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THE MODAL LOGIC OF INNER MODELS

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Abstract. Using techniques developed by Hamkins, Reitz and the second author, we determine the modal logic of inner models.

§1. Introduction. In [9], Hamkins introduced the modal logic interpretation of forcing by considering the generic multiverse of models of set theory connected by the relation of being a forcing extension; in [12, 13], Hamkins and the second author studied this modal logic of forcing and the related modal logic of grounds, respectively. Various other aspects of the modal logic of forcing were considered in [4–7, 10, 11, 14, 16, 19, 20, 23]. The techniques used in [12] and further developed in [11] are by no means restricted to the generic multiverse, but can be applied to other collections of models of set theory with other accessibility relations (cf. [2] for an account of the general theory in second order set theory). In this paper, we apply them to the inner model multiverse to determine the modal logic of inner models and determine this modal logic to be S4.2Top, an extension of the well-known modal logic S4.2 by an additional axiom (Theorem 5.6). The results of this paper are included in the Master's thesis of the first author, written under the supervision of the second author [16, Chapter 4].

In Section 2, we shall collect the results from modal logic needed for the proof of our main theorem; in particular, we define the class of relevant structures, called *inverted lollipops*. In Section 3, we develop a general approach to modal logics of set-theoretic model constructions. As opposed to the relations of being a forcing extension or being a ground, we cannot expect that relations between models of set theory are definable in the language of set theory: we therefore need to work in second-order set theory to deal with these relations. Our Section 4 contains a recapitulation of the standard techniques for producing lower and upper bounds for modal logics (mostly following [11]) and introduces a new type of control statements: buttons and switches relative to a pure button that acts as a global button pushing all relative buttons at once. We prove the appropriate transfer theorem for these control statements, linking them to the inverted lollipops of Section 2 (Theorem 4.5). Finally, in Section 5, we combine all these components and prove our main theorem (Theorem 5.6).

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§2. Results from Modal Logic. In this section, we define the modal theory S4.2Top needed for our main result and prove the relevant characterisation theorem. We remind the reader of the following well-known modal axioms:

$$\begin{array}{rl} \mathsf{K} & \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi), \\ \mathsf{Dual} & \neg\Diamond\varphi \leftrightarrow \Box\neg\varphi, \\ \mathsf{T} & \Box\varphi \rightarrow \varphi, \\ \mathsf{4} & \Box\varphi \rightarrow \Box\Box\varphi, \\ .\mathsf{2} & \Diamond\Box\varphi \rightarrow \Box\Diamond\varphi. \end{array}$$

The modal theory S4.2 is the smallest class of formulas containing all substitution instances of the above axioms and closed under modus ponens and necessitation (in other words, the smallest *normal modal logic* containing the above axioms).

As usual, a preorder is a set P with a reflexive and transitive relation $\leq$; preorders carry a natural equivalence relation $\equiv$ defined by $x \equiv y :\iff x \leq y \leq x$. The $\equiv$-classes are called *clusters*. Taking the $\equiv$-quotient of a preorder enforces antisymmetry and makes the preorder into a partial order. We call a preorder a pre-Boolean algebra if its $\equiv$-quotient is a Boolean algebra. A preorder $(P, \leq)$ is called *directed* if for any y_0 and $y_1 \in P$, there is a $z \in P$ with $y_0 \leq z$ and $y_1 \leq z$. A sequence $(z_0, \ldots, z_n)$ is called a *forward path connecting z_0 and z_n* if for any $i < n$, we have $z_i \leq z_{i+1}$; it is called a *path connecting z_0 and z_n* if for any $i < n$, we have either $z_i \leq z_{i+1}$ or $z_{i+1} \leq z_i$; a preorder $(P, \leq)$ is called *connected* if for any $x, y \in P$ there is a path connecting x and y. We say that a preorder $(P, \leq)$ is *rooted* if there is a $p \in P$ such that for all $q \in P$, we have that $p \leq q$; such a p is also called a *root* of $(P, \leq)$; in general, there can be a cluster of roots, but in a rooted partial order, there is a unique root. We say that a cluster $C \subseteq P$ is *maximal* if for $v \in C$ and $w \geq v$, we have $w \in C$; and we say that a cluster $C \subseteq P$ is *the top cluster* if for any $v \in P$ and $w \in C$, we have that $v \leq w$. Clearly, a preorder that has a top cluster is directed; conversely, if a preorder is directed and connected and has a maximal cluster, then this is the top cluster. Furthermore, we say that a preorder is *topped* if it has a top cluster consisting of exactly one node; it is *sharp* if it is topped and remains directed after removal of the largest element; and, finally, it is an *inverted lollipop* if it is topped and after removal of the top element, the remainder is a pre-Boolean algebra. (Cf. Figure 1 to see three inverted lollipops and to get an idea why we chose that name.)

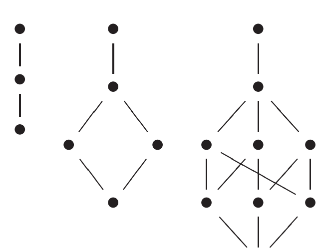

FIGURE 1. Three inverted lollipops.

