Uniqueness and solvability in the linearized two-dimensional problem of a body in a finite depth subcritical stream

O. MOTYGIN

Laboratory for Mathematical Modelling of Wave Phenomena,
Institute of Mechanical Engineering Problems, Russian Academy of Sciences,
V.O., Bol'shoy pr., 61, St. Petersburg, 199178, Russia

Uniqueness and solvability theorems are proved for the two-dimensional Neumann–Kelvin problem in the case when a body is totally submerged in a subcritical stream of finite depth fluid. A version of source method is developed to find a solution. The Green's identity coupling the solution with a solution of the problem with opposite stream direction is used to prove that the solution is unique.

1 Introduction

We consider a uniform stream of finite depth about an infinitely long, horizontal, totally submerged cylinder with generators orthogonal to the stream direction. It is assumed that this two-dimensional fluid motion is steady and can be described in the framework of linear surface-wave theory. The corresponding boundary value problem is often referred to as the two-dimensional Neumann–Kelvin problem.

The first solvability theorem for the Neumann–Kelvin problem was proved more than 50 years ago in Kochin [1] for the case when a cylinder, submerged in infinite depth fluid, has a sufficiently large or sufficiently small forward velocity U. Vainberg & Maz'ya [2] expand Kochin's result on solvability for all values of U with the possible exception of a finite number of values.

The case of finite depth fluid was treated in the classical work by Haskind [3], yet the unique solvability question was not considered there. In Vainberg & Maz'ya [2], unique solvability was established under the assumption that the kinetic energy of a solution to the homogeneous Neumann–Kelvin problem is finite. This proof obtained by means of the so-called Maz'ya's identity is applicable for both infinite and finite depth cases with some restrictions on the submerged contour. The same theorem but without geometrical restrictions was proved by Lahalle [4]. Unfortunately, in both schemes the assumption of finiteness of kinetic energy for a solution of the homogeneous problem is essential. Hence, the theorem is only directly applicable to the case of a supercritical stream of finite depth h (where g is gravity acceleration and $U^2 > gh$), when there are no waves at infinity.

The present paper is intended to complete the theory by giving the unique solvability theorem for the subcritical case of finite depth stream when a body moving with forward
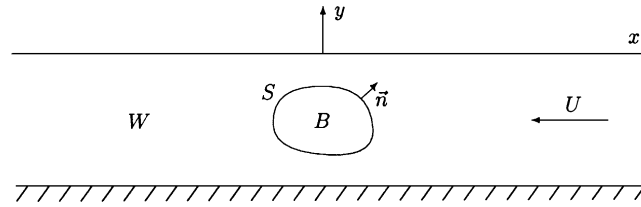
FIGURE 1

velocity $U^2 < gh$ produces a wave pattern at infinity downstream and the assumption on the finiteness of the kinetic energy does not hold. To prove the solvability we develop the above-mentioned scheme which was suggested by Kochin [1] and improved by Vainberg & Maz'ya [2]. The solvability is established for all values of U with the possible exception of a sequence tending to $\sqrt{gh}$. The method used by Kuznetsov & Maz'ya [5] when considering a surface-piercing cylinder yields the uniqueness of the problem under the same restriction on the forward velocity.

Now, the contents of the paper will be briefly summarized. In §2 we set up notation and introduce the two-dimensional Neumann–Kelvin problem. §3 is devoted to the study of the Green's function and establishes some auxiliary assertions on its properties. In §4 we apply the method of simple sources to reduce the problem to an integral equation and prove the solvability of the equation. In §5 an auxiliary problem with opposite stream direction is introduced and uniqueness is established by means of the Green's identity. The work has two appendices. The first is devoted to the Neumann problem in a layer of finite depth which appears as the zero limit case of the Neumann–Kelvin problem. The second appendix contains two auxiliary lemmas concerning properties of functions appearing in the Green's function.

2 Statement of the problem

The coordinate system is attached to the body and is taken so that the x-y plane is orthogonal to the horizontal generators of the cylinder, the mean free surface lies in the plane $y = 0$, and the x-axis is directed upstream (see Figure 1). The y-coordinate decreases with depth. Without loss of generality the depth of fluid can be assumed to be unity. Hence, the undisturbed stream is a strip $L = \{-\infty < x < +\infty, -1 < y < 0\}$. Let cylinder's cross-section be a bounded simply connected domain $B \subset L$, such that its boundary $S = \partial B$ is a $C^{1,\alpha}$-arc, $0 < \alpha < 1$. We denote by $W = L \setminus \overline{B}$ the domain occupied by fluid.

The Neumann–Kelvin problem for the velocity potential u (the NK problem for short) is stated as follows:

$$\nabla^2 u = 0 \quad \text{in} \quad W, \tag{2.1}$$

$$u_{xx} + \nu u_y = 0 \quad \text{when} \quad y = 0, \quad \nu \neq 1 \tag{2.2}$$

$$u_y = 0 \quad \text{when} \quad y = -1, \tag{2.3}$$

$$\partial u/\partial n = f \in C(S) \quad \text{on} \quad S, \tag{2.4}$$

$$\lim_{x \to +\infty} |\nabla u| = 0. \tag{2.5}$$

The Laplace equation follows from the assumptions that the fluid is incompressible and its motion is irrotational. The boundary condition (2.2) is a consequence of the linearized kinematic and dynamic conditions on the free surface of fluid. Here $v = gU^{-2}$ is the wave number. The condition (2.3) expresses the fact that the fluid is bounded by a rigid, impermeable, horizontal bottom. If the function f in (2.4) is equal to $U\cos(n,x)$, then S is rigid, impermeable contour. The relation (2.5) shows that there are no waves at infinity upstream. Besides, we shall demand that a solution to the above problem should be regular in sense of the following definition.

