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RATIO MONOTONICITY FOR TAIL PROBABILITIES IN THE RENEWAL RISK MODEL

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A renewal model in risk theory is considered, where $\overline{H}(u, y)$ is the tail of the distribution of the deficit at ruin with initial surplus u and $\overline{F}(y)$ is the tail of the ladder height distribution. Conditions are derived under which the ratio $\overline{H}(u, y)/\overline{F}(u + y)$ is nondecreasing in u for any $y \geq 0$. In particular, it is proven that if the ladder height distribution is stable and DFR or phase type, then the above ratio is nondecreasing in u. As a byproduct of this monotonicity, an upper bound and an asymptotic result for $\overline{H}(u, y)$ are derived. Examples are given to illustrate the monotonicity results.

1. INTRODUCTION

In this article, we consider a renewal model in actuarial risk theory (often referred to as a Sparre Andersen model), for which the number of claims follow a general renewal process. The distribution function (d.f.) of the deficit at ruin (also known as the severity of ruin) was introduced by Gerber, Goovaerts, and Kaas [11] and is an important concept of this model. In random walk terminology, it represents the overshoot over the value of the initial surplus for the model under consideration (see, e.g., Rolski, Schmidli, Schmidt, and Teugels [15, Sect. 5.1.4]). In practical terms, the

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deficit at ruin is a very useful measure for the stability of the risk process, generalizing the notion of probability of ruin.

Let $\psi(u)$ denote the probability of ruin and $\overline{H}(u, y)$ be the tail of the d.f. of the deficit at the time of ruin, where u is the initial surplus. There are many bounds and asymptotic results in the literature for these quantities. However, not much attention has been paid to the monotonicity of quantities associated with $\psi(u)$ and $\overline{H}(u, y)$. Dickson and dos Reis [7], as well as Willmot and Lin [20], studied the monotonicity of the ratio $\overline{H}(u, y)/\psi(u)$ in u in the classical risk model. Recently, Psarrakos and Politis [14] studied the monotonicity of the above ratio in the Sparre Andersen risk model. Another crucial quantity in this discussion is the ladder height d.f., $F(x)$, with tail $\overline{F}(x) = 1 - F(x)$. The aim of the present article is to derive conditions under which the ratio $\overline{H}(u, y)/\overline{F}(u + y)$ is nondecreasing in u. Note that, in general (see Psarrakos and Politis [14]),

$$\overline{H}(u, y)/\overline{F}(u + y) \geq \overline{H}(0, y)/\overline{F}(0 + y)$$

and so the ratio $\overline{H}(u, y)/\overline{F}(u + y)$ cannot be nonincreasing in u. Using the monotonicity of this ratio, we will also derive an upper bound and an asymptotic result for $\overline{H}(u, y)$.

The organization of this article is as follows. In Section 2 we review the renewal risk model under consideration and define the quantities in question. In Section 3 we recall the notions of stable, subexponential, and decreasing failure rate (DFR) distributions. When F is stable, we obtain an upper bound for the ratio $\overline{F^{*n}}(x)/\overline{F}(x)$ $(n = 2, 3, \ldots)$ as a function of the ratio $\overline{F^{*2}}(x)/\overline{F}(x)$. We also argue that if F is stable and DFR, then the ratio $\overline{H}(u, y)/\overline{F}(u + y)$ is nondecreasing in u for any $y \geq 0$. The monotonicity of this ratio is then used to study its asymptotic behavior as $u \to \infty$. In Section 4 we consider the classical model in light of notions and methods in linear algebra and nonnegative matrix theory. We prove that if the claim amount d.f. is phase type, then $\overline{H}(u, y)/\overline{F}(u + y)$ is again nondecreasing as a function of u. We conclude with examples to illustrate the monotonicity of $\overline{H}(u, y)/\overline{F}(u + y)$.

2. DETAILS AND BACKGROUND

We consider the (ordinary) renewal risk model, for which claims $Y_1, Y_2, \ldots$ arrive in a renewal process, whereby the interclaim times $T_1, T_2, \ldots$ are assumed to be independent identically distributed (i.i.d.) positive random variables with finite mean $\mathrm{E}(T_1)$. The claims are also i.i.d. positive random variables with d.f. P and finite mean $\mathrm{E}(Y_1)$. In the case for which P has a density, we denote this density by p. We further assume that the claims are independent of the claim-arrivals process. Then the surplus of the insurer at time t is given by

$$U(t) = u + ct - \sum_{i=1}^{N_t} Y_i,$$

where u is the initial surplus, c is the rate of premium income per unit time, and N_t is the number of claims until t.

