Canad. J. Math. Vol. 74 (3), 2022 pp. 858–886
http://dx.doi.org/10.4153/S0008414X21000122

Orbital integrals on $\mathrm{GL}_n \times \mathrm{GL}_n\backslash\mathrm{GL}_{2n}$

Hang Xue

Abstract. We study harmonic analysis on the symmetric space $\mathrm{GL}_n \times \mathrm{GL}_n\backslash\mathrm{GL}_{2n}$. We prove several standard results, e.g. Shalika germ expansion of orbital integrals, representability of the Fourier transform of orbital integrals and representability of spherical characters. These properties are not expected to hold for symmetric spaces in general.

1 Introduction

In the relative Langlands program, one often seeks to establish a comparison of two relative trace formulae in order to establish a connections between period integrals on the one hand, and special values of L-functions on the other. In [Guo96] such a result, as a generalization of Waldspurger's formula for toric periods, was conjectured for automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A}_F)$, with the period integrals corresponding to the subgroups $\mathrm{GL}_n(\mathbb{A}_F) \times \mathrm{GL}_n(\mathbb{A}_F)$ or $\mathrm{GL}_n(\mathbb{A}_E)$, where E/F is a quadratic extension of number fields. The case of $\mathrm{GL}_n(\mathbb{A}_F) \times \mathrm{GL}_n(\mathbb{A}_F)$ is referred to as "linear periods" and was first introduced and studied by Jacquet and his collaborators [FJ93, JR96]. This note seeks to establish the necessary analytic properties of relative orbital integrals arising from the geometric side of the corresponding relative trace formula to pursue this conjecture.

Let F be a p-adic field of characteristic zero and $\eta : F^\times \to \{\pm 1\}$ be a nontrivial quadratic character. Let $G = \mathrm{GL}_{2n,F}$ and $H = \mathrm{GL}_{n,F} \times \mathrm{GL}_{n,F}$ with an embedding

$$(h_1, h_2) \mapsto \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}, \quad h_1, h_2 \in \mathrm{GL}_{n,F}.$$

Put

$$\theta(g) = \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix} g \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix}.$$

Then, $H = \{g \in G \mid \theta(g) = g\}$. Let

$$S = \{g^{-1}\theta(g) \mid g \in G\} \subset G$$

This is a closed subvariety of G over F and H acts on S by conjugation. We prove some standard harmonic analysis results on S, e.g., density of regular semisimple orbital integrals, representability of Fourier transform of orbital integrals, representability of

Received by the editors April 20, 2020; revised November 12, 2020; accepted February 8, 2021.
Published online on Cambridge Core February 26, 2021.
This work is partially supported by the NSF grant DMS #1901862.
AMS subject classification: 22E35.
Keywords: orbital integral, spherical characters, symmetric spaces.

spherical characters, etc. Note that these results are *not* expected for general symmetric spaces, as indicated by various counterexamples of Rader and Rallis [RR96]. This means that the symmetric space S is of a particular good shape in this regard. Our argument follows closely the traditional route. The new ingredient is a detailed study of the nilpotent orbital integrals, which is needed in verifying the homogeneity properties of the nilpotent orbital integrals. This study leads to some very interesting linear algebra problems. One of them is the following: classify pairs of $n \times n$ matrices (A, B) with AB being nilpotent, up to the equivalence relation

$$(A, B) \sim (A', B') \Leftrightarrow \exists\, h_1, h_2 \in \mathrm{GL}_n(F), \quad \text{s.t. } A' = h_1^{-1} A h_2,\ B' = h_2^{-1} B h_1.$$

This innocent looking problem is in fact equivalent to the classification of nilpotent orbits and is (surprisingly) not easy, c.f. Section 3 for a solution.

Due to the very nature of the subject, this paper is leaning toward the technical side. We describe our results more precisely in the rest of the introduction for the convenience of future reference. The most applicable result perhaps is Theorem 1.5 which asserts that the spherical characters arising in this context are represented by locally integrable functions.

Elements of S are all of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a^2 = d^2 = 1_n + bc, \quad ab = bd, \quad dc = ca.$$

We say that an element $x \in S$ is θ-semisimple (respectively, θ-regular semisimple) if it is semisimple (respectively, regular semisimple) in $\mathrm{GL}_{n,F}$ (in the usual sense) and $\det(a^2 - 1_n) \neq 0$. We say that an element $x \in G$ is θ-semisimple (respectively, θ-regular semisimple) if its image in S is so.

Let $f \in C_c^\infty(G)$ and $g \in G$ be a θ-semisimple element. We define the θ-semisimple orbital integral

$$O(g, \eta, f) = \int_{(H\times H)_g \backslash H \times H} f(h_1 g h_2)\eta(\det h_2)\mathrm{d}h_1 \mathrm{d}h_2,$$

where $(H \times H)_g = \{(h, h') \in H \times H \mid hgh' = g\}$. This integral is absolutely convergent. Let $D(G)^{H\times H,\eta}$ be the space of left H-invariant and right (H, η)-invariant distributions on G. Then $O(g, \eta, \cdot) \in D(G)^{H\times H,\eta}$ for all θ-regular semisimple $g \in G$.

Theorem 1.1 *The set $\{O(g, \eta, \cdot) \mid g \in G \text{is } \theta\text{-regular semisimple}\}$ is weakly dense in $D(G)^{H\times H,\eta}$. This means that if $f \in C_c^\infty(G)$ and $O(g, \eta, f) = 0$ for all θ-regular semisimple $g \in G$, then $\lambda(f) = 0$ for all $\lambda \in D(G)^{H\times H,\eta}$.*

We also consider the tangent space of S at the point represented by the identity element in G. This is a vector space $\mathfrak{s}$ together with an action of the group H. By way of analogy with the group case, we will refer to it as the "Lie algebra" of S. Explicitly it can be described as follows. We have $\mathfrak{s} = M_{n,F} \times M_{n,F}$, considered as a subspace of $M_{2n,F}$ consisting of matrices of the form

$$\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}, \quad X, Y \in M_{n,F}.$$

The group H acts on $\mathfrak{s}$ by conjugation. An element in $\mathfrak{s}$ is θ-semisimple or θ-regular semisimple if it is so in $M_{2n,F}$. The locus of θ-semisimple and θ-regular semisimple elements in $\mathfrak{s}$ are denoted by $\mathfrak{s}_{\theta-\mathrm{ss}}$ and $\mathfrak{s}_{\theta-\mathrm{reg}}$ respectively.

Let $\gamma \in \mathfrak{s}_{\theta-\mathrm{ss}}$ and $f \in C_c^\infty(\mathfrak{s})$, we define an orbital integral

$$O(\gamma,\eta,f) = \int_{H_\gamma\backslash H} f(h^{-1}\gamma h)\eta(\det h)\mathrm{d}h,$$

where $H_\gamma = \{h \in H \mid h^{-1}\gamma h = \gamma\}$. The integral is absolutely convergent.

Let $D(\mathfrak{s})^{H,\eta}$ be the (H,η)-invariant distributions on $\mathfrak{s}$. Then $O(\gamma,\eta,\cdot) \in D(\mathfrak{s})^{H,\eta}$ for all θ-regular semisimple γ in $\mathfrak{s}$.

Theorem 1.2 *The set $\{O(\gamma,\eta,\cdot) \mid \gamma \in \mathfrak{s}_{\theta-reg}\}$ is weakly dense in $D(\mathfrak{s})^{H,\eta}$.*

Let us fix an H-invariant inner product on $\mathfrak{s}$ by $\langle\gamma,\delta\rangle = \mathrm{Tr}\gamma\delta$, where on the right hand side the product and the trace are taken in $M_{2n,F}$. Thus, we can speak of the Fourier transform of elements in $C_c^\infty(\mathfrak{s})$ and hence the Fourier transform of distributions on $\mathfrak{s}$. The following result is proved in [Zha15, Theorem 6.1].

Proposition 1.3 *Let $\gamma \in \mathfrak{s}$ be θ-regular semisimple. Then the Fourier transform of the distribution $O(\gamma,\eta,\cdot)$ is represented by a locally integrable (H,η)-invariant function on $\mathfrak{s}$. This function is locally constant on $\mathfrak{s}_{\theta-\mathrm{reg}}$.*

We will define "θ-nilpotent orbital integrals" in this note and prove the following result.

Proposition 1.4 *The Fourier transform of θ-nilpotent orbital integrals are represented by locally integrable functions on $\mathfrak{s}$. This function is locally constant on $\mathfrak{s}_{\theta-\mathrm{reg}}$.*

This proposition is the technical heart of the note. The hard part is that, as opposed to the case of the classical orbital integrals or the nonsplit analogue of this paper treating orbital integrals on $\mathrm{GL}_n(E)\backslash\mathrm{GL}_{2n}(F)$ [Guo98], the naive integration on the θ-nilpotent orbits is not absolutely convergent in our case and some subtle regularization process is needed to *define* "θ-nilpotent orbital integrals."

A standard consequence of this proposition is the representability of the relative spherical characters. Let π be an irreducible representation of G. Assume that $\mathrm{Hom}_H(\pi,\mathbb{C}) \neq 0$ and $\mathrm{Hom}_H(\widetilde{\pi},\eta) \neq 0$ where $\widetilde{\pi}$ is the contragredient of π. Fix nonzero elements $l \in \mathrm{Hom}_H(\pi,\mathbb{C})$ and $\widetilde{l}_\eta \in \mathrm{Hom}_H(\widetilde{\pi},\eta)$. Define a distribution on G by

$$J_\pi(f) = \sum_\varphi l(\pi(f)\varphi)\widetilde{l}_\eta(\widetilde{\varphi}), \quad f \in C_c^\infty(G).$$

Here, φ runs through a basis of π while $\widetilde{\varphi}$ runs through the dual basis. Then $J_\pi \in D(G)^{H\times H,\eta}$.

Theorem 1.5 *The distribution J_π is represented by a left H-invariant and right (H,η)-invariant locally integrable function on G.*

We end this introduction with a question. Let (G,H) be a general symmetric space in the sense that G is a reductive group over F and H is the fixed point in G of an involution. Rader and Rallis [RR96] showed using many counterexamples that the results in this note in general do not hold for (G,H). That is, regular semisimple orbital integrals might not be weakly dense in the space of all invariant distributions; the spherical characters might not be representable by a locally integrable functions. Apart from the case treated in this note, we only know that these good properties hold for the following pairs.

- The classical group case: $(H \times H, H)$. This is the celebrated result of Harish-Chandra.
- The Galois case: $(\mathrm{Res}_{E/F} H, H)$ where E/F is a quadratic field extension. This is due to Hakim [Hak94].
- The linear case: $(A^\times, B^\times)$ where E/F is a quadratic field extension and A is a central simple algebra over F containing E and B the centralizer of E in A. This is due to [Guo98] in if $A = M_{2n,F}$ and the general case follows from the same argument. It is unfortunate that no published proof is available.

The question is: Can you characterize symmetric spaces with these good properties in terms of their geometric properties or combinatorial invariants?

This note is organized as follows. We start with the semisimple descent of orbital integrals in Section 2. In Sections 3–7, we are going to work on the Lie algebra $\mathfrak{s}$. We study θ-nilpotent orbital integrals in Sections 3 and 4. We define all orbital integrals in Section 5. Then, we establish the Shalika germ expansion in Section 6 and prove that they are linearly independent in Section 7. Theorem 1.2 and Proposition 1.4 are also proved simultaneously with linear independence of Shalika germs. In Section 8, we deduce the results on the level of groups from the results on the Lie algebras. In particular, we prove Theorem 1.1. Finally in the last section, we prove Theorem 1.5, the local integrability of spherical characters.

Notation *We always take F to be a p-adic field of characteristic zero. Let $\mathfrak{o}_F$ be the ring of integers and ϖ_F a uniformizer.*

Let X be a scheme over F. Usually, we simply write X for $X(F)$ unless there are ambiguities. One notable exception is with the categorical quotient in which case we always distinguish the notation of the scheme from its set of F-points (see below). On the scheme X, we always use the Zariski topology while on the set of F-points $X(F)$ we always use the analytic topology.

Let G be an algebraic group over F and V be a G-variety over F, i.e., V admits an action of G. This action is sometimes denoted by $g \cdot v$ or gv where $g \in G$ and $v \in V$. If $x \in V$, we denote by G_x the stabilizer of x in G. If C is a subset of V and $g \in G$, then we let C^g the subset consisting of all elements of the form $g \cdot v$ where $v \in C$, and we let $C^G = \cup_{g \in G} C^g$. Thus, if $x \in V$, then x^G stands for the orbit of x. The adjoint action of G on its Lie algebra (or subgroup of G acting on subspaces of the Lie algebra of G) is denoted by Ad.

We denote by $q : V \to V//G$, or simply $V//G$, the categorical quotient. We should note that $(V//G)(F)$ is usually not the same as $V(F)//G(F)$ and we always write $V//G$ for the scheme instead of its F-points. A subset of U of $V(F)$ is called saturated if $U = q^{-1}(q(U))$.

We use capital letters to denote various groups and symmetric spaces. We use the corresponding Gothic letters to denote their Lie algebras, e.g., if G is an algebraic group, then without saying to the contrary, $\mathfrak{g}$ stands for the Lie algebra of G. Ele-

ments in the groups or symmetric spaces are usually denoted using lower case Latin letters, while elements in the Lie algebras are usually denoted by lower case Greek letters.

2 Semisimple descent

First, we consider some general setup. Let G be a reductive group over F, X be a G-variety over F and $x \in X$ be G-semisimple point, i.e., the orbit x^G of x is closed. We let N_x^X be the normal space of x^G at x. It admits a natural action of G_x and we call (H_x, N_x^X) the sliced representation at x. By [AG09], there exist the following data which we refer to as the analytic slice at x. We use analytic topology throughout.

(1) An G-invariant open neighborhood U of x^G in X with an G-equivariant retraction map $p : U \to x^G$.
(2) An G_x-equivariant embedding $\psi : p^{-1}(x) \to N_x^X$ with an open and saturated image such that $\psi(x) = 0$.

If $y \in p^{-1}(x)$ and $z = \psi(y)$, then we have

(1) $(G_x)_z \simeq G_y$ and $N_z^{N_x} \simeq N_y^X$ as representations of $(G_x)_z$ and G_y and
(2) y is G-semisimple in X if and only if z is G_x-semisimple in N_x^X.

The analytic slice at x is denoted by (U, p, ψ).