A *(Kripke) frame* is a relational structure (W, R) consisting of a set of nodes (or states or possible worlds) and a binary relation on them. A *Kripke model* (W, R, V) consists of a Kripke frame (W, R) together with a valuation V assigning a truth value to each propositional variable and each element $w \in W$. Semantics for Kripke models is defined as usual [1, Definition 1.20]. If $\mathcal{M} = (W, R, V)$ is a Kripke model and $w \in W$, we write $\mathcal{M}[w]$ for the submodel generated by w (i.e., the submodel of $\mathcal{M}$ that consists of all nodes that can be reached from w by a forward path via the relation R).

If (W, R) is a frame, a modal assertion is *valid for* (W, R) if it is true at all worlds of all Kripke models having (W, R) as a frame. If $\mathcal{C}$ is a class of frames, a modal theory is *sound with respect to* $\mathcal{C}$ if every assertion in the theory is valid for every frame in $\mathcal{C}$. A modal theory is *complete with respect to* $\mathcal{C}$ if every assertion valid for every frame in $\mathcal{C}$ is in the theory. Finally, a modal theory is *characterized by* $\mathcal{C}$ if it is both sound and complete with respect to $\mathcal{C}$ [15, p. 40].

We remind the reader of the technique called *filtration* (cf. [8, pp. 267–268]): if $\mathcal{M} = (W, R, V)$ is a Kripke model and Γ is a subformula-closed set of formulas, we can define an equivalence relation $\sim_\Gamma$ on W by saying that $u \sim_\Gamma v$ if and only if u and v agree on the truth values of all formulas in Γ. We define $W_\Gamma := W/{\sim_\Gamma}$ and

$$V_\Gamma(p, [w]_{\sim_\Gamma}) := \begin{cases} V(p, w) & \text{if } p \in \Gamma, \\ 0 & \text{otherwise,} \end{cases}$$

and say that a Kripke model $\mathcal{M}' = (W_\Gamma, R', V_\Gamma)$ is *a filtration of* $\mathcal{M}$ *with respect to* Γ if for all $\psi \in \Gamma$ and $w \in W$, we have $\mathcal{M}, w \vDash \psi$ if and only if $\mathcal{M}', [w]_{\sim_\Gamma} \vDash \psi$. Note that in general, there can be more than one filtration with respect to a given Γ.

THEOREM 2.1 (Filtration Theorem). *For every Kripke model $\mathcal{M} = (W, R, V)$ and every subformula-closed set of formulas Γ, there is a filtration $\mathcal{M}' = (W_\Gamma, R', V')$ of $\mathcal{M}$ with respect to Γ. One such filtration is the* minimal filtration *defined by $U\ R'\ V$ if and only if there are $u \in U$ and $v \in V$ such that $u\ R\ v$. Furthermore, if Γ was finite, then so is W_Γ.*

We also remind the reader of the technique of *canonical models* (cf. [1, Chapter 4.2]): If Λ is a normal modal theory, then we can construct the *canonical model* $(W^\Lambda, R^\Lambda, V^\Lambda)$ satisfying all of the formulas in Λ where W^Λ is the set of maximal Λ-consistent sets of formulas, R^Λ is the *canonical accessibility relation*, and V^Λ is the *canonical valuation* defined by $V^\Lambda(\varphi, w) := 1$ if and only if $\varphi \in w$.

THEOREM 2.2 (Canonical Model Theorem). *Any normal modal theory Λ is complete with respect to its canonical model. That is, for any $\varphi \notin \Lambda$, there is a node $w \in W^\Lambda$ such that $\neg\varphi \in w$.*

It is easy to see that if $\Lambda \supseteq \mathsf{S4.2}$ is a modal theory, then the Kripke frame of its canonical model must be reflexive, transitive and directed (i.e., a directed preorder). Furthermore, it is well-known that the class of finite directed preorders characterises S4.2; Hamkins and the second author have observed that the class of finite pre-Boolean algebras characterises S4.2 [12, Theorem 11].

We now go beyond S4.2 and introduce the following axiom Top as

$$\Diamond((\Box\varphi \leftrightarrow \varphi) \wedge (\Box\neg\varphi \leftrightarrow \neg\varphi));$$

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this formula is equivalent over S4.2 to the negation of the *cofinal subframe formula* for the one element frame of a single reflexive node.[1] The modal theory S4.2Top is defined to be the smallest normal modal theory containing S4.2 and Top.

LEMMA 2.3. *Every Kripke model on a finite rooted directed preorder is bisimilar to a Kripke model on a finite pre-Boolean algebra.*

This argument is implicit in the proof of [12, Theorem 11], but since the bisimulation was not explicitly given in [12], we include the proof here. Remember from [12, p. 1802] that a partial order is called a *baled tree* if it has a maximal element and the order after removing the maximal element is a tree. We observe that baled trees are lattices, and so least upper bounds of finite sets of nodes exist. A preorder is called a *baled pretree* if its $\equiv$-quotient is a baled tree. In baled pretrees, we still have least upper bounds, but they are only defined up to $\equiv$-equivalence, so the least upper bound of any finite subset of a baled pretree is a cluster.