We assume throughout our discussion that $\mathrm{E}(Y_1) < c\mathrm{E}(T_1)$, so that ruin is not certain to occur. Moreover, we write $c = (1+\theta)\mathrm{E}(Y_1)/\mathrm{E}(T_1)$, where $\theta > 0$ is the relative safety loading. Let T denote the time of ruin (i.e., the time that the surplus becomes negative for the first time) and note that T is a defective random variable. The probability of ruin is then defined by

$$\psi(u) = \Pr(T < \infty | U(0) = u). \tag{1}$$

It is well known (see Rolski et al. [15, p. 251]) that

$$\psi(u) = \Pr(L > u) = \sum_{n=1}^{\infty}(1-\phi)\phi^n \overline{F^{*n}}(u), \tag{2}$$

where $\phi = \psi(0)$ is the probability of a drop in surplus below its initial level, L is the maximal aggregate loss, $F(y) = 1 - \overline{F}(y)$ is the ladder height d.f. (i.e., the d.f. of the amount of the drop in surplus, given that a drop below its initial surplus occurs), and $F^{*n}(u) = 1 - \overline{F^{*n}}(u)$ is the n-fold convolution of F with itself. We define

$$\overline{H}(u,y) = \Pr(|U(T)| > y, T < \infty | U(0) = u), \tag{3}$$

which is the tail of the defective distribution of the deficit at ruin. By the results of Willmot [18], we know that $\overline{H}(u,y)$ satisfies the defective renewal equation

$$\overline{H}(u,y) = \phi \int_0^u \overline{H}(u-t,y)dF(t) + \phi\overline{F}(u+y), \tag{4}$$

whose solution is

$$\overline{H}(u,y) = \frac{\phi}{1-\phi}\int_{0-}^{u} \overline{F}(u+y-t)dH(t), \tag{5}$$

where $H(u) = 1 - \psi(u)$ is the probability of nonruin starting with capital surplus u.

As we are interested in studying the monotonicity of the ratio $\overline{H}(u,y)/\overline{F}(u+y)$ in u, we recall that Embrechts and Veraverbeke [9] studied the asymptotic behavior of the ratio $\psi(u)/\overline{P_e}(u)$ in the renewal model and proved in the case where P_e is a subexponential distribution that

$$\lim_{u\to\infty}\frac{\psi(u)}{\overline{P_e}(u)} = \frac{\phi}{1-\phi}.$$

Here, $\overline{P_e}(u) = 1 - P_e(u) = \int_u^\infty \overline{P}(t)dt/\int_0^\infty \overline{P}(t)dt$ is the tail of the equilibrium d.f. of P. Note that in the classical risk model, in which the number of claims N_t is a Poisson

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process given by the parameter λ, the ladder height d.f., F, satisfies $F = P_e$. Recently, Psarrakos [13] generalized this result to

$$\lim_{u\to\infty} \frac{\overline{H}(u,y)}{\overline{P_e}(u+y)} = \frac{\phi}{1-\phi},$$

for any $y \geq 0$. Note also that $\overline{H}(u,y)$ is a generalization of $\psi(u)$, since $\overline{H}(u,0) = \psi(u)$.

3. MONOTONICITY FOR STABLE AND DFR DISTRIBUTIONS

A random variable X is called *stable* (see, e.g., Samorodnitsky and Taqqu [16]) if for any two positive numbers A and B, there exist a positive number C and a real number D such that $AX_1 + BX_2 =^d CX + D$, where X_1 and X_2 are independent copies of X. The *index a* of a stable random variable is the value a such that $C^a = A^a + B^a$; thus, $a \in [0, 2]$. A stable random variable can be characterized by four parameters: $a \in [0, 2]$, $\sigma > 0$, $\beta \in [-1, 1]$, and $\mu \in (-\infty, \infty)$, where a is the index, σ is the *scale parameter*, β is the *skewness parameter*, and μ is the *location parameter*. By Bingham, Goldie, and Teugels [4, Eq. (8.3.12)] (see also Daley, Omey, and Vesilo [5]), stable distributions with support $(0, \infty)$ and index $a \in (0, 2)$ form a subclass of the subexponential distributions; that is,

$$\lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n \tag{6}$$

for any positive integer $n \geq 2$. Next, we consider the ladder height d.f. F with finite mean μ.

