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ON THE SUM OF PARTS IN THE PARTITIONS OF n INTO DISTINCT PARTS

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Abstract

We investigate the sum of the parts in all the partitions of n into distinct parts and give two infinite families of linear inequalities involving this sum. The results can be seen as new connections between partitions and divisors.

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1. Introduction

A partition of a positive integer n is a sequence of positive integers whose sum is n. The order of the summands is unimportant when writing the partitions of n, but for consistency the summands will be written in nonincreasing order (see [1]).

A famous theorem of Euler asserts that there are as many partitions of n into distinct parts as there are partitions into odd parts. As usual, we denote by $Q(n)$ the number of integer partitions of n into distinct parts. For example, $Q(7) = 5$ because the five partitions of 7 into distinct parts are $7,\ 6+1,\ 5+2,\ 4+3,\ 4+2+1$ and the five partitions of 7 into odd parts are $7,\ 5+1+1,\ 3+3+1,\ 3+1+1+1+1,\ 1+1+1+1+1+1+1$. The generating function of $Q(n)$ is given by

$$\sum_{n=0}^{\infty} Q(n)q^n = \frac{1}{(q;q^2)_\infty} = (-q;q)_\infty$$

and the expansion starts

$$(-q;q)_\infty = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 5q^7 + 6q^8 + 8q^9 + \cdots .$$

Here and throughout the paper, we use the customary q-series notation:

$$(a;q)_n = \begin{cases} 1 & \text{for } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{for } n > 0; \end{cases}$$
$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n.$$

Because the infinite product $(a;q)_\infty$ diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume $|q| < 1$.

We consider the following functions.

DEFINITION 1.1. Let n be a nonnegative integer. We define:

(1) $s_o(n)$ to be the sum of the odd parts in all the partitions of n into an odd number of distinct parts;
(2) $s_e(n)$ to be the sum of the odd parts in all the partitions of n into an even number of distinct parts;
(3) $s(n) = s_o(n) - s_e(n)$.

The partitions of 9 into an odd number of distinct parts are 9, $6+2+1$, $5+3+1$ and $4+3+2$, while the partitions of 9 into an even number of distinct parts are $8+1$, $7+2$, $6+3$ and $5+4$. So $s_o(9) = 9+1+(5+3+1)+3 = 22$, $s_e(9) = 1+7+3+5 = 16$ and $s(9) = 6$.

The following result shows that the function $s(n)$ is closely related to $nQ(n)$, the sum of the parts in all the partitions of n into distinct parts.

THEOREM 1.2. *For* $n > 0$,

$$nQ(n) + 2\sum_{j=1}^{\infty}(-1)^j(n-j^2)Q(n-j^2) = s(n).$$

For example, the case $n = 9$ of this theorem is

$$9 \cdot Q(9) - 2 \cdot 8 \cdot Q(8) + 2 \cdot 5 \cdot Q(5) = 72 - 96 + 30 = 6.$$

By Theorem 1.2, we can easily deduce that $s(n)$ and $nQ(n)$ have the same parity. In addition, it is known (see [5, Corollary 4.7]) that $Q(n)$ is odd if and only if n is a generalised pentagonal number. Consequently, $s(n)$ is odd if and only if n is an odd generalised pentagonal number.

Upon reflection, one expects that there might be a more general result where Theorem 1.2 is a limiting case. For what follows, we consider the notion of modular partition which was introduced by MacMahon (see [1, page 13]). For a positive integer k, any positive integer n can be uniquely written as $ka + s$ with $a \geqslant 0$ and $1 \leqslant s \leqslant k$. The k-modular partitions are a modification of the Ferrers graph so that n is represented by a row of a boxes with k in each of them and one box with s in it. For example, Figure 1 displays the 2-modular Ferrers graph of the partition $9+7+3+3$ with shape $5+4+2+2$.

The m-rectangle of a partition is defined to be the largest $(m+j)\times j$ rectangle contained in the Ferrers graphs of the partition (see [4]). An m-Durfee rectangle is referred to as a Durfee square when $m = 0$. In Figure 1, the 2-Durfee rectangle of the partition is the rectangle of size 2×4.