Let us now specialize to the case $X = \mathfrak{s}$ or S with the conjugation action of H.

First consider the case $X = \mathfrak{s}$. The categorical quotient $\mathfrak{s}//H$ is an n-dimensional affine space over F. The canonical map $\mathfrak{s} \to \mathfrak{s}//H$ is given by

$$\begin{pmatrix} & a \\ b & \end{pmatrix} \to \operatorname{Tr} \wedge^i ab, \quad i = 1, \dots, n.$$

More precisely it maps $\begin{pmatrix} & a \\ b & \end{pmatrix}$ to the coefficients of the characteristic polynomial of ab. Each fiber of $\mathfrak{s} \to \mathfrak{s}//H$ is a collection of of finitely many orbits.

Let $\gamma \in \mathfrak{s}_{\theta-\mathrm{ss}}$ and $G_\gamma = \{g \in G \mid g^{-1}\gamma g = \gamma\}$ be its stabilizer in G and then $H_\gamma = H \cap G_\gamma$. Let $\mathfrak{g}_\gamma$, $\mathfrak{h}_\gamma$ be the Lie algebras of them respectively. The involution θ preserves G_γ, and hence $\mathfrak{g}_\gamma$. Let $\mathfrak{s}_\gamma$ be the (-1)-eigenspace of θ in $\mathfrak{g}_\gamma$. Then, $\mathfrak{g}_\gamma = \mathfrak{h}_\gamma \oplus \mathfrak{s}_\gamma$ and H_γ acts on $\mathfrak{s}_\gamma$. By [AG09, Proposition 7.2.1], the sliced representation at x is isomorphic to $(H_\gamma, \mathfrak{s}_\gamma)$. By [JR96], up to conjugation by H, the θ-semisimple element γ takes the following form

$$\gamma = \begin{pmatrix} & & X & \\ & & & 0_r \\ 1_{n-r} & & & \\ & 0_r & & \end{pmatrix},$$

where $X \in \mathrm{GL}_{n-r}(E)$. It is not hard to check that the symmetric pair (G_γ, H_γ) is of the form

$$(G_1, H_1) \times (G_2, H_2),$$

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

where

$$G_1 \simeq \left\{ x = \begin{pmatrix} a & Xc \\ c & a \end{pmatrix} \in \mathrm{GL}_{2n-2r}(E) \;\middle|\; aX = Xa,\ Xc = cX \right\},$$

and

$$H_1 \simeq \left\{ h = \begin{pmatrix} a & \\ & a \end{pmatrix} \;\middle|\; aX = Xa \right\}.$$

The symmetric space (G_2, H_2) is isomorphic to $(\mathrm{GL}_{2r}, \mathrm{GL}_r \times \mathrm{GL}_r)$. The sliced representation $\mathfrak{s}_y$ is isomorphic to $\mathfrak{s}_1 \times \mathfrak{s}_2$ on which $H_1 \times H_2$ acts componentwise. The action of H_2 on $\mathfrak{s}_2$ is of the same shape as the action of H on $\mathfrak{s}$, but of a smaller size. The space $\mathfrak{s}_1$ with the action by H_1 is indeed isomorphic to the usual conjugation action of H_1 on its Lie algebra.

We now consider the case $X = S$. Let $g \in G$ be θ-semisimple and $x = g^{-1}\theta(g) \in S$. The centralizer G_x is stable under the involution θ and the fixed point of θ is precisely H_x. Then (G_x, H_x) form a symmetric space. Let $S_x = \{g^{-1}\theta(g) \mid g \in G_x\}$ and $\mathfrak{s}_x$ be its tangent space at 1. Then we have $\mathfrak{g}_x = \mathfrak{h}_x \oplus \mathfrak{s}_x$. Again by [AG09, Proposition 7.2.1], the sliced representation at x is isomorphic to $(H_x, \mathfrak{s}_x)$. According to [JR96, Proposition 4.1], x is H-conjugate to an element of the form

$$\begin{pmatrix} a & & & a-1_r & & \\ & 1_s & & & & \\ & & -1_{n-r-s} & & & \\ a+1_r & & & a & & \\ & & & & 1_s & \\ & & & & & -1_{n-r-r} \end{pmatrix},$$

where $a \in \mathrm{GL}_r(F)$ is semisimple in the usual sense and $\det(a^2 - 1_r) \neq 0$. Then it follows that the symmetric space (G_x, H_x) is a product

$$(G_1, H_1) \times (G_2, H_2) \times (G_3, H_3),$$

where (G_2, H_2) and (G_3, H_3) are of the same shape of (G, H) but of smaller sizes and

$$G \simeq \left\{ \begin{pmatrix} b & (a+1_r)c \\ (a-1_r)c & b \end{pmatrix} \;\middle|\; ab = ba,\ ac = ca \right\},\ H \simeq \left\{ \begin{pmatrix} b & \\ & b \end{pmatrix} \;\middle|\; ab = ba \right\}.$$

The sliced representation $\mathfrak{s}_x$ is isomorphic to $\mathfrak{s}_1 \times \mathfrak{s}_2 \times \mathfrak{s}_3$ where $H_1 \times H_2 \times H_3$ acts componentswise. Here, $(H_1, \mathfrak{s}_1)$ is isomorphic to the adjoint action of H_1 on its Lie algebra, and $(H_2, \mathfrak{s}_2)$, $(H_3, \mathfrak{s}_3)$ are of the same shape as $(H, \mathfrak{s})$ but of smaller sizes.

The following proposition connects the orbital integrals on S or $\mathfrak{s}$ near a θ-semisimple point x to the orbital integrals on the sliced representation at x. This procedure will be referred to as semisimple descent.

Proposition 2.1 *Let $X = \mathfrak{s}$ or S and $x \in X$ be θ-semisimple. There exists an open neighborhood $\omega_x \subset \psi(p^{-1}(x))$ of $0 \in N_x^X$ with the following property: if $f \in C_c^\infty(X)$, then there is an $f_x \in C_c^\infty(N_x^X)$ so that for all θ-regular semisimple $z \in \omega_x$, $z = \psi(y)$ with*

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

$y \in p^{-1}(x)$, *we have*

$$\int_{H_y\backslash H} f(h^{-1}yh)\eta(\det h)\mathrm{d}h = \int_{H_z\backslash H_x} f_x(h^{-1}zh)\eta(\det h)\mathrm{d}h \tag{2.1}$$

Proof This is stated in [Zha15, Proposition 5.20]. We give a short proof here as we will make use of the explicit construction (not merely the existence) of f_x later.

As usual the proof begins with the following compactness result.

<u>*Claim*</u>. Let $\omega_x \subset \psi(p^{-1}(x))$ be a saturated subset whose image in $(X_x//H_x)(F)$ is relatively compact. Let $\omega \subset X$ be a compact subset. Then the closure of

$$\{h \in H \mid \psi^{-1}(\omega_x)^h \cap \omega \neq \varnothing\}$$

is compact in $H_x\backslash H$.

The proof of this claim is clear. We consider the diagram

$$\begin{array}{ccc} H \times_{H_x} p^{-1}(x) & \xrightarrow{\ i\ } & X \times (N_x^X//H_x) \\ \downarrow{\scriptstyle j} & & \\ H_x\backslash H & & \end{array}$$

The horizontal arrow is a closed embedding. The set in the claim is contained in the compact set $ji^{-1}(\omega \times \omega_y)$.

With this claim at hand, we proceed as follows. Let $f \in C_c^\infty(X)$ and $\omega = \operatorname{supp} f$. Let C be an open compact subset of $H_x\backslash H$ which contains the closure of the set in the claim. Choose a function $\alpha \in C_c^\infty(H)$ such that

$$\int_{H_x} \alpha(hg)\mathrm{d}h = \mathbf{1}_C(g).$$

Put

$$f_x(z) = \int_H f(h^{-1}\psi^{-1}(z)h)\eta(\det h)\alpha(h)\mathrm{d}h, \quad z \in \omega_y.$$

Then, $f_x \in C_c^\infty(\omega_x)$ and we view f_x as a function on N_x^X. Let $z \in \omega_x$ be θ-regular semisimple and $y = \psi^{-1}(z) \in p^{-1}(x)$. Then $y \in X$ is θ-regular semisimple and $(H_x)_z = H_y$. A little computation gives

$$\int_{(H_x)_z\backslash H_x} f_x(h^{-1}zh)\eta(\det h)\mathrm{d}h = \int_{H_y\backslash H} f(h^{-1}yh)\eta(\det h)\mathrm{d}h.$$

This proves the proposition. ∎

3 The nilpotent cone

Let $\mathcal{N} \subset \mathfrak{s}$ be the nilpotent cone, i.e., the closed subvariety of $\mathfrak{s}$ consisting of all elements whose orbit closure contains $0 \in \mathfrak{s}$. Elements or orbits contained in $\mathcal{N}$ are called θ-nilpotent.

Lemma 3.1 *The nilpotent cone consists of elements in $\mathfrak{s}$ that are nilpotent in $\mathfrak{g}$ in the usual sense.*

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

Proof An element $\xi = \begin{pmatrix} & X \\ Y & \end{pmatrix} \in \mathfrak{s}$ is contained in the nilpotent cone if and only if its image in the categorical quotient $\mathfrak{s}//H$ is 0. The later condition means that the coefficients of the characteristic polynomial of XY are all zero (except for the leading one), i.e., XY is nilpotent. This is again equivalent to that ξ is nilpotent in $\mathfrak{g}$. ∎

To analyze the θ-nilpotent orbits, it would be better to use a more canonical formulation. Let $V = V^+ \oplus V^-$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space with homogeneous components $V^\pm$ and $\dim V^\pm = n$. Then, we have

$$\mathfrak{s} \simeq \mathrm{Hom}(V^+, V^-) \oplus \mathrm{Hom}(V^-, V^+), \quad H \simeq \mathrm{GL}(V^+) \times \mathrm{GL}(V^-).$$

The nilpotent cone in $\mathfrak{s}$ consists of pairs of endomorphism $\xi = (X, Y) \in \mathrm{End}(V)$, $X \in \mathrm{Hom}(V^+, V^-)$ and $Y \in \mathrm{Hom}(V^-, V^+)$ such that XY and hence YX are both nilpotent. This condition is equivalent to saying that $\xi = (X, Y) \in \mathrm{End}(V)$ is nilpotent.

Let $\theta \in H$ be the element which acts on $V^\pm$ by ± 1. Then θ acts on $\mathfrak{gl}(V)$ by sending $Z \in \mathfrak{gl}(V)$ to $\mathrm{Ad}(\theta)Z = \theta Z \theta$. It is clear that $\mathfrak{h}$ and $\mathfrak{s}$ are eigenspaces of $\mathrm{Ad}(\theta)$ of eigenvalue 1 and -1, respectively.

Let $\xi = (X, Y) \in \mathcal{N}$. Then we have a filtration on V given by

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{s-1} \subset W_s = V, \quad W_i = \mathrm{Ker}\xi^i. \tag{3.1}$$

We may view V as an $F[\xi]$-module and V is a direct sum of indecomposable $F[\xi]$-submodules. By [KP79, Section 4], one can choose the generators of these submodules to be *homogeneous*. More concretely, let U be such an indecomposable submodule of dimension a over F. Then, we can choose a *homogeneous* element $u \in U$ so that

$$u, \xi u, \xi^2 u, \cdots \xi^{a-1} u$$

form a F-basis of U. It follows that for each i, we have

$$W_i = W_i^+ \oplus W_i^-, \quad W_i^\pm = W_i \cap V^\pm.$$

Therefore, we have two filtrations

$$0 = W_0^\pm \subset W_1^\pm \subset W_2^\pm \subset \cdots \subset W_{s-1}^\pm \subset W_s^\pm = V^\pm. \tag{3.2}$$

Note that while the filtration (3.1) is strictly increasing, these two filtration might not be strictly increasing.

We put $r_i^? = \dim W_i^?/W_{i-1}^?$ where ? stands for $+, -$, or empty. Note that ξ induces an injective map $W_{i+1}/W_i \to W_i/W_{i-1}$ for $i = 1, \dots, s-1$. It follows that $r_i \geq r_{i+1}$ for all i. Moreover, since ξ induces injective maps $W_{i+1}^\pm/W_i^\pm \to W_i^\mp/W_{i-1}^\mp$, we conclude that $r_i^\pm \geq r_{i+1}^\mp$ for all i. By suitably choosing bases of these successive quotients and lifting them to $V^\pm$, we may assume that the maps $W_{i+1}^\pm/W_i^\pm \to W_i^\mp/W_{i-1}^\mp$ induced by ξ are all of the form $\begin{pmatrix} 1_{r_{i+1}^\pm} \\ 0 \end{pmatrix}$, where 0 stands for the zero matrix of size $(r_i^\mp - r_{i+1}^\pm) \times r_{i+1}^\pm$.

Let $P = MN$ be the parabolic subgroup of $\mathrm{GL}(V)$ stabilizing the filtration (3.1), and $P^+ = M^+N^+$ be the parabolic subgroup of H stabilizing both filtrations (3.2).

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

We have

$$M^+ \simeq \prod_{i=0}^{s-1} \mathrm{GL}(W_{i+1}^+/W_i^+) \times \prod_{i=0}^{s-1} \mathrm{GL}(W_{i+1}^-/W_i^-).$$

Lemma 3.2 *We have*

$$P \cap H = P^+, \quad M \cap H = M^+, \quad N \cap H = N^+.$$

Proof It follows from the definition that $P \cap H \supset P^+$. If $h \in H \cap P$, then $h(W_i^\pm) \subset W_i$. But $h(W_i^\pm) \subset V^\pm$. It follows that $h(W_i^\pm) \subset W_i \cap V^\pm = W_i^\pm$. This proves $P \cap H = P^+$. One can similarly prove the other two equalities. ■

Lemma 3.3 *The following assertions hold.*

(1) *We have*

$$\mathrm{Ad}(N^+)\xi = \xi + [\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{s}, \tag{3.3}$$

where $[-,-]$ stands for the Lie algebra bracket of $\mathfrak{n}$.

(2) *For any $h \in H$, if $\mathrm{Ad}(h)(\mathfrak{n} \cap \mathfrak{s}) \subset \mathfrak{n} \cap \mathfrak{s}$, then $h \in P^+$.*

Proof By [How74, Lemma 2(b)], $\mathrm{Ad}(N)\xi = \xi + [\mathfrak{n}, \mathfrak{n}]$. Note that $\mathrm{Ad}(\theta)\xi = -\xi$. Then both sides of (3.3) are (-1)-eigenspaces of $\mathrm{Ad}(\theta)$. This proves the first assertion.