Definition 2.1 *We say that a function u is regular if u belongs to $C(\overline{W}) \cap C^2(W)$ and has regular normal derivative (e.g. see Maz'ya [6, Ch. 1, §1]) on $\partial W = S \cup \{y = 0\} \cup \{y = -1\}$.*

Proposition 2.1 *Let u be a regular solution of the NK problem and let L' be a compact subset of L. Then*

$$\int_{L' \backslash B} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}y < \infty.$$

Proof Obviously, it suffices to consider the case when $B \subset L'$. Let B' be a compact set, such that $B \subset B' \subset L'$. Let $S' = \partial B'$ and $\partial L'$ be smooth contours and $S \cap S' = \emptyset$. Applying the Green identity, we arrive at

$$\int_{L' \backslash B'} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}y = \int_{S' \cup \partial L'} u \frac{\partial u}{\partial n} \, \mathrm{d}s.$$

The regularity of u guarantees that as $\mathrm{dist}\{S, S'\} \to 0$ the integral in the right-hand side of the last formula has a finite limit, which completes the proof. $\qquad\square$

The above consideration also justifies Green's formula for any pair of regular functions on subsets of W. The latter is needed to use the scheme suggested in Theorem 2.1 of Kuznetsov [7] to obtain the asymptotics at infinity of a solution of the NK problem. We write

$$D(k) = k - v \tanh k.$$

Then, as $|x| \to \infty$ and $\pm x > 0$,

$$u(x,y) = \theta(-x)\{2x + \theta(v-1)\cosh\lambda(y+1)(\mathscr{A}\sin\lambda x + \mathscr{B}\cos\lambda x)\} + C_\pm + \phi_\pm(x,y), \quad (2.6)$$

where θ is the Heaviside function, $\phi_\pm = O(|x|^{-1})$, $|\nabla\phi_\pm| = O(|x|^{-2})$, and λ denotes the only positive root of the equation

$$D(\lambda) = 0, \tag{2.7}$$

which exists when $v > 1$. The last equation can be treated as the dispersion relation.

The coefficients of the above expansion satisfy the following relationships:

$$2(1-v) = v \int_S \frac{\partial u}{\partial n} \, \mathrm{d}s, \quad (C_+ - C_-)(1-v) = v \int_S \left(x \frac{\partial u}{\partial n} - u \frac{\partial x}{\partial n} \right) \mathrm{d}s,$$

$$\mathscr{A} = \frac{2v}{\lambda(v - \cosh^2\lambda)} \int_S \left\{ u \frac{\partial}{\partial n} [\cosh\lambda(y+1)\cos\lambda x] - \frac{\partial u}{\partial n} \cosh\lambda(y+1)\cos\lambda x \right\} \mathrm{d}s,$$

and the expression for $\mathscr{B}$ can be obtained by replacing $\cos\lambda x \to -\sin\lambda x$ in that for $\mathscr{A}$.

The asymptotics changes its form at the critical value $v = 1$. A body moving in supercritical regime $v < 1$ does not induce waves, and in view of Proposition 2.1 the above expression for $\mathscr{Q}$ yields that the integral of kinetic energy $\int_W |\nabla u|^2 \, dx \, dy$ is finite for a solution with homogeneous Neumann data. In this work we treat the motion in subcritical regime $v > 1$ when there exist waves at infinity downstream and, obviously, the finiteness of the integral of kinetic energy does not hold.

3 On the Green's function of the Neumann–Kelvin problem

The Green function $G(x, y; \xi, \eta)$ ($=G(z, \zeta)$ where $z = x + iy$, $\zeta = \xi + i\eta$), is the velocity potential of a source placed at $\zeta \in L$. This function satisfies the following problem:

$$-\nabla^2_{x,y} G = \delta(|z - \zeta|) \quad \text{when} \quad z, \zeta \in L, \tag{3.1}$$

$$G_{xx} + v G_y = 0 \quad \text{when} \quad y = 0, \quad v \neq 1 \tag{3.2}$$

$$G_y = 0 \quad \text{when} \quad y = -1, \tag{3.3}$$

$$\lim_{x \to +\infty} |\nabla_{x,y} G| = 0. \tag{3.4}$$

The solution of the problem (3.1)–(3.4) was derived by Haskind [3]. Treatment of the function can also be found in Kuznetsov [7] and Vainberg & Maz'ya [2]. The Green's function is given by the following expression:

$$-(2\pi)^{-1} \Bigg\{ \log|z - \zeta| + \int_0^{+\infty} \Big[v \cosh k(y + \eta + 1) + k \sinh k(y + \eta + 1)$$

$$+ (v + k) e^{-k} \cosh k(y - \eta) \Big] \frac{\cos k(x - \xi)}{k \, D(k) \cosh k} \, dk + w(z, \zeta) \Bigg\}, \tag{3.5}$$

where

$$w(z, \zeta) = \frac{\pi v}{1 - v}(x - \xi) + \theta(v - 1) \frac{2\pi v \cosh \lambda(y + 1) \cosh \lambda(\eta + 1)}{\lambda(\cosh^2 \lambda - v)} \sin \lambda(x - \xi).$$

We write $G(z, \zeta)$ as follows

$$G(z, \zeta) = -(2\pi)^{-1} \big\{ \log|z - \zeta| + g(v, z, \zeta) \big\}. \tag{3.6}$$

To regularize the integral term in (3.5) containing the poles k^{-2}, k^{-1} and $(k - \lambda)^{-1}$ we use the scheme analogous to that suggested in Appendix A. Thus, extracting the poles from the integral we arrive at

$$g(v, z, \zeta) = \frac{1 + v}{2(v - 1)} \log|z - \overline{\zeta}| + \frac{1}{2(v - 1)} \log\big|(z - \zeta)^2 + 1\big| + \tfrac{1}{2} \log|z - \overline{\zeta} + 2i|$$

$$+ \frac{(v + \lambda) e^{\lambda h}}{2\lambda \, D'(\lambda) \cosh \lambda} \text{Re} \Big\{ s\big(\lambda(z - \overline{\zeta})\big) + \frac{v - \lambda}{v + \lambda} s\big(\lambda(\overline{z} - \zeta - 2i)\big)$$

$$+ e^{-\lambda} s\big(\lambda(z - \zeta - i)\big) + e^{-\lambda} s\big(\lambda(\overline{z} - \overline{\zeta} - i)\big) \Big\} + \frac{v}{v - 1} \beta(z, \zeta)$$

