^h} \leqslant L \left\| |u(t)-v(t)| \left(\mathrm{e}^{\kappa|u_1(t)+u_2(t)|^p} + \mathrm{e}^{\kappa|v_1(t)+v_2(t)|^p} \right) \right\|_{L^h}.$$

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Using the inequality $(a+b)^q < 2^{p-1}(a^q+b^q)$ for $a,b>0, q>1$, we obtain

$$\begin{aligned}
&\|G(u(t))-G(v(t))\|_{L^h} \\
&\quad \leqslant L \left\| |u(t)-v(t)| \left(\mathrm{e}^{\kappa 2^{p-1}|u_1^p(t)+u_2^p(t)|} + \mathrm{e}^{\kappa 2^{p-1}|v_1^p(t)+v_2^p(t)|} \right) \right\|_{L^h} \\
&\quad \leqslant L\, \mathrm{e}^{\kappa 2^{p-1}\mathscr{C}_T^p} \left\| |u(t)-v(t)| \left(\mathrm{e}^{\kappa 2^{p-1}|u_2^p(t)|} + \mathrm{e}^{\kappa 2^{p-1}|v_2^p(t)|} \right) \right\|_{L^h}.
\end{aligned}$$

Then, it follows from lemmas 3.5 and 2.7 that

$$\begin{aligned}
&\|G(u(t))-G(v(t))\|_{L^h} \\
&\quad \leqslant C\,\|u(t)-v(t)\|_{\Xi} \sum_{j=0}^{\infty} \left(3h\kappa 2^{p-1} \left[\max_{w\in\{u_2,v_2\}} \left\{ \sup_{t\in(0,T]} \|w(t)\|_{\Xi} \right\} \right]^p \right)^j. \qquad (3.17)
\end{aligned}$$

Then, by (3.16), the following nonlinear estimate holds

$$\|G(u(t))-G(v(t))\|_{L^h} \leqslant C\,\|u(t)-v(t)\|_{\Xi}. \qquad (3.18)$$

Now, we define the following set

$$\mathbf{O} := \left\{ u \in C\left((0,T];L^p(\mathbb{R}^N)\right) \cap L^\infty_{a,b}\left((0,T];L_0^{\Xi}(\mathbb{R}^N)\right) \,\Big|\, \sup_{t\in(0,T]} t^a\,\mathrm{e}^{-bt}\,\|u(t)\|_{\Xi} < \infty \right\}$$

and a mapping $\mathscr{H}$ that maps $\mathbf{O}$ to itself, formulated by

$$\mathscr{H}u(t) = \mathbb{Z}_{1,\alpha}(t)u_0 + \int_0^t \mathbb{Z}_{2,\alpha}(t-\tau)G(u(\tau))\,\mathrm{d}\tau. \qquad (3.19)$$

Since $u_0 \in L_0^{\Xi}(\mathbb{R}^N)$, lemma 2.1 ensures the continuity on $(0,T]$ of the first term on the RHS of (3.19). Furthermore, combining (3.18) and the same argument as in theorem 3.1, we have

$$\mathscr{H}u - \mathbb{Z}_{1,\alpha}u_0 \in C\left((0,T];L^p(\mathbb{R}^N)\right).$$

In addition, for any $u\in\mathbf{O}$, $\mathscr{H}u$ is completely bounded in time. Notice also that $u \in L^\infty_{a,b}\left((0,T];L_0^{\Xi}(\mathbb{R}^N)\right)$. Then, the Young convolution inequality and lemma 2.3 imply that

$$\begin{aligned}
&\|\mathscr{H}u(t)-\mathbb{Z}_{1,\alpha}(t)u_0\|_{L^\infty} \\
&\quad \leqslant \int_0^t \|\mathbb{Z}_{2,\alpha}(t-\tau)G(u(\tau))\|_{L^\infty}\,\mathrm{d}\tau \\
&\quad \leqslant \int_0^t (t-\tau)^{\alpha\left(1-\frac{N}{4h}\right)-1}\|G(u(\tau))\|_{L^h}\,\mathrm{d}\tau, \quad \text{for } h > \max\left\{\frac{N}{4}, p\right\}.
\end{aligned}$$

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By using (3.18), we obtain

$$\begin{aligned}\|\mathscr{H}u(t)-\mathbb{Z}_{1,\alpha}(t)u_0\|_{L^\infty} &\leqslant C\sup_{t\in(0,T]}\|u(t)\|_\Xi\left(\int_0^t(t-\tau)^{\alpha\left(1-\frac{N}{4h}\right)-1}\,\mathrm{d}\tau\right)\\ &=\frac{CT^{\alpha\left(1-\frac{N}{4h}\right)}}{\alpha\left(1-\frac{N}{4h}\right)}\sup_{t\in(0,T]}\|u(t)\|_\Xi\,.\end{aligned}$$

Likewise, we also have

$$\begin{aligned}\|\mathscr{H}u(t)-\mathbb{Z}_{1,\alpha}(t)u_0\|_{L^p} &\leqslant C\sup_{t\in(0,T]}\|u(t)\|_\Xi\left(\int_0^t(t-\tau)^{\alpha-1}\,\mathrm{d}\tau\right)\\ &=CT^\alpha\alpha^{-1}\sup_{t\in(0,T]}\|u(t)\|_\Xi\,.\end{aligned}$$

From two results above, lemmas 2.8 and 2.9, the conclusion $\mathscr{H}u\in L^{\infty,*}\left((0,T];L_0^\Xi(\mathbb{R}^N)\right)$ can be drawn through the following one

$$\begin{aligned}\|\mathscr{H}u(t)\|_\Xi &\leqslant \|\mathbb{Z}_{1,\alpha}(t)u_0\|_\Xi+\|\mathscr{H}u(t)-\mathbb{Z}_{1,\alpha}(t)u_0\|_\Xi\\ &\leqslant \|u_0\|_\Xi+C\left(T^{\alpha\left(1-\frac{N}{4h}\right)}+T^\alpha\right)\sup_{t\in(0,T]}\|u(t)\|_\Xi\,.\end{aligned}$$

We note that $\mathbf{O}$ is a complete metric space with the metric

$$d_{\mathbf{O}}(u,v):=\sup_{t\in(0,T]}t^a\,\mathrm{e}^{-bt}\,\|u(t)-v(t)\|_\Xi\,.$$

The remaining part of this proof is to show that $\mathscr{H}$ is a strict contraction on $\mathbf{O}$ with respect to the above metric. Indeed, for any $u,v\in\mathbf{O}$, we have

$$\begin{aligned}&t^a\,\mathrm{e}^{-bt}\,\|\mathscr{H}u(t)-\mathscr{H}v(t)\|_{L^p+L^\infty}\\ &\quad\leqslant\int_0^t(t-\tau)^{\alpha-1}\,\|\mathbb{Z}_{2,\alpha}(t-\tau)\left[G(u(\tau))-G(v(\tau))\right]\|_{L^p+L^\infty}\,\mathrm{d}\tau\\ &\quad\leqslant t^a\,\mathrm{e}^{-bt}\int_0^t(t-\tau)^{\alpha-1}\,\|G(u(\tau))-G(v(\tau))\|_{L^p+L^\infty}\,\mathrm{d}\tau\\ &\quad\leqslant Cd_{\mathbf{O}}(u,v)t^a\int_0^t\tau^{-a}\left[(t-\tau)^{\alpha-1}+(t-\tau)^{\alpha\left(1-\frac{N}{4h}\right)-1}\right]\mathrm{e}^{-b(t-\tau)}\,\mathrm{d}\tau.\end{aligned}\tag{3.20}$$