PROOF. Let $\mathcal{M} = (W, \leq, V)$ be a Kripke model on a finite directed preorder. By unravelling as in the proof of [12, Lemma 6.5], we can assume without loss of generality that $(W, \leq)$ is a baled pretree.

We start by considering the case where $(W, \leq)$ is a baled tree. Let r be the root of W, $S := W \backslash \{r\}$, and write $S = \{s_0, \dots, s_{n-1}\}$. Let $\mathbb{B}$ be the Boolean algebra of all subsets of S. For each nonempty $a \subseteq S$, let w_a be the least upper bound of a and $w_\varnothing := r$. We define a valuation on $\mathbb{B}$ by $V'(a, p) = 1$ if and only if $\mathcal{M}, w_a \vDash p$, and with this, $\mathcal{M}' := (\mathbb{B}, \subseteq, V')$. This means that in $\mathcal{M}'$, each $a \in \mathbb{B}$ looks like the node w_a in $\mathcal{M}$. It is straightforward to check that the relation $B \subseteq W \times \mathbb{B}$ defined by $(w, a) \in B$ if and only if $w = w_a$ is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$.

Now, if $(W, \leq)$ is a baled pretree, then the argument is similar except that we may have clusters of $\equiv$-equivalent nodes. We let R be the cluster of roots, $S = \{S_0, \dots, S_{n-1}\}$ the set of clusters other than the root cluster and $\mathbb{B}$ the Boolean algebra of subsets of S. For each $A \in \mathbb{B}$, let w_A be the least upper bound of $\bigcup A \subseteq W$. As mentioned above, w_A is not a node in W, but a cluster. We now copy this cluster w_A to A. The rest of the proof remains the same. ⊣

LEMMA 2.4. *Every Kripke model on a frame that is a finite topped preorder is bisimilar to a Kripke model on a frame that is a finite sharp preorder.*

PROOF. Let $(W, \leq)$ be the finite topped preorder that is the frame of our Kripke model and let t be its largest node. Build a Kripke model by adding an extra node t' above t which has the same valuation as t. It is easy to see that the two models are bisimilar, where the bisimulation is the identity on all the nodes which are not t, and t in the old model is matched to both t and t' in the new model. ⊣

LEMMA 2.5. *Every Kripke model on a frame that is a finite rooted sharp preorder is bisimilar to a Kripke model on a frame that is a finite inverted lollipop.*

PROOF. If $(W, \leq)$ is the finite sharp preorder underlying the Kripke model and t its largest element, then $(W \backslash \{t\}, \leq)$ is a finite directed preorder. By Lemma 2.3, this suborder is bisimilar to a pre-Boolean algebra, and therefore the entire frame is bisimilar to an inverted lollipop. ⊣

[1] A cofinal subframe formula for a finite frame $\mathcal{F} = (W, R)$ is a formula $\varphi_\mathcal{F}$ such that $\varphi_\mathcal{F}$ is invalid in any frame $\mathcal{G}$ if and only if $\mathcal{F}$ is cofinally subreducible to $\mathcal{G}$; cf. [3] for a discussion of these concepts.

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THEOREM 2.6. *The following classes characterise* S4.2Top:

1. *the class of finite topped preorders,*
2. *the class of sharp preorders,*
3. *the class of inverted lollipops.*

PROOF. In all three cases, soundness is easy to check. It is sufficient to show completeness for the class of finite topped preorders: if this is established, then the completeness with respect to the class of sharp preorders follows from Lemma 2.4; this in turn implies completeness with respect to the class of inverted lollipops by Lemma 2.5.[2] So, let us focus on completeness for the class of finite topped preorders: let φ be a formula which is not in S4.2Top. Let $\mathcal{M} = (W, R, V)$ be the canonical model of S4.2Top.

By Theorem 2.2, let $w \in W$ be a node such that $\varphi \notin w$ and let $\mathcal{M}_\Phi = (W_\Phi, R_\Phi, V_\Phi)$ be the minimal filtration of $\mathcal{M}[w]$ with respect to the finite subformula-closed set

$$\Phi = \{\psi \mid \psi \text{ is a subformula of } \varphi\}.$$

By Theorem 2.1, $\mathcal{M}_\Phi$ is a finite, rooted, connected, and directed preorder and φ is not valid in (W_Φ, R_Φ). Since $\mathcal{M}_\Phi$ is finite, connected and directed, it must have a top cluster. So, it is sufficient to show that this top cluster consists of one element.

Elements of W_Φ are $\sim_\Phi$-equivalence classes of maximal S4.2Top-consistent sets of formulas. Suppose towards a contradiction that there are two distinct elements of the maximal cluster $s, t \in W_\Phi$. Since $s \neq t$, there must be $v, w \in W$ such that $v \in s$ and $w \in t$, and $\psi \in \Phi$ such that $\mathcal{M}, v \models \psi$ and $\mathcal{M}, w \models \neg\psi$, and hence by Theorem 2.1, $\mathcal{M}_\Phi, s \models \psi$ and $\mathcal{M}_\Phi, t \models \neg\psi$.