LEMMA 3.1 (Daley et al. (2007)): *If F is stable, then the ratio $\overline{F^{*n}}(x)/\overline{F}(x)$ is nondecreasing in x, for any $n \geq 2$.*

In what follows, given a stable distribution F, we obtain an upper bound for the ratio

$$\overline{F^{*n}}(x)/\overline{F}(x), \quad n = 2, 3, \ldots$$

as a function of the ratio $\overline{F^{*2}}(x)/\overline{F}(x)$.

PROPOSITION 3.2: *If F is stable, then for any $x \geq 0$ and any $n \geq 2$,*

$$\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \leq \sum_{k=0}^{n-3} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1\right]^k + \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1\right]^{n-2}. \tag{7}$$

(By convention, $\sum_{k=0}^{s}[\overline{F^{*2}}(x)/\overline{F}(x) - 1]^k = 0$ for any $s < 0$.)

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PROOF: By Lemma 3.1, the ratio $\overline{F^{*n}}(x)/\overline{F}(x)$ is nondecreasing in x, for any $n \geq 2$. Thus,

$$
\begin{aligned}
\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} &= 1 + \int_0^x \frac{\overline{F^{*(n-1)}}(x-t)}{\overline{F}(x-t)} \frac{\overline{F}(x-t)}{\overline{F}(x)}\, dF(t) \\
&\leq 1 + \frac{\overline{F^{*(n-1)}}(x)}{\overline{F}(x)} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right] \\
&\leq 1 + \left\{ 1 + \frac{\overline{F^{*(n-2)}}(x)}{\overline{F}(x)} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right] \right\} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right] \\
&= 1 + \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right] + \frac{\overline{F^{*(n-2)}}(x)}{\overline{F}(x)} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right]^2 \\
&\leq 1 + \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right] + \left\{ 1 + \frac{\overline{F^{*(n-3)}}(x)}{\overline{F}(x)} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right] \right\} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right]^2 \\
&= 1 + \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right] + \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right]^2 + \frac{\overline{F^{*(n-3)}}(x)}{\overline{F}(x)} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right]^3 \\
&\vdots \\
&\leq 1 + \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right] + \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right]^2 + \cdots + \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right]^{n-3} \\
&\quad + \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right]^{n-2}
\end{aligned}
$$

and the result follows. ∎

Remark 3.3: We note that

$$\sum_{k=0}^{n-3} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right]^k = \sum_{k=0}^{n-3} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \right]^j.$$

By Fubini's theorem (see Apostol [1, p. 413]), we have

$$\sum_{k=0}^{n-3} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right]^k = \sum_{k=0}^{n-3} \gamma_{k,n} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \right]^k,$$

where

$$\gamma_{k,n} = \sum_{i=k}^{n-3} \binom{i}{k} (-1)^{i-k}.$$

Thus, (7) can be rewritten as

$$\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \leq \sum_{k=0}^{n-3} \gamma_{k,n} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \right]^k + \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right]^{n-2}.$$

Note that by (6), the upper bound in (7) is asymptotically (i.e., as $x \to \infty$) sharp.

In the sequel we need the notion of *decreasing* (resp., *increasing*) *failure rate* (DFR; resp., IFR). A d.f. $G(y)$ is said to be DFR (IFR) if $\overline{G}(x+y)/\overline{G}(x)$ is nondecreasing (nonincreasing) in x for all fixed $y \geq 0$ (see Fagiuoli and Pellerey [10]).

The next lemma is a special case of a result of Shanthikumar [17] and is a key ingredient to our discussion.

LEMMA 3.4 (Shanthikumar [17]): *If F is DFR, then the probability of nonruin $1 - \psi$ is also DFR.*

LEMMA 3.5: *If F is stable, then $\psi(u)/\overline{F}(u)$ is nondecreasing.*

PROOF: Dividing (2) by $\overline{F}(u)$ implies that

$$\frac{\psi(u)}{\overline{F}(u)} = (1-\phi)\phi + \sum_{n=2}^{\infty} (1-\phi)\phi^n \frac{\overline{F^{*n}}(u)}{\overline{F}(u)}.$$

By this relation and Lemma 3.1, $\psi(u)/\overline{F}(u)$ is nondecreasing. ∎

LEMMA 3.6 (Psarrakos and Politis [14]): *If F is DFR, then $\overline{H}(u, y)/\psi(u+y)$ is nondecreasing in u.*

