

## SIMPLE PROOF FOR GOOFSPIEL: THE GAME OF PURE STRATEGY

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### Abstract

A simple proof is given for the best strategy when the game of pure strategy is played.

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Following Ross (1983), we examined the game of pure strategy referred to as Goofspiel and played by two players using a normal deck of cards. The description is taken almost literally from Ross (1983).

First, the 13 clubs are discarded. Of the remaining 39 cards, the 13 hearts are given to Player I, the 13 diamonds to Player II, and the 13 spades are placed face down in the center. The spades are shuffled and the top one is turned face up. At this point the two players choose one of their cards and then simultaneously discard them. The one who discards the higher card wins from the other an amount equal to the value of the upturned spade. In the case of a draw (both players discarded the same card) then neither wins, the three upturned cards are thrown away and a new spade is upturned. After 13 plays, there are no remaining cards and the game ends.

Consider the game under the assumption that Player II discards his cards in a random manner and that Player I knows that this random strategy was adopted by Player II. In this situation Ross proves that the best way for Player I to play is always to match the upturned spade. In a more general setting there are  $N$  different cards  $V_1, \dots, V_N$  (instead of 13) having values  $V_1 \leq V_2 \leq \dots \leq V_N$  (the hand of player I), simultaneously  $Y_1 \leq Y_2 \leq \dots \leq Y_N$  (the hand of Player II), and  $S_1 \leq S_2 \leq \dots \leq S_N$  the spades. The game is played as before.

*Theorem.* If Player II discards in completely random manner, then the strategy maximizing Player I's expected winning is the one that discards the card having value  $V_i$  whenever the upturned spade has value  $S_i$ ,  $i = 1, \dots, N$ .

*Proof.* Observe that given any two sets of numbers  $A$  and  $B$  of equal cardinality  $A = \{a_i\}_{i=1}^n$ ,  $B = \{b_i\}_{i=1}^n$  and let  $\pi, \psi$  be any two permutations of  $\{1, 2, \dots, n\}$ . Denote  $a_\pi = (a_{\pi(1)}, a_{\pi(2)}, \dots, a_{\pi(n)})$ ,  $b_\psi = (b_{\psi(1)}, b_{\psi(2)}, \dots, b_{\psi(n)})$  scalar product

$$a_\pi \cdot b_\psi = \sum_{i=1}^n a_{\pi(i)} b_{\psi(i)}.$$

Then

$$\max_{\pi, \psi} a_\pi \cdot b_\psi = \sum_{i=1}^n a_i b_i, \quad \text{where } a_1 \leq a_2 \leq \dots \leq a_n; b_1 \leq b_2 < \dots \leq b_n.$$

Given  $N$  cards in the game, denote the probability that Player I wins given he discards the card  $V_i$  by  $P_i$ . Clearly  $P_1 \leq P_2 \leq \dots \leq P_N$ . These probabilities are independent of the upturned value of the spade card  $S_i$ , and are non-increasing with  $N$ . Thus, the scalar product of the

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ordered sequence of the spades vector (the  $\{S_i\}_{i=1}^N$ ) with the win probabilities takes the maximal value when matched in order.

This optimal strategy proof does not use an induction argument or the interchange argument of the proof in Ross (1983).

### Reference

Ross, S. M. (1983) *Introduction to Stochastic Dynamic Programming*. Academic Press, New York.