$$+ w(z, \zeta) + \tfrac{1}{2} \sum_{j=1}^4 \int_0^{+\infty} t_j(k, v) e^{k Y_j} \cos k(x - \xi) \, dk. \tag{3.7}$$

where

$$D'(\lambda) = D'(k)\big|_{k=\lambda} = 1 - v + v^{-1}\lambda, \quad s(\lambda z) = v.p. \int_0^{+\infty} \frac{e^{-ikz}}{k - \lambda}\, dk, \qquad (3.8)$$

$$Y_{2i+j+1} = (-1)^i \left(y + \tfrac{1}{2}\right) + (-1)^j \left(\eta + \tfrac{1}{2}\right) - 1, \quad i, j = 0, 1, \qquad (3.9)$$

$$\beta(z, \zeta) = -\frac{1}{2} \sum_{j=1}^4 \left[Y_j \log|x - \xi + iY_j| + (x - \xi)\arctan\frac{x - \xi}{Y_j} \right], \qquad (3.10)$$

and the analytic functions of $k \in \mathbf{R}$, $t_i(k, v)$ are expressed as follows:

$$t_{(5\pm3)/2} = \frac{(v \mp k)e^k}{k\, D(k)\cosh k} + \frac{(v \mp \lambda)e^\lambda}{\lambda(\lambda - k)\, D'(\lambda)\cosh \lambda} + \frac{v}{k^2(v - 1)} + \frac{v \mp 1}{k(v - 1)},$$

$$t_2 = t_3 = \frac{v + k}{k\, D(k)\cosh k} + \frac{v + \lambda}{\lambda(\lambda - k)\, D'(\lambda)\cosh \lambda} + \frac{v}{k^2(v - 1)} + \frac{1}{k(v - 1)}. \qquad (3.11)$$

Now we prove two lemmas concerning properties of the function $g(v, z, \zeta)$, which is involved in the representation (3.6) as the component depending on the velocity. These lemmas establish smoothness of g and behaviour of the function as $v \to \infty$. The proofs of the assertions are based on the considerations of Appendix B.

Lemma 3.1 *The function $g(v, z, \zeta)$ depends analytically upon parameters v, z and ζ when $v > 1$ and $z, \zeta \in L$.*

Proof Let $v \in [v_1, v_2]$, where $1 < v_1 < v_2$. Using (3.11) it is easy to show that $|t_i(k, v)| \leqslant C(v_1, v_2, k_*)k^{-1}$, when $k \geqslant k_* > v_2$. Clearly, $Y_i(y, \eta) < 0$ when $z, \zeta \in L'$, where L' is a compact subset of the strip L. Hence, the integrals in (3.7) converge uniformly with respect to v, z and ζ in these compact sets. In view of Lemma B.2 the latter guarantees the needed analytic property of the last term in the right-hand side of (3.7). Finally, the proof is completed by referring to (3.8) and (3.10). □

Lemma 3.2 *Let $z, \zeta \in L'$, where L' is a compact subset of L. Then, the following estimate is true:*

$$\sup\{|\nabla_{x,y}(g(v, z, \zeta) - g_0(z, \zeta))| : z, \zeta \in L'\} = O(v^{-1}), \quad \text{as} \quad v \to \infty, \qquad (3.12)$$

where the function $g_0(z, \zeta)$ is defined in Appendix A (see (A 11) and (A 9)).

Proof The presentation of $g(v, z, \zeta)$ given in (3.7) involves the terms $s(\lambda(x - \xi + iY_j))$. Consider the asymptotic behaviour of derivatives of these functions as $v \to \infty$. We have

$$s'(z) = -z^{-1} - i\, s(z) = -z^{-1} + ie^{-iz}\text{Ei}\,(iz),$$

where the exponential integral $\text{Ei}\,(z)$ is obtained with the help of 8.212.5 in Gradshteyn & Ryzhik [8]. The asymptotic expansion of $\text{Ei}\,(z)$ 8.215 in Gradshteyn & Ryzhik [8] leads to the estimate

$$\left|\nabla_{x,y}\, s(\lambda(x - \xi + iY_j))\right| = O(\lambda^{-1}).$$

Under the assumption of the assertion the inequalities $Y_j \leqslant d_j$, where d_j are negative constants, hold when $z, \zeta \in L'$. Hence, the form of the remainder term in the expansion

of Ei (z) guarantees that the above estimate is uniform with respect to $z, \zeta \in L'$. Finally, note that $\lambda(v) \sim v$ as $v \to +\infty$.

The contribution of the last terms in (3.7) and (A 11) to the expression $\partial_z(g - g_0)$ is equal to $\frac{1}{2} \sum_{j=1}^{4} (-1)^j I_j$, where

$$I_j = i \int_0^{+\infty} [t_j(k, v) - \tau_j(k)] k\, e^{k(Y_j + i\xi - ix)}\, dk.$$

By (B 1) we arrive at

$$|I_j| \leqslant \sup\{|t_j(k, v) - \tau_j(k)| : k > 0\} \int_0^{+\infty} k\, e^{-kd_j}\, dk = O(v^{-1}).$$

We omit the obvious asymptotic analysis of the other terms in the expression $\nabla_{x,y}(g - g_0)$. $\qquad\square$

4 Solvability of the problem

We seek a solution in form of the single layer potential

$$u(z) = \int_S \mu(\zeta) G(z, \zeta)\, ds_\zeta \qquad (4.1)$$

with an unknown density $\mu \in C(S)$. Due to the representation (3.6), where the function $g(v, z, \zeta)$ is analytic by Lemma 3.1, the theory of harmonic potentials is applicable.

Obviously, the potential (4.1) satisfies (2.1)–(2.3), (2.5). By theorem 2 in Maz'ya [6, ch. 1, § 1], the potential (4.1) is regular in sense of Definition 2.1 and the boundary condition (2.4) leads to the following integral equation:

$$-\mu(z) + (T\mu)(z) = 2f(z), \qquad (4.2)$$

where

$$(T\mu)(z) = 2 \int_S \mu(\zeta) \frac{\partial G}{\partial n_z}(z, \zeta)\, ds_\zeta$$

and the operator T is compact in $L_2(S)$ (e.g. see Maz'ya [6, ch. 1, § 1]).