Using the substitution technique, the latter inequality becomes

$$\begin{aligned}&t^a\,\mathrm{e}^{-bt}\,\|\mathscr{H}u(t)-\mathscr{H}v(t)\|_{L^p+L^\infty}\\ &\quad\leqslant Cd_{\mathbf{O}}(u,v)t^\alpha\int_0^1\tau^{-a}(1-\tau)^{\alpha-1}\,\mathrm{e}^{-bt(1-\tau)}\,\mathrm{d}\tau\\ &\qquad+Cd_{\mathbf{O}}(u,v)t^{\alpha\left(1-\frac{N}{4h}\right)}\int_0^1\tau^{-a}(1-\tau)^{\alpha\left(1-\frac{N}{4h}\right)-1}\,\mathrm{e}^{-bt(1-\tau)}\,\mathrm{d}\tau.\end{aligned}$$

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Choosing $0 < a < \alpha < 1$ and $4h > N$, we use lemma 3.7 to obtain

$$\begin{cases} \lim\limits_{b\to\infty} \sup\limits_{t\in(0,T]} t^{\alpha} \int_0^1 \tau^{-a}(1-\tau)^{\alpha-1} \, \mathrm{e}^{-bt(1-r)} \, \mathrm{d}\tau = 0, \\ \lim\limits_{b\to\infty} \sup\limits_{t\in(0,T]} t^{\alpha\left(1-\frac{N}{4h}\right)} \int_0^1 \tau^{-a}(1-\tau)^{\alpha\left(1-\frac{N}{4h}\right)-1} \, \mathrm{e}^{-bt(1-r)} \, \mathrm{d}\tau = 0. \end{cases}$$

Hence, there exists a large number b_0 such that the following holds

$$t^a \, \mathrm{e}^{-b_0 t} \left\| \mathscr{H}u(t) - \mathscr{H}v(t) \right\|_{L^p + L^\infty} \leqslant \mathcal{L}(\log 2)^{-p} \, \mathrm{d}_{\mathbf{O}}(u,v),$$

where $\mathcal{L}$ is a positive constant less than 1. This implies that

$$d_{\mathbf{O}}(\mathscr{H}u, \mathscr{H}v) \leqslant \mathcal{L} \, \mathrm{d}_{\mathbf{O}}(u,v).$$

Therefore, $\mathscr{H}$ has a unique fixed point in $\mathbf{O}$. Therefore, we can conclude that there exists a unique solution to problem (P1).

Next, let us verify the correctness of the statement (3.15) as $N < 4p$. Firstly, if $t \leqslant 1$, we have

$$t^a \, \mathrm{e}^{-b_0 t} \left\| \mathbb{Z}_{1,\alpha}(t) u_0 \right\|_{L^p} \leqslant C \left\| \mathbb{Z}_{1,\alpha}(t) u_0 \right\|_{L^p} \leqslant C \left\| u_0 \right\|_{L^p} \leqslant C \left\| u_0 \right\|_{\Xi}.$$

In addition, if $t > 1$, by using the Cauchy inequality and lemma 2.9, we obtain

$$\begin{aligned} t^a \, \mathrm{e}^{-b_0 t} \left\| \mathbb{Z}_{1,\alpha}(t) u_0 \right\|_{L^p} &\leqslant C \left\| \mathbb{Z}_{1,\alpha}(t) u_0 \right\|_{\Xi} \\ &\leqslant C t^{a - \frac{\alpha N}{4p}} \left[\log\left(2 t^{\frac{-\alpha N}{8}} \right) \right]^{\frac{-1}{p}} \left\| u_0 \right\|_{L^p} \leqslant C \left\| u_0 \right\|_{\Xi}, \end{aligned}$$

where we have used the assumption that $a \leqslant \frac{\alpha N}{4p}$. On the other hand, if $u \in L^\infty\left((0,T]; L_0^{\Xi}(\mathbb{R}^N) \right)$, we can use (3.18) to derive the following estimate

$$\left\| G(u(t)) \right\|_{L^p} \leqslant C \left\| u(t) \right\|_{\Xi}.$$

It follows immediately that

$$\begin{aligned} & t^a \, \mathrm{e}^{-b_0 t} \int_0^t (t-\tau)^{\alpha-1} \left\| \mathbb{Z}_{2,\alpha}(t)(t-\tau) G(u(\tau)) \right\|_{L^p} \, \mathrm{d}\tau \\ & \quad \leqslant C t^a \, \mathrm{e}^{-b_0 t} \int_0^t (t-\tau)^{\alpha-1} \left\| \mathbb{Z}_{2,\alpha}(t-\tau) u(\tau) \right\|_{\Xi} \, \mathrm{d}\tau. \end{aligned}$$

Thanks to the assumption that $N < 4p$, lemma 2.9 implies

$$\begin{aligned} t^a \, \mathrm{e}^{-b_0 t} \left\| u(t) - \mathbb{Z}_{1,\alpha}(t) u_0 \right\|_{L^p} \leqslant C t^a \, \mathrm{e}^{-b_0 t} & \int_0^t (t-\tau)^{\alpha\left(1-\frac{N}{4p}\right)-1} \\ & \times \left[\log\left(1 + (t-\tau)^{\frac{-\alpha N}{4}} \right) \right]^{\frac{-1}{p}} \left\| u(\tau) \right\|_{L^p} \, \mathrm{d}\tau. \end{aligned}$$

Thanks to the Hölder inequality,

$$\begin{aligned} & t^{a}\,\mathrm{e}^{-b_0 t}\left\|u(t)-\mathbb{Z}_{1,\alpha}(t)u_0\right\|_{L^p} \\ & \leqslant C\left(t^{2a}\int_0^t (t-\tau)^{\alpha\left(1-\frac{N}{4p}\right)-1}\left[\log\left(1+(t-\tau)^{\frac{-\alpha N}{4}}\right)\right]^{\frac{-2}{p}}\tau^{-2a}\,\mathrm{e}^{-2b_0(t-\tau)}\,\mathrm{d}\tau\right)^{\frac{1}{2}} \\ & \quad\times\left(\int_0^t (t-\tau)^{\alpha\left(1-\frac{N}{4p}\right)-1}\left[\tau^{a}\,\mathrm{e}^{-b_0\tau}\left\|u(\tau)\right\|_{L^p}\right]^2\,\mathrm{d}\tau\right)^{\frac{1}{2}} \end{aligned}$$

To deal with the log term, let us denote by γ the infimum of the set $\{z>0 : z > 2\log(1+z)\}$. If $t^{\frac{-\alpha N}{4}} > \gamma$, the results will be covered by the opposite case, for this reason, we only need to consider the case $t^{\frac{-\alpha N}{4}} > \gamma$. As $t^{\frac{-\alpha N}{4}} > \gamma$, thanks to lemma 3.7, the two following claims will be obtained.

