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FINITELY GENERATED δ-SUPPLEMENTED MODULES ARE AMPLY δ-SUPPLEMENTED

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Abstract

Let R be a commutative ring. It is shown that if an R-module M is a sum of δ-local submodules and a semisimple projective submodule, then every finitely generated submodule of M is δ-supplemented. From this result, we conclude that finitely generated δ-supplemented modules over commutative rings are amply δ-supplemented.

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1. Introduction

Throughout this paper R will denote an associative commutative ring with identity and all modules are unital R-modules. Recall that a submodule N of a module M is said to be *δ-small* in M, written $N \ll_\delta M$, provided $M \neq N + X$ for any proper submodule X of M with M/X singular. Let L be a submodule of a module M. A submodule K of M is called a *δ-supplement* of L in M provided $M = L + K$ and $M \neq L + X$ for any proper submodule X of K with K/X singular—equivalently, $M = L + K$ and $L \cap K \ll_\delta K$. The module M is called *δ-supplemented* if every submodule of M has a δ-supplement in M. On the other hand, the submodule N is said to have ample δ-supplements in M if every submodule L of M with $M = N + L$ contains a δ-supplement of N in M. The module M is called amply δ-supplemented if every submodule of M has ample δ-supplements in M. Let $\mathbf{P}$ be the class of all singular simple modules. Let M be any module. As in [7], let $\delta(M) = Rej_M(\mathbf{P}) = \bigcap\{N \leq M \mid M/N \in \mathbf{P}\}$. It is shown in [7, Lemma 1.5(1)] that $\delta(M) = \sum\{N \leq M \mid N \ll_\delta M\}$.

As in [3], a module M is said to be *δ-local* if $\delta(M) \ll_\delta M$ and $\delta(M)$ is a maximal submodule of M. For an R-module M, let $\mathrm{ann}(M) = \{r \in R \mid rM = 0\}$ and for any $x \in M$, let $\mathrm{ann}(x) = \{r \in R \mid rx = 0\}$. Let L be a cyclic δ-local module. Then $L \cong R/a$ with $a = \mathrm{ann}(L)$. Since L is δ-local, there exists a maximal ideal m of R such that $a \subseteq m$ and $\delta(R/a) = m/a$. In this case, we call the module L *m-δ-local*.

2. Main results

We begin with a lemma taken from [7, Lemmas 1.2, 1.3 and 1.5].

LEMMA 2.1. *Let M be a module.*

(1) *A submodule $N \le M$ is δ-small if and only if, for all submodules $X \le M$, if $M = X + N$, then $M = X \oplus Y$ for a semisimple projective submodule Y with $Y \subseteq N$.*
(2) *For submodules N and L of M, $N + L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.*
(3) *If $K \ll_\delta M$ and $f : M \to N$ is a homomorphism, then $f(K) \ll_\delta N$. In particular, if $K \ll_\delta M \subseteq N$, then $K \ll_\delta N$.*
(4) *If $M = \bigoplus_{i \in I} M_i$, then $\delta(M) = \bigoplus_{i \in I} \delta(M_i)$.*
(5) *If M is finitely generated, then $\delta(M) \ll_\delta M$.*

It is well known that if a is an ideal of R, then a is essential in R if and only if R/a is a singular R-module (see, for example, [4, p. 32]).

Let S be a simple R-module. Then S is either singular or projective, but not both (see [4, Proposition 1.24]). Therefore S is either δ-local or projective.

A submodule L of a module M is called *small* in M if $L + X \neq M$ for every proper submodule X of M. Let N be a submodule of a module M. A submodule K of M is called a *supplement* of N in M provided $M = N + K$ and $N \cap K$ is small in K. The module M is called *supplemented* if every submodule of M has a supplement in M. On the other hand, a submodule N of a module M has *ample supplements* in M if every submodule L such that $M = N + L$ contains a supplement of N in M. The module M is called *amply supplemented* if every submodule has ample supplements in M.

The following example shows that a δ-supplemented module need not be amply δ-supplemented.