For a fixed $k \geqslant 1$ and any $n \geqslant 0$, Merca *et al.* [8] recently defined $M_{o,k}(n)$ to be the number of partitions of n into odd parts such that all odd numbers less than or

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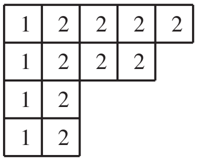

FIGURE 1. The 2-modular Ferrers graph of $9 + 7 + 3 + 3$.

equal to $2k + 1$ occur as parts at least once and the parts below the $(k + 2)$-Durfee rectangle in the 2-modular graph are strictly less than the width of the rectangle. For example, take $k = 2$. Then the partition $11 + 11 + 7 + 7 + 5 + 3 + 1$ is counted by $M_{o,2}(45)$. However, the partition $11 + 11 + 11 + 5 + 3 + 3 + 1$ is not counted by $M_{o,2}(41)$ because its 4-Durfee rectangle is of size 2×6 and the third part of length 11 that goes below the Durfee rectangle forms a row of length 6.

We have the following truncated form of Theorem 1.2, where we denote by $\sigma_o(n)$ the sum of the odd positive divisors of n.

THEOREM 1.3. *For $k, n > 0$,*

$$(-1)^k\Big(nQ(n) + 2\sum_{j=1}^{k}(-1)^j(n-j^2)Q(n-j^2) - s(n)\Big) = \sum_{j=1}^{n}\sigma_o(j)M_{o,k}(n-j).$$

An immediate consequence of this theorem is the following infinite family of linear inequalities.

COROLLARY 1.4. *For any positive integers k and n,*

$$(-1)^k\Big(nQ(n) + 2\sum_{j=1}^{k}(-1)^j(n-j^2)Q(n-j^2) - s(n)\Big) \geqslant 0,$$

with strict inequality if $n > (k + 1)^2$.

For example,

$$nQ(n) - 2(n-1)Q(n-1) \leqslant s(n),$$
$$nQ(n) - 2(n-1)Q(n-1) + 2(n-4)Q(n-4) \geqslant s(n).$$

When $k \to \infty$, Theorem 1.3 yields Theorem 1.2. Therefore, we will provide a proof only for Theorem 1.3. Relevant to Theorem 1.3, it would be very appealing to have combinatorial interpretations of

$$\sum_{j=1}^{n}\sigma_o(j)M_{o,k}(n-j).$$

2. Proof of Theorem 1.3

Firstly, we provide the following generating function for $nQ(n)$.

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LEMMA 2.1. *For* $|q| < 1$,

$$\sum_{n=1}^{\infty} nQ(n)q^n = \frac{1}{(q;q^2)_\infty} \sum_{k=1}^{\infty} \frac{(2k-1)q^{2k-1}}{1-q^{2k-1}}.$$

PROOF. We want the generating function for partitions into odd parts where z keeps track of the number of parts equal to $2k-1$. This is

$$\frac{1}{1-zq^{2k-1}} \prod_{\substack{n=1\\ n\neq k}}^{\infty} \frac{1}{1-q^{2n-1}} = \frac{1}{(q;q^2)_\infty} \cdot \frac{1-q^{2k-1}}{1-zq^{2k-1}}.$$

Let $t(k,n)$ be the total number of $(2k-1)$s in all the partitions of n into odd parts. Then

$$\sum_{n=1}^{\infty} t(k,n)q^n = \frac{d}{dz}\bigg|_{z=1} \frac{(1-q^{2k-1})}{(q;q^2)_\infty(1-zq^{2k-1})} = \frac{q^{2k-1}}{1-q^{2k-1}} \cdot \frac{1}{(q;q^2)_\infty}.$$

Since

$$nQ(n) = \sum_{k=1}^{\infty} (2k-1)t(k,n),$$

we can write

$$\sum_{n=1}^{\infty} nQ(n)q^n = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (2k-1)t(k,n)q^n = \frac{1}{(q;q^2)_\infty} \sum_{k=1}^{\infty} \frac{(2k-1)q^{2k-1}}{1-q^{2k-1}},$$

which completes the proof. □

According to [6, Corollary 4.3],

$$\frac{q^k}{1-q^k}(q;q)_\infty = \sum_{n=0}^{\infty} (t_o(k,n) - t_e(k,n))q^n$$

where $t_o(k,n)$ (respectively, $t_e(k,n)$) denotes the number of ks in all the partitions of n into an odd (respectively, even) number of distinct parts. It is clear that

