Centralizers of derived-from-Anosov systems on $\mathbb{T}^3$: rigidity versus triviality

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Abstract. In this paper, we study the centralizer of a partially hyperbolic diffeomorphism on $\mathbb{T}^3$ which is homotopic to an Anosov automorphism, and we show that either its centralizer is virtually trivial or such diffeomorphism is smoothly conjugate to its linear part.

1. *Introduction*

Let M be a closed C^∞ Riemannian manifold. Given a diffeomorphism $f \in \mathrm{Diff}^r(M)$ and $r \in [1, \infty]$, $s \in [0, r]$, the C^s-*centralizer* of f is defined as

$$\mathcal{Z}^s(f) = \{g \in \mathrm{Diff}^s(M) : g \circ f = f \circ g\}.$$

By definition, f is conjugate to itself by any element $g \in \mathcal{Z}^s(f)$. In other words, the centralizer of f is in fact the group of symmetries of f, where 'symmetries' can be interpreted classically as: changes of coordinates do not break the dynamics of the system. The centralizer of f always contains the integer powers of f. One says that f has *trivial*

C^s-*centralizer* if $\mathcal{Z}^s(f) = \{f^n\}_{n\in\mathbb{Z}}$, and f has *virtually trivial C^s-centralizer* if the cyclic group $<f> := \{f^n\}_{n\in\mathbb{Z}}$ is a finite index subgroup of $\mathcal{Z}^s(f)$.

The centralizers of diffeomorphisms play important roles in several topics of dynamical systems. For instance, people attempt to classify diffeomorphisms up to differentiable conjugacies, especially in the study of circle diffeomorphisms [**He**]. On the other hand, the property of the centralizer of a diffeomorphism can give some consequences in foliation theory; see, for instance, [**B**]. Moreover, the centralizer of a diffeomorphism is closely related to the study of higher-rank abelian actions on manifolds; see, for instance, [**DK**, **H**, **HW**].

Smale [**Sm1**, **Sm2**] proposed the famous conjecture that typical diffeomorphisms have trivial centralizer, and considered it as one of the mathematical problems of this century.

Conjecture 1.1. There exists a residual subset $\mathcal{U} \subseteq \text{Diff}^r(M)$, such that every $f \in \mathcal{U}$ has trivial C^r-centralizer.

The history of this conjecture goes back to the work of Kopell [**Ko**] proving that $\text{Diff}^r(S^1)$ ($r \geq 2$) admits an open and dense subset in which each element has trivial centralizer. This conjecture has been solved for the case $r = 1$ in [**BCW**, **BCVW**]. It is still wide open for general case when $r > 1$. Palis and Yoccoz studied Anosov diffeomorphisms [**PaYo2**] and proved that C^∞-open dense Anosov diffeomorphisms have trivial centralizer. See also [**Fi**, **PaYo1**, **RV**] for other related results.

Since hyperbolic systems are not dense, it is natural to study the centralizer problem for $C^r, r > 1$, diffeomorphisms under a weaker hyperbolicity assumption. In 1970s, Brin and Pesin proposed the notion of partial hyperbolicity to weaken the notion of hyperbolicity; see §2.1 for a precise definition. There are only some partial results in this direction. Burslem [**Bur**] showed that C^1-open and dense partially hyperbolic diffeomorphisms have discrete centralizer. Recently, under a volume-preserving assumption, [**DWX**] showed that for some classical C^1 open class of C^r partially hyperbolic diffeomorphisms, the centralizer is either small (virtually trivial) or exceptionally large (contains a non-discrete Lie group); see related results in [**BG**]. It is still unknown whether the results in [**DWX**] or [**BG**] can be generalized to the general non-volume-preserving case.

In this paper we will classify the centralizer for a classical open class of partially hyperbolic systems on $\mathbb{T}^3$, without assuming a volume-preserving condition. More precisely, we consider derived-from-Anosov (DA) systems on $\mathbb{T}^3$. For a partially hyperbolic diffeomorphism f on $\mathbb{T}^3$, we denote its linear part $L_f : \pi_1(\mathbb{T}^3) = \mathbb{Z}^3 \to \mathbb{Z}^3$ induced by f on the fundamental group of $\mathbb{T}^3$ by L_f. If $L_f \in \text{GL}(3, \mathbb{Z})$ is Anosov (hyperbolic), that is, f is homotopic to an Anosov automorphism, then f is called a *DA diffeomorphism*.

The study of partially hyperbolic DA diffeomorphisms originated from Mañé [**M**]. Partially hyperbolic DA diffeomorphisms have been studied extensively in their topological aspects (see. for instance. [**BBI1**, **Ha**, **HaPo**, **Po**]) as well as their measure-theoretic aspects (see, for instance, [**BFSV**, **GS**, **HaU**, **PTV**, **U**, **VY**]).

In this paper we essentially classify the centralizer of any $C^r, 1 < r \leq \infty$, partially hyperbolic DA diffeomorphisms on $\mathbb{T}^3$.

THEOREM 1.2. *Let $f \in \mathrm{Diff}^{\infty}(\mathbb{T}^3)$ be a partially hyperbolic derived-from-Anosov diffeomorphism. Then one has the following dichotomy:*

- *either the C^{∞} centralizer of f is virtually trivial, and*

$$\#\{g \in \mathcal{Z}^{\infty}(f) : g \text{ is homotopic to the identity}\} \leq |\det(L_f - \mathrm{Id})|;$$

- *or f is C^{∞}-conjugate to L_f, thus $\mathcal{Z}^{\infty}(f) \cong \mathcal{Z}^{\infty}(L_f)$.*

For the centralizer of the partially hyperbolic DA diffeomorphisms with lower regularity, we get the following theorem. And Theorem 1.2 is a direct corollary of this theorem.