THEOREM 3.7: *If F is DFR and stable, then $\overline{H}(u, y)/\overline{F}(u+y)$ is nondecreasing in u.*

PROOF: The result follows by Lemmas 3.5 and 3.6 and the fact that

$$\frac{\overline{H}(u,y)}{\overline{F}(u+y)} = \frac{\overline{H}(u,y)}{\psi(u+y)} \frac{\psi(u+y)}{\overline{F}(u+y)}.$$ ∎

In the following theorem we use the notation

$$\Lambda_u(y) = \int_u^{u+y} \frac{\overline{F}(u+y-t)}{\overline{F}(u+y)} dF(t). \quad \textbf{(8)}$$

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THEOREM 3.8: *If, for any* $y \geq 0$, *the function* $\overline{H}(u,y)/\overline{F}(u+y)$ *is nondecreasing in* u *and*

$$\overline{F^{*2}}(u+y) \leq \frac{1+\phi+\phi\Lambda_u(y)}{\phi}\overline{F}(u+y), \tag{9}$$

then

$$\overline{H}(u,y) \leq \frac{\phi\overline{F}(u+y)}{1-\phi[\overline{F^{*2}}(u+y)/\overline{F}(u+y)]+\phi+\phi\Lambda_u(y)}. \tag{10}$$

PROOF: Dividing (4) by $\overline{F}(u+y)$, we have

$$\frac{\overline{H}(u,y)}{\overline{F}(u+y)} = \phi\int_0^u \frac{\overline{H}(u-t,y)}{\overline{F}(u+y-t)}\frac{\overline{F}(u+y-t)}{\overline{F}(u+y)}dF(t)+\phi.$$

By the fact that the function $\overline{H}(u,y)/\overline{F}(u+y)$ is nondecreasing in u for any $y \geq 0$, we obtain

$$\frac{\overline{H}(u,y)}{\overline{F}(u+y)} \leq \phi\frac{\overline{H}(u,y)}{\overline{F}(u+y)}\int_0^u \frac{\overline{F}(u+y-t)}{\overline{F}(u+y)}dF(t)+\phi. \tag{11}$$

Moreover,

$$\frac{\overline{F^{*2}}(u+y)}{\overline{F}(u+y)} = 1+\int_0^{u+y}\frac{\overline{F}(u+y-t)}{\overline{F}(u+y)}dF(t).$$

Keeping in mind the definition of the function $\Lambda_u(y)$ in (8), by the last expression we see that

$$\int_0^u \frac{\overline{F}(u+y-t)}{\overline{F}(u+y)}dF(y) = \frac{\overline{F^{*2}}(u+y)}{\overline{F}(u+y)}-1-\Lambda_u(y). \tag{12}$$

By (11), solving in terms of $\overline{H}(u,y)$, and using (12), the result follows. ■

THEOREM 3.9: *Under the assumptions of Theorem* 3.7, *it follows that*

$$\lim_{u\to\infty}\frac{\overline{H}(u,y)}{\overline{F}(u+y)} = \frac{\phi}{1-\phi}.$$

PROOF: Since the function $\overline{F}(u+y-t)$ is nondecreasing in $t \in [0,u]$, (5) implies

$$\overline{H}(u+y) \geq \frac{\phi}{1-\phi}\overline{F}(u+y)\int_{0-}^u dH(t) = \frac{\phi}{1-\phi}\overline{F}(u+y)[1-\psi(u)].$$

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Thus,

$$\lim_{u\to\infty} \frac{\overline{H}(u+y)}{\overline{F}(u+y)} \geq \frac{\phi}{1-\phi}. \qquad \textbf{(13)}$$

Again, by the monotonicity (nondecreasing) of $\overline{F}(u+y-t)$ in $t \in [u, u+y]$, we have

$$\overline{F}(y)\left[\frac{\overline{F}(u)}{\overline{F}(u+y)} - 1\right] \leq \Lambda_u(y) \leq \frac{\overline{F}(u)}{\overline{F}(u+y)} - 1,$$

and letting $u \to \infty$,

$$\lim_{u\to\infty} \Lambda_u(y) = 0. \qquad \textbf{(14)}$$

Thus, by (10) and (14), we take

$$\lim_{u\to\infty} \frac{\overline{H}(u+y)}{\overline{F}(u+y)} \leq \frac{\phi}{1-\phi}. \qquad \textbf{(15)}$$

Furthermore, if F is stable, then

$$\frac{\overline{F^{*2}}(u+y)}{\overline{F}(u+y)} \leq 2 \leq \frac{1+\phi+\phi\,\Lambda_u(y)}{\phi},$$

where the first inequality holds by Lemma 3.1 and the definition of subexponential distribution. By (13) and (15), the result follows. ■

The next result follows immediately from Theorems 3.8 and 3.9 for $y = 0$.