EXAMPLE 2.2. Let R be an incomplete discrete valuation ring with field of fractions Q. Then the R-module $Q \oplus Q$ is supplemented but not amply supplemented by [8, Theorem 2.2]. Let m be the maximal ideal of R. Clearly m is essential in R. Thus the simple R-module R/m is singular. Hence every simple R-module is singular. So R has no simple projective R-modules. Now let N be a δ-small submodule of an R-module M and let X be a submodule of M with $N + X = M$. By Lemma 2.1(1), $M = Y \oplus X$ for a projective semisimple submodule Y with $Y \subseteq N$. This clearly forces $Y = 0$ and $X = M$. So N is small in M. Consequently, the R-module $Q \oplus Q$ is δ-supplemented but not amply δ-supplemented.

LEMMA 2.3.

(1) *If a is a proper ideal of R, then the module R/a is semisimple projective if and only if $a = \bigcap_{i=1}^{k} m_i$ is a finite intersection of nonessential maximal ideals m_i $(1 \le i \le k)$.*
(2) *For any module M, $M \ll_\delta M$ if and only if M is semisimple projective.*

PROOF. (1) This follows from the Chinese remainder theorem.

(2) This follows from Lemma 2.1(1). □

Proposition 2.4. *Let a be an ideal of R. The following conditions are equivalent:*

(i) *R/a is an m-δ-local module;*
(ii) *m is essential in R and is the only essential maximal ideal of R which contains a.*

Proof. (i) $\Rightarrow$ (ii) Suppose that m is not essential in R. Then there exists a simple ideal b of R such that $m \oplus b = R$. Clearly, $(m/a) \oplus (b + a)/a = R/a$. Thus $(b + a)/a \nsubseteq \delta(R/a)$ since $\delta(R/a) = m/a$. Hence the simple module $b \cong (b + a)/a$ is singular by Lemmas 2.3(2) and 2.1(3). That is, R/m is a singular R-module. Therefore m is essential in R, a contradiction. Now suppose that R has an essential maximal ideal $m' \neq m$ with $a \subseteq m'$. Then $(R/a)/(m'/a) \cong R/m'$ is a singular R-module. So $\delta(R/a) = m/a \subseteq m'/a$, a contradiction.

(ii) $\Rightarrow$ (i) Note first that $R/m \cong (R/a)/(m/a)$ is singular. Moreover, for any maximal ideal m' of R with $m' \neq m$ and $a \subseteq m'$, $R/m' \cong (R/a)/(m'/a)$ is projective. Therefore $\delta(R/a) = m/a$. By Lemma 2.1(5), $\delta(R/a) \ll_\delta R/a$. So R/a is m-δ-local. □

Let m be a maximal ideal of R and let M be an R-module. Consider the subset $K_{\delta_m}(M) \subseteq M$ of elements $x \in M$ such that:

(i) $x = 0$; or
(ii) m is essential in R and is the only essential maximal ideal of R which contains $\mathrm{ann}(x)$; or
(iii) $\mathrm{ann}(x) = \bigcap_{i=1}^{k} m_i$ such that each m_i $(1 \leq i \leq k)$ is a nonessential maximal ideal of R.

Proposition 2.5. *Let x be an element of a module M and let m be a maximal ideal of R. The following are equivalent:*

(i) *$x \in K_{\delta_m}(M)$;*
(ii) *$R/\mathrm{ann}(x)$ is m-δ-local or semisimple projective.*

Proof. This is a consequence of Lemma 2.3 and Proposition 2.4. □

Proposition 2.6. *A factor module of an m-δ-local module is either m-δ-local or semisimple projective.*

Proof. Let a be an ideal of R such that the R-module R/a is m-δ-local. Let b be an ideal of R with $a \subseteq b$. Note that $\delta(R/b) \ll_\delta R/b$ by Lemma 2.1(5). Consider the canonical epimorphism $\pi : R/a \to R/b$. We have $\pi(m/a) = (m + b)/b \ll_\delta R/b$ by Lemma 2.1(3). If $b \subseteq m$, then $m/b \ll_\delta R/b$. Therefore $m/b \subseteq \delta(R/b)$. This implies that $\delta(R/b) = R/b \ll_\delta R/b$ or $\delta(R/b) = m/b \ll_\delta R/b$. If $b \nsubseteq m$, then $R/b \ll_\delta R/b$. Therefore R/b is semisimple projective or m-δ-local (see Lemma 2.3(2)). □

Proposition 2.7. *Let M be an R-module and let m be a maximal ideal of R. Then $K_{\delta_m}(M)$ is a submodule of M.*

Proof. (1) Let us show that $K_{\delta_m}(M)$ is closed under multiplication by elements of R. Let $x \in K_{\delta_m}(M)$ and let $r \in R$. Let $a = \mathrm{ann}(x)$ and let $b = \mathrm{ann}(rx)$. Note that $a \subseteq b$.