THEOREM 1.3. *Let $f \in \mathrm{Diff}^r(\mathbb{T}^3)$ $(r > 1)$ be a partially hyperbolic derived-from-Anosov diffeomorphism. Then f satisfies one of the following properties:*

- *the C^r centralizer of f is virtually trivial and*

$$\#\{g \in \mathcal{Z}^r(f) : g \text{ is homotopic to the identity}\} \leq |\det(L_f - \mathrm{Id})|;$$

- *f is Anosov, and $C^{r-\varepsilon}$-conjugate to L_f for any $\varepsilon > 0$.*

Unlike in [**BG**, **DWX**], the proofs of our main results do not depend on the measure-theoretic properties of the center foliation. Instead we study the accessibility and topological information of the stable and unstable foliations, which allow us to get rid of the volume-preserving assumption in [**BG**, **DWX**].

Remark 1.4.

(1) In the second case of Theorem 1.3, the loss of regularity comes from Journé's theorem [**J**]. If r is not an integer, then f is C^r-conjugate to L_f. Thus $\mathcal{Z}^r(f) \cong \mathcal{Z}^r(L_f) = \mathcal{Z}^{\infty}(L_f)$, which is virtually $\mathbb{Z}^2$.

(2) When $\mathcal{Z}^r(f)$ is virtually trivial, one has $\#\{g \in \mathcal{Z}^r(f) : g \text{ is homotopic to the}$ identity$\} \leq \#(\mathcal{Z}^r(f)/<f>)$, and the inequality could be strict provided that $\mathcal{Z}^r(f)$ contains elements homotopic to $-\mathrm{Id}_{\mathbb{T}^3}$, or g with $g^{n_0} = f$ and $n_0 > 1$.

(3) From the proof, we can see that the dichotomy in both theorems comes from whether f is accessible. If f is accessible, then $\mathcal{Z}^r(f)$ is virtually trivial. Otherwise, f is forced to be smoothly conjugate to its linear part L_f.

(4) The reason why we discuss virtual triviality rather than triviality of the centralizer is that this property is more likely to be robust; for example, the class of systems which satisfies the dichotomy in our paper forms an open subset in the group of C^r diffeomorphisms on $\mathbb{T}^3$. See [**DWX**] and references therein for more results on 'virtual triviality of centralizer or rigidity' for partially hyperbolic diffeomorphisms.

In particular, our result implies that the centralizer of every diffeomorphism constructed by Mañé in [**M**] is virtually trivial. A direct corollary is that C^r-open densely, the C^r-centralizer of a partially hyperbolic diffeomorphism which is homotopic to an Anosov automorphism on $\mathbb{T}^3$ is trivial, which can be achieved by perturbing different fixed points to have different center Lyapunov exponents.

Question 1.5. Does Theorem 1.3 hold when $r = 1$?

We do not know the answer to this question. Our proof strongly relies on the recent results in [**GS**, **HaS**] where $r > 1$ is crucial in their arguments.

Remark 1.6. If f is C^1-smooth and accessible, then $\mathcal{Z}^1(f)$ is virtually trivial. See Remark 3.9 and Theorem 3.16 in §3. As accessible partially hyperbolic diffeomorphisms with one-dimensional center form a C^1-open and dense subset [**Di**, **DoW**, **HHU**], it follows that for a C^1-open dense set of partially hyperbolic DA diffeomorphisms on $\mathbb{T}^3$, the C^1 centralizer is trivial.

Question 1.7. Suppose that M is a closed 3-manifold and $\mathrm{PH}^r(M)$ is the set of C^r partially hyperbolic diffeomorphisms on M. Let $\mathcal{U}^r \subset \mathrm{PH}^r(M)$ be defined by

$$\mathcal{U}^r := \{f \in \mathrm{PH}^r(M) : \mathcal{Z}^r(f) \text{ is trivial}\}.$$

Does $\mathcal{U}^r$ contain an open dense subset in $\mathrm{PH}^r(M)$ for every $r > 1$?

2. *Preliminaries*
In this section we collect the notions and results used in this paper.

2.1. *Domination and partial hyperbolicity.* A Df-invariant splitting $TM = E \oplus F$ is *dominated*, if there exists $N \in \mathbb{N}$ such that

$$\|Df^N|_{E(x)}\| \cdot \|Df^{-N}|_{F(f^n(x))}\| \le \tfrac{1}{2} \quad \text{for any } x \in M.$$

$\dim(E)$ is called the index of the dominated splitting.

   The following well-known result tells us that the dominated bundles are invariant under the diffeomorphisms in the centralizer.

PROPOSITION 2.1. [**DWX**, Lemma 13] *Let $f \in \mathrm{Diff}^1(M)$ admit a dominated splitting of the form $TM = E \oplus F$. Then for any $g \in \mathcal{Z}^1(f)$, one has $Dg(E) = E$ and $Dg(F) = F$.*

   A diffeomorphism $f \in \mathrm{Diff}^r(M)$ is *partially hyperbolic* if there exist a Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ and $N \in \mathbb{N}$ such that the following statements hold.
- *Uniform contraction and expansion.* For any $x \in M$, one has

$$\|Df^N|_{E^s(x)}\| \le \tfrac{1}{2} \quad \text{and} \quad \|Df^{-N}|_{E^u(x)}\| \le \tfrac{1}{2}.$$

- *Domination.* For any $x \in M$, one has

$$\|Df^N|_{E^s(x)}\| \cdot \|Df^{-N}|_{E^c(f^N(x))}\| \le \tfrac{1}{2},$$
$$\|Df^N|_{E^c(x)}\| \cdot \|Df^{-N}|_{E^u(f^N(x))}\| \le \tfrac{1}{2}.$$

It is clear that the set of all C^r partially hyperbolic diffeomorphisms is an open subset of $\mathrm{Diff}^r(M)$ in the C^r-topology. If either the strong stable bundle E^s or the strong unstable bundle E^u is trivial, we say f is *weakly partially hyperbolic*.