By [How74, Lemma 2(d)], if $g \in G$ and $\mathrm{Ad}(g)\xi \subset \mathfrak{n}$, then $g \in P$. Note that $\xi \in \mathfrak{n} \cap \mathfrak{s}$. Then the second assertion follows from Lemma 3.2. ■

Lemma 3.4 *The P^+-orbit of ξ in $\mathfrak{s}$ is an (Zariski) open subset of $\mathfrak{n} \cap \mathfrak{s}$ consisting of elements Z with the properties that*

$$Z|_{W_{i+1}^\pm/W_i^\pm} : W_{i+1}^\pm/W_i^\pm \to W_i^\mp/W_{i-1}^\mp, \quad i = 1, \ldots, s-1$$

is injective.

Proof Since $\mathrm{Ad}(N^+)\xi$ is the coset $\xi + [\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{s}$ in $\mathfrak{n} \cap \mathfrak{s}$, it is enough to consider the image of $\mathrm{Ad}(M^+)\xi$ in

$$\mathfrak{n} \cap \mathfrak{s}/[\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{s},$$

which is isomorphic to

$$\bigoplus_{i=1}^{s-1} \mathrm{Hom}(W_{i+1}^+/W_i^+, W_i^-/W_{i-1}^-) \oplus \bigoplus_{i=1}^{s-1} \mathrm{Hom}(W_{i+1}^-/W_i^-, W_i^+/W_{i-1}^+).$$

As explained before, ξ induces an injective map $W_{i+1}^\pm/W_i^\pm \to W_i^\mp/W_{i-1}^\mp$ for all i and with suitable choice of basis, this map is represented by the matrix $\begin{pmatrix} 1_{r_{i+1}^\pm} \\ 0 \end{pmatrix}$. Moreover, by choosing suitable bases, any injective map $W_{i+1}^\pm/W_i^\pm \to W_i^\mp/W_{i-1}^\mp$ can be represented by a matrix of this form. It follows that the image of $\mathrm{Ad}(P^+)\xi$ in $\mathrm{Hom}(W_{i+1}^\pm/W_i^\pm, W_i^\mp/W_{i-1}^\mp)$ is the subset of all injective maps. This proves the lemma. ■

We thus have the following classification of θ-nilpotent orbits.

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

Lemma 3.5 *The set of θ-nilpotent orbits in $\mathcal{N}$ is in one-to-one correspondence with the set of two sequences of integers $r_i^{\pm}$, $i = 1, \dots, s$, such that*

$$n = r_1^{\pm} + \cdots + r_s^{\pm}, \quad r_1^{\pm} \geq r_2^{\mp} \geq r_3^{\pm} \geq \cdots, \quad r_1^{+} + r_1^{-} > r_2^{+} + r_2^{-} > \cdots > r_s^{+} + r_s^{-} > 0. \tag{3.4}$$

Proof To each $\xi \in \mathcal{N}$, we have constructed as above two sequences of vector space $W_i^{\pm}$, $i = 1, \dots, s$. We simply put $r_i^{\pm} = \dim W_i^{\pm}/W_{i-1}^{\pm}$ and they satisfy (3.4).

Conversely, given any two sequences of integers $r_i^{\pm}$ satisfying (3.4), one can find an element $\xi \in \mathcal{N}$ so that $\dim W_i^{\pm}/W_{i-1}^{\pm} = r_i^{\pm}$. This can be achieved as follows. We are going to write $\mathfrak{s}$ explicitly as matrices of the form $\begin{pmatrix} & X \\ Y & \end{pmatrix}$ as before. First, write X as a blocked matrix where rows correspond to the partition $n = r_1^{+} \cdots + r_s^{+}$ and columns correspond to the partition $n = r_1^{-} + \cdots + r_s^{-}$. Similarly, write Y as a blocked matrix where rows correspond to the partition $n = r_1^{-} + \cdots + r_s^{-}$ and columns correspond to the partition $n = r_1^{+} \cdots + r_s^{+}$. Then, ξ is the matrix of following form. All the block entries of X and Y are zero except for the $(i, i+1)$ entry. The $(i, i+1)$ entry of X and Y are of size $r_i^{+} \times r_{i+1}^{-}$ and $r_i^{-} \times r_{i+1}^{+}$, respectively and we have $r_i^{\pm} \geq r_{i+1}^{\mp}$. The $(i, i+1)$ entry of X and Y are of the form $\begin{pmatrix} 1_{r_{i+1}^{\pm}} \\ 0 \end{pmatrix}$ where $1_{r_{i+1}^{\pm}}$ stands for the identity matrix of size r_{i+1}^{+} in X and size r_{i+1}^{-} in Y, and 0 stands for the zero matrix. It is not hard to check that this ξ is the desired θ-nilpotent matrix. ■

We now study the stabilizer M_{ξ}^{+} of ξ in M^{+}. If the H-orbit represented by ξ were to support an (H, η)-invariant distribution, then $\eta \circ \det$ would have to be trivial on M_{ξ}^{+}.

We have two chains of injective maps induced by the element ξ:

$$W_s^{\varepsilon}/W_{s-1}^{\varepsilon} \hookrightarrow \cdots \hookrightarrow W_3^{\mp}/W_2^{\mp} \hookrightarrow W_2^{\pm}/W_1^{\pm} \hookrightarrow W_1^{\mp}, \tag{3.5}$$

where $\varepsilon = +$ or $-$ according to the parity of s. For each i, the map $W_{i+1}^{\pm}/W_i^{\pm} \to W_i^{\mp}/W_{i-1}^{\mp}$ is either an isomorphism or (genuine) injective and it is an isomorphism if and only if $\dim W_{i+1}^{\pm}/W_i^{\pm} = \dim W_i^{\mp}/W_{i-1}^{\mp}$. We call the integer i a jump if $\dim W_{i+1}^{\pm}/W_i^{\pm} < \dim W_i^{\mp}/W_{i-1}^{\mp}$ (either the $+$ one or the $-$ one, the inequality does not have to hold for both filtrations). To unify treatment, we call s a jump if $\dim W_s^{\varepsilon}/W_{s-1}^{\varepsilon} \neq 0$.

Lemma 3.6 *Suppose that the orbit represented by ξ supports an (H, η)-invariant distribution. Then all jumps are even integers, i.e., we have the strict inequality $r_i^{\varepsilon} > r_{i+1}^{-\varepsilon}$ ($\varepsilon = +$ or $-$) in (3.4) only when i is even.*

Proof Let i be the smallest jump in one of the chains of injective maps (3.5), say the one ends with W_1^{+}. The last a few terms in the filtration looks like

$$W_{i+1}^{(-1)^i}/W_i^{(-1)^{i-1}} \hookrightarrow W_i^{(-1)^{i-1}}/W_{i-1}^{(-1)^{i-2}} \xrightarrow{\sim} \cdots \xrightarrow{\sim} W_1^{+},$$

where the leftmost arrow is injective but not an isomorphism. We construct a basis of V as follows. Choose linearly independent elements in $W_s^{\pm}$ so that its image in $W_s^{\pm}/W_{s-1}^{\pm}$ is a basis. Then the image under ξ of these elements in $W_{s-1}^{\mp}$ are also linearly independent. We extend them to a set of linearly independent elements in $W_{s-1}^{\mp}$ so that the image in $W_{s-1}^{\mp}/W_{s-2}^{\mp}$ forms a basis. We repeat this process for all $W_j^{\pm}$'s. Then,

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

we get a basis of V. Among elements in this basis, we can find $w_j \in W_j^{(-1)^{j-1}}$ so that $\xi(w_j) = w_{j-1}$, $j = 1, \dots, i$, and w_i is not in the image of W_{i+1}^- under ξ. Choose $\lambda \in F^\times$ with $\eta(\lambda) = -1$ and let $h \in \mathrm{GL}(M^+)$ be an element such that it acts as multiplication by λ on $w_1, \dots, w_i$ and trivially on all elements of the basis of V. Then by construction $h \in M_\xi^+$ and $\eta(\det h) = (-1)^i$. Since $\eta \circ \det$ is trivial on M_ξ^+, i has to be even. We may repeat this process for all other jumps. ∎

Example 3.7 To facilitate understanding, we suggest the following example. Let us consider the case $n = 4$ and the nilpotent element $\xi \in \mathfrak{s}$ given by the following matrix.

$$\xi = \left(\begin{array}{cccc|cccc} & & & & 0 & 1 & & \\ & & & & & 0 & 1 & \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \\ \hline 0 & 0 & 1 & 0 & & & & \\ & 0 & 0 & 0 & & & & \\ & & 0 & 1 & & & & \\ & & & 0 & & & & \end{array}\right), \quad V^+ = \begin{pmatrix} * \\ * \\ * \\ * \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad V^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ * \\ * \\ * \\ * \end{pmatrix}.$$

Simple computation gives

$$W_1^+ = \begin{pmatrix} * \\ * \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad W_2^+ = \begin{pmatrix} * \\ * \\ * \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad W_3^+ = \begin{pmatrix} * \\ * \\ * \\ * \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad W_1^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ * \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad W_2^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ * \\ * \\ * \\ 0 \end{pmatrix}, \quad W_3^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ * \\ * \\ * \\ * \end{pmatrix}.$$

Moreover, $(r_1^+, r_2^-, r_3^+) = (2, 2, 1)$ and $(r_1^-, r_2^+, r_3^-) = (1, 1, 1)$. The elements w and h that we chose in the proof of Lemma 3.6 is

$$w = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad h = \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \lambda & & & & & & \\ & & 1 & & & & & \\ & & & \lambda & & & & \\ \hline & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \lambda & \\ & & & & & & & 1 \end{array}\right), \quad \det h = \lambda^3.$$

It is straightforward to check that h commutes with ξ. According to our terminology, in the sequence $r_1^+ = r_2^- > r_3^+$, 2 and 3 are jumps, which are not all even. The orbit represented by ξ does not support any (H, η)-invariant distribution.

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

4 Nilpotent orbital integrals

In this section, we are going to show that the necessary condition in Lemma 3.6 that a nilpotent orbital integral supports an (H, η)-invariant distribution is also sufficient. Moreover, these (H, η)-invariant distributions extend to an (H, η)-invariant distribution on $\mathfrak{s}$.

Let us keep the notation from Section 3. Let $\mathcal{O}$ be a θ-nilpotent orbit in $\mathfrak{s}$ represented by an element ξ. Then attached to ξ is a parabolic subgroup $P^+ = M^+N^+$ of H. We also have two sequences of integers $r_1^{\pm} \geq r_2^{\mp} \geq r_3^{\pm} \geq \cdots$. We assume that all the jumps in these two sequences are even integers. By Lemma 3.6, this is a necessary condition for $\mathcal{O}$ to support an (H, η)-invariant distribution.

Let $2i_1 < \cdots < 2i_a$ be the set of all jumps in the sequence $r_1^+ \geq r_2^- \geq \cdots$. Let $2j_1 < \cdots < 2j_b$ be the set of all jumps in the sequence $r_1^- \geq r_2^+ \geq \cdots$. Note that we either have $2i_a = s$ and $W_s^-/W_{s-1}^- \neq 0$, or $2i_a < s$ and all $W_{i+1}^{\varepsilon}/W_i^{\varepsilon} = 0$ if $i \geq 2i_a$, where ε is an appropriate sign. We have a similar assertion for the jump $2j_b$. Then the space $\mathfrak{n} \cap \mathfrak{s}/[\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{s}$ is isomorphic to

$$\bigoplus_{i=1}^{2i_a} \mathrm{Hom}(W_{i+1}^{(-1)^i}/W_i^{(-1)^i}, W_i^{(-1)^{i-1}}/W_{i-1}^{(-1)^{i-1}})$$
$$\oplus \bigoplus_{i=1}^{2j_b} \mathrm{Hom}(W_{i+1}^{(-1)^{i+1}}/W_i^{(-1)^{i+1}}, W_i^{(-1)^i}/W_{i-1}^{(-1)^i}).$$

Let us define some determinant functions. Let us write an element in $\mathfrak{n} \cap \mathfrak{s}/[\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{s}$ as a sequence

$$m = (x_1, \ldots, x_{2i_a}; y_1, \ldots, y_{2j_b}),$$

with

$$x_i \in \mathrm{Hom}(W_{i+1}^{(-1)^i}/W_i^{(-1)^i}, W_i^{(-1)^{i-1}}/W_{i-1}^{(-1)^{i-1}}),$$
$$y_i \in \mathrm{Hom}(W_{i+1}^{(-1)^{i+1}}/W_i^{(-1)^{i+1}}, W_i^{(-1)^i}/W_{i-1}^{(-1)^i}).$$

Note that if i is odd, then both $r_{i+1}^{\pm} = r_i^{\mp}$ by the assumption that all jumps are even integers. Moreover,

$$\xi|_{W_{i+1}^{\pm}/W_i^{\pm}} : W_{i+1}^{\pm}/W_i^{\pm} \to W_i^{\mp}/W_{i-1}^{\mp}$$

is an isomorphism. To shorten notation, we put $\xi_i^{\mp} = \xi|_{W_{i+1}^{\pm}/W_i^{\pm}}$. Define

$$\det{}_{2i-1}^{+}(x_{2i-1}) = \det x_{2i-1}(\xi_{2i-1}^{+})^{-1}, \quad \det{}_{2i-1}^{-}(y_{2i-1}) = \det y_{2i-1}(\xi_{2i-1}^{-})^{-1},$$

and

$$\det{}_{\mathfrak{n}}(m) = \det{}_1^{+}(x_1)\det{}_3^{+}(x_3)\cdots\det{}_{2j_a-1}^{+}(x_{2j_a-1})\det{}_1^{-}(y_1)\det{}_3^{-}(y_3)\cdots\det{}_{2j_b-1}^{-}(y_{2j_b-1}).$$

Lemma 4.1 *For $p \in P^+$ and $u \in \mathfrak{n} \cap \mathfrak{s}$, we have*

$$\eta(\det{}_{\mathfrak{n}}(pup^{-1})) = \eta(\det p)\eta(\det{}_{\mathfrak{n}} u).$$

Proof This follows from the definition of $\det_{\mathfrak{n}}$. ∎

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

Let $\mathfrak{n}'$ be the subspace of $\mathfrak{n} \cap \mathfrak{s}$ generated by $[\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{s}$ and

$$\bigoplus_{i \text{ even}} \operatorname{Hom}(W_{i+1}^{+}/W_i^{+}, W_i^{-}/W_{i-1}^{-}) \oplus \bigoplus_{i \text{ even}} \operatorname{Hom}(W_{i+1}^{-}/W_i^{-}, W_i^{+}/W_{i-1}^{+}).$$

Let $f \in C_c^\infty(\mathfrak{s})$, we define a function $\widetilde{f} \in C_c^\infty(\mathfrak{n} \cap \mathfrak{s}/\mathfrak{n}')$ as

$$\widetilde{f}(m) = \int_{\mathfrak{n}'} f(m+u)\mathrm{d}u. \tag{4.1}$$

Before we proceed, let us recall the following result due to Godement and Jacquet [GJ72, Theorem 3.3] (taking the representation π to be $\eta \circ \det$). Note that the holomorphy is a consequence of the fact that E/F is a quadratic extension of nonarchimedean local fields and η is nontrivial.