By Proposition 2.5, R/a is m-δ-local or semisimple projective. Note that $R/b \cong (R/a)/(b/a)$. From Proposition 2.6 it follows that R/b is m-δ-local or semisimple projective. So $rx \in K_{\delta_m}(M)$ by Proposition 2.5.

(2) Let us show that $K_{\delta_m}(M)$ is an additive subgroup of M. Let $x_1, x_2 \in K_{\delta_m}(M)$, $a_1 = \operatorname{ann}(x_1)$, $a_2 = \operatorname{ann}(x_2)$ and $a = \operatorname{ann}(x_1 - x_2)$. Then $a_1 \cap a_2 \subseteq a$.

If $x_1 = 0$ or $x_2 = 0$, then of course $x_1 - x_2 \in K_{\delta_m}(M)$.

Suppose that R/a_1 is m-δ-local and R/a_2 is m-δ-local or semisimple projective. Since m/a_1 is essential in R/a_1, $m/(a_1 \cap a_2)$ is essential in $R/(a_1 \cap a_2)$ by [4, Proposition 1.1]. Moreover, if $a_1 \cap a_2 \subseteq m'$ for some maximal ideal $m' \neq m$, then $a_1 \subseteq m'$ or $a_2 \subseteq m'$. Therefore m' is not essential in R (see Lemma 2.3 and Proposition 2.4). Thus $R/(a_1 \cap a_2)$ is m-δ-local by Proposition 2.4. Since

$$R/a \cong \frac{R/(a_1 \cap a_2)}{a/(a_1 \cap a_2)},$$

R/a is m-δ-local or semisimple projective by Proposition 2.6. So $x_1 - x_2 \in K_{\delta_m}(M)$ by Proposition 2.5.

Assume that both of R/a_1 and R/a_2 are semisimple projective. Note that $Rx_1 + Rx_2$ is semisimple projective. Since $R(x_1 - x_2) \leq Rx_1 + Rx_2$, $R/a \cong R(x_1 - x_2)$ is semisimple projective. Thus $x_1 - x_2 \in K_{\delta_m}(M)$. □

Recall that a submodule N of a module M is called *cofinite* if M/N is finitely generated.

Proposition 2.8. *Let M be a left module over any ring (not necessarily commutative). Suppose that every finitely generated submodule of M is δ-supplemented. Then every cofinite submodule of M has ample δ-supplements.*

Proof. Let N be a cofinite submodule of M and let L be a submodule of M such that $M = N + L$. Then there exists a finitely generated submodule $F \leq L$ such that $M = N + F$. Consider the submodule $N \cap F \leq F$. By assumption, there exists a submodule $K \leq F \leq L$ such that $(N \cap F) + K = F$ and $N \cap K \ll_\delta K$. Since $N + K = M$, K is a δ-supplement of N in M. This completes the proof. □

Theorem 2.9. *Let m be a maximal ideal of R and M be a module such that $K_{\delta_m}(M) = M$. Then every finitely generated submodule of M is δ-supplemented.*

Proof. Let $x \in M$. Then $Rx \cong R/\operatorname{ann}(x)$ is m-δ-local or semisimple projective by Proposition 2.5. Therefore every finitely generated submodule of M is a finite sum of δ-local submodules and simple projective submodules. The result follows from [3, Proposition 3.5]. □

Corollary 2.10. *Let m be a maximal ideal of R. Let M be a module such that $K_{\delta_m}(M) = M$. Then every cofinite submodule of M has ample δ-supplements.*

Proof. This follows by Proposition 2.8 and Theorem 2.9. □

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For any commutative ring R, let $\mathrm{Soc}_P(R)$ denote the sum of all simple projective ideals of R.