   Not every manifold supports a partially hyperbolic diffeomorphism. For instance, there are no partially hyperbolic diffeomorphisms on $\mathbb{S}^3$ [**BI**].

2.2. *Dynamical coherence.* For a partially hyperbolic diffeomorphism f, by [**HPS**], there always exist f-invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ tangent to the bundles E^s and E^u respectively, and such foliations are unique. f is called *dynamically coherent* if there exist f-invariant foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ tangent to $E^{cs} := E^s \oplus E^c$ and $E^{cu} := E^c \oplus E^u$, respectively. By taking the intersection of $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$, one gets an invariant foliation $\mathcal{F}^c$ tangent to E^c. The dynamical coherence of a partially hyperbolic DA diffeomorphism on $\mathbb{T}^3$ has been substantially investigated; see, for instance, [**BBI1**, **BBI2**, **BI**, **FiPoSa**, **HaPo**, **Po**].

Two transverse foliations $\mathcal{F}, \mathcal{G}$ on $\mathbb{R}^3$ have *global product structure* if for any $x, y \in \mathbb{R}^3$, the leaf $\mathcal{F}(x)$ intersects the leaf $\mathcal{G}(y)$ at a unique point. Following [**Fe**, **Br**], one says that a foliation $\mathcal{F}$ on $\mathbb{R}^3$ is *quasi-isometric* if there exist $a, b > 0$ such that for any $x \in \mathbb{R}^3$ and $y \in \mathcal{F}(x)$, one has $\mathrm{d}_\mathcal{F}(x, y) \le a \cdot \mathrm{d}(x, y) + b$, where $\mathrm{d}_\mathcal{F}(\cdot, \cdot)$ denotes the distance on the leaves of $\mathcal{F}$ and $\mathrm{d}(\cdot, \cdot)$ denotes the Euclidean distance. A foliation $\mathcal{F}$ that satisfies this property is also called *quasi-geodesic*.

The following result gives the dynamical coherence of a partially hyperbolic hyperbolic DA diffeomorphism and further geometrical properties of the invariant foliations.

THEOREM 2.2. [**BI**, **Ha**, **Po**] *Let $f \in \mathrm{Diff}^1(\mathbb{T}^3)$ be a partially hyperbolic diffeomorphism with the partially hyperbolic splitting $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$. Assume that L_f is Anosov. Then one has the following statements:*

- *f has unique foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$, respectively;*
- *the lifts of the foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^u$ to $\mathbb{R}^3$ have global product structure;*
- *the lifts of the foliations $\mathcal{F}^s$, $\mathcal{F}^c$, $\mathcal{F}^u$ to $\mathbb{R}^3$ are quasi-isometric;*
- *each leaf of $\mathcal{F}^c$ is dense in $\mathbb{T}^3$;*
- *L_f has simple spectrum.*

Remark 2.3. [**BI**, Key Lemma 2.1] gives the existence of two-dimensional foliations transverse to E^u and E^s respectively which is exactly the assumption of [**Po**, Theorem 1.2].

Notation 2.4. Throughout this paper, for any foliation $\mathcal{F}$ on $\mathbb{T}^3$, we will denote by $\tilde{\mathcal{F}}$ the lift of $\mathcal{F}$ to $\mathbb{R}^3$. We denote by $\mathrm{d}_{\tilde{\mathcal{F}}}(\cdot, \cdot)$ the distance in a $\tilde{\mathcal{F}}$-leaf, and

$$\tilde{\mathcal{F}}_r(x) = \{y \in \tilde{\mathcal{F}}(x) : \mathrm{d}_{\tilde{\mathcal{F}}}(x, y) < r\}.$$

And we assume the center Lyapunov exponent of L_f is larger than zero, that is, the stable dimension of L_f as an Anosov diffeomorphism is 1. Otherwise, we only need to consider f^{-1}.

As a consequence of the global product structure for the lifted foliations, one has the following result whose proof can be found in [**Po**, Proposition 6.8].

COROLLARY 2.5. *Let f be as in the assumption of Theorem 2.2. Then there exists a constant $K > 0$ such that for any $r > 0$, any $x \in \mathbb{R}^3$, any $y \in \tilde{\mathcal{F}}_r^u(x)$ and any $w \in \tilde{\mathcal{F}}^{cs}(x)$, one has that $\tilde{\mathcal{F}}_{r+K}^u(w) \cap \tilde{\mathcal{F}}^{cs}(y) \ne \emptyset$.*

The analogous result with respect to the strong stable and center unstable foliations holds.

Combining with the result from the previous section, one has the following corollary.

COROLLARY 2.6. *Let f be a C^1-partially hyperbolic diffeomorphism homotopic to a linear Anosov. Assume that f has the splitting of the form $TM = E^s \oplus E^c \oplus E^u$, then for any $g \in \mathcal{Z}^1(f)$, each invariant foliation $\mathcal{F}^*$ of f is invariant under g for $* = s, cs, c, cu, u$.*

Proof. By the classical stable manifold theorem and Theorem 2.2, there exist unique invariant foliations $\mathcal{F}^s$, $\mathcal{F}^u$, $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ tangent to E^s, E^u, E^{cs}, and E^{cu} respectively. For any $g \in \mathcal{Z}^1(f)$, by Proposition 2.1, one has that $g(\mathcal{F}^*)$ is an f-invariant foliation tangent to E^* for $* = s, u, cs, cu$; therefore, one has $g(\mathcal{F}^*) = \mathcal{F}^*$ for $* = s, u, cs, cu$. Finally, on has

$$g(\mathcal{F}^c) = g(\mathcal{F}^{cs}) \cap g(\mathcal{F}^{cs}) = \mathcal{F}^{cs} \cap \mathcal{F}^{cs} = \mathcal{F}^c. \qquad \square$$

For a diffeomorphism on the torus, if its linear part is Anosov, then it is semi-conjugate to its linear part.