Claim 1. If $\tau \leqslant t-\gamma^{\frac{-4}{\alpha N}}$, we have $(t-\tau)^{\frac{\alpha N}{4}} < \gamma$. This one implies that

$$\log\left(1+(t-\tau)^{\frac{-\alpha N}{4}}\right) > \frac{(t-\tau)^{\frac{-\alpha N}{4}}}{2}, \quad \text{for } 0<\tau\leqslant t-\gamma^{\frac{-4}{\alpha N}}.$$

Based on the above inequality, one can derive that

$$\begin{aligned} & t^{2a}\int_0^{t-\gamma^{\frac{-4}{\alpha N}}} (t-\tau)^{\alpha\left(1-\frac{N}{4p}\right)-1}\left[\log\left(1+(t-\tau)^{\frac{-\alpha N}{4}}\right)\right]^{\frac{-2}{p}}\tau^{-2a}\,\mathrm{e}^{-2b_0(t-\tau)}\,\mathrm{d}\tau \\ & \quad\leqslant Ct^{2a}\int_0^{t-\gamma^{\frac{-4}{\alpha N}}} (t-\tau)^{\alpha\left(1+\frac{N}{4p}\right)-1}\tau^{-2a}\,\mathrm{e}^{-2b_0(t-\tau)}\,\mathrm{d}\tau \\ & \quad\leqslant Ct^{\alpha\left(1+\frac{N}{4p}\right)}\int_0^1 (1-r)^{\alpha\left(1+\frac{N}{4p}\right)-1}\tau^{-2a}\,\mathrm{e}^{-2b_0t(1-\tau)}\,\mathrm{d}\tau \leqslant C. \end{aligned} \tag{3.21}$$

Claim 2. On the contrary, if $t-r<\gamma^{\frac{-4}{\alpha N}}$, we have

$$\begin{aligned} & t^{2a}\int_{t-\gamma^{\frac{-4}{\alpha N}}}^{t} (t-\tau)^{\alpha\left(1-\frac{N}{4p}\right)-1}\left[\log\left(1+(t-\tau)^{\frac{-\alpha N}{4}}\right)\right]^{\frac{-2}{p}}\tau^{-2a}\,\mathrm{e}^{-2b_0(t-\tau)}\,\mathrm{d}r \\ & \quad\leqslant Ct^{2a}[\log(1+\gamma)]^{\frac{-2}{p}}\int_{t-\gamma^{\frac{-4}{\alpha N}}}^{t} (t-\tau)^{\alpha\left(1-\frac{N}{4p}\right)-1}\tau^{-2a}\,\mathrm{e}^{-2b_0(t-\tau)}\,\mathrm{d}\tau \\ & \quad\leqslant t^{\alpha\left(1-\frac{N}{4p}\right)}\int_0^1 (1-r)^{\alpha\left(1-\frac{N}{4p}\right)-1}\tau^{-2a}\,\mathrm{e}^{-2b_0t(1-\tau)}\,\mathrm{d}\tau \leqslant C. \end{aligned}$$

Combining the above results, the triangle inequality and the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$ we deduce

$$t^{2a}\,\mathrm{e}^{-2b_0 t}\left\|u(t)\right\|_{L^p}^2 \leqslant C\left\|u_0\right\|_{\Xi}^2 + C\int_0^t (t-\tau)^{\alpha\left(1-\frac{N}{4p}\right)-1}\left[\tau^{a}\,\mathrm{e}^{-b_0\tau}\left\|u(\tau)\right\|_{L^p}\right]^2\,\mathrm{d}\tau.$$

Now, we are in the position to apply the Grönwall inequality to achieve the desired result

$$\left\|u(t)\right\|_{L^p} \leqslant Ct^{-a}\,\mathrm{e}^{b_0 t}\left\|u_0\right\|_{\Xi}. \qquad \square$$

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4. Time-fractional Cahn–Hilliard on the unbounded domain $\mathbb{R}^N$

In this section, we consider the following time-fractional Cahn–Hilliard on $\mathbb{R}^N$

$$\begin{cases} \partial^\alpha_{0|t} u(t,x) + \Delta^2 u(t,x) - \Delta F(t,x,u) = 0, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \\ u(0,x) = u_0(x), & \text{in} \quad \mathbb{R}^N. \end{cases} \tag{P2}$$

Lemma 4.1 (see [**35**]). *If* $1 \leqslant b \leqslant p \leqslant d$ *and* $v \in L^b(\mathbb{R}^N) \cap L^d(\mathbb{R}^N)$, *then* $v \in L^p(\mathbb{R}^N)$ *where*

$$\|u\|_{L^p} \leqslant \|u\|^\alpha_{L^b} \|u\|^{1-\alpha}_{L^d}, \quad \frac{1}{p} = \frac{\alpha}{b} + \frac{1-\alpha}{d} \tag{4.1}$$

We first consider the following linear problem

$$\begin{cases} D^\alpha_t \widehat{u}(t,\xi) + |\xi|^4 \widehat{u}(t,\xi) = \widehat{\Delta F}(t,\xi), & \text{in} \quad (0,T] \times \mathbb{R}^N, \\ \widehat{u}(0,\xi) = \widehat{u}_0(\xi), & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{4.2}$$

As in the previous section, by the Duhamel principle, the solution to problem (4.2) is given by

$$\widehat{u}(t,\xi) = E_{\alpha,1}(-t^\alpha |\xi|^4) \widehat{u}_0(\xi) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-\tau)^\alpha |\xi|^4) \widehat{\Delta F}(\tau,\xi) \, \mathrm{d}\xi \tag{4.3}$$

To deal with the source term in the form of ΔH, we have to do some revisions to the generalized formula of solution. Applying the fact that $\frac{\mathrm{d}}{\mathrm{d}x}(f(x) * g(x)) = (\frac{\mathrm{d}}{\mathrm{d}x} f(x)) * g(x)$, we immediately have that

$$\begin{aligned} \Delta \mathbb{Z}_{i,\alpha}(t) v(x) &= \Delta \Big(\mathbb{K}_{i,\alpha}(t,x) * v(t,x) \Big) = \Big(\Delta v(t,x) \Big) * \mathbb{K}_{i,\alpha}(t,x) \\ &= \int_{\mathbb{R}^N} \Delta \mathbb{K}_{i,\alpha}(t,x-y) v(t,y) \, \mathrm{d}y, \quad i = 1,2. \end{aligned}$$

Next, we show that u satisfies the following equality

$$u(t,x) = \underbrace{\mathbb{Z}_{1,\alpha}(t) u_0(x)}_{\mathscr{I}_1(t,x)} + \underbrace{\int_0^t \Delta \mathbb{Z}_{2,\alpha}(t-s) F(u(s,x)) \, \mathrm{d}s}_{\mathscr{I}_2(t,x)}.$$

Using (2.2), we infer that the Fourier transform of the first quantity $\mathscr{I}_1(t,x)$ is given by

$$\widehat{\mathscr{I}_1}(t,\xi) = \widehat{\mathbb{K}_{1,\alpha} * u_0} = \widehat{\mathbb{K}_{1,\alpha}} \widehat{u_0} = E_{\alpha,1}(-t|\xi|^4) \widehat{u}_0(\xi).$$