$$s_o(n) = \sum_{k=1}^{\infty} (2k-1)t_o(2k-1,n)$$

and

$$s_e(n) = \sum_{k=1}^{\infty} (2k-1)t_e(2k-1,n).$$

So we deduce that

$$\begin{aligned}(q;q)_\infty \sum_{k=1}^{\infty} \frac{(2k-1)q^{2k-1}}{1-q^{2k-1}} &= \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} ((2k-1)t_o(2k-1,n) - (2k-1)t_e(2k-1,n))q^n \\ &= \sum_{n=0}^{\infty} (s_o(n) - s_e(n))q^n = \sum_{n=0}^{\infty} s(n)q^n. \qquad (2.1)\end{aligned}$$

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The following theta identity is often attributed to Gauss [1, page 23, Equation (2.2.13)]:

$$1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}. \tag{2.2}$$

Recently, Merca *et al*. considered this identity and provided in [8, Theorem 1.1] the following result: for $k > 0$,

$$\begin{aligned}
&\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\Big(1 + 2\sum_{j=1}^{k}(-1)^j q^{j^2}\Big)\\
&\quad = 1 + 2\sum_{j=1}^{\infty}(-1)^j q^{2j^2} + 2(-1)^k q^{(k+1)^2}(-q;q^2)_\infty \sum_{j=0}^{\infty}\frac{q^{(2k+2j+3)j}}{(q^2;q^2)_j(q;q^2)_{k+j+1}},
\end{aligned}$$

where $(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty$. By this identity, considering (2.2), we obtain

$$\begin{aligned}
&\frac{1}{(q;q^2)_\infty}\Big(1 + 2\sum_{j=1}^{k}(-1)^j q^{j^2}\Big)\\
&\quad = \frac{(q^2;q^2)_\infty}{(-q,q^2)_\infty(-q^2;q^2)_\infty} + 2(-1)^k q^{(k+1)^2}\sum_{j=0}^{\infty}\frac{q^{(2k+2j+3)j}}{(q^2;q^2)_j(q;q^2)_{k+j+1}}\\
&\quad = (q;q)_\infty + (-1)^k \sum_{n=0}^{\infty} M_{o,k}(n)q^n,
\end{aligned} \tag{2.3}$$

where we have invoked [8, Theorem 3.1] for the generating function of $M_{o,k}(n)$. Multiplying both sides of (2.3) by

$$\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}},$$

we obtain

$$\begin{aligned}
&\Big(\sum_{n=1}^{\infty} nQ(n)q^n\Big)\Big(1 + 2\sum_{n=1}^{k}(-1)^n q^{n^2}\Big)\\
&\quad = (q;q)_\infty \sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} + (-1)^k\Big(\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}\Big)\Big(\sum_{n=0}^{\infty} M_{o,k}(n)q^n\Big)\\
&\quad = \sum_{n=1}^{\infty} s(n)q^n + (-1)^k\Big(\sum_{n=1}^{\infty}\sigma_o(n)q^n\Big)\Big(\sum_{n=0}^{\infty} M_{o,k}(n)q^n\Big),
\end{aligned}$$

where we have invoked the Lambert series expansion (see [7]),

$$\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} = \sum_{n=1}^{\infty}\sigma_o(n)q^n, \quad |q| < 1.$$

The proof follows easily by considering Cauchy's multiplication of two power series.

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3. Partitions into distinct parts and odd divisors

Taking into account Euler's pentagonal number theorem

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} \tag{3.1}$$

and relation (2.1), we deduce that the partition function $s(n)$ is closely related to the sum of odd positive divisors function $\sigma_o(n)$, that is,

$$s(n) = \sum_{j=-\infty}^{\infty} (-1)^j \sigma_o(n - j(3j-1)/2).$$

Thus, this relation and Theorem 1.2 can be seen as connections between partitions into distinct parts and odd positive divisors, that is,

$$nQ(n) + 2\sum_{j=1}^{\infty}(-1)^j (n-j^2)Q(n-j^2) = \sum_{j=-\infty}^{\infty} (-1)^j \sigma_o(n - j(3j-1)/2).$$

DEFINITION 3.1. Let n be a positive integer. We define

$$S(n) := \sum_{j=0}^{\infty} \sigma_o(n - j(j+1)/2),$$

where $\sigma_o(k) = 0$ when k is a nonpositive integer.