THEOREM 2.7. [**Fr**, **W2**] *Let $f \in \mathrm{Diff}^1(\mathbb{T}^d)$ and assume that L_f is Anosov. Consider a lift F of f to the universal cover $\mathbb{R}^d$. Then there exists a unique continuous surjective map $H : \mathbb{R}^d \to \mathbb{R}^d$ such that:*
- $H \circ F = L_f \circ H$;
- $H(x + z) = H(x) + z$, for any $z \in \mathbb{Z}^d$ and any $x \in \mathbb{R}^d$.

As a consequence, one has the following corollary.

COROLLARY 2.8. *Let $f \in \mathrm{Diff}^1(\mathbb{T}^d)$ whose linear part L_f is Anosov and F be a lift of f to $\mathbb{R}^d$. Then there exists a continuous surjective map $H : \mathbb{R}^d \to \mathbb{R}^d$ such that:*
- $H \circ F = L_f \circ H$;
- $H(x + z) = H(x) + z$ for any $x \in \mathbb{R}^d$ and any $z \in \mathbb{Z}^d$;
- *for any $g \in \mathcal{Z}^1(f)$ and any lift G of g to $\mathbb{R}^d$, if $F \circ G = G \circ F$, then $H \circ G = L_g \circ H$.*

Proof. Let $H : \mathbb{R}^d \to \mathbb{R}^d$ be the continuous surjective map given by Theorem 2.7 such that $H \circ F = L_f \circ H$ and $H - \mathrm{Id}_{\mathbb{R}^d}$ is $\mathbb{Z}^d$-periodic. Consider the map $\widehat{H} = L_g^{-1} \circ H \circ G$ which satisfies that $\widehat{H} - \mathrm{Id}_{\mathbb{R}^d}$ is $\mathbb{Z}^d$-periodic. Then one has

$$\widehat{H} \circ F = L_g^{-1} \circ H \circ G \circ F = L_g^{-1} \circ H \circ F \circ G = L_g^{-1} \circ L_f \circ H \circ G$$
$$= L_f \circ L_g^{-1} \circ H \circ G = L_f \circ \widehat{H}.$$

By the uniqueness property in Theorem 2.7, one has $\widehat{H} = H$ which gives $H \circ G = L_g \circ H$. $\qquad \square$

Furthermore, the semi-conjugation preserves certain foliations.

THEOREM 2.9. [**Ha**, **HaPo**, **Po**, **U**] *Let f be a C^1-partially hyperbolic diffeomorphism on $\mathbb{T}^3$ which is homotopic to an Anosov automorphism L_f with two positive Lyapunov exponents. Denote by $\mathcal{F}^*$ and $\mathcal{W}^*$ the foliations of f and L_f respectively, for $* = s, u, cu, cs, c$. Let h be the semi-conjugacy between f and L_f, in formula: $L_f \circ h = h \circ f$. Then one has the following properties.*

(1) *For any $x \in \mathbb{T}^3$ and $* = cu, cs, c$, one has $h(\mathcal{F}^*(x)) = \mathcal{W}^*(h(x))$.*
(2) *For any $x \in \mathbb{T}^3$, the map $h|_{\mathcal{F}^s(x)} : \mathcal{F}^s(x) \to \mathcal{W}^s(h(x))$ is a homeomorphism.*
(3) *For any $x \in \mathbb{T}^3$, the pre-image $h^{-1}(h(x))$ is a segment (could be trivial) contained in $\mathcal{F}^c(x)$. In particular, for any center leaf $\mathcal{W}^c(y)$, there exist at most countably many points whose pre-images under h are non-trivial center segments.*

Remark 2.10.
(1) This result is obtained in [**Ha**, **U**] assuming absolute partial hyperbolicity, and it is extended to general partially hyperbolic setting in [**Po**, Appendix A] (see also [**HaPo**, §3]).
(2) The last item implies that f must have fixed points.
(3) Each center leaf of f is dense in $\mathbb{T}^3$.

2.3. *Centralizer of linear Anosov automorphisms.* The following result comes from [**AP**] (see also [**W1**]).

THEOREM 2.11. [**AP**] *Let L be a linear Anosov map on $\mathbb{T}^3$ and h be a homeomorphism on $\mathbb{T}^3$. If $h \circ L = L \circ h$, then h is affine.*

In particular, Adler and Palais' result implies that for each Anosov diffeomorphism f on $\mathbb{T}^3$, there exists a homeomorphism h on $\mathbb{T}^3$ such that for any $g \in \mathcal{Z}^0(f)$, one has that $h \circ g \circ h^{-1}$ is affine. Corollary 2.8 tells us that such results hold for semi-conjugacy to Anosov case.

The following result gives the rank of the group of linear automorphisms commuting with an Anosov automorphism, and it comes from [**DWX**, **KKS**].