Lemma 4.2 *Let $\varphi \in C_c^\infty(M_n(F))$. Put*

$$Z(s, \eta, \varphi) = \int_{\mathrm{GL}_n(F)} \varphi(h)|\det h|^s \eta(\det h)\mathrm{d}h,$$

where dh stands for the multiplicative measure on $\mathrm{GL}_n(F)$. Then this integral is convergent if $\Re s \gg 0$ and has a meromorphic continuation to the whole complex plane. It is holomorphic at all $s \in \mathbb{R}$.

The function $\widetilde{f}$ is a function in the variables

$$m = (x_1, x_3, \dots, x_{2j_a-1}; y_1, y_3, \dots, y_{2j_b-1}).$$

Let $\underline{s} = (s_1, s_3, \dots, s_{2j_a-1})$ and $\underline{t} = (t_1, t_3, \dots, t_{2j_b-1})$ be complex numbers. Put

$$\begin{aligned}\det{}_{\mathfrak{n},\underline{s},\underline{t}}(m) = &|\det{}_1^+(x_1)|^{s_1}|\det{}_3^+(x_3)|^{s_3}\cdots|\det{}_{2j_a-1}^+(x_{2j_a-1})|^{s_{2j_a-1}}\\ &|\det{}_1^-(y_1)|^{t_1}|\det{}_3^-(y_3)|^{t_3}\cdots|\det{}_{2j_b-1}^-(y_{2j_b-1})|^{t_{2j_b-1}}.\end{aligned}$$

Consider the integral

$$Z(\underline{s}, \underline{t}, \eta, \widetilde{f}) = \int \widetilde{f}(m)\eta(\det{}_{\mathfrak{n}}(m)) \det{}_{\mathfrak{n},\underline{s},\underline{t}}(m)\mathrm{d}m,$$

where the domain of integration is $\mathfrak{n} \cap \mathfrak{s}/\mathfrak{n}$, which is identified with

$$\bigoplus_{i \text{ odd}} \operatorname{Hom}(W_{i+1}^{-}/W_i^{-}, W_i^{+}/W_{i-1}^{+}) \oplus \bigoplus_{i \text{ odd}} \operatorname{Hom}(W_{i+1}^{+}/W_i^{+}, W_i^{-}/W_{i-1}^{-}).$$

By Lemma 4.2, the integral $Z(\underline{s}, \underline{t}, \eta, \widetilde{f})$ is convergent when the real part of s_i and t_is are large enough and $Z(\underline{s}, \underline{t}, \eta, \widetilde{f})$ has meromorphic continuation to $\mathbb{C}^{i_a+j_b}$, which is holomorphic at the points where all s_i and t_i's are integers. We define

$$\widetilde{\mu}_{\mathcal{O}}(f) = Z(\underline{s}, \underline{t}, \eta, \widetilde{f})\Big|_{\substack{s_i = r_i^-, \text{ for all } i\\ t_i = r_i^+, \text{ for all } i}}.$$

The point is that for the variable coming from one of the decreasing sequences, we evaluate this integral at the point given by the corresponding integer in *the other* sequence.

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

Lemma 4.3 *For any $f \in C_c^\infty(\mathfrak{s})$, and any $p \in P^+$, we have*

$$\widetilde{\mu}_{\mathcal{O}}(\mathrm{Ad}(p)f) = \delta_{P^+}(p)\eta(\det p)\widetilde{\mu}_{\mathcal{O}}(f). \tag{4.2}$$

Proof The invariance by elements in N^+ is straightforward to check. One has to prove (4.2) for elements in M^+. We may even assume that $m \in \mathrm{GL}(W_{i+1}^+/W_i^+)$. The other cases can be derived from this one or follow from the same argument.

Elementary computation shows that

$$\delta_{P^+}(m) = |\det m|^{-(r_i^+ \cdots + r_1^+) + r_{i+2}^+ + \cdots + r_s^+}.$$

If i is odd, then in computing the integration over $\mathfrak{n}'$, after changing variables, we obtain

$$|\det m|^{-(r_{i-1}^- + \cdots + r_1^-) + r_{i+2}^- + \cdots + r_s^-}.$$

In computing the integration $Z(\underline{s}, \underline{t}, \eta, \widetilde{f})$, by changing the variable, we obtain another term

$$|\det m|^{-r_i^+}\eta(\det m).$$

Note that we have $r_1^\pm = r_2^\mp$, $r_3^\pm = r_4^\mp$, etc. Thus, we conclude

$$-(r_i^+ \cdots + r_1^+) + r_{i+2}^+ + \cdots + r_s^+ = -(r_{i-1}^- + \cdots + r_1^-) + r_{i+2}^- + \cdots + r_s^- + (-r_i^+).$$

This proves (4.2) when i is odd. The case i being even is similar. ∎

Let us now choose an open compact subgroup K of H so that $H = P^+K$. Let us put

$$f_K(\gamma) = \int_K f(\gamma^k)\eta(\det k)\mathrm{d}k, \quad \mu_{\mathcal{O}}(f) = \widetilde{\mu}_{\mathcal{O}}(f_K). \tag{4.3}$$

Proposition 4.4 *The distribution on $\mathfrak{s}$ given by $f \mapsto \mu_{\mathcal{O}}(f)$ is (H,η)-invariant. Moreover, the linear form $\mu_{\mathcal{O}}$ extends the (H,η)-invariant distribution on $\mathcal{O}$ to an (H,η)-invariant distribution on $\mathfrak{s}$ supported on $\mathcal{O}$.*

Proof The first assertion follows from Lemma 4.3 and [How74, Proposition 4]. Even though [How74, Proposition 4] does not involve the extra character η, the same argument goes through without change.

If f is a compactly supported function on $\mathcal{O}$, so is f_K. By Lemma 3.4, the support of $\widetilde{f_K}$ defined by (4.1) is a compact subset of

$$\prod_{i \text{ odd}} \mathrm{GL}_{r_i^+}(F) \times \prod_{i \text{ odd}} \mathrm{GL}_{r_i^-}(F).$$

It follows that the integral $Z(\underline{s}, \underline{t}, \eta, \widetilde{f_K})$ is convergent for all $\underline{s}$ and $\underline{t}$. When evaluated at $s_i = r_i^-$ and $t_i = r_i^+$, this convergent integral gives precisely the (convergent) integral of f along the orbit $\mathcal{O}$. This proves the second assertion. ∎

Corollary 4.5 *A θ-nilpotent orbit $\mathcal{O}$ supports an (H,η)-invariant distribution if and only if the necessary condition in Lemma 3.6 is satisfied. If $\mathcal{O}$ supports an (H,η)-invariant distribution, then this distribution extends to an (H,η)-invariant distribution on $\mathfrak{s}$.*

Proof This is merely a combination of Lemma 3.6 and Proposition 4.4. ∎

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

In the following, we call a θ-nilpotent orbit that supports an (H,η)-invariant distribution or any element contained in it *visible*. We let $\mathcal{N}_0$ be the subset of $\mathcal{N}$ consisting of visible θ-nilpotent orbits. From the discussion above, the set

$$\{\mu_{\mathcal{O}} \mid \mathcal{O} \subset \mathcal{N}_0\}$$

is a natural basis of the space of (H,η)-invariant distributions on $\mathfrak{s}$ supported on $\mathcal{N}$.

Let us put $d_{\mathcal{O}} = \dim N^+$.

Lemma 4.6 *Let $f \in C_c^\infty(\mathfrak{s})$ and for any $t \in F^\times$ we put $f_t(X) = f(t^{-1}X)$. Let $\mathcal{O} \subset \mathcal{N}_0$ then we have*

$$\mu_{\mathcal{O}}(f_t) = |t|^{d_{\mathcal{O}}}\eta(t)^n\mu_{\mathcal{O}}(f), \quad \mu_{\mathcal{O}}(\widehat{f_t}) = |t|^{2n^2-d_{\mathcal{O}}}\eta(t)^n\mu_{\mathcal{O}}(\widehat{f})$$

Proof We just need to prove the first equality. The second one on the Fourier transform follows from the first one easily. Suppose that $\mathcal{O}$ is represented by ξ and gives rise to the sequences of integers $r_1^\pm \geq r_2^\mp \geq \cdots$. It follows from the definition of $\mu_{\mathcal{O}}$ that

$$\mu_{\mathcal{O}}(f_t) = |t|^{\dim \mathfrak{n}'+2(r_1^+r_1^-+r_3^+r_3^-+\cdots)}\eta(t)^n\mu_{\mathcal{O}}(f).$$

It is thus enough to prove that

$$\dim N^+ = \dim \mathfrak{n}' + 2(r_1^+r_1^- + r_3^+r_3^- + \cdots). \tag{4.4}$$

We have

$$\dim N^+ = \sum_{i=1}^{n}\sum_{j\geq i+1} r_i^+r_j^+ + r_i^-r_j^-. \tag{4.5}$$

To organize the terms on the right hand side of (4.4) into a better form, let us write $2(r_1^+r_1^- + r_3^+r_3^- + \cdots)$ as

$$r_1^+r_2^+ + r_3^+r_4^+ + \cdots + r_1^-r_2^- + r_3^-r_4^- + \cdots$$

Then the right hand side becomes

$$\sum_{i \text{ odd}}\left(r_i^+r_{i+1}^- + r_i^-r_{i+1}^+ + \sum_{j\geq i+2}(r_i^+r_j^- + r_i^-r_j^+)\right) + \sum_{i \text{ even}}\sum_{j\geq i+1}(r_i^+r_j^- + r_i^-r_j^+). \tag{4.6}$$

Let i be an integer. In computing the dimension of N^+, the terms involving r_i^+ are $r_i^+(r_{i+1}^+ + r_{i+2}^+r_{i+3}^+ + \cdots)$. If i is odd, then on the right hand side of (4.4), the terms involving r_i^+ are

$$r_i^+r_{i+1}^+ + r_i^+(r_{i+2}^- + r_{i+3}^- + \cdots).$$

If i is even, then we have

$$r_i^+(r_{i+1}^+ + r_{i+2}^+ + \cdots).$$

Note that we have $r_1^\pm = r_2^\mp$, $r_3^\pm = r_4^\mp$ etc. So we conclude that for a fixed i, the terms in (4.5) and in (4.6) involving r_i^+ coincide. Similarly, we can conclude that the terms involving r_i^- coincide. Thus we conclude that (4.5) and (4.6) are the same, i.e., the identity (4.4) holds. This proves the lemma. ∎

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

Again to facilitate understanding, we suggest the following example.

Example 4.7 Let $\mathcal{O}$ be the nilpotent orbit represented by

$$\xi = \left(\begin{array}{cccc|cccc} & & & & 0 & & 1 & \\ & & & & & 0 & & 1 \\ & & & & & & 0 & \\ & & & & & & & 1 \\ \hline 0 & 1 & & & & & & \\ & 0 & 1 & & & & & \\ & & 0 & 1 & & & & \\ & & & 0 & & & & \end{array}\right)$$

We have $r_1^+ = r_2^- = r_3^+ = r_4^- = 1$ and $r_1^- = r_2^+ = 2 > r_3^- = r_4^+ = 0$. So this orbit is visible. The spaces $[\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{s}$, $\mathfrak{n} \cap \mathfrak{s}/[\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{s}$, and $\mathfrak{n}'$ look like the following, respectively

$$\left(\begin{array}{cccc|cccc} & & & & 0 & & & * \\ & & & & & 0 & & * \\ & & & & & & 0 & * \\ & & & & & & & 0 \\ \hline 0 & & & * & & & & \\ & 0 & & * & & & & \\ & & 0 & & & & & \\ & & & 0 & & & & \end{array}\right) \quad \left(\begin{array}{cccc|cccc} & & & & 0 & & * & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & * \\ \hline 0 & * & * & & & & & \\ & * & * & & & & & \\ & & 0 & * & & & & \\ & & & 0 & & & & \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} & & & & 0 & & & * \\ & & & & & 0 & & * \\ & & & & & & 0 & * \\ & & & & & & & 0 \\ \hline 0 & & & * & & & & \\ & 0 & & * & & & & \\ & & 0 & * & & & & \\ & & & 0 & & & & \end{array}\right)$$

In this case, direct computation shows that we have $\mu_{\mathcal{O}}(f_t) = |t|^{10}\mu_{\mathcal{O}}(f)$. This is compatible with Lemma 4.6.

5 Orbital integrals

In this section, we define all orbital integrals on $\mathfrak{s}$, not necessarily θ-semisimple or θ-nilpotent.