For example, for $n = 6$,

$$S(6) = \sigma_o(6) + \sigma_o(5) + \sigma_o(3) = (1+3) + (1+5) + (1+3) = 14.$$

The following result provides another connection between partitions into distinct parts and odd positive divisors.

THEOREM 3.2. *For $n \geqslant 0$,*

$$\sum_{j=-\infty}^{\infty} (-1)^j (n - j(3j-1))Q(n - j(3j-1)) = S(n).$$

The case $n = 6$ of this theorem reads as follows

$$6 \cdot Q(6) - 4 \cdot Q(4) - 2 \cdot Q(2) = 24 - 8 - 2 = 14.$$

In [2], while investigating the truncated Euler's pentagonal number theorem (3.1), Andrews and Merca introduced the partition function $M_k(n)$, which counts the number of partitions of n where k is the least positive integer that is not a part and there are more parts greater than k than there are parts less than k. For example, $M_3(18) = 3$ because the three partitions in question are $5+5+5+2+1, 6+5+4+2+1, 7+4+4+2+1$.

There is a more general result where Theorem 3.2 is the limiting case $k \to \infty$.

THEOREM 3.3. *For $k, n > 0$,*

$$(-1)^{k-1}\Big(\sum_{j=-(k-1)}^{k}(-1)^j(n-j(3j-1))Q(n-j(3j-1))-S(n)\Big)=\sum_{j=0}^{\lfloor n/2\rfloor}M_k(j)S(n-2j).$$

PROOF. In [2], Andrews and Merca proved a truncated form of (3.1): for any $k \geqslant 1$,

$$\frac{(-1)^{k-1}}{(q;q)_\infty}\sum_{n=-(k-1)}^{k}(-1)^n q^{n(3n-1)/2}=(-1)^{k-1}+\sum_{n=k}^{\infty}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix}, \tag{3.2}$$

where

$$\begin{bmatrix}n\\k\end{bmatrix}=\begin{cases}\dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0\leqslant k\leqslant n,\\ 0 & \text{otherwise.}\end{cases}$$

The series on the right-hand side of (3.2) is the generating function for $M_k(n)$, that is,

$$\sum_{n=0}^{\infty}M_k(n)q^n=\sum_{n=k}^{\infty}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix}. \tag{3.3}$$

By (3.2), with q replaced by q^2,

$$(-1)^{k-1}\Big(\sum_{n=-(k-1)}^{k}(-1)^n q^{n(3n-1)}-(q^2;q^2)_\infty\Big)=(q^2;q^2)_\infty\sum_{n=0}^{\infty}M_k(n)q^{2n}. \tag{3.4}$$

Multiplying both sides of this identity by

$$\sum_{n=1}^{\infty}nQ(n)q^n=\frac{1}{(q;q^2)_\infty}\sum_{n=1}^{\infty}\sigma_o(n)q^n,$$

we obtain

$$\begin{aligned}(-1)^{k-1}\Big(\Big(\sum_{n=1}^{\infty}nQ(n)q^n\Big)\Big(\sum_{n=-(k-1)}^{k}(-1)^n q^{n(3n-1)}\Big)-\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=1}^{\infty}\sigma_o(n)q^n\Big)\\ =\Big(\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=1}^{\infty}\sigma_o(n)q^n\Big)\Big(\sum_{n=0}^{\infty}M_k(n)q^{2n}\Big)\end{aligned}$$

which yields

$$\begin{aligned}(-1)^{k-1}\Big(\Big(\sum_{n=1}^{\infty}nQ(n)q^n\Big)\Big(\sum_{n=-(k-1)}^{k}(-1)^n q^{n(3n-1)}\Big)-\sum_{n=1}^{\infty}S(n)q^n\Big)\\ =\Big(\sum_{n=1}^{\infty}S(n)q^n\Big)\Big(\sum_{n=0}^{\infty}M_k(n)q^{2n}\Big),\end{aligned}$$

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taking into account the following theta identity of Gauss [1, page 23, Equation (2.2.13)],

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{3.5}$$

The proof follows easily by considering Cauchy's multiplication of two power series. □

An immediate consequence of Theorem 3.3 is the following infinite family of linear inequalities.