Remark 4.1 If the equation (4.2) is solvable in $L_2(S)$ and $f \in C(S)$, the solution μ belongs to $C(S)$ (e.g. see theorem 3 in Maz'ya [6, ch. 1, § 1]).

Theorem 4.1 *For sufficiently large v the NK problem is solvable for any $f \in C(S)$.*

Proof By (3.6), (A 8) and (A 10) we have

$$\|T - T_0; L_2(S)\|^2 \leqslant \int_S \int_S \left| \frac{\partial}{\partial n_z} \big(g(v, z, \zeta) - g_0(z, \zeta)\big) \right|^2 ds_\zeta ds_z.$$

The estimate (3.12) provides that

$$\|T - T_0; L_2(S)\| = O(v^{-1}).$$

In view of properties of the operator T_0 established in Theorem A.2 the last estimate completes the proof. $\qquad\square$

Lemma 3.1 guarantees that the operator T depends analytically on the parameter $v > 1$. Hence, by the theorem on invertibility of an operator-function depending on a parameter (see Trofimov [9]), the resolvent of (4.2) represents a meromorphic function of v. Due to Theorem 4.1, poles of the resolvent cannot accumulate near the limit value $v = \infty$. Thus, we arrive at the following assertion.

Theorem 4.2 *For all values of $v > 1$ with possible exception of a sequence tending to 1, the NK problem is solvable for any $f \in C(S)$.*

We can improve the smoothness of the potential (4.1) provided that the contour and the right-hand side of (2.4) are smoother than they were defined initially. For example, if $S \in C^{2,\alpha}$ and $f \in C^{1,\alpha}(S)$, then the potential (4.1) can be extended from W to $\overline{W}$ so as to belong to $C^{2,\alpha}(\overline{W})$.

It is well-known that T is a continuous operator, $T : C(S) \mapsto C^{0,\alpha}(S)$ (e.g. see Colton & Kress [10]). Rewriting (4.2) in the form $\mu(z) = (T\mu)(z) - 2f(z)$, we see that $\mu \in C^{0,\alpha}(S)$ when $\mu \in C(S)$ and $f \in C^{1,\alpha}(S)$. Moreover, $T\mu \in C^{1,\alpha}(S)$ when $\mu \in C^{0,\alpha}(S)$ (e.g. see Theorem 2.22 in Colton & Kress [10]). Thus, $\mu \in C^{1,\alpha}(S)$. Further, in view of (3.6) it is sufficient to consider the potential

$$\phi(z) = \int_S \Gamma(z,\zeta)\mu(\zeta)\,\mathrm{d}s_\zeta,$$

where $\Gamma(z,\zeta) = \log|z - \zeta|$. Since $\nabla_{x,y}\Gamma = -\nabla_{\xi,\eta}\Gamma$, we write

$$\nabla\phi(z) = -\int_S \mu(\zeta)\vec{n}(\zeta)\frac{\partial\Gamma}{\partial n_\zeta}\,\mathrm{d}s_\zeta - \int_S \mu(\zeta)\vec{\tau}(\zeta)\frac{\partial\Gamma}{\partial\tau_\zeta}\,\mathrm{d}s_\zeta,$$

$$= -\int_S \mu(\zeta)\vec{n}(\zeta)\frac{\partial\Gamma}{\partial n_\zeta}\,\mathrm{d}s_\zeta + \int_S \Gamma\frac{\partial\mu\vec{\tau}}{\partial\tau_\zeta}\,\mathrm{d}s_\zeta, \quad z \in W,$$

where $\vec{n}(\zeta)$ is the unit normal and $\vec{\tau}(\zeta)$ is the unit vector tangent to S in point ζ. The first (second) term in the right-hand side of the last formula represents a double (single) layer potential with the density belonging to $C^{1,\alpha}(S)$ $(C^{0,\alpha}(S))$. By theorem 2.23 (2.17) in Colton & Kress [10], derivatives of the potential can be extended from W to $\overline{W}$ so as to belong to $C^{0,\alpha}(\overline{W})$.

5 Uniqueness of the problem

Following the method suggested by Kuznetsov & Maz'ya [5] we consider an auxiliary problem (the ANK problem) with opposite direction of flow. Denote by u' a regular solution of this problem, satisfying (2.1)–(2.4), and the following condition at infinity:

$$\lim_{x \to -\infty} |\nabla u'| = 0.$$

We define the functional

$$J(u, u') = \int_S \left[u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n} \right]\,\mathrm{d}s$$

and prove that the NK problem and the ANK problem are in some sense 'adjoint'.

Lemma 5.1 *Let u and u' are solutions to* NK *and* ANK *problems and let the functions f and f' in condition* (2.4) *for u and u' have the zero mean value on S. Then the relationship* $J(u, u') = 0$ *holds.*

Proof We denote by R_d a rectangle containing B, $R_d = \{|x| < d, -1 < y < 0\}$. Let $W_d = R_d \setminus \overline{B}$ be the subdomain occupied by fluid and let $\vec{n}$ be directed into W_d. Due to the assumption on regularity of u and u' (see Definition 2.1) we have

$$\int_{\partial W_d} \left(u \frac{\partial u'}{\partial n} - u' \frac{\partial u}{\partial n} \right) \, ds = 0.$$

By the condition (2.3) we have

$$J = \int_{-d}^{+d} \left[u u'_y - u' u_y \right]_{y=0} \, dx + \int_{-1}^{0} \left[u u'_x - u' u_x \right]_{x=d} \, dy + \int_{-1}^{0} \left[u' u_x - u u'_x \right]_{x=-d} \, dy. \quad (5.1)$$

Further, we consider integrals along the straight segments which form the integral in the right-hand side of (5.1).