Let $\gamma \in \mathfrak{s}$ and $\gamma = \gamma_s + \gamma_n$ be the Jordan decomposition of γ in $\mathfrak{g}$, γ_s being semisimple and γ_n being nilpotent (in the usual sense). Since $\theta(\gamma_s)$ and $\theta(\gamma_n)$ are still semisimple and nilpotent respectively in $\mathfrak{g}$ and $\theta(\gamma) = -\gamma$, we conclude that $\gamma_s, \gamma_n \in \mathfrak{s}$. Note that $\gamma_s\gamma_n = \gamma_n\gamma_s$, we conclude that $\gamma_n \in \mathfrak{s}_{\gamma_s}$ and is θ-nilpotent in $\mathfrak{s}_{\gamma_s}$. Assume that γ_n is visible in $\mathfrak{s}_{\gamma_s}$ and its orbit is denoted by $\mathcal{O}_{\gamma_n}$. Let $f \in C_c^\infty(\mathfrak{s})$ and $h \in H$. Let us define a function

$$f_1(h) = \mu_{\mathcal{O}_{\gamma_n}}(f(h^{-1}(\gamma_s + \cdot)h)).$$

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

Lemma 5.1 *As a function in $h \in H$, f_1 is compactly supported on $H_{\gamma_s}\backslash H$.*

Proof If for some $h \in H$, $f_1(h) \neq 0$, then there is some $y \in H_{\gamma_s}$ such that $h^{-1}(\gamma_s + y^{-1}\gamma_n y)h \in \operatorname{supp} f$ which is a compact set. Note that $h^{-1}\gamma_s h$ is θ-semisimple in $\mathfrak{s}$ and $h^{-1}y^{-1}\gamma_n yh$ is θ-nilpotent in $\mathfrak{s}$. So $h\gamma_s h^{-1}$ is the semisimple part of $h^{-1}(\gamma_s + y^{-1}\gamma_n y)h$ and hence lies in some compact subset C of $\mathfrak{s}$. As the orbit of γ_s is closed, it follows that y lies in some compact subset of $H_{\gamma_s}\backslash H$. This proves the lemma. ∎

It follows from the definition that $f_1(yh) = \eta(\det y)f_1(h)$ if $y \in H_{\gamma_s}$. We then put

$$O(\gamma, \eta, f) = \int_{H_{\gamma_s}\backslash H} f_1(h)\eta(\det h)\mathrm{d}h.$$

This integral is absolutely convergent. It is not hard to check that if the restriction f to the orbit of γ is compactly supported, then $O(\gamma, \eta, f)$ agrees with the integral on the orbit of γ.

We now connect the orbital integral on $\mathfrak{s}$ with the orbital integral on $\mathfrak{s}_{\gamma_s}$. We keep the notation from (the proof of) Proposition 2.1 in Section 2. We have the analytic slice (U, p, ψ) at γ. Let $f \in C_c^\infty(\mathfrak{s})$ and we have constructed an $f_{\gamma_s} \in C_c^\infty(\mathfrak{s}_{\gamma_s})$. According to the definition, we have

$$f_{\gamma_s}(\xi) = \int_H f(h^{-1}\psi^{-1}(\xi)h)\eta(\det h)\alpha(h)\mathrm{d}h, \quad \xi \in \omega_\gamma.$$

Lemma 5.2 *We have $\mu_{\mathcal{O}_{\gamma_n}}(f_{\gamma_s}) = O(\gamma, \eta, f)$.*

Proof When we restrict it to the nilpotent cone in $\mathfrak{s}_{\gamma_s}$, the function f_{γ_s} equals

$$\int_H f(h^{-1}(\gamma_s + \cdot)h)\eta(\det h)\alpha(h)\mathrm{d}h.$$

From this and the definition of $O(\gamma, \eta, f)$ we conclude that

$$\begin{aligned}\mu_{\mathcal{O}_{\gamma_n}}(f_{\gamma_s}) &= \int_H f_1(h)\eta(\det h)\alpha(h)\mathrm{d}h = \int_{H_{\gamma_s}\backslash H} f_1(h)\eta(\det h)\mathbf{1}_C(h)\mathrm{d}h \\ &= O(\gamma, \eta, f). \qquad (5.1)\end{aligned}$$

∎

We finish the definition of orbital integrals with the following lemma.

Lemma 5.3 *If γ_n is not visible in $\mathfrak{s}_{\gamma_s}$, then the orbit if γ in $\mathfrak{s}$ does not support any (H, η)-invariant distribution.*

Proof An obvious necessary condition that the orbit represented by γ supports an (H, η)-invariant distribution is $\eta(\det h) = 1$ if $h \in H_\gamma$. If $h \in H_\gamma$, i.e., $h^{-1}\gamma h = \gamma$, then $h^{-1}\gamma_s h + h^{-1}\gamma_n h = \gamma_s + \gamma_n$. As $h^{-1}\gamma_s h$ is θ-semisimple and $h^{-1}\gamma_n h$ is θ-nilpotent, we conclude $h^{-1}\gamma_s h = \gamma_s$ and $h^{-1}\gamma_n h = \gamma_n$ by the uniqueness of the Jordan decomposition. Therefore, H_γ is a subgroup of H_{γ_s} that stabilizes γ_n. Then the condition $\eta(\det h) = 1$ if $h \in H_\gamma$ is precisely that γ_n represents a visible θ-nilpotent orbit in $\mathfrak{s}_{\gamma_s}$. ∎

6 The germ expansion

We study an analogue of the Shalika germ expansion in this section.

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

Proposition 6.1 *There is a unique (H, η)-invariant real valued function $\Gamma_{\mathcal{O}}$ on $\mathfrak{s}_{\theta-\mathrm{reg}}$ for each nilpotent orbit $\mathcal{O} \subset \mathcal{N}_0$ with the following properties.*

(1) *For any $f \in C_c^\infty(\mathfrak{s})$, there is an H-invariant neighborhood U_f of $0 \in \mathfrak{s}$ such that*

$$O(\gamma, f) = \sum_{\mathcal{O} \subset \mathcal{N}_0} \Gamma_{\mathcal{O}}(\gamma)\mu_{\mathcal{O}}(f). \tag{6.1}$$

for all θ-regular semisimple $\gamma \in U_f$.

(2) *For all $t \in F^\times$ and all $\xi \in \mathfrak{s}_{\theta-\mathrm{reg}}$, we have*

$$\Gamma_{\mathcal{O}}(t\gamma) = |t|^{-d_{\mathcal{O}}}\eta(t)^n \Gamma_{\mathcal{O}}(\gamma).$$

Proof It follows from [RR96, Proposition 1.2] that there are functions $\Gamma'_{\mathcal{O}}$ on $\mathfrak{s}_{\theta-\mathrm{reg}}$ for each $\mathcal{O} \subset \mathcal{N}_0$ with property (1). Note that if $\Gamma''_{\mathcal{O}}$ is another set of functions satisfying (1), then $\Gamma'_{\mathcal{O}}$ and $\Gamma''_{\mathcal{O}}$ have the same germs at $0 \in \mathfrak{s}$ (i.e., they equal in a small neighborhood of 0). We first explain that $\Gamma'_{\mathcal{O}}$ can be chosen to be real valued, at least when γ is close to $0 \in \mathfrak{s}$. In fact, since $\mu_{\mathcal{O}}$'s form a basis of (H, η)-invariant distributions on $\mathfrak{s}$ that are supported on $\mathcal{N}$, for each $\mathcal{O} \subset \mathcal{N}_0$ we can find a function $f_{\mathcal{O}}$ so that $\mu_{\mathcal{O}}(f_{\mathcal{O}'}) = \delta_{\mathcal{O},\mathcal{O}'}$ (Kronecker delta). It is obvious that $f_{\mathcal{O}}$'s can be chosen to be real valued. For this particular choice, we have $O(\gamma, f_{\mathcal{O}}) = \Gamma'_{\mathcal{O}}(\gamma)$ when γ lies in a small neighborhood of 0. Indeed, this can be taken as the definition of $\Gamma'_{\mathcal{O}}(\gamma)$. As $f_{\mathcal{O}}$ is real, it follows that $\Gamma'_{\mathcal{O}}(\gamma)$ can be taken to be real. We need to prove that among these functions, we can choose a unique $\Gamma_{\mathcal{O}}$ for each $\mathcal{O} \subset \mathcal{N}$ with property (2).

Let $t \in F^\times$ be fixed. We claim that as a function of γ, $\Gamma_{\mathcal{O}}(t\gamma)$ and $|t|^{-d_{\mathcal{O}}}\eta(t)^n\Gamma_{\mathcal{O}}(\gamma)$ have the same germs at 0. Indeed, on the one hand, we have

$$O(\gamma, f_t) = \sum_{\mathcal{O} \subset \mathcal{N}_0} \Gamma'_{\mathcal{O}}(\gamma)|t|^{d_{\mathcal{O}}}\eta(t)^n \mu_{\mathcal{O}}(f)$$

when γ lies in a small neighborhood (depending on f and t) of $0 \in \mathfrak{s}$. On the other hand,

$$O(\gamma, f_t) = O(t^{-1}\gamma, f) = \sum_{\mathcal{O} \subset \mathcal{N}_0} \Gamma'_{\mathcal{O}}(t^{-1}\gamma)\mu_{\mathcal{O}}(f).$$

when γ lies in a small neighborhood (depending on f and t) of $0 \in \mathfrak{s}$. Comparing these two, we conclude that $\Gamma'_{\mathcal{O}}(t\gamma)$ and $|t|^{-d_{\mathcal{O}}}\eta(t)^n\Gamma'_{\mathcal{O}}(\gamma)$ have the same germs at $\gamma = 0$.

Thus, we put

$$\Gamma_{\mathcal{O}}(\gamma) = \lim_{t \to 0} |t|^{d_{\mathcal{O}}}\eta(t)^n\Gamma'_{\mathcal{O}}(t\gamma).$$

It is straightforward to check that $\Gamma_{\mathcal{O}}(\gamma)$ does satisfy property (2). Of course, in order that $\Gamma_{\mathcal{O}}$ satisfies property (2), it has to be of this form. Thus, this function is unique. ∎

The function $\Gamma_{\mathcal{O}}$ in the lemma is called the Shalika germ indexed by $\mathcal{O}$.

We now consider the Shalika germ expansion around an arbitrary θ-semisimple element $\gamma \in \mathfrak{s}$. We keep the notation from Section 2. The space $\mathfrak{s}_\gamma$ with an action of H_γ is isomorphic to $\mathfrak{s}_1 \times \mathfrak{s}_2$ with an action of $H_1 \times H_2$, where the action of H_1 on $\mathfrak{s}_1$ is isomorphic to the conjugation of H_1 on its Lie algebra and the action of H_2 on $\mathfrak{s}_2$ is of the same shape as the action of H on $\mathfrak{s}$ but of a smaller size. Note that according to

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

the decomposition $\mathfrak{s} = \mathfrak{s}_1 \times \mathfrak{s}_2$, $\gamma = (\gamma^{(1)}, 0)$ where $\gamma^{(1)} \in \mathfrak{s}_1$ is a central element in $\mathfrak{s}_1$. A θ-nilpotent orbit in $\mathfrak{s}_\gamma$ is of the form $\mathcal{O}^{(1)} \times \mathcal{O}^{(2)}$ where $\mathcal{O}^{(1)}$ is a nilpotent orbit in $\mathfrak{s}_1$ (in the usual sense) and $\mathcal{O}^{(2)}$ is a θ-nilpotent oribt in $\mathfrak{s}_2$. The orbit $\mathcal{O}$ is visible in $\mathfrak{s}_\gamma$ if and only if $\mathcal{O}^{(2)}$ is visible in $\mathfrak{s}_2$. Let $\{\mathcal{O}_1, \dots, \mathcal{O}_r\}$ be the set of nilpotent orbits in $\mathfrak{s}_\gamma$. We thus have the Shalika germs on $\mathfrak{s}_\gamma$, indexed by the θ-nilpotent orbits in $\mathfrak{s}_\gamma$, which on $\mathfrak{s}_1$ is given by the one defined in [Kot05, Section 17] and on $\mathfrak{s}_2$ is given by the one we have just defined. Let $\{\xi_1, \dots, \xi_r\}$ be a complete set of representatives of θ-nilpotent elements in $\mathfrak{s}_\gamma$ and $\xi_i \in \mathcal{O}_i$. We denote the Shalika germ on $\mathfrak{s}_\gamma$ indexed by $\mathcal{O}_i$ by Γ_i^γ.

Corollary 6.2 *Let $f \in C_c^\infty(\mathfrak{s})$. Then there is a neighborhood U_f of γ in $\mathfrak{s}_\gamma$ so that for any $\xi \in U_f \cap \mathfrak{s}_{\theta-\mathrm{reg}}$, we have*

$$O(\xi, \eta, f) = \sum_{i=r}^{r} \Gamma_i^\gamma(\xi) O(\gamma + \xi_i, \eta, f).$$

Proof Let us keep the notation from Section 2 Proposition 2.1. We have constructed an $f_{\gamma_s} \in C_c^\infty(\mathfrak{s}_{\gamma_s})$. Apply Proposition 6.1 (germ expansion on $\mathfrak{s}_2$ near 0) and [Kot05, Theorem 17.5] (germ expansion on $\mathfrak{s}_1$ near a central element), we have

$$O^{\mathfrak{s}_\gamma}(\xi, \eta, f_\gamma) = \sum_{i=1}^{r} \Gamma_i^\gamma(\xi) \mu_{\mathcal{O}_i}^{\mathfrak{s}_\gamma}(f_\gamma),$$

where the upper script $\mathfrak{s}_\gamma$ indicates that these are orbital integrals on the space $\mathfrak{s}_\gamma$. Applying Proposition 2.1 to the left hand side and the equality (5.1) to the right hand side, we obtain the desired result in the corollary. ∎

7 Linear independence of Shalika germs

The goal of this section is to prove the linear independence of Shalika germs that we defined in the last section and the density of θ-regular semisimple integrals on $\mathfrak{s}$ simultaneously. We follow the argument of [Kot05, Section 27] closely.

Lemma 7.1 *The set of all orbital integrals is weakly dense in $D(\mathfrak{s})^{H,\eta}$.*

Proof Recall that weak density means that if all orbital integrals of $f \in C_c^\infty(\mathfrak{s})$ vanish, then $D(f) = 0$ for all (H, η)-invariant distributions D on $\mathfrak{s}$. For any space V on which H acts, we let

$$V_{H,\eta} = V / \{h.v - \eta(h)v \mid h \in H,\ v \in V\}$$

be the (H, η)-coinvariance. Then, $D(f) = 0$ for all (H, η)-invariant distributions D on $\mathfrak{s}$ means that the image of f in $C_c^\infty(\mathfrak{s})_{H,\eta}$ is zero.