It is not difficult to verify that the Fourier transformation of the given second quantity $\mathscr{I}_2(t,x)$ is as follows:

$$\begin{aligned}\widehat{\mathscr{I}_2}(t,\xi) &= \int_{\mathbb{R}^N} \mathrm{e}^{-i<\xi,x>} \int_0^t \Delta\mathbb{Z}_{2,\alpha}(t-\tau)F(u(\tau,x))\,\mathrm{d}\tau\,\mathrm{d}x \\ &= \int_0^t \Big(\int_{\mathbb{R}^N} \mathrm{e}^{-i<\xi,x>}\mathbb{K}_{2,\alpha}(t-\tau)\Delta F(u(\tau,x))\,\mathrm{d}x \Big)\,\mathrm{d}\tau \\ &= \int_0^t \mathcal{F}\Big(\Big(\Delta F(u(\tau,x))\Big) * \mathbb{K}_{2,\alpha}(t-\tau,x)\Big)\,\mathrm{d}\tau \\ &= \int_0^t \widehat{\mathbb{K}_{2,\alpha}}(t-\tau,\xi)\widehat{\Delta F}(u(\tau,x))\,\mathrm{d}\tau\end{aligned}$$

where we have used the formula $\mathcal{F}(f*g) = \widehat{f}\widehat{g}$, and from the following fact

$$\widehat{\mathbb{K}_{2,\alpha}}(t-\tau,\xi) = (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-(t-\tau)^{\alpha}|\xi|^4\big),$$

we arrive at the following equality

$$\widehat{\mathscr{I}_2}(t,\xi) = \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^{\alpha}|\xi|^4)\widehat{\Delta F}(u(s,\xi))\,\mathrm{d}\xi.$$

We establish the local well-posedness of solutions for problem (P2) with small initial data in $\mathbb{R}^N$ by using Kato's method (see [**29**]). More precisely, our main result in this section can be stated as follows.

THEOREM 4.2. *Let $\alpha > \frac{1}{2}$ and $\mathcal{E} > 0$ be a sufficientl y small constant. Assume that*

$$\|u_0\|_{L^\infty} \leqslant \mathcal{E} \quad \textit{and} \quad \max_{|z|\leqslant\mathcal{E}} \sum_{k=1}^{L} \Big|D^kF(z)\Big| = \mathcal{A}.$$

Then, problem (P2) has a unique solution $u(t,x)$ on the strip

$$\mathscr{P}_{T_0} = \Big\{(t,x) : 0 < t \leqslant T_0,\ x \in \mathbb{R}^N\Big\}$$

such that

$$\|u(t)\|_{L^\infty} \leqslant 2\mathcal{E}, \quad 0 \leqslant t \leqslant T_0. \tag{4.4}$$

Here T_0 is given by

$$\begin{aligned}T_0 \leqslant \min\Bigg\{1, \Bigg(\frac{\alpha\Gamma(\alpha/2+1)}{4\Big(\int_{\mathbb{R}^N}\Big|\overline{\mathscr{B}}_2(z)\Big|dz\Big)\mathcal{A}\Gamma\left(\frac{3}{2}\right)\Gamma(\alpha/2)}\Bigg)^{\frac{2}{\alpha}}, \\ \times\Bigg(\frac{\alpha\Gamma(\alpha/2+1)}{4\Big(\int_{\mathbb{R}^N}\Big|\overline{\mathscr{B}}_2(z)\Big|dz\Big)\mathcal{A}\Gamma\left(\frac{3}{2}\right)}\Bigg)^{\frac{2}{\alpha}}\Bigg\}.\end{aligned} \tag{4.5}$$

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In addition, for each partition $0 < t_1 < t_2 < \cdots < t_N < t \leqslant T_0$, it holds the following estimate

$$\|D^k u(t)\|_{L^\infty} \leqslant (t-t_k)^{-\frac{\alpha k}{4}} \mathcal{Q}_k(\mathcal{E},\alpha,t_k-t_1,t-t_k), \quad t_k < t \leqslant T_0,$$

where $\mathcal{Q}_k$ is a continuous increasing function of $t-t_k$.

Proof. Let us consider the following integral

$$u(t,x) = \int_{\mathbb{R}^N} \mathbb{K}_{1,\alpha}(t,x-y)u_0(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}^N} \mathbb{K}_{2,\alpha}(t-\tau,x-y)F(u(\tau,y))\,\mathrm{d}\tau. \tag{4.6}$$

First, we show that for T_0 defined by (4.5), the integro-differential equation (4.6) admits a unique continuous solution $u(t,x)$ on the strip $\mathscr{P}_{T_0}$. We apply the successive approximations method given in [**29**]. Let us consider the following sequence

$$\begin{aligned} v_0(t,x) &= u_0(x),\\ v_n(t,x) &= \int_{\mathbb{R}^N} \mathbb{K}_{1,\alpha}(t,x-y)u_0(y)\,\mathrm{d}y\\ &\quad + \int_0^t \int_{\mathbb{R}^N} \Delta\mathbb{K}_{2,\alpha}(t-\tau,x-y)F(v_{n-1}(\tau,y))\,\mathrm{d}y\,\mathrm{d}\tau \qquad (4.7)\\ &= \mathbb{Z}_{1,\alpha}(t)u_0(x) + \int_0^t \Delta\mathbb{Z}_{2,\alpha}(t-\tau)F(v_{n-1}(\tau,x))\,\mathrm{d}\tau. \qquad (4.8)\end{aligned}$$

It is easy to see that v_n is well defined on $[0,\infty)\times\mathbb{R}^N$. For $n=0$, we have immediately that $\|v_0\|_{L^\infty} = \|u_0\|_{L^\infty} \leqslant \mathcal{E}$. Thus, we apply inequality (2.9) for $p=1$ to derive

$$\left\|\mathbb{K}_{1,\alpha}(t)\right\|_{L^1(\mathbb{R}^N)} \leqslant \frac{\left(\int_{\mathbb{R}^N}\left|\overline{\mathscr{B}}_0(z)\right|\mathrm{d}z\right)\Gamma(1)}{\Gamma(1)} \leqslant 1, \tag{4.9}$$

where we have used the fact that $\int_{\mathbb{R}^N}\left|\overline{\mathscr{B}}_0(z)\right|\mathrm{d}z = \int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N} \mathrm{e}^{iz\vartheta}\,\mathrm{e}^{-|\vartheta|^4}\mathrm{d}\vartheta\right|\mathrm{d}z = 1$. Now, we apply induction method to show the following inequality

$$|||v_j|||_{\mathscr{P}_{T_0}} = \sup_{(t,x)\in\mathscr{P}_{T_0}} |v_j(t,x)| \leqslant 2\mathcal{E}, \quad j \geqslant 1. \tag{4.10}$$

For $n=1$, we recall the result based on the Young convolution inequality and (4.9) as follows

$$\begin{aligned}\left\|\int_{\mathbb{R}^N} \mathbb{K}_{1,\alpha}(t,\cdot-y)u_0(y)\,\mathrm{d}y\right\|_{L^\infty} &= \left\|\mathbb{K}_{1,\alpha}(t)*u_0\right\|_{L^\infty}\\ &\leqslant \|\mathbb{K}_{1,\alpha}(t)\|_{L^1(\mathbb{R}^N)}\|u_0\|_{L^\infty} \leqslant \|u_0\|_{L^\infty} \leqslant \mathcal{E}. \qquad (4.11)\end{aligned}$$

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Let us continue to verify that

$$\begin{aligned}
&\left\| \int_0^t \Delta\mathbb{Z}_{2,\alpha}(t-\tau)F(v_0(\tau,x))\,\mathrm{d}\tau \right\|_{L^\infty} \\
&\quad = \left\| \int_0^t \int_{\mathbb{R}^N} \Delta\mathbb{K}_{2,\alpha}(t-\tau,x-y)F(v_0(\tau,y))\mathrm{d}y\,\mathrm{d}\tau \right\|_{L^\infty} \\
&\quad \leqslant \int_0^t \left\| \int_{\mathbb{R}^N} \Delta\mathbb{K}_{2,\alpha}(t-\tau,x-y)F(v_0(\tau,y))\mathrm{d}y \right\|_{L^\infty} \mathrm{d}\tau \\
&\quad \leqslant \int_0^t \|\Delta\mathbb{K}_{2,\alpha}(t-\tau,x)\|_{L^1(\mathbb{R}^N)} \|F(v_0(\tau,x))\|_{L^\infty}\,\mathrm{d}\tau.
\end{aligned}$$