Since $\mathcal{M}_\Phi \models$ S4.2Top, the ψ-instance of Top is true at every node of $\mathcal{M}_\Phi$, so in particular at s:

$$\mathcal{M}_\Phi, s \models \Diamond((\Box\psi \leftrightarrow \psi) \wedge (\Box\neg\psi \leftrightarrow \neg\psi)).$$

Thus, we find u with $s \; R \; u$ such that

$$\mathcal{M}_\Phi, u \models (\Box\psi \leftrightarrow \psi) \wedge (\Box\neg\psi \leftrightarrow \neg\psi). \tag{*}$$

Since s was in the maximal cluster, so is u, and thus s, t, and u are all R_Φ-accessible from each other; but now (*) implies that s and t must agree with u (and hence with each other) about the truth value of ψ which they do not. ⊣

§3. Modal logics of set-theoretic model constructions. In the following, we denote the language of propositional modal logic with a countable set of propositional variables by $\mathcal{L}_\Box$ and the first-order language of set theory by $\mathcal{L}_\in$. In [12, 13], the crucial idea for the analysis of the modal logic of forcing and the modal logic of grounds was that an assignment of sentences of $\mathcal{L}_\in$ to the propositional variables yields a translation of all formulas in $\mathcal{L}_\Box$ into sentences of $\mathcal{L}_\in$ by interpreting $\Box$ as "in all forcing extensions" or "in all grounds". These two cases are rather special since the meta-quantifiers "for all forcing extensions" and "for all grounds" are

[2]Note that we do not need to demand "rooted" in the statement of the theorem since the truths at a node in the model only depends on the generated subframe which is rooted.

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expressible in $\mathcal{L}_\in$.[3] In general, this is not true and thus the translations for other relations between models of set theory do not give formulas in $\mathcal{L}_\in$. The natural setting for this is second-order set theory. In this paper, we emulate second-order set theory by restricting our attention to transitive set models of ZFC inside a fixed meta-universe $\mathbf{V}$. We write $\mathrm{tsmst}(M)$ for the $\mathcal{L}_\in$-formula stating that M is a transitive set modelling ZFC.

In this situation, fix an *assignment* T assigning a sentence of $\mathcal{L}_\in$ to every propositional variable and let Γ be an arbitrary $\mathcal{L}_\in$-formula in two free variables. By recursion in the surrounding set-theoretic universe $\mathbf{V}$, we define for any transitive set model $M \vDash \mathsf{ZFC}$,

$$\begin{aligned}
M \vDash^T_\Gamma p &:\iff M \vDash T(p),\\
M \vDash^T_\Gamma \varphi \wedge \psi &:\iff M \vDash^T_\Gamma \varphi \text{ and } M \vDash^T_\Gamma \psi,\\
M \vDash^T_\Gamma \varphi \vee \psi &:\iff M \vDash^T_\Gamma \varphi \text{ or } M \vDash^T_\Gamma \psi,\\
M \vDash^T_\Gamma \neg\varphi &:\iff \text{not } M \vDash^T_\Gamma \varphi, \text{ and}\\
M \vDash^T_\Gamma \Box\varphi &:\iff \forall N((\Gamma(M,N) \wedge \mathrm{tsmst}(N)) \to N \vDash^T_\Gamma \varphi).
\end{aligned}$$

For every transitive set $M \vDash \mathsf{ZFC}$, we can now define

$$\begin{aligned}
\mathsf{ML}\Gamma^M &:= \{\varphi \in \mathcal{L}_\Box \,;\, \text{for all assignments } T, \text{ we have } M \vDash^T_\Gamma \varphi\} \text{ and}\\
\mathsf{ML}\Gamma &:= \bigcap\{\mathsf{ML}\Gamma^M \,;\, \mathrm{tsmst}(M)\}.
\end{aligned}$$

We define two particular cases of Γ:

$$\begin{aligned}
M \sqsupseteq_{\mathrm{IM}} N &:\iff N \subseteq M \wedge \mathrm{Ord}^M = \mathrm{Ord}^N \text{ and}\\
M \sqsupseteq_{\mathrm{G}} N &:\iff \exists \mathbb{B} \in N \exists G (N \vDash \text{“}\mathbb{B} \text{ is a complete Boolean algebra”},\\
&\qquad G \text{ is } \mathbb{B}\text{-generic over } N, \text{ and } M = N[G]).
\end{aligned}$$

For notational simplicity, we write MLIM and MLG for $\mathsf{ML}{\sqsupseteq_{\mathrm{IM}}}$ and $\mathsf{ML}{\sqsupseteq_{\mathrm{G}}}$, respectively. By general forcing theory, we have that $M \sqsupseteq_{\mathrm{G}} N$ implies $M \sqsupseteq_{\mathrm{IM}} N$, but more is true:

PROPOSITION 3.1. *Let M_0, M_1, and M_2 be transitive set models of* ZFC. *If $M_0 \sqsupseteq_{\mathrm{IM}} M_1$, $M_1 \sqsupseteq_{\mathrm{IM}} M_2$, and $M_0 \sqsupseteq_{\mathrm{G}} M_2$, then $M_0 \sqsupseteq_{\mathrm{G}} M_1$; i.e., if M_1 is an inner model in M_0, M_2 is an inner model in M_1 and M_0 is a forcing extension of M_2, then M_0 is a forcing extension of M_1.*

In order to prove Proposition 3.1, we need the following well known theorem due to Grigorieff [17, Lemma 15.43]:

THEOREM 3.2 (Grigorieff). *Let M_0 be a model of* ZFC. *Let $\mathbb{B} \in M_0$ be a complete atomless Boolean algebra. Let H be M_0-generic for $\mathbb{B}$ and $M_0[H]$ the corresponding generic extension. Let M_1 be an inner model of $M_0[H]$ such that $M_0 \subseteq M_1 \subseteq M_0[H]$. Then there is a complete atomless Boolean subalgebra $\mathbb{C}$ of $\mathbb{B}$ in M_0 such that $M_1 = M_0[\mathbb{C} \cap H]$.*

PROOF OF PROPOSITION 3.1. Let M_0, M_1, and M_2 be as in the statement. Theorem 3.2 gives that $M_1 \sqsupseteq_{\mathrm{G}} M_2$, and the Boolean algebra witnessing this is

[3] In the case of “for all forcing extensions”, this is just the Forcing Theorem [17, Theorem 14.6]; in the case of “for all grounds”, this is the Laver–Woodin theorem; cf. [18, 24].

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a complete atomless Boolean subalgebra of the Boolean algebra witnessing $M_2 \sqsupseteq_{\mathrm{G}} M_0$. By [17, Exercise 16.4], this implies $M_1 \sqsupseteq_{\mathrm{G}} M_0$. ⊣

§4. Lower and upper bounds. As before, let Γ be a formula in two variables; the formula Γ defines a (possibly) class-sized relation on the meta-universe $\mathbf{V}$. In order to determine $\mathsf{ML}\Gamma$, we give lower and upper bounds. If W is a model of set theory, we call the transitive closure of W under Γ the *multiverse of W generated by* Γ. The following easy observations about lower bounds are essentially due to [11, Theorem 7] and [19]:

PROPOSITION 4.1. *Let Γ be a relation between models of set theory.*

1. *If Γ is reflexive, then* T *is in* $\mathsf{ML}\Gamma$.
2. *If Γ is transitive, then* $\mathsf{4}$ *is in* $\mathsf{ML}\Gamma$.
3. *If Γ is directed, then* $.2$ *is in* $\mathsf{ML}\Gamma$.
4. *If Γ is topped, then* Top *is in* $\mathsf{ML}\Gamma$.

PROPOSITION 4.2. *The relation $\sqsupseteq_{\mathrm{IM}}$ is reflexive and transitive. For any transitive set model $M \models \mathsf{ZFC}$, the relation $\sqsupseteq_{\mathrm{IM}}$ restricted to the multiverse of M generated by $\sqsupseteq_{\mathrm{IM}}$ is directed and topped. Hence,* $\mathsf{S4.2Top} \subseteq \mathsf{MLIM}$.

PROOF. It is easy to see that reflexivity and transitivity hold. Since every relation that is topped is directed, we only need to show that $\sqsupseteq_{\mathrm{IM}}$ restricted to the generated subframe of M is topped: for this, consider $\mathbf{L}^M$ which is equal to $\mathbf{L}_{\mathrm{Ord}^M}$ (where $\mathbf{L}$ is the constructive universe inside our ambient set-theoretic universe). The model $\mathbf{L}^M$ is an inner model of every transitive set model of height Ord^M. ⊣

Since it makes our definitions considerably easier, from now on, assume that the (possibly class-sized) relation Γ is reflexive and transitive. The following is [11, Definition 8]:

DEFINITION 4.3. Let $(F, \leq_F)$ be a rooted Kripke frame with root w_0. A Γ-*labelling* of F for a model of set theory W is an assignment to each node w in F a set-theoretic statement Φ_w such that

1. The statements Φ_w form a mutually exclusive partition of truth in the multiverse of W generated by Γ. That is, if W' is in the multiverse of W generated by Γ, then W' satisfies exactly one of the Φ_w,
2. any W' in the multiverse of W generated by Γ in which Φ_w is true satisfies $\Diamond_\Gamma \Phi_u$ if and only if $w \leq_F u$, and
3. $W \models \Phi_{w_0}$.

The formal assertion of these properties is called the Jankov–Fine formula for F (cf. [12]). The next theorem, which is [11, Lemma 9] (which itself generalises a result from [12]), is our main technique to calculate upper bounds for $\mathsf{ML}\Gamma$.