Let us consider the categorical quotient

$$q : \mathfrak{s} \to \mathfrak{s}//H.$$

Let x be an element in $(\mathfrak{s}//H)(F)$. Restriction a function $f \in C_c^\infty(\mathfrak{s})$ to the fiber $q^{-1}(x)$ gives a surjective H-equivariant map

$$C_c^\infty(\mathfrak{s}) \to C_c^\infty(q^{-1}(x)).$$

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

Passing to the (H, η)-coinvariance, we obtain a surjective map

$$C_c^\infty(\mathfrak{s})_{H,\eta} \to C_c^\infty(q^{-1}(x))_{H,\eta}.$$

As x ranges over all points in $(\mathfrak{s}//H)(F)$, we obtain a map

$$C_c^\infty(\mathfrak{s})_{H,\eta} \to \prod_{x\in(\mathfrak{s}//H)(F)} C_c^\infty(q^{-1}(x))_{H,\eta}. \tag{7.1}$$

By [Kot05, Lemma 27.1] this map is injective.

Recall that $\mathfrak{s}//H$ is identified with an n-dimensional affine space over F. When x is the origin of $(\mathfrak{s}//H)(F)$, the fiber $q^{-1}(x)$ is the nilpotent cone $\mathcal{N}$. Thus the dual space of $C_c^\infty(q^{-1}(0))_{H,\eta}$ is finite dimensional and a basis is provided by all visible θ-nilpotent orbital integrals. The case of general $x \in (\mathfrak{s}//H)(F)$ is quite similar. The dual space of $C_c^\infty(q^{-1}(x))_{H,\eta}$ is finite dimensional and a basis is provided by orbital integrals where the orbits are contained $q^{-1}(x)$ and support (H, η)-invariant distributions.

It follows that if $f \in C_c^\infty(\mathfrak{s})$ so that all orbital integrals vanish, then its image in $C_c^\infty(q^{-1}(x))_{H,\eta}$ vanishes for all $x \in (\mathfrak{s}//H)(F)$. Thus, the image of f in $C_c^\infty(\mathfrak{s})_{H,\eta}$ also vanishes by the injectivity of (7.1). This is equivalent to saying that $D(f) = 0$ for all (H, η)-invariant distribution D on $\mathfrak{s}$. ∎

Lemma 7.2 *The functions $\Gamma_{\mathcal{O}}$'s for $\mathcal{O} \subset \mathcal{N}_0$ are linearly independent if and only if their restrictions to an arbitrary small neighborhood of $0 \in \mathfrak{s}$ are still linearly independent.*

Proof Let U be a small neighborhood of $0 \in \mathfrak{s}$. We may assume that U is a lattice in $\mathfrak{s}$, or in other words, U is an $\mathfrak{o}_F$-module.

Now, we use homogeneity of Shalika germs. The additive semigroup of non-negative integers acts on $U \cap \mathfrak{s}_{\theta-\mathrm{reg}}$, with j acting by multiplication by the scalar ϖ_F^{2j}, and therefore acts on the space of functions on $U \cap \mathfrak{s}_{\theta-\mathrm{reg}}$ (the action of j transforming a function $f(X)$ into $f(\varpi_F^{2j}X)$). By homogeneity of Shalika germs, c.f. Proposition 6.1, the restriction of $\Gamma_{\mathcal{O}}$ to $U \cap \mathfrak{s}_{\theta-\mathrm{reg}}$ transforms under the character $j \mapsto |\varpi_F|^{-jd_{\mathcal{O}}}\eta(\varpi_F)^j$ on our semigroup. But in any representation of our semigroup, vectors transforming under distinct characters are linearly independent. Thus, in order to prove linear independence of the restrictions of Shalika germs to $U \cap \mathfrak{s}_{\theta-\mathrm{reg}}$, it is enough to fix a non-negative integer d and prove linear independence of the restrictions of the Shalika germs for all θ-nilpotent orbits with $d_{\mathcal{O}} = d$. But all these germs are homogeneous of the same degree, namely d, so it is clear that any dependence relation that holds on the subset $U \cap \mathfrak{s}_{\theta-\mathrm{reg}}$ will also hold on the whole set $\mathfrak{s}_{\theta-\mathrm{reg}}$. ∎

The following lemma relates the linear independence of the Shalika germs to the density of θ-regular semisimple orbital integrals.

Lemma 7.3 *The following assertions are equivalent.*

(1) *The Shalika germs $\Gamma_{\mathcal{O}}$, $\mathcal{O} \subset \mathcal{N}_0$ are linearly independent.*
(2) *The θ-nilpotent orbital integrals $\mu_{\mathcal{O}}$'s lie in the weak closure of the set of θ-regular semisimple orbital integrals in $D(\mathfrak{s})^{H,\eta}$.*

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

Proof (1) ⇒ (2). Let $f \in C_c^\infty(\mathfrak{s})$ and assume that the θ-regular semisimple orbital integrals $O(\gamma, \eta, f)$ are all zero. Then it follows from the Shalika germ expansion that

$$\sum_{\mathcal{O} \subset \mathcal{N}} \mu_{\mathcal{O}}(f)\Gamma_{\mathcal{O}}(\gamma) = 0$$

for any θ-regular semisimple $\gamma \in U_f$ where U_f is a small neighborhood of $0 \in \mathfrak{s}$. Since $\Gamma_{\mathcal{O}}$'s are linear independent, by the previous lemma, they are linearly independent even when restricted to U_f. Thus we conclude that $\mu_{\mathcal{O}}(f) = 0$ for all $\mathcal{O}$.

(2) ⇒ (1). Suppose that we have a linear relation

$$\sum_{\mathcal{O} \subset \mathcal{N}_0} a_{\mathcal{O}}\Gamma_{\mathcal{O}}(\gamma) = 0, \quad \text{for all } \gamma \in \mathfrak{s}_{\theta-\mathrm{reg}}.$$

As $\mu_{\mathcal{O}}$'s form a basis of the space of (H, η)-invariant distributions on $\mathfrak{s}$ supported on $\mathcal{N}$, we may choose a test function $f \in C_c^\infty(\mathfrak{s})$ so that $\mu_{\mathcal{O}}(f) = a_{\mathcal{O}}$ for all $\mathcal{O} \subset \mathcal{N}$. Thus, using the Shalika germ expansion, we conclude that there is a small neighborhood U_f of $0 \in \mathfrak{s}$ so that

$$O(\gamma, \eta, f) = \sum_{\mathcal{O} \subset \mathcal{N}_0} \mu_{\mathcal{O}}(f)\Gamma_{\mathcal{O}}(\gamma) = \sum_{\mathcal{O} \subset \mathcal{N}_0} a_{\mathcal{O}}\Gamma_{\mathcal{O}}(\gamma) = 0$$

for all θ-regular semisimple $\gamma \in U_f$. The set $(U_f)^H$ contains an open and closed neighborhood V of $\mathcal{N}$. Let $f' = f\mathbf{1}_V$. Then, we have that $O(\gamma, \eta, f') = 0$ for all θ-regular semisimple γ. Moreover, since V is an open and closed neighborhood of $\mathcal{N}$, we have that $\mu_{\mathcal{O}}(f) = \mu_{\mathcal{O}}(f')$ for all $\mathcal{O} \subset \mathcal{N}$. Now by assertion (2), the θ-nilpotent orbital integrals $\mu_{\mathcal{O}}$'s all lie in the weak closure of the θ-regular semisimple orbital integrals. Since $O(\gamma, \eta, f') = 0$ for θ-regular semisimple γ, we conclude that $a_{\mathcal{O}} = \mu_{\mathcal{O}}(f) = \mu_{\mathcal{O}}(f') = 0$. This proves (1). ∎

The next lemma allows us to use induction.

Lemma 7.4 *Let $\gamma \in \mathfrak{s}_{\theta-ss}$. Suppose that $\Gamma_{\mathcal{O}}^{\gamma}$'s are linearly independent. Then for all ξ whose θ-semisimple part is γ the orbital integral $O(\xi, \eta, f)$ lies in the weak closure of the set of all θ-regular semisimple orbital integrals.*

Proof Let $\mathcal{N}_\gamma$ be the nilpotent cone of $\mathfrak{s}_\gamma$ and for each nilpotent orbit $\mathcal{O} \subset \mathcal{N}_\gamma$, we fix an element $\xi_{\mathcal{O}} \in \mathcal{O}$. Then by the Shalika germ expansion at γ, there is a small neighborhood U_f of γ in $\mathfrak{s}_\gamma$, so that for all θ-regular semisimple $\xi \in U_f$,

$$O(\xi, \eta, f) = \sum_{\mathcal{O} \subset \mathcal{N}_{\gamma,0}} \Gamma_{\mathcal{O}}^{\gamma}(\xi) O(\gamma + \xi_{\mathcal{O}}, \eta, f).$$

As $\Gamma_{\mathcal{O}}^{\gamma}$'s are linearly independent and they remain linearly independent when restricted to U_f, we conclude that if $O(\xi, \eta, f) = 0$ for all θ-regular semisimple $\xi \in U_f$, we have $O(\gamma + \xi_{\mathcal{O}}, \eta, f) = 0$ for all $\mathcal{O} \subset \mathcal{N}_{\gamma,0}$. This proves the lemma. ∎

We now prove the linear independence of Shalika germs and the density of θ-regular semisimple orbital integrals *simultaneously*.

Theorem 7.5 *The following assertions hold.*

(1) *The Shalika germs $\Gamma_{\mathcal{O}}$'s, $\mathcal{O} \subset \mathcal{N}_0$, are linearly independent.*
(2) *The set of θ-regular semisimple orbital integrals are weakly dense in $D(\mathfrak{s})^{H,\eta}$.*

Proof We argue by induction on n, i.e., the size of $\mathfrak{s}$.

First we show that, under the inductive hypothesis, the two assertions in the proposition are equivalent. In fact, if the second assertion holds, then the first holds by Lemma 7.3. If the first assertion holds, then θ-nilpotent orbital integrals lie in the weak closure of θ-regular semisimple orbital integrals. When combined with the induction hypothesis and Lemma 7.4, this implies that all orbital integrals lie in the weak closure of the θ-regular semisimple orbital integrals. This proves that two assertions in the proposition are equivalent. We will prove the second assertion under the induction hypothesis.

Put

$$C_1 = \{f \in C_c^\infty(\mathfrak{s}) \mid \text{all orbital integrals of } f \text{ vanish}\};$$
$$C_2 = \{f \in C_c^\infty(\mathfrak{s}) \mid \text{all } \theta\text{-regular semisimple orbital integrals of } f \text{ vanish}\}.$$

By Lemma 7.4 and the induction hypothesis, the set C_2 consists of all functions $f \in C_c^\infty(\mathfrak{s})$ such that all orbital integrals, except the θ-nilpotent orbital integrals, vanish. Thus, the dual space of C_2/C_1 is spanned by all $\mu_{\mathcal{O}}$'s, $\mathcal{O} \subset \mathcal{N}_0$.

By Lemma 7.1, C_1 consists of all $f \in C_c^\infty(\mathfrak{s})$ such that $I(f) = 0$ for all (H, η)-invariant distribution I. Thus, it is clear that C_1 is closed under the Fourier transform. Since the Fourier transform of θ-regular semisimple orbital integrals are represented by (H, η)-invariant locally integrable functions on $\mathfrak{s}_{\theta-\mathrm{reg}}$ by Proposition 1.3, we conclude that C_2 is also preserved under the Fourier transform. Thus Fourier transform induces an isomorphism of C_2/C_1 onto itself. Therefore, the dual space of C_2/C_1 is also spanned by $\widehat{\mu_{\mathcal{O}}}$'s.

By Lemma 4.6, $\mu_{\mathcal{O}}$ and $\widehat{\mu_{\mathcal{O}}}$ have the homogeneity properties

$$\mu_{\mathcal{O}}(f_t) = |t|^{d_{\mathcal{O}}} \mu_{\mathcal{O}}(f), \quad \widehat{\mu_{\mathcal{O}}}(f_t) = |t|^{2n^2 - d_{\mathcal{O}}} \widehat{\mu_{\mathcal{O}}}(f).$$

The proof of Lemma 4.6, or more precisely (4.5) shows that $d_{\mathcal{O}} < n^2$ for all $\mathcal{O} \subset \mathcal{N}_0$. Therefore, $d_{\mathcal{O}} < 2n^2 - d_{\mathcal{O}'}$ for any $\mathcal{O}, \mathcal{O}' \subset \mathcal{N}_0$. We thus have two spanning sets of the dual space of C_2/C_1, all being homogeneous, but with different scaling factors from each set. Therefore, $C_2/C_1 = 0$ and this proves the proposition. ■

Corollary 7.6 *The Fourier transform of $\mu_{\mathcal{O}}$ is represented by a locally integrable function in $\mathfrak{s}$ for all $\mathcal{O} \subset \mathcal{N}_0$.*

Proof We need to make use of Howe's finiteness theorem for $\mathfrak{s}$, established by Rader and Rallis in [RR96, Theorem 6.7]. We do not need the statement this theorem here, but rather a standard consequence of it, i.e., the uniformity of the germ expansion. Let $L \subset \mathfrak{s}$ be a lattice, i.e., an open compact subgroup of $\mathfrak{s}$. Then Howe's finitenss theorem implies that there is a neighborhood U_L such that the germ expansion

$$O(\gamma, \eta, f) = \sum_{\mathcal{O} \subset \mathcal{N}_0} \Gamma_{\mathcal{O}}(\gamma) \mu_{\mathcal{O}}(f)$$

holds for all $f \in C_c^\infty(\mathfrak{s}/L)$ and all θ-regular semisimple $\gamma \in U_L$.

Now let $L \subset \mathfrak{s}$ be a lattice. There is a lattice L' in $\mathfrak{s}$ (in fact the dual lattice of L) so that $\widehat{f} \in C_c^\infty(\mathfrak{s}/L')$ for all $f \in C_c^\infty(L)$. Therefore, there is a neighborhood U_L of $0 \in \mathfrak{s}$

so that

$$O(\gamma,\eta,\widehat{f}) = \sum_{\mathcal{O}\subset\mathcal{N}_0} \Gamma_{\mathcal{O}}(\gamma)\mu_{\mathcal{O}}(\widehat{f})$$

holds for all θ-regular semisimple $\gamma \in U_L$ and all $f \in C_c^\infty(L)$. By Theorem 7.5 and Lemma 7.2, $\Gamma_{\mathcal{O}}$'s, $\mathcal{O} \subset \mathcal{N}_0$, when restricted to $U_{L'}$, are linearly independent. Therefore, we can choose a θ-regular semisimple $\gamma_{\mathcal{O}}$ for each $\mathcal{O} \subset \mathcal{N}_0$ so that matrix

$$(\Gamma_{\mathcal{O}}(\gamma_{\mathcal{O}'}))_{\mathcal{O},\mathcal{O}'\subset\mathcal{N}_0}$$

is invertible. We then conclude that there are constants $c_{\mathcal{O}}$, so that

$$\mu_{\mathcal{O}}(\widehat{f}) = \sum_{\mathcal{O}'\subset\mathcal{N}_0} c_{\mathcal{O}'} O(\gamma_{\mathcal{O}'},\eta,\widehat{f})$$

holds for all $f \in C_c^\infty(L)$.