Lemma 2.11. *Let a module $M = \bigoplus_{i\in I} M_i$ be a direct sum of submodules M_i ($i \in I$). Assume that for all $i \neq j$ in I, $\mathrm{ann}(M_i) + \mathrm{ann}(M_j)$ is a direct summand of R and $R/(\mathrm{ann}(M_i) + \mathrm{ann}(M_j))$ is semisimple. Then, for every submodule N of M,*

$$N \subseteq \bigoplus_{i\in I} ((N \cap M_i) + \mathrm{Soc}_P(R)M_i).$$

Proof. Let N be a submodule of M. Let $x \in N$. Then there exist a positive integer n, distinct elements $i_j \in I$ $(1 \leq j \leq n)$ and elements $x_j \in M_{i_j}$ $(1 \leq j \leq n)$ such that $x = x_1 + \cdots + x_n$. If $n = 1$, then $x = x_1 \in N \cap M_{i_1} + \mathrm{Soc}_P(R)M_{i_1}$. Suppose that $n \geq 2$. By hypothesis, there exists a semisimple ideal A_{1n} of R such that

$$R = (\mathrm{ann}(M_{i_1}) + \mathrm{ann}(M_{i_n})) \oplus A_{1n}.$$

So there exist elements r, s and t in R such that $rx_1 = 0$, $sx_n = 0$, $t \in A_{1n}$ and $1_R = r + s + t$. So

$$\begin{aligned} sx &= sx_1 + sx_2 + \cdots + sx_n \\ &= sx_1 + sx_2 + \cdots + sx_{n-1} \\ &= (1_R - r - t)x_1 + sx_2 + \cdots + sx_{n-1} \\ &= (1_R - t)x_1 + sx_2 + \cdots + sx_{n-1}. \end{aligned}$$

Note that $sx \in N$, $(1_R - t)x_1 \in M_{i_1}$ and $sx_j \in M_{i_j}$ $(2 \leq j \leq n-1)$. By induction on n, $(1_R - t)x_1 \in N \cap M_{i_1} + \mathrm{Soc}_P(R)M_{i_1}$. Thus $x_1 \in N \cap M_{i_1} + \mathrm{Soc}_P(R)M_{i_1} + A_{1n}M_{i_1}$. Clearly $A_{1n} \subseteq \mathrm{Soc}_P(R)$. Hence $x_1 \in N \cap M_{i_1} + \mathrm{Soc}_P(R)M_{i_1}$. In the same manner we can prove that $x_j \in N \cap M_{i_j} + \mathrm{Soc}_P(R)M_{i_j}$ $(2 \leq j \leq n)$. □

Lemma 2.12. *Let M be any module. Then $\mathrm{Soc}_P(R)M$ is a semisimple projective module. In particular, $\mathrm{Soc}_P(R)M \subseteq \mathrm{Soc}(M)$.*

Proof. Let S be a simple projective ideal of R. Then there exists a maximal ideal m of R such that $S \cong R/m$. Let $x \in M$ and let $\alpha \in S$. It is clear that $m(\alpha x) = 0$. So $m \subseteq \mathrm{ann}(\alpha x)$. Thus $R(\alpha x) = 0$ or $R(\alpha x) \cong R/m$ is simple projective. It follows that SM is semisimple projective. Therefore $\mathrm{Soc}_P(R)M$ is semisimple projective. □

Proposition 2.13. *Let a module $M = M_1 \oplus \cdots \oplus M_n$ be a finite direct sum of submodules M_i $(1 \leq i \leq n)$, for some positive integer $n \geq 2$. Assume that for all $1 \leq i < j \leq n$, $\mathrm{ann}(M_i) + \mathrm{ann}(M_j)$ is a direct summand of R and $R/(\mathrm{ann}(M_i) + \mathrm{ann}(M_j))$ is semisimple. If finitely generated submodules of M_i are δ-supplemented for all $1 \leq i \leq n$, then finitely generated submodules of M are δ-supplemented.*

Proof. Let N be a finitely generated submodule of M. By Lemma 2.11,

$$N \subseteq \bigoplus_{i=1}^{n}((N \cap M_i) + \mathrm{Soc}_P(R)M_i).$$

That is,

$$N \subseteq \left(\bigoplus_{i=1}^{n}(N \cap M_i)\right) + \mathrm{Soc}_P(R)M.$$

Therefore

$$N + \mathrm{Soc}_P(R)M = \left(\bigoplus_{i=1}^{n}(N \cap M_i)\right) + \mathrm{Soc}_P(R)M.$$

Since N is finitely generated, there exist finitely generated submodules $K_i \leq N \cap M_i$ $(1 \leq i \leq n)$ such that

$$N + \mathrm{Soc}_P(R)M = \left(\bigoplus_{i=1}^{n} K_i\right) + \mathrm{Soc}_P(R)M.$$