THEOREM 4.4. *Suppose that $w \mapsto \Phi_w$ is a Γ-labelling of a finite rooted Kripke frame F with root w_0 for a model of set theory W. Then for any model M based on F, there is an assignment of the propositional variables to set theoretic assertions $p \mapsto \psi_p$ such that for any modal assertion $\varphi(p_0, p_1, \ldots, p_n)$,*

$$(M, w_0) \Vdash \varphi(p_0, p_1, \ldots, p_n) \textit{ iff } W \models \varphi(\psi_{p_0}, \psi_{p_1}, \ldots, \psi_{p_n}).$$

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In particular, any modal assertion that fails at w_0 in M also fails in W under this Γ-interpretation. Consequently, the modal logic of Γ over W is contained in the modal logic of assertions valid in F.

We use so-called *control statements* in order to prove the existence of labellings as in Theorem 4.4. In [12] and [11], the notions of *buttons* and *switches* were introduced; here, we need conditional variants of these.

Let M be a transitive set such that $M \models \mathsf{ZFC}$. A *pure Γ-button over M* is a sentence σ of $\mathcal{L}_\in$ such that for every assignment T with $T(p) = \sigma$, we have that $M \models^T_\Gamma \Box\Diamond\Box p$ and $M \models^T_\Gamma \Box(p \to \Box p)$. We say that σ is *unpushed in M* if $M \models \neg\sigma$. In the following, fix an unpushed pure Γ-button σ over M.

We say that τ is called a *Γ-σ-switch over M* if, for any transitive set $M' \models \mathsf{ZFC} \wedge \neg\sigma$ such that $\Gamma(M, M')$, there are transitive sets N and N' such that $\Gamma(M', N)$, $\Gamma(M', N')$, $N \models \mathsf{ZFC} \wedge \neg\sigma \wedge \tau$, and $N' \models \mathsf{ZFC} \wedge \neg\sigma \wedge \neg\tau$. Equivalently, for any assignment T such that $T(p) = \sigma$ and $T(q) = \tau$, we have $M \models^T_\Gamma \Box(\neg p \to \Diamond(\neg p \wedge q) \wedge \Diamond(\neg p \wedge \neg q))$.

A statement τ is called a *pure Γ-σ-button over M* if, for any assignment T such that $T(p) = \sigma$ and $T(q) = \tau$, we have $M \models^T_\Gamma \Box(p \to q) \wedge \Box((\neg p \wedge \neg q) \to \Diamond(\neg p \wedge q)) \wedge \Box(q \to \Box q)$.

A finite family $\mathcal{S} = \{s_0, \dots, s_n, b_0, \dots, b_m\}$ of Γ-σ-switches $s_0, \dots, s_n$ and pure Γ-σ-buttons $b_0, \dots, b_m$ over M is called *independent* if for any transitive set $M' \models \mathsf{ZFC}$ such that $\Gamma(M, M')$ the following hold:

1. for any $0 \leq i \leq n$, if $M' \models \neg\sigma$, then there are transitive sets N and N' such that $\Gamma(M', N)$, $\Gamma(M', N')$, $N \models \mathsf{ZFC} \wedge \neg\sigma \wedge s_i$, $N' \models \mathsf{ZFC} \wedge \neg\sigma \wedge \neg s_i$, and for any $c \in \mathcal{S} \backslash \{s_i\}$, $M' \models c$ if and only if $N \models c$ if and only if $N' \models c$;
2. for any $0 \leq i \leq m$, if $M' \models \neg\sigma$, then there is a transitive set N such that $\Gamma(M', N)$, $N \models \mathsf{ZFC} \wedge \neg\sigma \wedge b_i$, and for any $c \in \mathcal{S} \backslash \{b_i\}$, $M' \models c$ if and only if $N \models c$.

We omit Γ from the notation when it is clear from the context. The proof of the following theorem is similar to that of [11, Theorem 13]:

Theorem 4.5. *Let Γ be a formula defining a reflexive and transitive relation between transitive sets $N \models \mathsf{ZFC}$, and let M be a model of ZFC such that there is an unpushed pure button σ in M. Suppose that there are arbitrarily large finite families of mutually independent unpushed pure σ-buttons and σ-switches over M. Then*

(1) *any inverted lollipop can be Γ-labelled over M, and*
(2) $\mathsf{ML}\Gamma^M$ *is contained within* $\mathsf{S4.2Top}$.

Proof. We show (1). Claim (2) then follows from the conjunction of Theorems 2.6 and 4.4.

Let L be a frame which is an inverted lollipop. Let F be the quotient partial order of L under the natural equivalence relation. Then F is a finite Boolean algebra with a single extra node on top; in particular, F is a topped partial order. Therefore, the partial order of nonmaximal elements of F is isomorphic to the power set algebra $\wp(A)$ for some finite set A. We fix such a set A.

With each element $a \in F$ which is not maximal, there is associated a cluster $w^a_1, w^a_2, \dots, w^a_{k_a}$ of worlds of L. By adding dummy nodes to each cluster, we may assume that there is some natural number m such that for each nonmaximal $a \in F$, the sizes k_a of the complete clusters at node a are the same, and equal to 2^m.