Due to the restriction imposed in the assertion upon the functions f and f', the coefficient $\mathscr{D}$ in the asymptotics (2.6) of u is equal to zero and, analogously,

$$u' \sim c'_{\pm} + \theta(x) \cosh \lambda(y+1) \left(\mathscr{A}' \sin \lambda x + \mathscr{B}' \cos \lambda x \right), \quad \text{as } |x| \to \infty, \quad \pm x > 0.$$

Further, using the boundary condition (2.2) we obtain

$$\begin{aligned}
\int_{-d}^{+d} \left[u u'_y - u' u_y \right]_{y=0} \, dx &= v^{-1} \int_{-d}^{+d} \left[u' u_{xx} - u u'_{xx} \right]_{y=0} \, dx \\
&= v^{-1} \left[u'(x,0) u_x(x,0) - u(x,0) u'_x(x,0) \right]_{x=-d}^{x=+d} \\
&= -\lambda v^{-1} c_+ \cosh \lambda \left(\mathscr{A}' \cos \lambda d - \mathscr{B}' \sin \lambda d \right) \\
&\quad - \lambda v^{-1} c'_- \cosh \lambda \left(\mathscr{A} \cos \lambda d + \mathscr{B} \sin \lambda d \right) + O\left(d^{-1} \right).
\end{aligned}$$

According to the asymptotics of u and u' at infinity, the sum of the two last terms in the right-hand side of (5.1) is equal to

$$c_+ \sinh \lambda \left(\mathscr{A}' \cos \lambda d - \mathscr{B}' \sin \lambda d \right) + c'_- \sinh \lambda \left(\mathscr{A} \cos \lambda d + \mathscr{B} \sin \lambda d \right) + O\left(d^{-1} \right).$$

By (2.7), $J(u, u') = O(d^{-1})$ and taking limit as $d \to \infty$ we arrive at $J(u, u') = 0$.   □

We denote by $G'(z, \zeta)$ the Green's function for the problem with the opposite flow direction. It is easily seen that for the ANK problem Theorem 4.2 is also true and guarantees that the integral equation

$$-\mu(z) + 2 \int_S \mu(\zeta) \frac{\partial G'}{\partial n_z}(z, \zeta) \, ds_\zeta = f'$$

is uniquely solvable when $v \in \mathscr{V}'$, where the set $\mathscr{V}'$ contains all values of $v > 1$ with possible exception of a sequence tending to 1. We use analogous notation $\mathscr{V}$ so that the equation (4.2) is uniquely solvable when $v \in \mathscr{V}$. Further we shall establish that $\mathscr{V} = \mathscr{V}'$.

Theorem 5.1 *If $v \in \mathscr{V}$ and u is a solution of the* NK *problem with $f = 0$ in* (2.4)*, then u is constant.*

Proof Let $v \in \mathscr{V}'$ and u' be a solution of the ANK problem with arbitrary Neumann data on S orthogonal to constant ($\int_S f' \, ds = 0$). Taking into account the homogeneous Neumann condition for the function u and Lemma 5.1 we have

$$\int_S u f' \, ds = 0.$$

Since the function f' is arbitrary, we obtain $u = \text{const}$ on S. Further, the uniqueness theorem for the Cauchy problem for the Laplace equation yields that $u = \text{const}$ in W.

Now we prove that $\mathscr{V} = \mathscr{V}'$. In fact, the above proof states the uniqueness of the NK problem for $v \in \mathscr{V}'$. The standard method of the theory of potentials (e.g. see Theorem A.2) allows us to prove the solvability of the equation (4.2) for $v \in \mathscr{V}'$. Thus, by the definition of $\mathscr{V}$, $\mathscr{V}' \subset \mathscr{V}$. Using the same arguments with the NK problem as the 'adjoint' for the ANK problem we get $\mathscr{V} \subset \mathscr{V}'$. $\qquad\square$

6 Summary and conclusions

In this work we have studied the linearized two-dimensional boundary value problem which describes the forward subcritical motion of a cylinder immersed in a fluid of finite depth. It is proved that if the contour of the body S belongs to $C^{1,\alpha}$ and the Neumann data f is continuous, then for all values of the wave number $v > 1$ with possible exception of a sequence tending to the critical value 1, the problem has a unique solution, which is continuous in $\overline{W}$, satisfies the boundary conditions on the free surface and on the bottom in the classical sense and has regular normal derivative on the contour of body. Moreover, it is shown that if $S \in C^{2,\alpha}$ and $f \in C^{1,\alpha}(S)$ then this solution belongs to $C^{2,\alpha}(\overline{W})$.

Acknowledgements

Appendix A On the Neumann problem in a layer of finite depth

In the appendix we consider the Neumann problem, which appears as the zero speed limit case of the NK problem and describes the motion of a body in a layer of unit depth with rigid bottom and top. A solution to the problem u_0 satisfies

$$\nabla^2 u_0 = 0 \quad \text{in} \quad W, \tag{A 1}$$

$$\partial u_0 / \partial n = f \quad \text{on} \quad S, \tag{A 2}$$

$$\partial u_0 / \partial y = 0 \quad \text{when} \quad y = 0, \ y = -1, \tag{A 3}$$

$$\lim_{x \to +\infty} |\nabla u_0| = 0. \tag{A 4}$$

Here the notations introduced in Figure 1 are used, the contour of body S is a $C^{1,\alpha}$-arc, $f \in C(S)$ and u_0 is assumed to be regular in sense of Definition 2.1. In Werner [11], a theorem on unique solvability was proved for a problem which differs from the problem

under consideration by condition at infinity (A 4). We give this theorem in form needed for purpose of the present work.