By Proposition 1.3, there is a locally constant function $K_{\gamma_{\mathcal{O}'}}$ on $\mathfrak{s}_{\theta-\mathrm{reg}}$ which is locally integrable on $\mathfrak{s}$ so that the distribution on $\mathfrak{s}$ given by $f \mapsto O(\gamma_{\mathcal{O}'},\eta,\widehat{f})$ is represented by $K_{\gamma_{\mathcal{O}'}}$. It follows that for all $f \in C_c^\infty(L)$ we have

$$\mu_{\mathcal{O}}(\widehat{f}) = \int_{\mathfrak{s}} f(\gamma)\left(\sum_{\mathcal{O}'\subset\mathcal{N}_0} c_{\mathcal{O}'} K_{\gamma_{\mathcal{O}'}}(\gamma)\right)\mathrm{d}\gamma.$$

We put $K_{\mathcal{O},L}(\gamma) = \sum_{\mathcal{O}'\subset\mathcal{N}_0} c_{\mathcal{O}'}K_{\gamma_{\mathcal{O}'}}(\gamma)$ for $\gamma \in \mathfrak{s}_{\theta-\mathrm{reg}}$. This function is locally constant on $\mathfrak{s}_{\theta-\mathrm{reg}}$ and is locally integrable on $\mathfrak{s}$.

We now choose another lattice L_1 so that $L \subset L_1$. Then we get another function $K_{\mathcal{O},L'}$. We claim that $K_{\mathcal{O},L_1}(\gamma) = K_{\mathcal{O},L}(\gamma)$ if $\gamma \in L$ and is θ-regular semisimple. In fact both functions, when restricted to L, represent the distribution $f \mapsto \mu_{\mathcal{O}}(\widehat{f})$. Then, we conclude by the local constancy of them.

It follows that there is a well-defined function $K_{\mathcal{O}}$ on $\mathfrak{s}$, which is locally constant on $\mathfrak{s}_{\theta-\mathrm{reg}}$ and locally integrable on $\mathfrak{s}$, so that $K_{\mathcal{O}}(\gamma) = K_{\mathcal{O},L}(\gamma)$ if L is a lattice in $\mathfrak{s}$ and $\gamma \in L$. It is then clear that $K_{\mathcal{O}}$ represents the distribution $f \mapsto \mu_{\mathcal{O}}(\widehat{f})$ on $\mathfrak{s}$. ■

8 Density of regular semisimple orbital integrals

We explain how to establish the results on the level of G in this section.

We fix an H-invariant neighborhood ω of $0 \in \mathfrak{s}$ and a neighborhood Ω of $1 \in S$ so that the the exponential (rational) map $\exp : \mathfrak{s} \to \Omega$ is defined and is a homeomorphism. Let $f \in C_c^\infty(G)$. We put $\widetilde{f}(g^{-1}\theta(g)) = \int_H f(hg)\mathrm{d}h$ and $f_\natural \in C_c^\infty(\omega)$ given by $f_\natural(\gamma) = \widetilde{f}(\exp(\gamma))$. We extend $f_\natural$ to a function on $\mathfrak{s}$ via extension by zero.

We consider the $H \times H$ action on G by left and right multiplication and the conjugation action of H on S. We say that an element $x \in S$ or rather the H-orbit of x is θ-unipotent if it is unipotent in G. We say that $g \in G$ is θ-unipotent if $x = g^{-1}\theta(g)$ is so in S. Let $Y \subset S$ be the variety of θ-unipotent elements in S. By [JR96, Lemma 4.1], the exponential map induces an H-equivariant isomorphism $Y \to \mathcal{N}$ and thus induces a bijection on the set of H-orbits in Y and that in $\mathcal{N}$. Let $u_1,\ldots,u_r,u_{r+1},\ldots,u_s$ be a complete set of representatives of θ-unipotent orbits in G. Let $\mathcal{O}_i$ be the θ-nilpotent orbits in $\mathfrak{s}$ represented by $\exp^{-1}(u_i^{-1}\theta(u_i))$ and we may label these u_i's so that $\mathcal{O}_i$

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

is visible precisely when $1 \le i \le r$. Therefore u_i represents a θ-unipotent orbit in G which supports a left H-invariant and right (H,η)-invariant distribution precisely when $1 \le i \le r$. We denote this distribution by $O(u_i,\eta,\cdot)$ and call it a θ-unipotent orbital integral on G. If $f \in C_c^\infty(G)$, we have

$$O(u_i,\eta,f) = \mu_{\mathcal{O}_i}(f_\natural).$$

We call the θ-unipotent elements $u_1,\dots,u_r$ or their orbits visible.

The following proposition is the Shalika germ expansion of orbital integrals on G.

Proposition 8.1 *Let $f \in C_c^\infty(G)$. There is a neighborhood $U_f \subset \Omega$ of $1 \in S$ so that if $g \in G$ is a θ-regular element in G with $g^{-1}\theta(g) \in U_f$, $g^{-1}\theta(g) = \exp(X)$ where $X \in \omega$, then*

$$O(g,\eta,f) = \sum_{i=1}^{r} \Gamma_{\mathcal{O}_i}(X) O(u_i,\eta,f). \tag{8.1}$$

Proof This follows from the definition of $O(u_i,\eta,f)$ and the Shalika germ expansion on $\mathfrak{s}$, i.e. Proposition 6.1. ∎

Remark 8.2 Due to the lack of a symmetric space version of the Howe's finiteness theorem, we are not able to obtain the "uniformity" of the Shalika germ expansion on G. This is however well expected. More precisely let K be an open compact subgroup of K, we expect that there is an open neighborhood U_K of $1 \in G$ so that the germ expansion (8.1) holds for all $f \in C_c^\infty(K\backslash G/K)$.

Let $x \in S$ be θ-semisimple. Let $\mathcal{N}_x \subset \mathfrak{s}_x$ be the nilpotent cone and the map $\mathcal{N}_x \to S_x$, $\xi \mapsto x\exp(\xi)$ is H_x-equivariant and induces a bijection between the θ-nilpotent orbits in $\mathfrak{s}_x$ and the orbits in S_x represented an element y such that the semisimple part of y is x. Let us recall the explicit construction of an analytic slice of $x \in S$ given in [Zha15, Section 5.3]. Recall first from Section 2 that an analytic slice at $x \in S$ is a triple (U,p,ψ), where U is an H-invariant neighborhood of the orbit x^H in S, the map $p : U \to x^H$ is an H-equivariant retraction and ψ is an H_x-equivariant embedding of $p^{-1}(x)$ into the normal space N_x^S of x^H in S at x. By [Zha15, Section 5.3], we can take $p^{-1}(x)$ to be a small neighborhood of x in S_x and identify N_x^S with $\mathfrak{s}_x$. The map $\xi \mapsto x\exp(\xi)$ define an H_x-equivariant homeomorphism from a neighborhood of $0 \in N_x^S$ to $p^{-1}(x)$ and we can and will take ψ to be the inverse of this map.

Let $g \in G$ and $x = g^{-1}\theta(g)$. Let $x = x_s x_u = x_u x_s$ be the Jordan decomposition of x in G (with obvious notation). Then one checks readily that $x_s, x_u \in S$. Let $\mathcal{O} \subset \mathcal{N}_{x_s}$ be a visible θ-nilpotent orbit and assume that x_u is contained in the image of $\mathcal{O}$ under the exponential map. Let $f \in C_c^\infty(G)$ and $\widetilde{f} \in C_c^\infty(S)$. We define $f_1 \in C^\infty(H)$ by

$$f_1(h) = \mu_{\mathcal{O}}^{\mathfrak{s}_{x_s}}(\widetilde{f}(h^{-1}(x_s\exp(\cdot))h)).$$

The right hand side is interpreted as follows. Fix an $h \in H$ and an H-invariant neighborhood U of $0 \in \mathfrak{s}_x$. We assume that U is compact modulo H and the exponential map is defined on U. Define a compactly supported function on U by $\xi \mapsto \widetilde{f}(h^{-1}(x_s\exp(\xi))h)$ when $\xi \in U$ and extend it to $\mathfrak{s}_x$ by zero. It is clear that the orbital integral is independent of the choice of U as it depends only on the value of the integrand when $\xi \in \mathcal{N}_x$. Then the right hand side stands for the orbital integral of this function along $\mathcal{O}$ on $\mathfrak{s}_{x_s}$. The same proof of Lemma 5.1 shows that the image of supp f_1 in $H_{x_s}\backslash H$ is compact. We put

$$O(g,\eta,f)=\int_{H_x\backslash H} f_1(h)\eta(\det h)\mathrm{d}h.$$

The same argument the proof of Lemma 5.2 gives that

$$O(g,\eta,f)=\mu_{\mathcal{O}}^{\mathfrak{s}_{x_s}}(\widetilde{f}_{x_s}), \tag{8.2}$$

where $\widetilde{f}_{x_s}$ is the function constructed in Proposition 2.1 from $\widetilde{f}$. Again the same proof of Lemma 5.3 gives that if x_n is not contained in the image of a visible θ-unipotent orbit, then the orbit of g does not support any distribution that is left H-invariant and right (H,η)-invariant.

Theorem 8.3 *The set of θ-regular semisimple orbital integrals is weakly dense in the set of left H-invariant and right (H,η)-invariant distributions on G.*

Proof As in the case of invariant distributions on $\mathfrak{s}$, the set of all orbital integrals is weakly dense in $D(G)^{H\times H,\eta}$. This is the symmetric space version of Lemma 7.1 and can be proved by the same argument. Thus we need to prove that if $f\in C_c^\infty(G)$, and $O(g,\eta,f)=0$ for all θ-regular semisimple $g\in G$, then all orbital integrals of f vanish. Let $x\in S$ be θ-semisimple. We have a function $\widetilde{f}\in C_c^\infty(S)$ and we let $\widetilde{f}_x\in C_c^\infty(\mathfrak{s}_x)$ be the function constructed in Proposition 2.1. Then by Proposition 2.1, all θ-regular semisimple orbital integrals of $\widetilde{f}_x$ near $0\in\mathfrak{s}_x$ vanish. Thus, by Theorem 7.5 (apply for $\mathfrak{s}_x$), we conclude that all θ-nilpotent orbital integrals of $\widetilde{f}_x$ vanish. By the equality (8.2), we conclude that $O(g,\eta,f)=0$ if the θ-semisimple part of $g^{-1}\theta(g)$ is x. This shows that all orbital integrals of f vanish and proves the theorem. ∎

Remark 8.4 It is expected that the orbital integrals $O(g,\eta,\cdot)$ on G are all tempered distributions, i.e. they extend continuously to the Harish-Chandra Schwartz space on G, c.f. [Clo91]. The proof of this would rely on the "uniformity" of Shalika germ expansions, which in term rely on the Howe's finiteness theorem on the symmetric spaces, c.f. Remark 8.2.

9 Spherical characters

We prove the local integrability of spherical characters in this section. This is a standard consequence of the germ expansion and the local integrability of the Fourier transform of θ-nilpotent orbital integrals.

Let π be an irreducible admissible representation of G. Assume that $\mathrm{Hom}_H(\pi,\mathbb{C})\neq 0$ and $\mathrm{Hom}_H(\widetilde{\pi},\eta)\neq 0$ where $\widetilde{\pi}$ is the contragredient of π. Fix nonzero elements $l\in\mathrm{Hom}_H(\pi,\mathbb{C})$ and $\widetilde{l}_\eta\in\mathrm{Hom}_H(\widetilde{\pi},\eta)$. Define a distribution on G by

$$J_\pi(f)=\sum_{\varphi} l(\pi(f)\varphi)\widetilde{l}_\eta(\widetilde{\varphi}),\quad f\in C_c^\infty(G).$$

Here, φ runs through a basis of π while $\widetilde{\varphi}$ runs through the dual basis. Then, $J_\pi\in D(G)^{H\times H,\eta}$.

To state the germ expansion for J_π, let us recall the following setup. Let $x = g^{-1}\theta(g) \in S$ be a θ-semisimple element. Consider the map

$$H \times G_x \times H \to G, \quad (h_1, g, h_2) \mapsto h_1 x g h_2.$$

Let U_x be the open subset in G_x consisting of elements $g \in G_x$ such that the above map is submersive at $(1, g, 1)$. Let Ω_x be the image of $H \times U_x \times H$ in G. Then, U_x is a bi-H_x-invariant neighborhood of 1 in G_x and Ω_x is a bi-H-invariant neighborhood of x in G. By standard theory of Harish-Chandra, there is a surjective map

$$C_c^\infty(H \times U_x \times H) \to C_c^\infty(\Omega_x), \quad \alpha \mapsto f_\alpha, \tag{9.1}$$

with the property that

$$\int_{H\times U_x \times H} \alpha(h_1, g, h_2)\beta(h_1 x g h_2) \mathrm{d}h_1 \mathrm{d}g \mathrm{d}h_2 = \int_{\Omega_x} f_\alpha(g)\beta(g)\mathrm{d}g,$$

for all $\beta \in C_c^\infty(\Omega_x)$. According to [JR96, Section 5.1, p. 103], there is a unique left H_x-invariant and right (H_x, η)-invariant distribution J_x on G_x such that

$$J_\pi(f_\alpha) = J_x(\beta_\alpha),$$

for all $\alpha \in C_c^\infty(H \times U_x \times H)$, where

$$\beta_\alpha(g) = \int_H \int_H \alpha(h_1, g, h_2)\eta(\det h_2)\mathrm{d}h_1 \mathrm{d}h_2, \quad g \in G_x.$$

Recall that we have defined a test function $\beta_{\alpha,\natural} \in C_c^\infty(\mathfrak{s}_x)$ at the beginning of Section 8. The germ expansion of J_π refers to the following theorem.