By taking $p=1$ and $k=2$ into inequality (2.7), we obtain the following bound immediately

$$\left\|\Delta\mathbb{K}_{2,\alpha}(t,x)\right\|_{L^1(\mathbb{R}^N)} = \left\|D^2\mathbb{K}_{2,\alpha}(t,x)\right\|_{L^1(\mathbb{R}^N)} \leqslant \frac{\left(\int_{\mathbb{R}^N}\left|\overline{\mathscr{B}}_2(z)\right|\mathrm{d}z\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)} t^{\frac{\alpha}{2}-1}, \tag{4.12}$$

which combined with the condition

$$\|F(v_0(s,x))\|_{L^\infty} \leqslant \mathcal{A} \sup_{(t,x)\in\mathscr{S}_0} |v_0(t,x)|, \tag{4.13}$$

imply

$$\begin{aligned}
&\left\| \int_0^t \Delta\mathbb{Z}_{2,\alpha}(t-s)F(v_0(s,x))\,\mathrm{d}s \right\|_{L^\infty} \\
&\quad \leqslant \frac{\left(\int_{\mathbb{R}^N}\left|\overline{\mathscr{B}}_2(z)\right|\mathrm{d}z\right)\mathcal{A}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)} \sup_{(t,x)\in\mathscr{S}_0} |v_0(t,x)| \int_0^t (t-s)^{\alpha/2-1}\,\mathrm{d}s \\
&\quad \leqslant \frac{\left(\int_{\mathbb{R}^N}\left|\overline{\mathscr{B}}_2(z)\right|\mathrm{d}z\right)\mathcal{A}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)} \frac{2t^{\alpha/2}}{\alpha} 2\mathcal{E} \\
&\quad \leqslant \frac{4\left(\int_{\mathbb{R}^N}\left|\overline{\mathscr{B}}_2(z)\right|\mathrm{d}z\right)\mathcal{A}\Gamma\left(\frac{3}{2}\right)|T_0|^{\alpha/2}\mathcal{E}}{\alpha\Gamma\left(1+\frac{\alpha}{2}\right)} \leqslant \mathcal{E}.
\end{aligned} \tag{4.14}$$

Estimates (4.11) and (4.14) yield that

$$\begin{aligned}
\sup_{(t,x)\in\mathscr{S}_0} |v_1(t,x)| &\leqslant \left\|\mathbb{T}_{1,\alpha}(t)u_0\right\|_{L^\infty} + \left\| \int_0^t \Delta\mathbb{T}_{2,\alpha}(t-\tau)F(v_0(\tau,x))\,\mathrm{d}s \right\|_{L^\infty} \\
&\leqslant \mathcal{E}+\mathcal{E} = 2\mathcal{E},
\end{aligned}$$

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where T_0 is given by (4.5). Let us assume that $\sup_{(t,x)\in\mathscr{S}_0}|v_n(t,x)|\leqslant 2\mathcal{E},\ n\geqslant 1$. Then for any $(t,x)\in\mathscr{S}_0$, it follows

$$
\begin{aligned}
|v_{n+1}(t,x)| &\leqslant \left\|\int_{\mathbb{R}^N}\mathbb{K}_{1,\alpha}(t,x-y)u_0(y)\,\mathrm{d}y\right\|_{L^\infty} \\
&\quad + \left\|\int_0^t\int_{\mathbb{R}^N}\Delta\mathbb{K}_{2,\alpha}(t,x-y)F(v_n(\tau,y))\mathrm{d}y\,\mathrm{d}\tau\right\|_{L^\infty} \\
&\leqslant \big\|\mathbb{K}_{1,\alpha}(t,x)\big\|_{L^1(\mathbb{R}^N)}\|u_0\|_{L^\infty} \\
&\quad + \int_0^t \big\|\Delta\mathbb{K}_{2,\alpha}(t,x)\big\|_{L^1(\mathbb{R}^N)}\big\|F(v_n(\tau,x))\big\|_{L^\infty}\,\mathrm{d}\tau \\
&\leqslant \mathcal{E} + \frac{\left(\int_{\mathbb{R}^N}\left|\overline{\mathscr{B}}_2(z)\right|\mathrm{d}z\right)\mathcal{A}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)}\sup_{(t,x)\in\mathscr{S}_0}|v_n(t,x)|\int_0^t(t-\tau)^{\alpha/2-1}\,\mathrm{d}\tau \\
&\leqslant \mathcal{E} + \frac{4\left(\int_{\mathbb{R}^N}\left|\overline{\mathscr{B}}_2(z)\right|\mathrm{d}z\right)\mathcal{A}\Gamma\left(\frac{3}{2}\right)|T_0|^{\alpha/2}\mathcal{E}}{\alpha\Gamma\left(1+\frac{\alpha}{2}\right)} \leqslant 2\mathcal{E},
\end{aligned}
$$

where it follows from (4.5) that $\frac{4(\int_{\mathbb{R}^N}|\overline{\mathscr{B}}_2(z)|\mathrm{d}z)\mathcal{A}\Gamma(\frac{3}{2})|T_0|^{\alpha/2}}{\alpha\Gamma(1+\frac{\alpha}{2})}\leqslant 1$. The latter inequality is true for $j=n$ and, by induction, we deduce that (4.10) holds for any $j\geqslant 0$.

In the following, we show that the following estimate holds for $j\geqslant 0$

$$
\sup_{x\in\mathbb{R}^N}|v_{j+1}(t,x)-v_j(t,x)|\leqslant\frac{(\mathcal{C}_\alpha t^{\frac{\alpha}{2}})^{j+1}}{\Gamma(\frac{\alpha j}{2}+\frac{\alpha}{2}+1)}, \tag{4.15}
$$

where $\mathcal{C}_\alpha$ is given by

$$
\mathcal{C}_\alpha = 4\Big(\int_{\mathbb{R}^N}\left|\overline{\mathscr{B}}_2(z)\right|\mathrm{d}z\Big)\mathcal{A}\Gamma\left(\frac{3}{2}\right)\max\left(\frac{\mathcal{E}}{\alpha},\ \frac{\Gamma(\alpha/2)}{2\Gamma(\alpha/2+1)}\right). \tag{4.16}
$$

Indeed, for $j=0$, using (4.12) and (4.13), we find that

$$
\begin{aligned}
\sup_{x\in\mathbb{R}^N}\left|v_1(t,x)-v_0(t,x)\right| &\leqslant \left\|\int_0^t\Delta\mathbb{K}_{2,\alpha}(t-\tau)F(v_0(\tau,x))\,\mathrm{d}s\right\|_{L^\infty} \\
&\leqslant \int_0^t\|\Delta\mathbb{K}_{2,\alpha}(t-\tau,x)\|_{L^1(\mathbb{R}^N)}\|F(v_0(\tau,x))\|_{L^\infty}\,\mathrm{d}\tau \\
&\leqslant \frac{2\mathcal{E}\left(\int_{\mathbb{R}^N}\left|\overline{\mathscr{B}}_2(z)\right|\mathrm{d}z\right)\mathcal{A}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)}\int_0^t(t-\tau)^{\alpha/2-1}\,\mathrm{d}\tau \\
&= \frac{4\mathcal{E}t^{\alpha/2}\left(\int_{\mathbb{R}^N}\left|\overline{\mathscr{B}}_2(z)\right|\mathrm{d}z\right)\mathcal{A}\Gamma\left(\frac{3}{2}\right)}{\alpha\Gamma\left(1+\frac{\alpha}{2}\right)}\leqslant\frac{t^{\alpha/2}\mathcal{C}_\alpha}{\Gamma(\alpha/2+1)},
\end{aligned}
$$