Denote by $G_0(z, \zeta)$ the Green's function of the problem (A 1)–(A 4), describing motion of a source placed at the point ζ. This function must satify (A 3), (A 4) and the source equation (3.1). Using the representation derived in Werner [11], we get

$$G_0(z, \zeta) = -(2\pi)^{-1} \log \left| \left(1 - e^{-\pi|x-\xi|+i\pi(y+\eta)}\right) \left(1 - e^{-\pi|x-\xi|+i\pi(y-\eta)}\right) \right|$$
$$+ \tfrac{1}{2}|x - \xi| + \tfrac{1}{2}(\xi - x). \tag{A 5}$$

As $|x| \to \infty$ and $|\zeta| \leqslant c < \infty$, the following asymptotic representation holds:

$$G_0(z, \zeta) = (x - \xi)\, \theta(-x) + O\left(e^{-\pi|x|}\right).$$

In view of the assumption on the regularity of u_0, the asymptotics of $u_0(z)$ at infinity can be obtained following theorem 2.1 in Kuznetsov [7]. This asymptotics has the same form as the above asymptotics of the Green function and is expressed as follows

$$u_0(z) = c_{\pm} - x\, \theta(-x) \int_S \frac{\partial u_0}{\partial n}\, ds + O\left(e^{-\pi|x|}\right), \quad |x| \to \infty, \quad \pm x > 0$$

where $c_+ - c_- = \int_S \left[u_0 \partial x/\partial n - x \partial u_0/\partial n\right]\, ds$.

Theorem A.1 *If a regular potential u_0 satisfies the homogeneous problem (A 1)–(A 4), then u_0 is constant.*

Proof In view of Proposition 2.1 the above asymptotics yields that $\int_W |\nabla u_0|^2\, dx\, dy$ is finite under the condition $f = 0$. Then, applying the Green's formula and taking into account (A 3), we get

$$\int_W |\nabla u_0|^2\, dx\, dy = \int_S u_0 \frac{\partial u_0}{\partial n}\, ds = 0.$$

which implies that $u_0 = \text{const}$ in W. □

Theorem A.2 *The problem (A 1)–(A 4) is solvable for any $f \in C(S)$.*

Proof We seek a solution in form of the single layer potential

$$u_0(z) = \int_S \mu(\zeta) G_0(z, \zeta)\, ds_\zeta \tag{A 6}$$

with an unknown density $\mu \in C(S)$. Due to the representation (A 10), where the function $g_0(z, \zeta)$ is obviously analytic, the theory of harmonic potentials is applicable to the solution.

It is easily seen that the potential (A 6) is regular and satisfies (A 1), (A 3)–(A 4). The boundary condition (A 2) leads to the following integral equation:

$$-\mu(z) + (\mathrm{T}_0 \mu)(z) = 2f(z), \tag{A 7}$$

where

$$(\mathrm{T}_0 \mu)(z) = 2 \int_S \mu(\zeta) \frac{\partial G_0}{\partial n_z}(z, \zeta)\, ds_\zeta. \tag{A 8}$$

The operator T_0 is compact in $L_2(S)$. Hence, the equation (A 7) is a Fredholm one and

to prove the assertion it suffices to prove that $\mu_0 = 0$, where $\mu_0 \in L_2(S)$ is a solution of homogeneous equation (A 7).

Consider the potential

$$V(z) = \int_S \mu_0(\zeta) G_0(z, \zeta) \, \mathrm{d}s_\zeta.$$

According to Remark 4.1 the density μ_0 is continuous. Then, the definition of μ_0 and Theorem 2 in Maz'ya [6, ch. 1, §1] guarantee that the potential V satisfies the conditions of Theorem A.1. Therefore, $V = \text{const}$ in $\overline{W}$. Moreover, the potential V is continuous in L and, hence, $V = \text{const}$ in B because of uniqueness of the Dirichlet problem in B. Thus, $\partial V / \partial n_i = 0$ and the jump relation yields that $\mu_0 = 0$. Finally, the equation (A 7) is uniquely solvable in $L_2(S)$ and $\mu \in C(S)$ when $f \in C(S)$ (see Remark 4.1). □

In the remainder of the Appendix we introduce a representation of the Green's function, which differs from the representation (A 5). This form is in use in §3. Applying the standard procedure based on the Fourier transformation with respect to x to the set of equations (A 3) and (3.1) we arrive at the expression

$$(2\pi)^{-1} \left\{ -\log|z - \zeta| + \int_0^{+\infty} \frac{\left[\cosh k(y + \eta + 1) + \mathrm{e}^{-k} \cosh k(y - \eta) \right] \cos k(x - \xi) \, \mathrm{d}k}{k \sinh k} \right\}.$$

The integral term in the last formula contains poles k^{-1}, k^{-2} and should be regularized. For this purpose we rewrite this term as follows

$$I(z, \zeta) = \tfrac{1}{2} \Bigg[\alpha_1(x - \xi, Y_1) + \alpha_1(x - \xi, Y_4) + \sum_{j=1}^{4} \alpha_2(x - \xi, Y_j)$$
$$- \sum_{j=1}^{4} \int_0^{+\infty} \tau_j(k) \mathrm{e}^{kY_j} \cos k(x - \xi) \, \mathrm{d}k \Bigg],$$

where Y_i are given by (3.9),

$$\alpha_1(x, y) = \int_0^{+\infty} \frac{\mathrm{e}^{ky} \cos kx - \mathrm{e}^{-k}}{k} \, \mathrm{d}k = -\log|x + iy|,$$

$$\alpha_2(x, y) = \int_0^{+\infty} \frac{\mathrm{e}^{ky} \cos kx - 1 - ky \mathrm{e}^{-k}}{k^2} \, \mathrm{d}k$$

and the functions

$$\tau_1(k) = \tau_4(k) = -\frac{\mathrm{e}^k}{k \sinh k} + \frac{1}{k^2} + \frac{1}{k}, \quad \tau_2(k) = \tau_3(k) = -\frac{1}{k \sinh k} + \frac{1}{k^2} \qquad \text{(A 9)}$$

are analytic in $\mathbf{R}$. Integrating the representation of $\alpha_2(x, y)$ by parts and using the relationship

$$\int_0^{+\infty} \frac{\mathrm{e}^{ky} \sin kx}{k} \, \mathrm{d}k = -\arctan \frac{x}{y}, \quad y < 0,$$

we get

$$\alpha_2(x, y) = y \left(1 - \log|x + iy| \right) + x \arctan \frac{x}{y}.$$

Due to the analytic properties of $\tau_i(k)$, the last term in definition of $I(z, \zeta)$ is small as $|x| \to \infty$. Hence, the asymptotics of $I(z, \zeta)$ can be easily established with help of the presentation of functions α_1 and α_2. Consider the function equal to $I(z, \zeta)$ minus the linear term in the asymptotics of $I(z, \zeta)$ as $x \to +\infty$. Obviously, this potential satisfies the condition (A 4) along with the conditions (A 3) and (3.1). Finally, we write