COROLLARY 3.10:

(a) If the function $\psi(u)/\overline{F}(u)$ is nondecreasing in u and if for any $u \geq 0$,

$$\overline{F^{*2}}(u) \leq \frac{1+\phi}{\phi}\overline{F}(u),$$

then

$$\psi(u) \leq \frac{\phi\overline{F}(u)}{1-\phi[\overline{F^{*2}}(u)/\overline{F}(u)]+\phi}.$$

(b) If F is stable, then

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{F}(u)} = \frac{\phi}{1-\phi}.$$

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4. MONOTONICITY FOR PHASE-TYPE DISTRIBUTIONS

In this section, we consider the classical model of risk theory with Poisson arrivals and phase-type claim size distribution. Our study will be aided by notions and methods of linear algebra and nonnegative matrix theory. The reader is directed to Berman and Plemmons [3], as well as Horn and Johnson [12] for background. The main references on phase-type d.f. in risk theory are the books of Rolski et al., [15, Chap. 8] and Asmussen [2, Chap. 8]. Early references on phase-type distributions are Drekic, Dickson, Stanford, and Willmot [8] and Willmot, Dickson, Drekic and Stanford [14] as well as references therein.

Before we proceed with the description of the risk theory model, some matrix theory notation, terminology, and preliminaries are in order.

- By **1** we denote a column vector of all ones.
- The *spectral radius* of a matrix $B \in \mathbb{R}^{m\times m}$ is defined and denoted by

$$\rho(B) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } B\}.$$

- When $b_{ij} \geq 0$ for all i and j, we call $B = [b_{ij}] \in \mathbb{R}^{m\times m}$ a *nonnegative* matrix and denote it by $B \geq 0$. We write $B \leq 0$ if $-B \geq 0$ and write $B \geq C$ if $B - C \geq 0$. We use similar notation for vectors.
- When $a_{ij} \leq 0$ for all $i \neq j$, we refer to $A = [a_{ij}] \in \mathbb{R}^{m\times m}$ as a *Z-matrix*.
- An *M-matrix* is a Z-matrix all of whose eigenvalues lie in the right-half plane. Equivalently, an M-matrix is one of the form $sI - B$, where $B \geq 0$ and $s \geq \rho(B)$. For the theory of Z-matrixes and M-matrixes, see Berman and Plemmons [3, Chapter 5].
- When

$$|a_{ii}| \geq \sum_{i\neq j} a_{ij}, \quad i = 1, 2, \ldots, m, \tag{16}$$

  we refer to $A = [a_{ij}] \in \mathbb{R}^{m\times m}$ as a *diagonally dominant* matrix.
- We call $A = [a_{ij}] \in \mathbb{R}^{m\times m}$ a *subintensity matrix* if $a_{ij} \geq 0$ for all $i \neq j$, $a_{ii} \leq 0$ for $i = 1, 2, \ldots, m$ and if A is diagonally dominant with at least one inequality in (16) being strict.
- The matrix exponential e^{xA} and its derivative relative to $x \in \mathbb{R}$ are given, respectively, by

$$e^{xA} = \sum_{k=0}^{\infty} \frac{x^k A^k}{k!} \quad \text{and} \quad \frac{de^{xA}}{dx} = \sum_{k=1}^{\infty} \frac{x^{k-1}A^k}{(k-1)!} = Ae^{xA}.$$

Next, we consider the existence and sign of the inverse of a subintensity matrix as well as a comparison result on matrix exponentials. These results will be quoted later in our analysis.

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LEMMA 4.1: *Let A be an invertible subintensity matrix. Then* $A^{-1} \leq 0$.