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which means that (4.15) holds for $j = 0$. Suppose that (4.15) holds for $j \leqslant n-1$, where $n \geqslant 2$ is a positive integer. From (4.8), we find that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^N}\Big|v_n(t,x)-v_{n-1}(t,x)\Big| \\
&\leqslant \left\| \int_0^t \Delta\mathbb{K}_{2,\alpha}(t-\tau)\Big(F(v_{n-1}(\tau,x)-F(v_{n-2}(s,x)\Big)\,\mathrm{d}\tau \right\|_{L^\infty} \\
&\leqslant \int_0^t \Big\|\Delta\mathbb{K}_{2,\alpha}(t-\tau,x)\Big\|_{L^1(\mathbb{R}^N)}\Big\|F(v_{n-1}(\tau,x)-F(v_{n-2}(\tau,x)\Big\|_{L^\infty}\,\mathrm{d}\tau \\
&\leqslant \frac{2\mathcal{E}\Big(\int_{\mathbb{R}^N}\Big|\overline{\mathscr{B}}_2(z)\Big|\mathrm{d}z\Big)\mathcal{A}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)}\int_0^t (t-\tau)^{\alpha/2-1}\|v_{n-1}(\tau,.)-v_{n-2}(\tau,.)\|_{L^\infty}\,\mathrm{d}\tau \\
&\leqslant \frac{2\mathcal{E}\Big(\int_{\mathbb{R}^N}\Big|\overline{\mathscr{B}}_2(z)\Big|\mathrm{d}z\Big)\mathcal{A}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)}\int_0^t \frac{(t-\tau)^{\alpha/2-1}(\mathcal{C}_\alpha t^{\frac{\alpha}{2}})^{n-1}}{\Gamma(\frac{\alpha n}{2}-\frac{\alpha}{2}+1)}\,\mathrm{d}\tau. \qquad (4.17)
\end{aligned}$$

Noting that $\Gamma(\alpha) \leqslant 1$, we observe the following:

$$\begin{aligned}
&\underline{TheRHSof(4.17)} \\
&= \frac{2\mathcal{E}\Big(\int_{\mathbb{R}^N}\Big|\overline{\mathscr{B}}_2(z)\Big|\mathrm{d}z\Big)\mathcal{A}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)}\frac{(\mathcal{C}_\alpha)^{n-1}}{\Gamma(\frac{\alpha n}{2}-\frac{\alpha}{2}+1)}\int_0^t (t-\tau)^{\alpha/2-1}\tau^{\frac{\alpha n-\alpha}{2}}\,\mathrm{d}\tau \\
&= \frac{2\mathcal{E}\Big(\int_{\mathbb{R}^N}\Big|\overline{\mathscr{B}}_2(z)\Big|\mathrm{d}z\Big)\mathcal{A}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)}\frac{(\mathcal{C}_\alpha)^{n-1}}{\Gamma(\frac{\alpha n}{2}-\frac{\alpha}{2}+1)}t^{\frac{\alpha n}{2}}\frac{\Gamma(\alpha/2)\Gamma(\frac{\alpha n-\alpha}{2}+1)}{\Gamma(\frac{\alpha n}{2}+1)} \\
&\leqslant \frac{2\Gamma(\alpha/2)\mathcal{E}\Big(\int_{\mathbb{R}^N}\Big|\overline{\mathscr{B}}_2(z)\Big|\mathrm{d}z\Big)\mathcal{A}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)}\frac{(\mathcal{C}_\alpha)^{n-1}}{\Gamma(\frac{\alpha n}{2}+1)}t^{\frac{\alpha n}{2}} \leqslant \frac{(\mathcal{C}_\alpha t^{\alpha/2})^n}{\Gamma(\frac{\alpha n}{2}+1)},
\end{aligned}$$

where in the last inequality, we have used from (4.16) that

$$\frac{2\Gamma(\alpha/2)\mathcal{E}\Big(\int_{\mathbb{R}^N}\Big|\overline{\mathscr{B}}_2(z)\Big|\mathrm{d}z\Big)\mathcal{A}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)} \leqslant \mathcal{C}_\alpha.$$

By the induction method, we derive that the estimate (4.15) holds for any $j \geqslant 1$. It follows from (4.15) that

$$\begin{aligned}
\|\!|v_{j+1}-v_j|\!\|_{\mathscr{P}_{T_0}} &= \sup_{(t,x)\in\mathscr{P}_{T_0}}\Big|v_{j+1}(t,x)-v_j(t,x)\Big| \\
&\leqslant \sup_{0\leqslant t\leqslant T_0}\frac{(\mathcal{C}_\alpha t^{\frac{\alpha}{2}})^{j+1}}{\Gamma(\frac{\alpha j}{2}+\frac{\alpha}{2}+1)} = \frac{(\mathcal{C}_\alpha T_0^{\frac{\alpha}{2}})^{j+1}}{\Gamma(\frac{\alpha j}{2}+\frac{\alpha}{2}+1)}. \qquad (4.18)
\end{aligned}$$

Since (4.18), we deduce that for $m > n$

$$\begin{aligned}\left\|\left|v_m - v_n\right|\right\|_{\mathscr{P}_{T_0}} &= \sup_{(t,x)\in\mathscr{P}_{T_0}} \Big|v_m(t,x) - v_n(t,x)\Big| \leqslant \sum_{j=n}^{m} \big\|v_{j+1}(t,x) - v_j(t,x)\big\|_{L^\infty} \\ &\leqslant \sum_{j=n}^{m} \frac{(\mathcal{C}_\alpha T_0^{\frac{\alpha}{2}})^{j+1}}{\Gamma(\frac{\alpha j}{2} + \frac{\alpha}{2} + 1)}. \end{aligned} \tag{4.19}$$

In the next step, we claim that the infinite sum $\sum_{j=1}^{\infty} \frac{(\mathcal{C}_\alpha T_0^{\frac{\alpha}{2}})^{j+1}}{\Gamma(\frac{\alpha j}{2}+\frac{\alpha}{2}+1)}$ is convergent. Due to the definition of T_0 as in (4.5), we find that $T_0^{\alpha/2} \leqslant \frac{\alpha\Gamma(\alpha/2+1)}{4(\int_{\mathbb{R}^N} |\overline{\mathscr{B}}_2(z)|\mathrm{d}z)\mathcal{A}\Gamma(\frac{3}{2})\Gamma(\alpha/2)}$. By relying on (4.16), we can easily achieve that

$$(\mathcal{C}_\alpha T_0^{\alpha/2})^{j+1} \leqslant \left(4\Big(\int_{\mathbb{R}^N} \Big|\overline{\mathscr{B}}_2(z)\Big|\mathrm{d}z\Big)\mathcal{A}\Gamma\left(\frac{3}{2}\right) \frac{\Gamma(\alpha/2)}{2\Gamma(\alpha/2+1)} T_0^{\alpha/2}\right)^{j+1} \leqslant \left(\frac{\alpha}{2}\right)^{j+1}.$$