LEMMA 2.12. [**KKS**, Proposition 3.7] *Consider a matrix $L \in SL(n, \mathbb{Z})$ whose characteristic polynomial is irreducible over $\mathbb{Z}$. Then the group $\mathcal{G}(L) = \{L_1 \in SL(n, \mathbb{Z}) : LL_1 = L_1 L\}$ is abelian. Moreover, $\mathcal{G}(L)$ is virtually $\mathbb{Z}^{r+c-1}$, where r is the number of real eigenvalues of L and 2c is the number of complex eigenvalues of L.*

In our paper all the linear Anosov maps we consider have real simple spectrum. It is easy to see that these linear Anosov maps are irreducible in the sense that their characteristic polynomials are irreducible over $\mathbb{Z}$.† Therefore, their centralizers are virtually $\mathbb{Z}^2$ by Theorem 2.11 and Lemma 2.12.

COROLLARY 2.13. *Let $f \in \text{Diff}^1(\mathbb{T}^3)$ be a partially hyperbolic diffeomorphism with Anosov linear part $L_f \in GL(3, \mathbb{Z})$. Then the C^0-centralizer $\mathcal{Z}^0(L_f)$ of L_f is virtually $\mathbb{Z}^2$.*

Moreover, the non-trivial elements in the centralizer is also Anosov.

LEMMA 2.14. *Let $A \in SL(3, \mathbb{Z})$ be an Anosov automorphism. For any $B \in SL(3, \mathbb{Z})$, if $AB = BA$, then either $B = \text{Id}$ or B is Anosov.*

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† Otherwise, one should have ± 1 as the eigenvalue, contradicting to the hyperbolicity of the maps.

Proof. If $B \in SL(3, \mathbb{Z})$ has eigenvalues of modulus 1, then 1 is an eigenvalue of B since $\det(B) = 1$. And there exists a rational vector $0 \neq v$ such that $Bv = v$. For any eigenvector w of B with respective to 1, one has that $BAw = ABw = Aw$, which implies that the eigenvector space of B with respect to 1 is invariant under A; in particular. the rational vector Av is also an eigenvector of B with respect to the eigenvalue 1 and is not collinear to v, since A is Anosov and v is rational. Then the two-dimensional linear space generated by Av and v has rational slope and is contained in the eigenspace of B with respect to the eigenvalue 1. Once again, as A is Anosov (in particular, irreducible), the linear space generated by Av and v is not A-invariant, which implies that each vector in $\mathbb{R}^3$ is in the eigenspace of B with respect to eigenvalue 1. Hence B is the identity. $\square$

2.4. *Regularity.* Now, we collect some regularity lemmas showing that if a homeomorphism is differentiable along pairs of transverse foliations up to certain order, then the homeomorphism is differentiable.

LEMMA 2.15. [**J**] *Let M be a closed manifold and h be a homeomorphism on M. Assume that there exist two transverse continuous foliations $\mathcal{F}$ and $\mathcal{G}$ on M with C^r-leaves, and h is uniformly C^r when restricted to leaves of $\mathcal{F}$ and $\mathcal{G}$. Then h is $C^{r-\varepsilon}$ for any $\varepsilon > 0$.*

Remark 2.16. If r is not an integer, then h is C^r. If r is an integer, then h is $C^{r-1+\text{Lip}}$.

One says that a foliation $\mathcal{F}$ with C^1-leaves is *expanding* for $f \in \text{Diff}^1(M)$ if $\mathcal{F}$ is f-invariant and there exists $N \in \mathbb{N}$ such that $\|Df^{-N}|_{T_x \mathcal{F}(x)}\| \leq \frac{1}{2}$, for any $x \in M$.

LEMMA 2.17. [**Go2**, Lemma 2.4] *Let f, g be two C^r-diffeomorphisms on a closed manifold M. Let $\mathcal{F}$, $\mathcal{G}$ be one-dimensional expanding foliations with C^r-leaves for f and g, respectively. Assume that there exists a homeomorphism h on M such that:*
- *$h \circ f = g \circ h$ and $h(\mathcal{F}) = \mathcal{G}$;*
- *h and its inverse are uniformly C^1 along the leaves of $\mathcal{F}$ and $\mathcal{G}$, respectively.*

Then h is uniformly C^r along the leaves of $\mathcal{F}$ and h^{-1} is uniformly C^r along the leaves of $\mathcal{G}$.

2.5. *Accessibility.* Given a partially hyperbolic diffeomorphism $f \in \text{Diff}^1(M)$ and a point $x \in M$, the *accessible class* $\text{Acc}(x)$ *of x* is defined as the set of points which can be joined to x by paths which are concatenations of paths in a strong stable or strong unstable manifold. By definition, $\text{Acc}(x)$ is saturated by strong stable and strong unstable leaves. One says that f is *accessible* if any two points $x, y \in M$ can be connected by a path γ which is a concatenation of paths in strong stable or strong unstable manifolds of f; in other words, the accessible class of a point is the whole manifold.

For a partially hyperbolic diffeomorphism f, the bundles E^s and E^u are *jointly integrable*, if there exists an f-invariant foliation tangent to $E^s \oplus E^u$ everywhere. In this case, we call f *su-integrable* or *su-jointly-integrable*.

It has been proved in [**Di**] that if f is accessible, then $E^s \oplus E^u$ is not jointly integrable. Moreover, if f has one-dimensional center, then there exists a point $x \in M$ with the local accessibility property as follows.