PROOF: Let A be as prescribed and note that $-A$ is an invertible Z-matrix with nonnegative diagonal entries. By Berman and Plemmons [3, Chap. 5, Thm. 2.3], we can first conclude that $-A$ is an invertible M-matrix and, in turn, that $(-A)^{-1} \geq 0$. ∎

LEMMA 4.2: *Let* $-A$ *be a Z-matrix. Then* $e^{xA} \geq 0$ *for all* $x \geq 0$. *Moreover, if* $B \geq A$, *then* $e^{xB} \geq e^{xA}$ *for all* $x \geq 0$.

PROOF: Note that since $-A$ is a Z-matrix, there exists $C \geq 0$ and $s > 0$ such that $A = C - sI$. As C and sI commute, we have

$$e^{xA} = e^{x(C-sI)} = e^{xC}e^{-xsI} = e^{xC}e^{-xs} = e^{-xs}\sum_{k=0}^{\infty}\frac{x^k C^k}{k!} \geq 0.$$

Moreover, if $B \geq A$, then $B = A + E = C + E - sI$ for some $E \geq 0$. Hence, for all $x \geq 0$,

$$e^{xB} = e^{x(C+E-sI)} = e^{x(C+E)}e^{-xs} \geq e^{xC}e^{-xs} = e^{xA},$$

completing the proof of the lemma. ∎

We now return to the risk theory model. Consider a *continuous-time Markov chain* (CTMC) with a single absorbing state 0 and m transient states. Let J_t denote the state of the CTMC at time t. The row vector β^T contains the probabilities b_j that the process starts in the various transient states $j = 1, 2, \ldots, m$.

We write $P = PH_m(\beta, A)$; that is, the claim size has a phase-type distribution with representation (β, A) of dimension m, having d.f.

$$P(x) = 1 - \beta^T e^{xA}\mathbf{1}, \qquad x \geq 0,$$

where β^T is a $1 \times m$ row vector of probabilities and A is the $m \times m$ transition matrix.

In what follows, we assume that A is an invertible subintensity matrix.

The ladder height d.f., the maximal aggregate loss, and the deficit at ruin are also phase-type distributions and their tails are given by

$$\overline{F}(u) = \hat{\beta}^T e^{Au}\mathbf{1}, \tag{17}$$

$$\psi(u) = \phi\hat{\beta}^T e^{u(A+\phi\alpha\hat{\beta}^T)}\mathbf{1}, \tag{18}$$

and

$$\overline{H}(u, y) = \phi\hat{\beta}^T e^{Ay} e^{u(A+\phi\alpha\hat{\beta}^T)}\mathbf{1}, \tag{19}$$

respectively, where $\hat{\beta}^T = -\mu^{-1}\beta^T A^{-1}$ and $\alpha = -A\mathbf{1}$.

Next, we derive the monotonicity, as a function of u, of the ratio

$$\frac{\overline{H}(u, y)}{\overline{F}(u + y)} = \frac{\phi\hat{\beta}^T e^{yA} e^{u(A+\phi\alpha\hat{\beta}^T)}\mathbf{1}}{\hat{\beta}^T e^{yA} e^{uA}\mathbf{1}}. \tag{20}$$

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THEOREM 4.3: *If the claim size $P = PH_m(\beta, A)$ has a phase-type distribution with representation (β, A), then the ratio $\overline{H}(u,y)/\overline{F}(u+y)$ in (20) is a nondecreasing function of $u \in [0,\infty]$.*

PROOF: Note that to establish that $\overline{H}(u,y)/\overline{F}(u+y)$ as defined in (20) is a nondecreasing function of u in $[0,\infty]$, it is enough to establish the following three claims:

1. $\hat{\beta} \geq 0$,
2. $e^{yA}e^{uA} \geq 0$ and $e^{yA}e^{u(A+\phi\alpha\hat{\beta}^T)} \geq 0$ for all $u \geq 0$,
3. $$\frac{de^{yA}e^{u(A+\phi\alpha\hat{\beta}^T)}}{du} \geq \frac{de^{yA}e^{uA}}{du}.$$

Indeed, if we establish claims 1–3, then for all $u \geq 0$, $\overline{H}(u,y)/\overline{F}(u+y)$ is a ratio of positive numbers whose numerator, as a function of u, increases at least as fast as the denominator.

We proceed with justification of the three claims in order.

Claim 1: By assumption, $\mu > 0$ and $\beta \geq 0$. Also by Lemma 4.1, $A^{-1} \leq 0$. Hence, $\hat{\beta} = -\mu^{-1}\beta^T A^{-1} \geq 0$.