$$G_0(z, \zeta) = -(2\pi)^{-1}\{\log|z - \zeta| + g_0(z, \zeta)\}. \tag{A 10}$$

Here

$$g_0(z, \zeta) = \pi(\xi - x) + \tfrac{1}{2}\log\left|z - \overline{\zeta}\right| + \tfrac{1}{2}\log\left|z - \overline{\zeta} + 2\mathrm{i}\right| + \beta(z, \zeta)$$

$$+ \frac{1}{2}\sum_{j=1}^{4}\int_0^{+\infty} \tau_j(k)\mathrm{e}^{kY_j}\cos k(x - \xi)\,\mathrm{d}k, \tag{A 11}$$

where the function β is defined by (3.10).

Appendix B

In the present appendix we establish properties of the functions $t_i(k, v)$ given by (3.11).

Lemma B.1 *As $v \to \infty$, the following estimates are true:*

$$\sup\{|t_i(k, v) - \tau_i(k)| : k > 0\} = O\left(v^{-1}\right), \quad i = 1, 2, 3, 4. \tag{B 1}$$

Proof We denote

$$v(k, \lambda) = t_1(k, v(\lambda)) - \tau_1(k),$$

where the substitution $v(\lambda) = \lambda/\tanh\lambda$ follows from (2.7), and it is worth pointing out that $v \to \infty$ as $\lambda \to \infty$. The proof is based on resummation of v with help of the following set of functions

$$n_j(k) = k^{-j}\left(\mathrm{e}^{-2k} - \sum_{\ell=0}^{j-1}\frac{(-2k)^\ell}{\ell!}\right), \quad j = 1, 2, 3, 4. \tag{B 2}$$

Besides, we shall use the notations

$$\sigma(k) = 1 - \mathrm{e}^{-2k}, \quad h_i(k, \lambda) = \mathrm{e}^{-2\lambda}n_i(k - \lambda), \quad i = 1, 2.$$

Then, by means of direct but tedious resummation we rewrite

$$v(k, \lambda) = \frac{R(k, \lambda)\tanh\lambda}{2\mathrm{e}^{6\lambda}\left(1 - \mathrm{e}^{2\lambda}\right)(2\lambda - \sinh 2\lambda)(\lambda - \tanh\lambda)Q(k, \lambda)}, \tag{B 3}$$

where

$$Q(k, \lambda) = n_1(k)\left[\left(1 + \mathrm{e}^{-2\lambda}\right)n_1(k) - 2h_1(k, \lambda)\right],$$

$$R(k, \lambda) = 4\lambda\mathrm{e}^{4\lambda}\left(1 - \mathrm{e}^{4\lambda}\right)a_1(k, \lambda) + \sum_{j=1}^{5}a_j(k, \lambda)\mathrm{e}^{2j\lambda}, \tag{B 4}$$

and

$$a_1 = \tfrac{8}{3} n_1 - 2 n_2 + 2(e^{-2k} - 2) n_3 - \sigma n_4, \quad a_2 = -2(4 + h_1)(2 n_2 + n_3),$$

$$a_3 = -16 h_1 - \left(\tfrac{88}{3} + 20 h_1\right) n_1 + (20 e^{-2k} - 8 h_1) n_2 + 2(8 + 2 e^{-2k} + h_1) n_3 + 2 \sigma n_4,$$

$$a_4 = -16 h_1 - 8 \sigma h_2 + 8 h_2 n_1 + (8 \sigma + 12 h_1) n_2 + 2(h_1 - 4 e^{-2k}) n_3,$$

$$a_5 = -\sigma(8 h_2 + n_4) - 4 \left(\tfrac{4}{3} + 3 h_1 + 2 h_2\right) n_1 + (4 e^{-2k} - 6) n_2 + 2(e^{-2k} - 2 - h_1) n_3.$$

It is easy to check that the function $(-1)^i n_i(k)$, $i = 1, 2, 3, 4$, is continuous, positive and monotonic decreasing. Hence, the following inequalities are true

$$|n_3(k)| \leqslant \tfrac{4}{3}, \quad |n_4(k)| \leqslant \tfrac{2}{3}, \quad |n_i(k)| \leqslant 2, \quad |h_i(k, \lambda)| < \lambda^{-i}, \quad i = 1, 2 \qquad \text{(B 5)}$$

when $k \geqslant 0$ and $\lambda > 0$. The latter leads to the estimate

$$|R(k, \lambda)| = O(e^{10\lambda}), \quad \text{as} \quad \lambda \to \infty, \qquad \text{(B 6)}$$

which is uniform with respect to $k \geqslant 0$.

Consider the function $Q(k, \lambda)$ when $k \in [0, 1]$. There is no loss of generality in assuming that $\lambda \geqslant 4$. Then, in view of (B 5) we have

$$Q(k, \lambda) \geqslant \left(1 - e^{-2}\right) \left(-2\lambda^{-1} - n_1(k)\right) \geqslant \left(1 - e^{-2}\right) \left(1 - e^{-2} - \tfrac{1}{2}\right) > 0.$$

The latter inequality, the representation (B 3) and the estimate (B 6) yield that

$$\sup\{|v(k, \lambda)| : 0 \leqslant k \leqslant 1\} = O\left(\lambda^{-1}\right) \quad \text{as} \quad \lambda \to \infty. \qquad \text{(B 7)}$$

To obtain the analogous estimate for the case $k \geqslant 1$ we consider the expressions $k^2 R(k, \lambda)$ and $k^2 Q(k, \lambda)$. By using the relationship $k h_1 = \lambda h_1 + e^{-2k} - e^{-2\lambda}$ we obtain

$$k^2 Q(k, \lambda) = \left(1 - e^{-2k}\right) \left[2\lambda h_1(k, \lambda) + \left(1 - e^{-2\lambda}\right) \left(1 + e^{-2k}\right)\right]$$

and assuming that $\lambda \geqslant 4$ we arrive at the inequality

$$\frac{k^2 Q(k, \lambda)}{\left(1 - e^{-2}\right)} > 2\lambda \frac{e^{-2} - e^{-2\lambda}}{1 - \lambda} + 1 - 2 e^{-2\lambda} > 1 - \tfrac{8}{3} e^{-2} \qquad \text{(B 8)}$$