Since $\alpha > 1/2$, we know that $\alpha j + \alpha + 1 > 2$ for $j \geqslant 1$. Due to the fact that the function $\Gamma(z)$ is increasing for $z > 2$, we find that $\Gamma(\alpha j + \alpha + 1) > \Gamma(2\alpha + 1)$. It follows from (4.20) that for $m > n \geqslant \overline{M}$

$$\begin{aligned}\big\|v_m(t,x) - v_n(t,x)\big\|_{L^\infty} &\leqslant \frac{1}{\Gamma(2\alpha+1)} \sum_{j=n}^{m} \left(\frac{\alpha}{2}\right)^{j+1} \leqslant \frac{1}{\Gamma(2\alpha+1)} \sum_{j=\overline{M}}^{\infty} \left(\frac{\alpha}{2}\right)^{j+1} \\ &\leqslant \frac{2}{(2-\alpha)\Gamma(2\alpha+1)} \left(\frac{\alpha}{2}\right)^{\overline{M}+1}.\end{aligned}$$

Now, given any $\epsilon > 0$, we can pick $\overline{M}$, depending on ϵ, such that $\frac{2}{(2-\alpha)\Gamma(2\alpha+1)}(\frac{\alpha}{2})^{\overline{M}+1} < \epsilon$. Some of above observations allow us to conclude that the sequence $\{v_n\}$ is a Cauchy one in the space $L^\infty(\mathbb{R}^N)$. Therefore, there exists a function $v(t,x)$ which is the limitation of the sequence $\{v_n\}$ on the strip $\mathscr{P}_{T_0}$. It is obvious to see that v is a continuous solution of the integral equation (4.6) on the strip $\mathscr{P}_{T_0}$. Next, we examine the regularity of the solution u. We only need to derive the following estimation. For each $1 \leqslant k \leqslant L,\ n \geqslant 1$, there exists $\mathscr{Q}_k$ which is a continuous increasing function of $t - t_k$ such that, for each $0 < t_1 < t_2 < \cdots < t_N < t \leqslant T_0$, the following estimate holds true

$$\|D^k v_n(t,x)\|_{L^\infty} \leqslant (t-t_k)^{-\frac{\alpha k}{4}} \mathscr{Q}_k(\mathcal{E}, \alpha, t_k - t_1, t - t_k), \quad t_k < t \leqslant T_0.$$

From formula (4.8), we find that

$$\begin{aligned}Dv_n(t,x) &= D\mathbb{Z}_{1,\alpha}(t-t_1)v_n(t_1,x) \\ &\quad + \int_{t_1}^{t} D^3\mathbb{Z}_{2,\alpha}(t-\tau)F(v_{n-1}(\tau,x))\,\mathrm{d}\tau, \quad t_1 \leqslant t \leqslant T_0.\end{aligned} \tag{4.20}$$

This implies that

$$\sup_{x\in\mathbb{R}^N}\Big|Dv_n(t,x)\Big| \leqslant \sup_{x\in\mathbb{R}^N}\Big|D\mathbb{Z}_{1,\alpha}(t-t_1)v_n(t_1,x)\Big| + \sup_{x\in\mathbb{R}^N}\left|\int_{t_1}^t D^3\mathbb{Z}_{2,\alpha}(t-\tau)F(v_{n-1}(\tau,x))\,\mathrm{d}\tau\right|$$
$$= \underbrace{\sup_{x\in\mathbb{R}^N}\Big|D\Big(\mathbb{K}_{1,\alpha}(t-t_1,x)*v_n(t_1,x)\Big)\Big|}_{(I)} + \underbrace{\sup_{x\in\mathbb{R}^N}\int_{t_1}^t\Big|D^3\mathbb{Z}_{2,\alpha}(t-\tau)F(v_{n-1}(\tau,x))\Big|\,\mathrm{d}\tau}_{(II)}. \tag{4.21}$$

Using the fact that $\frac{\mathrm{d}}{\mathrm{d}x}(f(x)*g(x)) = (\frac{\mathrm{d}}{\mathrm{d}x}f(x))*g(x)$ and thanks to lemma (2.3), the term (I) is bounded by

$$(I) = \sup_{x\in\mathbb{R}^N}\Big|D\Big(\mathbb{K}_{1,\alpha}(t-t_1,x)*v_n(t_1,x)\Big)\Big| = \Big\|D\mathbb{K}_{1,\alpha}(t-t_1,.)*v_n(t_1,.)\Big\|_{L^\infty}$$
$$\leqslant \|D\mathbb{K}_{1,\alpha}(t-t_1,.)\|_{L^1(\mathbb{R}^N)}\|v_n(t_1,.)\|_{L^\infty} \leqslant 2\mathcal{E}\mathscr{C}_{1,1}(\alpha,N)(t-t_1)^{-\frac{\alpha}{4}} \tag{4.22}$$

Using (4.4) and the second part of lemma (2.3) with $p=1, k=3$, we find that the term (II) is bounded by

$$(II) \leqslant \int_{t_1}^t \big\|D^3\mathbb{Z}_{2,\alpha}(t-\tau)F(v_{n-1}(\tau,x))\big\|_{L^\infty}\,\mathrm{d}\tau$$
$$\leqslant \int_{t_1}^t \big\|D^3\mathbb{K}_{2,\alpha}(t-\tau,.)\big\|_{L^1(\mathbb{R}^N)}\|F(v_{n-1}(\tau,.))\|_{L^\infty}\,\mathrm{d}\tau$$
$$\leqslant \mathcal{A}\sup_{(t,x)\in[0,T_0]\times\mathbb{R}^N}|v_{n-1}(t,x)|\int_{t_1}^t\Big\|D^3\mathbb{K}_{2,\alpha}(t-\tau,.)\Big\|_{L^1(\mathbb{R}^N)}\,\mathrm{d}\tau$$
$$\leqslant 2\mathcal{E}\mathcal{A}\overline{\mathscr{C}_{3,1}}(\alpha,N)\int_{t_1}^t(t-\tau)^{\alpha-\frac{\alpha N}{4}-1+\frac{\alpha N}{4}-\frac{3\alpha}{4}}\,\mathrm{d}\tau = \frac{8\mathcal{E}\mathcal{A}\overline{\mathscr{C}_{3,1}}(\alpha,N)}{\alpha}(t-t_1)^{\frac{\alpha}{4}}. \tag{4.23}$$

Combining (4.21), (4.22) and (4.23), we deduce that there exists $\mathscr{Q}_1$ which is a continuous increasing function of $t-t_1$ such that

$$\sup_{x\in\mathbb{R}^N}\Big|Dv_n(t,x)\Big| \leqslant (t-t_1)^{\frac{-\alpha}{4}}\mathscr{Q}_1(\mathcal{E},\alpha,t-t_1). \tag{4.24}$$

From formula (4.8), we find that

$$D^2v_n(t,x) = D^2\mathbb{Z}_{1,\alpha}(t-t_2)v_n(t_1,x) + \int_{t_2}^t D^4\mathbb{Z}_{2,\alpha}(t-\tau)F(v_{n-1}(\tau,x))\,\mathrm{d}\tau, \quad t_2\leqslant t\leqslant T_0. \tag{4.25}$$