Claim 2: As $\phi > 0$ and $\alpha = -A\mathbf{1} \geq 0$ and since, by (1), $\hat{\beta} \geq 0$, we have that $A + \phi\alpha\hat{\beta}^T \geq A$. Hence, by Lemma 4.2, for all $u \geq 0$,

$$e^{u(A+\phi\alpha\hat{\beta}^T)} \geq e^{uA} \geq 0.$$

Claim 3: Let $E = \phi\alpha\hat{\beta}^T \geq 0$. We then have

$$\begin{aligned}
\frac{d}{du}e^{yA}e^{u(A+\phi\alpha\hat{\beta}^T)} &= e^{yA}\,(A+E)e^{u(A+E)} \\
&= e^{yA}[Ae^{u(A+E)} + Ee^{u(A+E)}] \\
&\geq e^{yA}Ae^{u(A+E)} \qquad \text{(since } Ee^{u(A+E)} \geq 0 \text{ and by Lemma 4.2)} \\
&\geq e^{yA}Ae^{uA} \qquad \text{(by Lemma 4.2)} \\
&= \frac{d}{du}e^{yA}\,e^{uA},
\end{aligned}$$

completing the proof of the theorem. ■

COROLLARY 4.4:

(i) The ratio $\psi(u)/\overline{F}(u)$ is a nondecreasing function of $u \in [0,\infty]$.

(ii) The ratio $\overline{H}(u,y)/\psi(u+y)$ is a nonincreasing function of $y \in [0,\infty]$ for any $u \geq 0$.

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PROOF:

(i) The result follows immediately from Theorem 4.3 for $y = 0$.

(ii) This statement admits a proof similar to the proof of Theorem 4.3. ■

Remark 4.5:

1. As in all cases in which one is using a phase-type distribution to model a purely continuous quantity with no discrete weight at 0, $\beta^T \mathbf{1} = 1$. In general, however, the components of β^T need not sum to 1 as the process might start in the absorbing state with probability b_0. Note that in our above analysis, we have not assumed or used that the entries of β are in $(0, 1)$ and add up to 1.
2. Theorem 4.3 holds in the ordinary and stationary renewal model of risk theory. For details on the stationary renewal model, where P is a phase-type d.f., see Willmot et al. [19].
3. Recall the notion of *irreducibility* of a square matrix (see Berman and Plemmons [3]): $A \in \mathbb{R}^{m\times m}$ is *irreducible* if and only if its directed graph is strongly connected. Let now A be a subintensity matrix. The assumption of invertibility of A used in Lemma 4.1 and in our above analysis can be substituted by irreducibility of A. This is because the conclusion of Lemma 4.1 that $A^{-1} \leq 0$ holds for every irreducible subintensity matrix. Indeed, for such a matrix A, by a celebrated result of Olga Taussky (see Horn and Johnson [12, Thm. 6.2.26 and Corr. 6.2.27], $-A$ is an invertible Z-matrix with nonnegative diagonal entries. It then follows by Berman and Plemmons [3, Chap. 6, Thm. 2.3] that $-A$ is an invertible M-matrix and $(-A)^{-1} \geq 0$.

Example 4.6: Suppose that the claim amount distribution is the mixture of exponentials with density

$$p(x) = \frac{1}{2}e^{-x} + e^{-2x}, \qquad x \geq 0,$$

and assume that the relative safety loading is $\theta = 0.6$. Such a distribution is phase type and DFR. The ladder height d.f. F is also a mixture of exponentials with tail

$$\overline{F}(x) = \frac{\int_x^\infty \overline{P}(t)dt}{\int_0^\infty \overline{P}(t)dt} = \frac{2}{3}e^{-x} + \frac{1}{3}e^{-2x},$$

the probability of ruin is given by $\psi(u) = 0.592581e^{-0.432479u} + 0.032419e^{-1.73419u}$, and the tail of the d.f. of the deficit at ruin is

$$\overline{H}(u, y) = 0.117504e^{-1.73419u-2y} + 0.0908294e^{-0.432479u-2y} - 0.0850836e^{-1.73419u-y} + 0.501751e^{-0.4324794u-y}.$$