By hypothesis, the K_i $(1 \leq i \leq n)$ are δ-supplemented. Since $\mathrm{Soc}_P(R)M$ is δ-supplemented, $N + \mathrm{Soc}_P(R)M$ is δ-supplemented by [6, Proposition 3.5]. As $\mathrm{Soc}_P(R)M$ is semisimple, N is a direct summand of $N + \mathrm{Soc}_P(R)M$. So N is δ-supplemented by [6, Proposition 3.6]. □

Lemma 2.14. *Let a, b, c and d be ideals of the ring R such that $a \subseteq c$ and $b \subseteq d$. If $c/a \ll_\delta R/a$ and $d/b \ll_\delta R/b$, then $(c \cap d)/(a \cap b) \ll_\delta R/(a \cap b)$.*

Proof. Let u be an ideal of R containing $a \cap b$ such that $(c \cap d)/(a \cap b) + u/(a \cap b) = R/(a \cap b)$ and R/u is singular. Then

$$((c \cap d) + a)/a + (u + a)/a = R/a \quad \text{and} \quad ((c \cap d) + b)/b + (u + b)/b = R/b.$$

Hence $(c/a) + (u + a)/a = R/a$ and $(d/b) + (u + b)/b = R/b$. Moreover,

$$\frac{R/a}{(u+a)/a} \cong \frac{R/u}{(u+a)/u} \quad \text{and} \quad \frac{R/b}{(u+b)/b} \cong \frac{R/u}{(u+b)/u}.$$

Thus $(R/a)/((u + a)/a)$ and $(R/b)/((u + b)/b)$ are singular modules by [4, Proposition 1.22(b)]. By hypothesis, $R = u + a = u + b$. So $R = RR = (u + a)(u + b) = u + ab$. But $ab \subseteq a \cap b \subseteq u$. Then $u = R$. This completes the proof. □

Proposition 2.15. *Let m_1 and m_2 be maximal ideals of R with $m_1 \neq m_2$. Let a module $M = M_1 \oplus M_2$ such that M_1 is a finite direct sum of cyclic m_1-δ-local submodules and M_2 is a finite direct sum of cyclic m_2-δ-local submodules. Then $a = \mathrm{ann}(M_1) + \mathrm{ann}(M_2)$ is a direct summand of R and R/a is semisimple.*

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Proof. Assume that $M_1 = \bigoplus_{i=1}^{k_1}(R/a_{1i})$ and $M_2 = \bigoplus_{i=1}^{k_2}(R/a_{2i})$, where the a_{1i} $(1 \le i \le k_1)$ and a_{2i} $(1 \le i \le k_2)$ are ideals of R such that R/a_{1i} $(1 \le i \le k_1)$ are m_1-δ-local modules and R/a_{2i} $(1 \le i \le k_2)$ are m_2-δ-local modules. Then $\mathrm{ann}(M_1) = \bigcap_{i=1}^{k_1} a_{1i}$ and $\mathrm{ann}(M_2) = \bigcap_{i=1}^{k_2} a_{2i}$. Note that $m_1/a_{1i} \ll_\delta R/a_{1i}$ $(1 \le i \le k_1)$ and $m_2/a_{2i} \ll_\delta R/a_{2i}$ $(1 \le i \le k_2)$. Therefore

$$m_1\Big/\Big(\bigcap_{i=1}^{k_1} a_{1i}\Big) \ll_\delta R\Big/\Big(\bigcap_{i=1}^{k_1} a_{1i}\Big) \quad \text{and} \quad m_2\Big/\Big(\bigcap_{i=1}^{k_2} a_{2i}\Big) \ll_\delta R\Big/\Big(\bigcap_{i=1}^{k_2} a_{2i}\Big)$$

by Lemma 2.14. That is, $m_1/\mathrm{ann}(M_1) \ll_\delta R/\mathrm{ann}(M_1)$ and $m_2/\mathrm{ann}(M_2) \ll_\delta R/\mathrm{ann}(M_2)$. By Lemma 2.1(3), $(m_1 + a)/a \ll_\delta R/a$ and $(m_2 + a)/a \ll_\delta R/a$. Thus

$$(m_1 + m_2 + a)/a \ll_\delta R/a$$

by Lemma 2.1(2). So $R/a \ll_\delta R/a$. By Lemma 2.3, R/a is semisimple projective. Therefore a is a direct summand of R by [2, Proposition 17.2]. □