Also, suppose that F has size $2^n + 1$, that is, the size of A is n, so there are n atoms in the Boolean algebra of nonmaximal elements of F. We can therefore think of the Boolean algebra of nonmaximal elements of F as the worlds w_j^a where $a \subseteq A$, and $j < 2^m$, with the order obtained by $w_j^a \leq w_i^c$ if and only if $a \subseteq c$. Also, since F consists exactly of this pre-Boolean algebra and a single extra node above every element of it, we can consider F as being made up of worlds w_j^a where $a \subseteq A$, and $j < 2^m$, with the order obtained by $w_j^a \leq w_i^c$ if and only if $a \subseteq c$, and a node t with $w_j^a < t$ for each a and j.

Associate with each element $i \in A$ an unpushed pure σ-button b_i such that the collection $\{b_i \mid i \in A\}$ form a mutually independent family with m-many σ-switches $s_0, s_1, \ldots, s_{m-1}$. Without loss of generality, we can assume that all the σ-switches are off in M. For $j < 2^m$, let $\bar{s}_j$ be the assertion that the pattern of switches corresponds to the binary digits of j (i.e., s_k is true if and only if the kth binary bit of j is 1). We associate the node w_j^a with the assertion

$$\Phi_{w_j^a} = \neg\sigma \wedge (\bigwedge_{i \in a} b_i) \wedge \bar{s}_j,$$

and we associate the node t with the assertion $\Phi_t = \sigma$. Now, if W is a model in the multiverse of Γ generated by M, and $W \models \Phi_{w_j^a}$, then by the mutual independence of buttons and switches combined with our remark that pushing these buttons cannot push the button σ, we see that $W \models \Diamond\Phi_{w_r^c}$ if and only if $a \subseteq c$.

Also, for any model W in the multiverse of M generated by Γ, if $W \models \neg\sigma$, then as σ is itself a button, it follows that $W \models \Diamond\Phi_t$. Therefore, if $W \models \Phi_{w_j^a}$, then $W \models \Diamond\Phi_t$. Also, since all of these buttons and switches are off in M, we have $M \models \Phi_{w_0^{\varnothing}}$. Thus, we have provided a Γ-labelling of this frame for W, hence demonstrating that we can label all inverted lollipops. The result follows. $\dashv$

§**5. The modal logic of inner models.** In [22], Reitz constructed a *bottomless* model which is not a set forcing extension of the constructible universe **L** and in which there is a class-sized descending sequence of grounds. Furthermore, we know that in this model, the relation $\sqsupseteq_{\mathrm{G}}$ is reflexive, transitive and directed. This was used in [13] to calculate MLG in that model to be S4.2; we are reusing parts of this proof in our main result.

Definition 5.1. Let κ be a regular cardinal. The forcing poset $\mathrm{Add}(\kappa)$ which adds a *Cohen subset of* κ is the following:

1. $p \in \mathrm{Add}(\kappa)$ if p is a function and there is a $\gamma < \kappa$ such that $\mathrm{dom}(p) = \gamma$ and $\mathrm{ran}(p) \subseteq \{0, 1\}$.
2. If $p, q \in \mathrm{Add}(\kappa)$, then $p \leq q$ if $p \supseteq q$.

Definition 5.2. Let $\mathrm{SuccR}^{\mathbf{L}}$ denote the class of successors of infinite regular cardinals in **L**. Define in **L** the following (class-sized) partial order with Easton support:

$$\mathbb{P} := \prod_{\gamma \in \mathrm{SuccR}^{\mathbf{L}}} \mathrm{Add}(\gamma).$$

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That is, $p \in \mathbb{P}$ if

1. p is a function such that $\text{dom}(p) \subseteq \text{SuccR}^{\mathbf{L}}$;
2. for each $\gamma \in \text{dom}(p)$, $p(\gamma) \in \text{Add}(\gamma)$;
3. for each regular cardinal γ, $|\text{dom}(p) \cap \gamma| < \gamma$.

The ordering is defined by $p \leq q$ if $p \supseteq q$.

Also, for each $p \in \mathbb{P}$ and each $\gamma \in \text{SuccR}^{\mathbf{L}}$, we can decompose p into three parts:

$$p_{<\gamma} = p \restriction [0, \gamma);$$
$$p_{\gamma} = p \restriction [\gamma, \gamma];$$
$$p_{>\gamma} = p \restriction (\gamma, \infty).$$

Using this decomposition, for each $\gamma \in \text{SuccR}^{\mathbf{L}}$, we can decompose $\mathbb{P}$ into three parts:

$$\mathbb{P}_{<\gamma} = \{p_{<\gamma} \mid p \in \mathbb{P}\};$$
$$\mathbb{P}_{\gamma} = \{p_{\gamma} \mid p \in \mathbb{P}\};$$
$$\mathbb{P}_{>\gamma} = \{p_{>\gamma} \mid p \in \mathbb{P}\}.$$

It is clear that $\mathbb{P} \cong \mathbb{P}_{<\gamma} \times \mathbb{P}_{\gamma} \times \mathbb{P}_{>\gamma}$.