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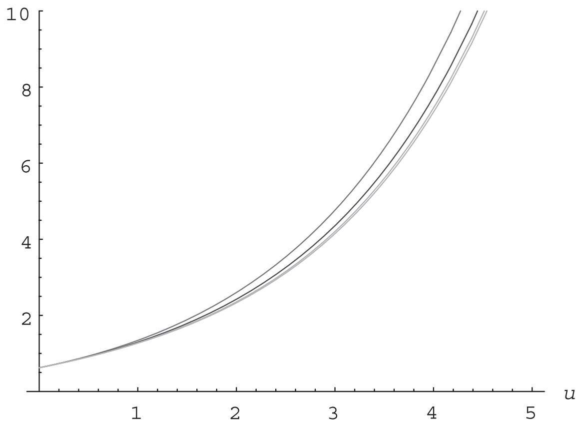

FIGURE 1. Monotonicity for $\overline{H}(u,y)/\overline{F}(u+y)$ in u for $y = 0, 1, 2, 3$.

We observe in Figure 1 that the ratio $\overline{H}(u,y)/\overline{F}(u+y)$ is nondecreasing in u. Specifically, in this figure we display[1]

$$\frac{\overline{H}(u,y)}{\overline{F}(u+y)} = \begin{cases} \text{red line for} & y = 0 \\ \text{blue line for} & y = 1 \\ \text{green line for} & y = 2 \\ \text{yellow line for} & y = 3. \end{cases}$$

By Psarrakos and Politis [14], if P is DFR, then for any $u \geq 0$, $\overline{H}(u,y)/\overline{F}(u+y)$ is nonincreasing in y. This explains why

$$\text{red line} \geq \text{blue line} \geq \text{green line} \geq \text{yellow line},$$

namely

$$\frac{\psi(u)}{\overline{F}(u)} \geq \frac{\overline{H}(u,1)}{\overline{F}(u+1)} \geq \frac{\overline{H}(u,2)}{\overline{F}(u+2)} \geq \frac{\overline{H}(u,3)}{\overline{F}(u+3)}.$$

Example 4.7: We refer here to an example from Dickson [6] in a renewal risk model: Let the interclaims arrivals have an Erlang $(2,2)$ d.f. with density $k(x) = 4xe^{-2x}$. Also assume that the claim amount d.f. is Erlang $(2,2)$ with density $p(x) = 4xe^{-2x}$. For $c = 1.1$, by Dickson [6], it follows that

$$\overline{F}(y) = e^{-2y} + 0.8217e^{-2y}y,$$
$$\psi(u) = 0.8841e^{-0.1818u} - 0.0109e^{-2.7892u},$$

[1] In case colors cannot be viewed in Figure 1, red, blue, green, and yellow refer respectively to the displayed curves in decreasing order of slope.

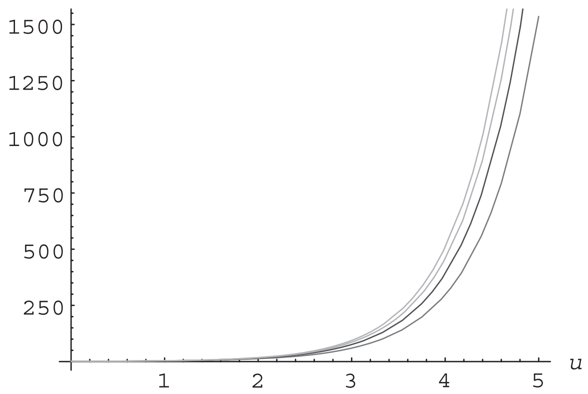

FIGURE 2. Monotonicity for $\overline{H}(u,y)/\overline{F}(u+y)$ in u for $y = 0, 1, 2, 3$.

and

$$\overline{H}(u,y) = 0.8841e^{-0.1818u-2y} - 0.0109e^{-2.7892u-2y} + 0.5003e^{-0.1818u-2y}y + 0.2172e^{-2.7892u-2y}y.$$

Erlang $(2,2)$ is a phase-type distribution. The colors[2] in Figure 2 represent the same choices of parameters as Example 1. Observe that the ratio $\overline{H}(u,y)/\overline{F}(u+y)$ is nondecreasing in u and

$$\text{red line} \leq \text{blue line} \leq \text{green line} \leq \text{yellow line}.$$

The latter inequalities follow by the fact that F is IFR and so for any $u \geq 0$, $\overline{H}(u,y)/\overline{F}(u+y)$ is nondecreasing in y (see Psarrakos and Politis [14]).

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[2] In case colors cannot be viewed in Figure 2, red, blue, green, and yellow refer respectively to the displayed curves in increasing order of slope.

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