LEMMA 2.18. [**Di**, **HHU**] *Let f be a C^1-partially hyperbolic diffeomorphism on M. If f is accessible and the center bundle is one-dimensional, then there exist $r_0 > 0$, $r_1 > 0$ which can be arbitrarily small, and $x \in M$ such that fir any center curve $I_{r_1}^c(x)$ centered at x of radius r_1, there exist $x^s, x^{su}, x^{sus} \in M$ and $x^c \in I_{r_1}^c(x)$ such that:*

- *$x^s \in \mathcal{F}_{r_0}^s(x)$ and $x^{su} \in \mathcal{F}_{r_0}^u(x^s)$,*
- *$x^{sus} \in \mathcal{F}_{r_0}^s(x^{su})$ and $x^c \in \mathcal{F}_{r_0}^u(x^{sus})$,*

where $\mathcal{F}_r^*(z)$ denotes the r-neighborhood of z in the leaf $\mathcal{F}^*(z)$ for $* = s, u$.

Moreover, let $I^c(x, x^c)$ denote the set of all points located between x and x^c in $I_{r_1}^c(x)$. Then $\mathrm{Acc}(x)$ contains an open set U close to x, that is,

$$U \subseteq \bigcup_{y \in I^c(x,x^c)} \bigcup_{z \in \mathcal{F}_{loc}^s(y)} \mathcal{F}_{loc}^u(z) \subseteq \mathrm{Acc}(x).$$

Each point in U can be connected to x by a local su-path contained in a small neighborhood of x.

If a diffeomorphism $f : M \to M$ is partially hyperbolic, and $\pi : \tilde{M} \to M$ is a covering map, then any lift $\tilde{f} : \tilde{M} \to \tilde{M}$ is also partially hyperbolic. And the partially hyperbolic splitting on $\tilde{M}$ is defined by pulling back the splitting on M:

$$T\tilde{M} = \pi^*(E^s) \oplus \pi^*(E^c) \oplus \pi^*(E^u).$$

The following result tells us that the accessibility is preserved under lifts of the manifold.

LEMMA 2.19. *Let $f \in \mathrm{Diff}^1(M)$ be an accessible partially hyperbolic diffeomorphism, and assume that the center bundle is one-dimensional. Consider a covering map $\pi : \tilde{M} \to M$ from a connected manifold $\tilde{M}$ to M, and a lift $\tilde{f}$ of f to $\tilde{M}$. Then $\tilde{f}$ is accessible.*

Proof. Notice that the lifts of the strong stable and unstable foliations to $\tilde{M}$ are the strong stable and unstable foliations of $\tilde{f}$. Since f is accessible, Lemma 2.18 shows that there exists $x \in M$ with the local accessibility property.

This implies for every $\tilde{x} \in \pi^{-1}(x)$, that the accessibility class $\mathrm{Acc}(\tilde{x})$ with respect to $\tilde{f}$ contains an open set close to $\tilde{x}$. If an accessible class contains an open set, then it is open. Thus $\mathrm{Acc}(\tilde{x})$ is open for every $\tilde{x} \in \pi^{-1}(x)$.

On the other hand, for every $\tilde{y} \in \tilde{M}$, since $\pi(y) \in M$ can be connected to x by an su-path, $\tilde{y}$ can be connected to a point $\tilde{x} \in \pi^{-1}(x)$ by an su-path. This implies

$$\tilde{M} = \bigcup_{\tilde{x} \in \pi^{-1}(x)} \mathrm{Acc}(\tilde{x}).$$

Since $\tilde{M}$ is connected and each $\mathrm{Acc}(\tilde{x})$ is open, we must have $\tilde{M} = \mathrm{Acc}(\tilde{x})$ for every $\tilde{x} \in \pi^{-1}(x)$. Thus $\tilde{f}$ is accessible. □

The following result gives equivalence conditions for the joint integrability of strong stable and unstable distributions.

THEOREM 2.20. ([**GS**, Theorem 1.1] and [**HaS**]) *Let f be a C^r ($r > 1$) partially hyperbolic diffeomorphism on $\mathbb{T}^3$ whose linear part L_f is Anosov. The following statements are equivalent.*

- *Strong stable and unstable distributions of f are jointly integrable.*
- *Each periodic orbit of f has same center Lyapunov exponent as L_f, and f is Anosov.*
- *f is not accessible.*

Remark 2.21.
- Under the volume-preserving assumption, Hammerlindl and Ures [**HaU**] proved that the first and third items are equivalent; in particular, *f* is topological Anosov;
- The equivalence of the first and second items is obtained in [**GS**] under the volume-preserving setting. Then the volume-preserving condition is removed by [**HaS**] and the third equivalent item is obtained in [**HaS**].

2.6. *Equivalent conditions for su-integrability of an Anosov map.* In this part, we collect the consequences of *su*-joint-integrability for Anosov diffeomorphisms on $\mathbb{T}^3$, which is proved in [**GRZ**, **GS**].

THEOREM 2.22. [**GS**, Theorem 5.1] *Let f be a C^r ($r > 1$) partially hyperbolic and Anosov diffeomorphism on $\mathbb{T}^3$. Let h be the conjugacy between f and L_f. Then the following statements are equivalent:*
- *f is su-integrable;*
- *f is not accessible;*
- *h preserves the strong stable and strong unstable foliations;*
- *the center Lyapunov exponent of any periodic point p of f coincides with the center Lyapunov exponent of L_f;*
- *h is differentiable along the center leaves of f.*

Remark 2.23. When the conjugacy preserves the strong foliations, one can show that *h* and h^{-1} is uniformly Hölder continuous along the leaves of strong foliations (see for instance Lemma 2.3 in [**GS**]).

Now, we state the following result which is essentially [**GS**, Proposition 4.1]. For completeness, we will give the proof in Appendix A.