THEOREM 5.3. *Assume* $\mathbf{V}=\mathbf{L}$. *Let γ be an infinite successor cardinal. Let $\mathbb{Q}_\gamma := \mathbb{P}_{<\gamma} \times \mathbb{P}_{>\gamma}$. Then forcing with $\mathbb{Q}_\gamma$ does not add a Cohen subset of γ.*

PROOF. This result is essentially due to McAloon [21]. Since $\mathbb{P}_{>\gamma}$ is clearly $\leq \gamma$-closed, we only need to show that forcing with $\mathbb{P}_{<\gamma}$ does not add a Cohen subset of γ. But for this, notice that by GCH and the fact that γ is the successor of a regular cardinal, it has the γ-c.c. and consequently does not add Cohen subsets of γ. ⊣

Fix any transitive set $M \vDash \text{ZFC}$, consider $\mathbb{P}$ to be defined in $\mathbf{L}^M$, and let G be $\mathbf{L}^M$-generic for $\mathbb{P}$, and let $R := \mathbf{L}^M[G]$. If γ is an infinite successor cardinal of $\mathbf{L}^M$, we write $G_{>\gamma} := G \cap \mathbb{P}_{>\gamma}$. Let

$$\tau := \forall \kappa \in \text{SuccR}^{\mathbf{L}} \exists G \subseteq \kappa (G \text{ is an } \mathbf{L}\text{-Cohen subset of } \kappa) \text{ and}$$
$$\sigma := \forall \mathbb{B}[(\mathbb{B} \text{ is a complete atomless Boolean algebra}) \to (\|\tau\|_{\mathbb{B}} \neq 1_{\mathbb{B}})].$$

Clearly, $R \vDash \tau$; the statement σ expresses that τ cannot be forced by set forcing. Therefore, if $R \sqsupseteq_{\text{IM}} N$ with $N \vDash \sigma$, then any further inner model M with $N \sqsupseteq_{\text{IM}} M$ will satisfy $M \vDash \sigma$. Hence, σ is a pure $\sqsupseteq_{\text{IM}}$-button (and unpushed in R).

LEMMA 5.4. *Let N be a ground of R. Then in $\mathbf{L}^M$, there is an infinite successor cardinal γ such that $N \supseteq \mathbf{L}^M[G_{>\gamma}]$. In particular, $\mathbf{L}^M[G_{>\gamma}]$ is a ground (and hence, an inner model) of N.*

PROOF. This lemma is due to Reitz [22, proof of Theorem 23]. Towards a contradiction, suppose this is not so. Let $\mathbb{Q} \in N$ be a forcing poset and let H be $\mathbb{Q}$-generic over N such that $R = N[H]$. For some infinite successor cardinal γ large enough, let x be a name for $G_{>\gamma}$. Let $p \in \mathbb{Q}$ be such that

$$p \Vdash \text{“}x \text{ is } \mathbb{P}_{>\gamma}\text{-generic over } \mathbf{L}^M\text{”}.$$

By assumption, $G_{>\gamma}$ is a function of size Ord^M which is not in N, but in a set forcing extension of N by $\mathbb{Q}$. Therefore, working in N, for any $q \leq p$, q can decide only a set sized initial segment of $G_{>\gamma}$. However, in N, for every $\beta > \gamma$, there is a $r \leq p$

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such that r decides $G_{>\gamma}\upharpoonright(\gamma,\beta)$. Therefore, we can form a strictly increasing chain of length $\mathrm{Ord}^M = \mathrm{Ord}^N$ of conditions in $\mathbb{Q}$, thus contradicting that it is a set in N. ⊣

LEMMA 5.5. *In R, there are arbitrarily large finite independent families of σ-switches and pure σ-buttons. Consequently,* $\mathsf{MLIM}^R = \mathsf{S4.2Top}$.

PROOF. We use control statements from [13, Theorem 6]. Let b_n be the statement "there is no $\mathbf{L}^M$-generic subset of $\aleph_n^{\mathbf{L}^M}$". Partition the successor cardinals in $\mathbf{L}^M$ above $\aleph_\omega$ into $\aleph_0$-many classes, $\langle \Gamma_n \rangle_{n\in\omega}$ such that each class contains unboundedly many cardinals. Enumerate each class as $\Gamma_n = \{\gamma_\alpha^n \mid \alpha \in \mathrm{Ord}^M\}$. Let s_n be the statement "the least α such that there is an $\mathbf{L}^M$-generic subset of γ_α^n is even". In [13, Theorem 6], it was shown that these are $\sqsupseteq_{\mathrm{G}}$-switches over R which in our setting means that they are $\sqsupseteq_{\mathrm{IM}}$-$\sigma$-switches over R. We use Theorem 5.3 to see that the family of the b_n and s_n is an independent family of pure $\sqsupseteq_{\mathrm{IM}}$-$\sigma$-buttons and $\sqsupseteq_{\mathrm{IM}}$-$\sigma$-switches. As mentioned above, σ is a pure button; so we can appeal to Theorems 4.5 and 2.6 to see that we are done. ⊣

This, together with Proposition 4.2, gives us our main theorem:

THEOREM 5.6. *If there is a transitive set $M \models \mathsf{ZFC}$, then* $\mathsf{MLIM} = \mathsf{S4.2Top}$.

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