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By a similar argument as above, we find that

$$\begin{aligned}\sup_{x\in\mathbb{R}^N}\left|D^2 v_n(t,x)\right| &\leqslant \sup_{x\in\mathbb{R}^N}\left|D^2\mathbb{Z}_{1,\alpha}(t-t_2)v_n(t_2,x)\right| \\ &\quad + \sup_{x\in\mathbb{R}^N}\left|\int_{t_2}^{t} D^4\mathbb{Z}_{2,\alpha}(t-\tau)F(v_{n-1}(\tau,x))\,\mathrm{d}\tau\right| \\ &\leqslant \left\|D^2\mathbb{K}_{1,\alpha}(t-t_2,.)\right\|_{L^1(\mathbb{R}^N)}\|v_n(t_2,.)\|_{L^\infty} \\ &\quad + \int_{t_2}^{t}\left\|D^3\mathbb{K}_{2,\alpha}(t-\tau,.)\right\|_{L^1(\mathbb{R}^N)}\|DF(v_{n-1}(\tau,.))\|_{L^\infty}\,\mathrm{d}\tau \\ &\leqslant 2\mathcal{E}\mathscr{C}_{2,1}(\alpha,N)(t-t_2)^{-\frac{\alpha}{2}} + (12\mathcal{E}^2+1)\overline{\mathscr{C}_{3,1}}(\alpha,N) \\ &\quad \times \int_{t_2}^{t}\mathscr{Q}_1(\mathcal{E},\alpha,\tau-t_1)(\tau-t_1)^{\frac{-\alpha}{4}}(t-\tau)^{\frac{\alpha}{4}-1}\,\mathrm{d}\tau, \end{aligned} \tag{4.26}$$

where it follows from (4.24) that

$$\begin{aligned}\left\|DF(v_{n-1}(\tau,.))\right\|_{L^\infty} &\leqslant \|3v_{n-1}^2-1\|_{L^\infty}\|D(v_{n-1}(\tau,.))\|_{L^\infty} \\ &\leqslant (12\mathcal{E}^2+1)(\tau-t_1)^{\frac{-\alpha}{4}}\mathscr{Q}_1(\mathcal{E},\alpha,\tau-t_1).\end{aligned}$$

Now, we handle the integral term on the RHS of expression (4.26). It is noted that $(\tau-t_1)^{\frac{-\alpha}{4}} \leqslant (t_2-t_1)^{\frac{-\alpha}{4}}$ for any $\tau \geqslant t_2$, and we find that

$$\begin{aligned}&\int_{t_2}^{t}\mathscr{Q}_1(\mathcal{E},\alpha,\tau-t_1)(\tau-t_1)^{\frac{-\alpha}{4}}(t-\tau)^{\frac{\alpha}{4}-1}\,\mathrm{d}\tau \\ &\quad\leqslant \mathscr{Q}_1(\mathcal{E},\alpha,t_2-t_1)(t_2-t_1)^{\frac{-\alpha}{4}}\int_{t_2}^{t}(t-\tau)^{\frac{\alpha}{4}-1}\,\mathrm{d}\tau\end{aligned}$$

where we recall that $\mathscr{Q}_1$ is a continuous increasing function of $t-t_1$. Therefore, it follows from the latter above estimate that

$$\begin{aligned}&\int_{t_2}^{t}\mathscr{Q}_1(\mathcal{E},\alpha,\tau-t_1)(\tau-t_1)^{\frac{-\alpha}{4}}(t-\tau)^{\frac{\alpha}{4}-1}\,\mathrm{d}\tau \\ &\quad\leqslant \frac{4\mathscr{Q}_1(\mathcal{E},\alpha,t_2-t_1)(t_2-t_1)^{\frac{-\alpha}{4}}}{\alpha}(t-t_2)^{\alpha/4}.\end{aligned} \tag{4.27}$$

Combining (4.26) and (4.27), we arrive at

$$\sup_{x\in\mathbb{R}^N}\left|D^2 v_n(t,x)\right| \leqslant \mathscr{Q}_2(\mathcal{E},\alpha,t_2-t_1,t-t_2)(t-t_2)^{-\frac{\alpha}{2}},$$

where

$$\begin{aligned}\mathscr{Q}_2(\mathcal{E},\alpha,t_2-t_1,t-t_2) &= 2\mathcal{E}\mathscr{C}_{2,1}(\alpha,N) \\ &\quad + \frac{4\mathscr{Q}_1(\mathcal{E},\alpha,t_2-t_1)(t_2-t_1)^{\frac{-\alpha}{4}}}{\alpha}(t-t_2)^{3\alpha/4}.\end{aligned}$$

It is easy to verify that $\mathscr{Q}_2$ as above is a continuous increasing function of $t-t_2$. By a similar way as above, we can verify that $\mathscr{Q}_k$ as above is a continuous increasing function of $t - t_k$ for any k is a natural number such that $k \geqslant 2$. This completes our proof. □

Appendix A.

DEFINITION A.1. Let α be a complex number whose real part is positive. The Gamma function can be formulated as $\Gamma(\alpha) = \int_0^\infty \frac{x^{\alpha-1}}{e^x} \, dx$, and the M-Wright type function $\mathcal{M}_\alpha$ is given by $\mathcal{M}_\alpha(z) = \sum_{j=1}^\infty \frac{(-z)^m}{m!\Gamma[-\alpha m + (1-\alpha)]}$.

LEMMA A.2 (see [**13**, proposition 2] or [**31**, Appendix F]). *Let $\alpha \in (0,1)$ and $\theta > -1$. Then, the following properties holds*

$$\mathcal{M}_\alpha(\nu) \geqslant 0, \quad \forall \nu \geqslant 0, \quad \textit{and} \quad \int_0^\infty \nu^\theta \mathcal{M}_\alpha(\nu) d\nu = \frac{\Gamma(\theta+1)}{\Gamma(\theta\alpha+1)}, \quad \forall \theta > -1. \tag{A.1}$$

PROPOSITION A.1. *Let $a, b > 0$ and $q \geqslant 1$. Then the following inequality holds*

$$(a+b)^p \leqslant 2^{q-1}\left(a^q + b^q\right).$$

Proof. From the fact that $q(q-1)x^{q-2} \geqslant 0$ for any $x > 0$ and $q \geqslant 1$, we assert that the one variable function $f(x) = x^q, q > 0$ is a convex function. It follows that

$$f\left(\frac{\sum_{k=1}^M a_k}{M}\right) \leqslant \frac{\sum_{k=1}^M f(a_k)}{M}.$$

This one gives us the desired result. □

LEMMA A.3 (Young's convolution inequality). *Let $p, q, r \in [1, \infty]$ such that*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Then, the inequality $\|u * v\|_{L^r} \leqslant \|u\|_{L^p} \|v\|_{L^q}$ holds for every $u \in L^p(\mathbb{R}^N)$ and $v \in L^q(\mathbb{R}^N)$.

LEMMA A.4 (fractional Grönwall inequality). *Let m, n be positive constants and $\zeta \in (0,1)$. Assume that function $u \in L^{\infty,*}(0,T]$ satisfies the following inequality*

$$u(t) \leqslant m + n \int_0^t (t-\tau)^{\zeta-1} u(r) \, dr, \quad \textit{for all } t \in (0,T],$$

then, the result below is satisfied

$$u(t) \leqslant m E_{\zeta,1}\left(n\Gamma(\zeta)t^\zeta\right).$$

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