Further, we write the expression $k^2 R(k, \lambda)$ in the following form:

$$k^2 R(k, \lambda) = -8\lambda^2 e^{6\lambda} \left(1 + e^{2\lambda}\right) \left[2 h_1(k, \lambda) + e^{2\lambda} \sigma(k) h_2(k, \lambda)\right]$$

$$+ 4\lambda e^{4\lambda} \sum_{j=1}^{3} b_j(k, \lambda) \left(e^{2j\lambda} - 1\right) + \sum_{j=1}^{4} c_j(k, \lambda) \left(e^{2j\lambda} - e^{10\lambda}\right),$$

where

$$c_1 = \left(2 e^{-2k} - 4\right) n_1 - \sigma n_2, \quad c_3 = 12 \sigma h_1 + 8 e^{-2k} n_1 - 2 c_1 - c_2,$$

$$c_2 = -4 e^{-2k} h_1 - 2 h_1 n_1 - 8 n_1, \quad c_4 = -4 \sigma h_1 - 8 n_1 - 2 c_1 - 2 c_2 - c_3,$$

and

$$b_3 = e^{-2k} h_1 + 2 \sigma h_2, \quad b_1 = -5 e^{-2k} h_1, \quad b_2 = h_1 - c_1 - b_1 - b_3.$$

In view of (B 5) the representation of $k^2 R(k, \lambda)$ yields the uniform estimate

$$\left|k^2 R(k, \lambda)\right| = O(e^{10\lambda}) \quad \text{as} \quad \lambda \to \infty,$$

and by (B 3) and (B 8) we arrive at

$$\sup\{|v(k,\lambda)| : k \geqslant 1\} = O\left(\lambda^{-1}\right) \quad \text{as} \quad \lambda \to \infty.$$

Combining the latter and (B 7) proves the assertion for the case $i = 1$. The estimate (B 1) for $i = 2, 3, 4$ can be established by using the above presentation of v.

We write

$$t_i(k,v) - \tau_i(k) = \mathrm{e}^{-k} v(k,\lambda) + \frac{2\mathrm{e}^{2\lambda} q(k,\lambda)}{\lambda[\sinh(2\lambda) - 2\lambda]} + \frac{\tanh \lambda \left[4 n_1(k/2) + n_2(k/2)\right]}{4(\tanh \lambda - \lambda)}, \quad \text{(B 9)}$$

where $i = 2, 3$, and

$$q(k,\lambda) = \lambda \frac{\mathrm{e}^{-k} - \mathrm{e}^{-\lambda}}{k - \lambda}.$$

Obviously, $|q(k,\lambda)| \leqslant q(0,\lambda) = \mathrm{e}^{-\lambda}$ when $k \geqslant 0$ and $\lambda > 0$. Thus, the above estimate of v and the representation (B 9) guarantee that (B 1) is also true for $i = 2, 3$.

Likewise, we write

$$t_4(k,v) - \tau_4(k) = \frac{v - k}{v + k} v(k,v) - \Delta(k,\lambda), \quad \text{(B 10)}$$

where

$$\Delta(k,\lambda) = \frac{2 \tanh \lambda \, n_2(k)}{\rho(k,\lambda) \, n_1(k)} + \frac{\tanh \lambda \left[\left(3 - \mathrm{e}^{-2\lambda}\right)\left(1 - \mathrm{e}^{4\lambda}\right) + 2\lambda \left(\mathrm{e}^{4\lambda} + 4\mathrm{e}^{2\lambda} - 1\right)\right]}{\rho(k,\lambda)\left(1 + \mathrm{e}^{2\lambda}\right)(\sinh(2\lambda) - 2\lambda)(\lambda - \tanh \lambda)}$$

and $\rho(k,\lambda) = \lambda + k \tanh \lambda$. It remains to note that

$$|(v - k)/(v + k)| \leqslant 1, \quad |n_2(k)/n_1(k)| \leqslant 2, \quad \rho(k,\lambda) \geqslant \lambda,$$

when $k \geqslant 0$. The proof is complete. $\qquad\square$

The following lemma is based on the representations of the expressions $t_i(k,v) - \tau_i(k)$ obtained in proof of the preceding lemma.

Lemma B.2 *The functions $t_i(k,v)$, $i = 1, 2, 3, 4$ defined by (3.11) are continuous (analytic) functions of (k,v) when $v > 1$ and $k \geqslant 1$ $(k > 1)$.*

Proof Consider the function $t_1(k,v)$. It can be written as $v(k,v) + \tau_1(k)$, where $\tau_1(k)$ depends analytically on $k \in \mathbf{R}$. Analytic properties of the functions defined by (B 2) and the representations (B 3) and (B 4) yield that $t_1(k,v)$ is analytic in the domain where the denominator of (B 3) is not equal to zero. We write

$$Q(k,\lambda) = \frac{\lambda n_1(k) \left(1 + \mathrm{e}^{-2k}\right)\left(1 + \mathrm{e}^{-2\lambda}\right)}{k - \lambda} \left[\frac{\tanh k}{k} - \frac{\tanh \lambda}{\lambda}\right].$$

The expression in square brackets only vanishes if $k = \lambda$. At the same time, by (B 4), $Q(k,\lambda)$ is continuous at the point (λ,λ), when $\lambda > 0$ and

$$Q(\lambda,\lambda) = 4\mathrm{e}^{-2\lambda} n_1(\lambda) \left[1 - \frac{\sinh(2\lambda)}{2\lambda}\right] \neq 0.$$

Thus, the inequality $Q(k, \lambda) \neq 0$ is established when $k \geqslant 0$ and $\lambda > 0$. The latter proves the assertion for t_1. Further, referring to the representations (B 9) and (B 10) completes the proof.

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