Proposition 2.16. *Let m_1 be a maximal ideal of R and let a module $M = M_1 \oplus M_2$ be such that M_1 is a finite direct sum of cyclic m_1-δ-local submodules and M_2 is a finite direct sum of simple projective submodules. Then $b = \mathrm{ann}(M_1) + \mathrm{ann}(M_2)$ is a direct summand of R and R/b is semisimple.*

Proof. Assume that $M_1 = \bigoplus_{i=1}^{k_1}(R/b_{1i})$ and $M_2 = \bigoplus_{i=1}^{k_2}(R/m_{2i})$, where the b_{1i} $(1 \le i \le k_1)$ are ideals of R and m_{2i} $(1 \le i \le k_2)$ are maximal ideals of R such that R/b_{1i} $(1 \le i \le k_1)$ are cyclic m_1-δ-local and R/m_{2i} $(1 \le i \le k_2)$ are simple projective. Then $\mathrm{ann}(M_1) = \bigcap_{i=1}^{k_1} b_{1i}$, $\mathrm{ann}(M_2) = \bigcap_{i=1}^{k_2} m_{2i}$ and $b = (\bigcap_{i=1}^{k_1} b_{1i}) + (\bigcap_{i=1}^{k_2} m_{2i})$. Note that $m_1/b_{1i} \ll_\delta R/b_{1i}$ for all $1 \le i \le k_1$. It follows from Lemma 2.14 that $m_1/(\bigcap_{i=1}^{k_1} b_{1i}) \ll_\delta R/(\bigcap_{i=1}^{k_1} b_{1i})$. By Lemma 2.1(3), $(m_1 + b)/b \ll_\delta R/b$. Suppose that $b \subseteq m_1$. Then $\bigcap_{i=1}^{k_2} m_{2i} \subseteq m_1$. Hence $m_{2j} \subseteq m_1$ for some $1 \le j \le k_2$. We thus get $m_{2j} = m_1$. Therefore R/m_1 is projective. This contradicts the fact that m_1 is essential in R (see Proposition 2.4 and [4, Proposition 1.24]). It follows that $b \nsubseteq m_1$ and hence $m_1 + b = R$. Then $R/b \ll_\delta R/b$. By Lemma 2.3, R/b is semisimple projective. Therefore b is a direct summand of R by [2, Proposition 17.2]. □

Proposition 2.17. *Let a module $M = M_1 \oplus \cdots \oplus M_n$ be a finite direct sum of submodules M_i $(1 \le i \le n)$, for some positive integer n such that each M_i is either cyclic δ-local or simple projective. Then every finitely generated submodule of M is δ-supplemented.*

Proof. By rearranging the submodules M_i $(1 \le i \le n)$, we can suppose that $M = L_1 \oplus \cdots \oplus L_k$ such that for every $1 \le i \le k-1$, L_i is a finite direct sum of cyclic m_i-δ-local submodules and L_k is a finite direct sum of simple projective submodules, where $m_1, \ldots, m_{k-1}$ are distinct maximal ideals of R. Clearly, for every $1 \le i \le k-1$, $K_{\delta_{m_i}}(L_i) = L_i$ (see Propositions 2.5 and 2.7). By Theorem 2.9, finitely generated

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submodules of L_i $(1 \le i \le k)$ are δ-supplemented. By Propositions 2.13, 2.15 and 2.16, finitely generated submodules of M are δ-supplemented. □

LEMMA 2.18. *Let M be a left module over any ring (not necessarily commutative). Assume that every finitely generated submodule of M is δ-supplemented. If N is a homomorphic image of M, then every finitely generated submodule of N is δ-supplemented.*

PROOF. By assumption, there exists an epimorphism $f : M \to N$. Let K be a finitely generated submodule of N. Then there exist a positive integer n and elements $b_i \in N$ $(1 \le i \le n)$ such that $K = Rb_1 + \cdots + Rb_n$. Then there exist elements $a_i \in M$ $(1 \le i \le n)$ such that $f(a_i) = b_i$ $(1 \le i \le n)$. Therefore $K = f(Ra_1 + \cdots + Ra_n)$. Since $Ra_1 + \cdots + Ra_n$ is δ-supplemented, K is δ-supplemented by [6, Proposition 3.6]. □