COROLLARY 3.4. *For $k, n > 0$,*

$$(-1)^{k-1}\Big(\sum_{j=-(k-1)}^{k} (-1)^j (n-j(3j-1))Q(n-j(3j-1)) - S(n)\Big) \geqslant 0,$$

with strict inequality if $n > k(3k+1)/2$.

For example,

$$nQ(n) - (n-2)Q(n-2) \geqslant S(n),$$
$$nQ(n) - (n-2)Q(n-2) - (n-4)Q(n-4) + (n-10)Q(n-10) \leqslant S(n).$$

4. Concluding remarks and open problems

Considering an interplay between two classical theta identities of Gauss and a special case of the generalised Lambert series, we obtain decompositions of $s(n)$ and $S(n)$ in terms of the partition function $nQ(n)$. Our proofs of Theorems 1.3 and 3.3 rely on generating functions. Combinatorial proofs of these theorems would be very interesting.

Inspired by Theorem 3.2, we can consider another sum involving the odd positive divisors function $\sigma_o(n)$, namely,

$$T(n) := \sum_{j=-\infty}^{\infty} (-1)^j \sigma_0(n - 2j(3j-1)).$$

The generating function of $T(n)$ is given by

$$\sum_{n=0}^{\infty} T(n)q^n = (q^4;q^4)_\infty \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}$$

and the expansion starts

$$\sum_{n=0}^{\infty} T(n)q^n = q + q^2 + 4q^3 + q^4 + 5q^5 + 3q^6 + 4q^7 + 6q^9 + q^{10} + 2q^{12} - 5q^{13} - 2q^{14} + \cdots.$$

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Taking into account Lemma 2.1 and the theta series (3.5), it is not difficult to obtain the identity

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n=1}^{\infty} nQ(n)q^n = \sum_{n=0}^{\infty} T(n)q^n, \quad |q|<1. \tag{4.1}$$

In [3], Andrews and Merca introduced the partition function $MP_k(n)$, which counts the number of partitions of n in which the first part larger than $2k-1$ is odd and appears exactly k times and all other odd parts appear at most once. For example, $MP_2(19) = 10$ because the partitions in question are $9+9+1$, $9+5+5$, $8+5+5+1$, $7+7+3+2$, $7+7+2+2+1$, $7+5+5+2$, $6+5+5+3$, $6+5+5+2+1$, $5+5+3+2+2+2$, $5+5+2+2+2+2+1$. According to [3, Theorem 9], for $k \geqslant 1$,

$$(-1)^{k-1}\Big(\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{j=0}^{2k-1}(-q)^{j(j+1)/2} - 1\Big) = \sum_{n=0}^{\infty} MP_k(n)q^n.$$

Multiplying both sides of this identity by (4.1), we obtain

$$(-1)^{k-1}\Big(\Big(\sum_{n=1}^{\infty} nQ(n)q^n\Big)\Big(\sum_{j=0}^{2k-1}(-q)^{j(j+1)/2}\Big) - \sum_{n=0}^{\infty} T(n)q^n\Big) = \Big(\sum_{n=0}^{\infty} T(n)q^n\Big)\Big(\sum_{n=0}^{\infty} MP_k(n)q^n\Big).$$

In this way, for $k \geqslant 1$, there is a substantial amount of numerical evidence to conjecture that

$$\Big(\sum_{n=0}^{\infty} T(n)q^n\Big)\Big(\sum_{n=0}^{\infty} MP_k(n)q^n\Big)$$

has nonnegative coefficients. We have the following equivalent form of this conjecture.

CONJECTURE 4.1. For $k, n > 0$,

$$(-1)^{k-1}\Big(\sum_{j=0}^{2k-1}(-1)^{j(j+1)/2}(n-j(j+1)/2)Q(n-j(j+1)/2) - T(n)\Big) \geqslant 0.$$

For example,

$$nQ(n) - (n-1)Q(n-1) \geqslant T(n),$$
$$nQ(n) - (n-1)Q(n-1) - (n-3)Q(n-3) + (n-6)Q(n-6) \leqslant T(n).$$

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