Theorem 9.1 *Let the notation be as above. There are constants $c_\mathcal{O}$ for each visible H_x-orbit $\mathcal{O}$ in the nilpotent cone $\mathcal{N}_x$, such that*

$$J_\pi(f_\alpha) = \sum_{\mathcal{O}\subset\mathcal{N}_x} c_\mathcal{O}\widehat{\mu_\mathcal{O}}(\beta_{\alpha,\natural})$$

for all $\alpha \in C_c^\infty(H \times U_x \times H)$. The sum ranges over all visible θ-nilpotent orbits $\mathcal{O}$ in $\mathcal{N}_x$.

The distributions $\widehat{\mu_\mathcal{O}}$ are locally integrable functions on $\mathfrak{s}_x$ by Corollary 7.6. Therefore, Theorem 9.1 implies the following result.

Theorem 9.2 *The distribution J_π is represented by a left H-invariant and right (H, η)-invariant locally integrable function on G.*

Theorem 9.1 is almost proved in the literature. It is established in [RR96, Theorem 7.11] near the identity element and in [Hak94, Theorem 2] near all semisimple elements for the Galois symmetric pairs. The general case can be established essentially by the same argument and is given in [Guo98]. It is unfortunate that [Guo98] is never published. For completeness, we briefly outline the argument in the rest of this section and refer the readers to [Hak94, RR96] for details. The readers may want to have these papers at hand. The germ expansion holds for spherical characters on all symmetric spaces. The argument outlined below also works in the general setting. We remark that the references usually consider only bi-H-invariant distributions, but the argument works without change in our setup. We also remark that even though [Hak94] aims at proving results for the Galois symmetric pairs in odd residue characteristic, Sections

2–7 of it are actually devoted to results of general symmetric spaces in arbitrary residue characteristic. There are misprints in the second paragraph on page 3 of [Hak94], where Section 7 should be Section 8, and Section 5 later in the paragraph should be Section 6.

We introduce more notation. Let $L \subset \mathfrak{g}_x$ be a θ-stable lattice with a decomposition $L = L_+ \oplus L_-$ where $L_+ \subset \mathfrak{h}_x$ and $L_- \subset \mathfrak{s}_x$. Suppose that L is ee in the sense of [RR96, p. 158] (we do not need the precise definition). The exponential map maps L onto an open compact subgroup K of G_x. Write $K_+ = \exp L_+ \subset H_x$. We may also assume that L is small enough so that η is trivial on K_+. By [RR96, Corollary 7.3], (K, K_+) is a Gelfand pair.

Let us now recall Howe's parametrization of $\widehat{K}$, the set of irreducible smooth representation of K. We put $K^{1/2} = \exp(\frac{1}{2}L)$ and $K_+^{1/2} = \exp(\frac{1}{2}L_+)$. The various $1/2$ appearing here and below are all to take care of the complications arise in the case of residue characteristic two. They do not play any roles in the case of odd residue characteristic. Let $\mathcal{E}(K/K_+)$ be the set of irreducible representations of K with a nonzero K_+-fixed vector. If the residue characteristic is two, the group $K_+^{1/2}$ acts on $\mathcal{E}(K/K_+)$ by conjugation and we let $\mathcal{E}^{1/2}(K/K_+)$ be the set of $K_+^{1/2}$-orbits in $\mathcal{E}(K/K_+)$. If the residue characteristic is odd, then $K = K^{1/2}$, $K_+ = K_+^{1/2}$ and $\mathcal{E}(K/K_+) = \mathcal{E}^{1/2}(K/K_+)$. If $\mu \in \mathcal{E}(K/K_+)$ we define

$$\phi_\mu(k) = \langle \mu(k)e, e \rangle$$

where $e \in \mu$ is the unique unit K_+-fixed vector. Let $\mathfrak{d} = \{\mu_1, \dots, \mu_m\}$ is an orbit in $\mathcal{E}^{1/2}(K/K_+)$. We then put

$$\phi_{\mathfrak{d}} = \sum_{i=1}^{m} \phi_{\mu_i}.$$

Let $d(\mathfrak{d}) = \dim \mu$ for any $\mu \in \mathfrak{d}$. We view $\phi_{\mathfrak{d}}$ either as a bi-K_+-invariant function on K or a function on S_x supported in the image of K.

Let $L_-^\perp$ be the dual lattice of L_- in $\mathfrak{s}_x$, i.e., the annihilator of L_- with respect to $\langle -, - \rangle$. As L_- is stable under the action of $K_+^{1/2}$, so are $L_-^\perp$ and $\mathfrak{s}_x / L_-^\perp$. Thus, we can consider the $K_+^{1/2}$-orbits in $\mathfrak{s}_x / L_-^\perp$. If $\mathcal{O}$ is such an orbit, we let $\kappa_{\mathcal{O}} \in C_c^\infty(\mathfrak{s}_x)$ be the characteristic function of it.

Proposition 9.3 *There is a bijection between $\mathcal{E}^{1/2}(K/K_+)$ and the set of $K_+^{1/2}$-orbits in $\mathfrak{s}_x / L_-^\perp$ which sends $\mathfrak{d} \in \mathcal{E}^{1/2}(K/K_+)$ to $\mathcal{O}_{\mathfrak{d}} \subset \mathfrak{s}_x / L_-^\perp$. Under this bijection, we have*

$$d(\mathfrak{d}) \phi_{\mathfrak{d}}(\exp X) = (\mathrm{vol} L_-^\perp)^{-1} \widehat{\kappa_{\mathcal{O}_{\mathfrak{d}}}}(X)$$

for all $X \in L_-$.

This is the combination of Propositions 7.8 and 7.10(2) of [RR96].

Let $\mathcal{N}_{\mathfrak{g}_x}$ be the cone of nilpotent matrices in $\mathfrak{g}_x$. We fix any norm $\|\cdot\|$ on $\mathfrak{g}_x$. Let S_1 be the unit ball in $\mathfrak{g}_x$, and

$$V(\varepsilon) = \{X \in S_1 \mid \|X - n\| \le \varepsilon \text{ for some } n \in \mathcal{N}_{\mathfrak{g}_x} \cap S_1\}.$$

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

We also put

$$FV(\varepsilon) = \{\lambda X \mid \lambda \in F,\ X \in V(\varepsilon)\}.$$

Lemma 9.4 *Let $\varepsilon > 0$ be sufficiently small and $R > 0$ be sufficiently large. Let $\mathfrak{d} \in \mathcal{E}^{1/2}(K/K_+)$. Assume that $J_x(\overline{\phi_\mathfrak{d}}) \neq 0$ and that that there is a $Z \in \mathcal{O}_\mathfrak{d}$ with $\|Z\| \geq R$. Then $\mathcal{O}_\mathfrak{d} \subset FV(\varepsilon)$.*

This is [Hak94, Theorem 4].

Let $U_0 \subset \mathfrak{s}_x$ be an open and closed neighborhood of zero such that $\exp U_0$ is contained in the image of U_x. The spherical character J_x is left H_x-invariant and therefore can be viewed as a distribution on S_x. Restrict J_x to $\exp U_0$ and pull it back via the exponential map we obtain a distribution on U_0. Extending it by zero to all $\mathfrak{s}_x$. We denote this distribution by J_0. Note that we have $J_x(f) = J_0(f_\natural)$ for all $f \in C_c^\infty(G_x)$.

Lemma 9.5 *Let $\varepsilon > 0$ be sufficiently small and $R > 0$ be sufficiently large. Let $Z \in \mathfrak{s}_x$ with $\|Z\| > R$, and let $f_Z \in C_c^\infty(\mathfrak{s}_x)$ be the characteristic function of $Z + L_-^\perp$. If $\widehat{J_0}(f_Z) \neq 0$, then there is an $n \in \mathcal{N}_x$ so that $\|Z - n\| \leq \varepsilon\|n\|$.*

Proof Let $\mathcal{O}$ be the $K_+^{1/2}$-orbit of $-Z$ in $\mathfrak{s}_x/L_-^\perp$ and $\mathfrak{d} \in \mathcal{E}^{1/2}(K/K_+)$ be the $K_+^{1/2}$-orbit of representations corresponding to $\mathcal{O}$ as in Proposition 9.3. We have

$$\kappa_\mathcal{O} = \sum_{Z' \in \mathcal{O}/L_-^\perp} f_{Z'}.$$

Note that Z''s are all the K_+-orbit of $-Z$, J_0 is (H_x, η)-invariant and η is trivial when restricted to K_+. It follows that

$$d(\mathfrak{d}) J_x(\overline{\phi_\mathfrak{d}}) = (\mathrm{vol} L_-^\perp)^{-1} \cdot \#\mathcal{O}/L_-^\perp \cdot \widehat{J_0}(f_Z).$$

Therefore, if $\widehat{J_0}(f_Z) \neq 0$, then $J_x(\overline{\phi_\mathfrak{d}}) \neq 0$. As $\|Z\| > R$, by Lemma 9.4 we have $Z \in FV(\varepsilon)$. In other words, there is a $\lambda \in F^\times$ and $n_1 \in \mathcal{N}_{\mathfrak{g}_x}$ so that $\|\lambda^{-1}Z - n_1\| \leq \varepsilon$. Put $n = \lambda n_1 \in \mathcal{N}_{\mathfrak{g}_x}$, we conclude that $\|Z - n\| \leq \varepsilon\|n\|$. Following the same argument as in the second paragraph of the proof of [RR96, Theorem 7.11] we can even choose $n \in \mathcal{N}_x$. This proves the lemma. ∎

We now finish the proof of Theorem 9.1. Lemma 9.5 tells us that $\widehat{J_0} * \kappa_{L_-^\perp}$ is contained in

(9.2) $$C(\varepsilon, R) = B(0, R) \cup \{Z \in \mathfrak{s}_x \mid \|Z - n\| \leq \varepsilon\|n\| \text{ for some } n \in \mathcal{N}_x\}.$$

Here, $*$ stands for the usual convolution of functions and distributions. Let

$$J(\varepsilon, R, L_-^\perp) = \left\{(H, \eta)\text{-invariant distributions } D \text{ with } \operatorname{supp}\left(D|_{C_c^\infty(\mathfrak{s}_x/L_-^\perp)}\right) \subset C(\varepsilon, R)\right\}.$$

With this notation, we have $\widehat{J_0} * \kappa_{L_-^\perp} \in J(\varepsilon, R, L) * \kappa_{L_-^\perp}$. Let ω be an H_x-invariant open and closed subset of $\mathfrak{s}_x$, compact modulo H_x, and $D(\omega)^{H,\eta}$ be the space of (H_x, η)-invariant distributions supported on ω. By Howe's finiteness theorem [RR96, Theorem 6.8], if ε is small enough and R is large enough, then

$$J(\varepsilon, R, L_-^\perp) * \kappa_{L_-^\perp} = D(\omega)^{H,\eta} * \kappa_{L_-^\perp}.$$

https://doi.org/10.4153/S0008414X21000122 Published online by Cambridge University Press

By [RR96, Proposition 6.9], if L is sufficiently small and hence $L_-^\perp$ is sufficiently large, then $D(\omega)^{H,\eta} * \kappa_{L_-^\perp}$ is spanned by the θ-nilpotent orbital integrals. Therefore, we can find constants $c_{\mathcal{O}}$ so that

$$\widehat{J_0} * \kappa_{L_-^\perp} = \sum_{\mathcal{O} \subset \mathcal{N}_x} c_{\mathcal{O}} \left(\mu_{\mathcal{O}} * \kappa_{L_-^\perp} \right).$$

Theorem 9.1 follows by taking inverse Fourier transform. For details, see the last paragraph of the proof of [RR96, Theorem 7.11].

Acknowledgment I would like to sincerely thank the anonymous referee for a lot of suggestions which greatly improve the clarity and completeness of the manuscript.

References

[AG09] A. Avraham and D. Gourevitch, *Generalized Harish-Chandra descent, Gelfand pairs, and an Archimedean analog of Jacquet-Rallis's theorem*. Duke Math. J. **149**(2009), no. 3, 509–567. http://doi.org/10.1215/00127094-2009-044 With an appendix by the authors and Eitan Sayag.

[FJ93] S. Friedberg and H. Jacquet, *Linear periods*. J. Reine Angew. Math. **443**(1993), 91–139. http://doi.org/10.1515/crll.1993.443.91

[GJ72] R. Godement and H. Jacquet, *Zeta functions of simple algebras*. Lecture Notes in Mathematics, 260, Springer-Verlag, Berlin, Germany-New York, NY, 1972.

[Guo96] J. Guo, *On a generalization of a result of Waldspurger*. Can. J. Math. **48**(1996), no. 1, 105–142. http://doi.org/10.4153/CJM-1996-005-3

[Guo98] J. Guo, *Spherical characters on certain p-adic symmetric spaces*. MPIM Preprint, 1998. https://www.mpim-bonn.mpg.de/de/preprints?year$=$&number$=$&name$=$guo%2C\protect$\relax+$J&title=

[Hak94] J. Hakim, *Admissible distributions on p-adic symmetric spaces*. J. Reine Angew. Math. **455**(1994), 1–19. http://doi.org/10.1515/crll.1994.455.1

[How74] R. Howe, *The Fourier transform and germs of characters (case of Gl_n over a p-adic field)*. Math. Ann. **208**(1974), 305–322. http://doi.org/10.1007/BF01432155

[JR96] H. Jacquet and S. Rallis, *Uniqueness of linear periods*. Compos. Math. **102**(1996), no. 1, 65–123.

[Kot05] R. E. Kottwitz, *Harmonic analysis on reductive p-adic groups and Lie algebras*. In: Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., 4, American Mathematical Society, Providence, RI, 2005, pp. 393–522.

[KP79] H. Kraft and C. Procesi, *Closures of conjugacy classes of matrices are normal*. Invent. Math. **53**(1979), no. 3, 227–247. http://doi.org/10.1007/BF01389764

[Clo91] C. Laurent, *Invariant harmonic analysis on the Schwartz space of a reductive p-adic group*. In: Harmonic analysis on reductive groups (Brunswick, ME, 1989), Progr. Math., 101, Birkhäuser, Boston, MA, 1991, pp. 101–121, http://doi.org/10.1007/978-1-4612-0455-8_6

[RR96] C. Rader and S. Rallis, *Spherical characters onp-adic symmetric spaces*. Amer. J. Math. **118**(1996), no. 1, 91–178.

[Zha15] C. Zhang, *On the smooth transfer for Guo-Jacquet relative trace formulae*. Compos. Math. **151**(2015), no. 10, 1821–1877.

Department of Mathematics, The University of Arizona, Tucson, AZ 85721, USA
e-mail: xuehang@math.arizona.edu