THEOREM 2.24. *Let g be a C^r ($r > 1$) Anosov diffeomorphism on $\mathbb{T}^3$ and let $h \in$* Homeo($\mathbb{T}^3$) *such that $h \circ g = L_g \circ h$. Assume that:*
- *there exists a Dg-invariant continuous splitting $E^s \oplus E^c \oplus E^u$;*
- *g is uniformly contracting along E^s and is uniformly expanding along $E^c \oplus E^u$;*
- *there exist g-invariant foliations $\mathcal{F}^c$, $\mathcal{F}^u$ and $\mathcal{F}^{su}$ tangent to E^c, E^u and $E^s \oplus E^u$ respectively;*
- *L_g is partially hyperbolic;*
- *h sends $\mathcal{F}^c$, $\mathcal{F}^u$ to the center, strong unstable foliations of L_g respectively;*
- *the holonomy map given by $\mathcal{F}^u$ restricted to each unstable leaf between two local plaques tangent to E^c at a uniform bounded distance is uniformly C^1.*

Then the Lyapunov exponent along E^c of any periodic point p is the same as the center Lyapunov exponent of L_g, and h is uniformly C^1 along the leaves of $\mathcal{F}^c$.

Remark 2.25. In the statement of Theorem 2.24, we do not assume the splitting $E^c \oplus E^u$ is dominated.

3. *Centralizer of partially hyperbolic DA diffeomorphism: Proofs of Theorems 1.2 and 1.3*

In this section we give the proof of our main theorems. We denote by $\mathcal{L} : \text{Diff}^r(\mathbb{T}^3) \to \text{GL}(3, \mathbb{Z})$ the linearization operator, that is, for every $g \in \text{Diff}^r(\mathbb{T}^3)$, $\mathcal{L}(g) = L_g$ is the action induced by g on $\pi_1(\mathbb{T}^3) = \mathbb{Z}^3$. For every $f \in \text{Diff}^r(\mathbb{T}^3)$, $\mathcal{L}$ induces a group homomorphism

$$\mathcal{L} : \; \mathcal{Z}^r(f) \; \longrightarrow \; \text{GL}(3, \mathbb{Z}),$$
$$g \; \longmapsto \; L_g.$$

The image of $\mathcal{Z}^r(f)$ by $\mathcal{L}$ satisfies

$$\mathcal{L}(\mathcal{Z}^r(f)) = \{L_g : g \in \mathcal{Z}^r(f)\} \subseteq \text{GL}(3, \mathbb{Z}) \cap \mathcal{Z}^0(L_f),$$

which is a subgroup in both $\text{GL}(3, \mathbb{Z})$ and $\mathcal{Z}^0(L_f)$.

Let

$$\mathcal{Z}_0^r(f) = \{g \in \mathcal{Z}^r(f) : g \text{ is homotopic to the identity}\}.$$

Then $\mathcal{Z}_0^r(f)$ is subgroup of $\mathcal{Z}^r(f)$. In particular, one has

$$\mathcal{Z}^r(f)/\mathcal{Z}_0^r(f) \; \cong \; \mathcal{L}(\mathcal{Z}^r(f)).$$

In §3.1, we show that $\mathcal{L}(\mathcal{Z}^r(f))$ is abelian and virtually $\mathbb{Z}^2$ or $\mathbb{Z}$. Then we prove that $\mathcal{Z}_0^r(f)$ is finite in §3.2. The index of $\mathcal{Z}_0^r(f)$ satisfies

$$\#\mathcal{Z}_0^r(f) \le |\text{det}(L_f - \text{Id}_{\mathbb{R}^3})|.$$

Recall that $\mathcal{L}(\mathcal{Z}^r(f))$ always contains $< L_f > \cong \mathbb{Z}$ which is induced by $<f> \subseteq \mathcal{Z}^r(f)$. If $\mathcal{L}(\mathcal{Z}^r(f))$ is virtually $\mathbb{Z}$, then we show $\mathcal{Z}^r(f)$ is virtually trivial, which is the first case of Theorem 1.3. Finally, we discuss the case where $\mathcal{L}(\mathcal{Z}^r(f))$ is virtually $\mathbb{Z}^2$ in §3.3, and we show that f is smoothly conjugate to L_f in this case.

3.1. *Preliminary lemmas.* The lifts of two commutable diffeomorphisms may not be commutable. The following result tells us that the lifts of the centralizer of partially hyperbolic DA diffeomorphisms, up to finite iterates, are still in the centralizer of the lifted diffeomorphism.

LEMMA 3.1. *Let f be a C^1-partially hyperbolic diffeomorphism on $\mathbb{T}^3$ whose linear part L_f is Anosov, and let F be a lift of f to $\mathbb{R}^3$.*

Then for any $g \in \mathcal{Z}^1(f)$, there exist an integer $0 < l \le |\text{det}(L_f - \text{Id}_{\mathbb{R}^3})|$ and a lift $\widehat{G}$ of g^l such that $F \circ \widehat{G} = \widehat{G} \circ F$. Furthermore, if g is homotopic to the identity, then l can be chosen as a factor of $|\text{det}(L_f - \text{Id}_{\mathbb{R}^3})|$.

Proof. Let $p_1, \ldots, p_k \in \mathbb{T}^3$ be all the fixed points of L_f. It is classical that $k = |\text{det}(L_f - \text{Id}_{\mathbb{R}^3})|$. Let $H : \mathbb{R}^3 \to \mathbb{R}^3$ be the semi-conjugacy between F and L_f given by

Theorem 2.7, and let $h : \mathbb{T}^3 \to \mathbb{T}^3$ be the map induced by H. Then the set of fixed points of f is contained in $\bigcup_{i=1}^{k} h^{-1}(p_i)$ and each $h^{-1}(p_i)$ is f-invariant. By Theorem 2.9, each $h^{-1}(p_i)$ is a compact center segment (could be trivial). By the Brouwer fixed point theorem, f has fixed points in each $h^{-1}(p_i)$. Let $I_i \subset h^{-1}(p_i)$ be the shortest connected and compact center segment (could be trivial) containing all fixed points of f in $h^{-1}(p_i)$. Then the two endpoints of I_i are fixed points of f.