LEMMA 2.19. *If M is a δ-local module, then $M = N \oplus L$ such that N is a cyclic δ-local submodule and L is semisimple projective.*

PROOF. Let M be a δ-local module. Let $x \in M - \delta(M)$. As $\delta(M)$ is a maximal submodule of M, we have $\delta(M) + Rx = M$. Since $\delta(M) \ll_\delta M$, there exists a semisimple projective submodule $L \le \delta(M)$ such that $L \oplus Rx = M$ (see Lemma 2.1(1)). By Lemma 2.1(4), $\delta(M) = \delta(L) \oplus \delta(Rx)$. From Lemma 2.3(2) it follows that $\delta(L) = L$. Thus $\delta(M) = L \oplus \delta(Rx)$. Therefore $\delta(Rx)$ is a maximal submodule of Rx. Moreover, according to Lemma 2.1(5), $\delta(Rx) \ll_\delta Rx$. Consequently, $N = Rx$ is a cyclic δ-local module. □

THEOREM 2.20. *Let M be a module such that M is a sum of δ-local submodules and a semisimple projective submodule. Then every finitely generated submodule of M is δ-supplemented.*

PROOF. By Lemma 2.19, there is no loss of generality in assuming that $M = \sum_{i \in I} M_i$ such that M_i is either cyclic δ-local or a simple projective submodule of M. Let K be a finitely generated submodule of M. There exists a finite subset $J \subseteq I$ such that $K \le \sum_{i \in J} M_i$. By Proposition 2.17, every finitely generated submodule of the module $\bigoplus_{i \in J} M_i$ is δ-supplemented. Since $\sum_{i \in J} M_i$ is a homomorphic image of $\bigoplus_{i \in J} M_i$, K is δ-supplemented by Lemma 2.18. This completes the proof. □

COROLLARY 2.21. *Let M be a module such that M is a sum of δ-local submodules and a semisimple projective submodule. Then every cofinite submodule of M has ample δ-supplements.*

PROOF. This follows by Proposition 2.8 and Theorem 2.20. □

LEMMA 2.22. *Let N be a maximal submodule of a module M. If K is a δ-supplement of N in M, then K is either δ-local or semisimple projective.*

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Proof. By assumption, $N + K = M$ and $N \cap K \ll_\delta K$. Thus $N \cap K \subseteq \delta(K)$. Since $M/N \cong K/(N \cap K)$, $N \cap K$ is a maximal submodule of K. Hence $\delta(K) = N \cap K$ or $\delta(K) = K$. If $\delta(K) = N \cap K$, then K is δ-local. Now suppose $\delta(K) = K$. Then for every $x \in K - (N \cap K)$, $Rx + (N \cap K) = K$. Moreover, since $Rx \subseteq \delta(K)$ and $\delta(K) = \sum\{L \leq K \mid L \ll_\delta K\}$, $Rx \ll_\delta K$ by Lemma 2.1(2). Again by Lemma 2.1(2), $Rx + (N \cap K) = K \ll_\delta K$. Therefore K is semisimple projective by Lemma 2.3. □

A module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M.

Corollary 2.23. *Let M be a coatomic module. Suppose that every cofinite submodule of M has a δ-supplement in M. Then:*

(1) *every finitely generated submodule of M is δ-supplemented;*
(2) *every cofinite submodule of M has ample δ-supplements.*

Proof. (1) By [1, Theorem 2.9] and Lemma 2.22, M is a sum of δ-local submodules and a semisimple projective submodule. The result follows from Theorem 2.20.

(2) Use (1) and Proposition 2.8. □

Corollary 2.24. *Any finitely generated δ-supplemented module is amply δ-supplemented.*

Proof. This follows by Corollary 2.23. □

We conclude this paper by noting that there are some types of rings, not necessarily commutative, over which finitely generated δ-supplemented modules are amply δ-supplemented.

Example 2.25.

(1) It is easily seen that if a ring R is semisimple or right artinian, then all finitely generated modules are amply δ-supplemented.
(2) In [7], Zhou called a ring R *δ-semiperfect* if every left ideal I of R can be written as $I = Re \oplus S$, where $e^2 = e \in R$ and $S \subseteq \delta({}_RR)$. From [5, Theorem 3.3] and Proposition 2.8 it follows that finitely generated modules over δ-semiperfect rings are amply δ-supplemented.

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