Let $\pi : \mathbb{R}^3 \to \mathbb{T}^3$ be the canonical covering map. Without loss of generality, one can assume that p_1 is the projection of $0 \in \mathbb{R}^3$ under π, that is, $\pi(0) = p_1$.

Since $g \in \mathcal{Z}^1(f)$, by Corollary 2.6, g preserves the center foliation of f. As the set of fixed points of f is g-invariant, for each $i \in \{1, \ldots, k\}$, there exists $j \in \{1, \ldots, k\}$ such that $g(I_i) = I_j$ which defines a permutation on $\{1, \ldots, k\}$. Therefore, there exists $0 < l \le k$ such that $g^l(I_1) = I_1$. If g^l preserves the orientation of the center bundle, then the two endpoints of I_1 are fixed points of f and g^l. If g^l reverses the orientation of the center bundle, then g^l has a unique fixed point in I_1 which is also a fixed point of f since $g^l(I_1) = f(I_1) = I_1$ and $g \in \mathcal{Z}^1(f)$. To summarize, f and g^l have a common fixed point $q_1 \in I_1$. Notice that $\widehat{q_1} = H^{-1}(0) \cap \pi^{-1}(q_1)$ is a fixed point of F. Since q_1 is a fixed point of g^l, there exists a lift $\widehat{G}$ of g^l such that $\widehat{G}(\widehat{q_1}) = \widehat{q_1}$. Observe that $F \circ \widehat{G} \circ F^{-1} \circ \widehat{G}^{-1}$ is a lift of the identity map on $\mathbb{T}^3$ and has a fixed point $\widehat{q_1}$, hence $F \circ \widehat{G} = \widehat{G} \circ F$.

Now, we assume that g is homotopic to the identity. Let G be a lift of g to $\mathbb{R}^3$. Since $f \circ g = g \circ f$, there exists $n \in \mathbb{Z}^3$ such that $F \circ G = G \circ F + n$. Since L_f is Anosov, the linear map $L_f - \mathrm{Id}_{\mathbb{R}^3}$ is invertible. Let $m = (L_f - \mathrm{Id}_{\mathbb{R}^3})^{-1} n \in \mathbb{Q}^3$. Then there exists an integer $l > 0$ which is a factor of $|\det(L_f - \mathrm{Id}_{\mathbb{R}^3})|$ such that $l \cdot m \in \mathbb{Z}^3$. Since g is homotopic to the identity, then $F \circ G^l = G^l \circ F + ln$. Now, let $\widehat{G} = G^l - l \cdot m$ which is a lift of g^l, and one has

$$F \circ \widehat{G} = F \circ (G^l - lm) = F \circ G^l - L_f(lm) = F \circ G^l - ln - lm$$
$$= G^l \circ F - lm = \widehat{G} \circ F,$$

which ends the proof. $\qquad\square$

The following result discusses the existence of common fixed points for lifted dynamics.

LEMMA 3.2. *Let f be a C^1-partially hyperbolic diffeomorphism on $\mathbb{T}^3$ whose linear part L_f is Anosov, and let $g \in \mathcal{Z}^1(f)$. Assume that there exist a lift F of f to $\mathbb{R}^3$ and a lift G of g to $\mathbb{R}^3$ such that $F \circ G = G \circ F$. Then F and G have a common fixed point, that is, there exists $p \in \mathbb{R}^3$ such that $F(p) = G(p) = p$.*

Proof. By Corollary 2.6, the center foliation of F is G-invariant. Let $H : \mathbb{R}^3 \to \mathbb{R}^3$ be the continuous surjective map given by Corollary 2.8 such that:

- $H \circ F = L_f \circ H$ and $H \circ G = L_g \circ H$;
- $H - \mathrm{Id}_{\mathbb{R}^3}$ is $\mathbb{Z}^3$-periodic.

As L_f and L_g have a unique fixed point $0 \in \mathbb{R}^3$, all the fixed points of F and G are contained in $H^{-1}(0)$. By Theorem 2.9, $H^{-1}(0)$ is a compact and F-invariant center segment.

As F commutes with G, the set of fixed points of F is G-invariant and *vice versa*. If F reverses the orientation of the center bundle, by the fact that $\dim(E^c) = 1$, F has a unique fixed point in $H^{-1}(0)$ which is also a fixed point of G. If G reverses the orientation of the center bundle, one concludes analogously. If F and G preserve the orientation of the center bundle, the endpoints of $H^{-1}(0)$ are the fixed points of F and G, proving the existence of common fixed points. □

The following lemma tells us that the linearization of the centralizer is virtually $\mathbb{Z}$ or $\mathbb{Z}^2$.

LEMMA 3.3. *Let f be a partially hyperbolic diffeomorphism on $\mathbb{T}^3$ whose linear part is Anosov. Then the group $\{L_g \in GL(3, \mathbb{Z}) : g \in \mathcal{Z}^r(f)\}$ is abelian and is virtually $\mathbb{Z}$ or $\mathbb{Z}^2$.*

Proof. Since f is partially hyperbolic, its linear part L_f has real simple spectrum. By Lemma 2.12, the group

$$\{L_g \in GL(3, \mathbb{Z}) : g \in \mathcal{Z}^r(f)\} \subset \{B \in GL(3, \mathbb{Z}) : L_f B = B L_f\}$$

is abelian and virtually $\mathbb{Z}^2$ or $\mathbb{Z}$. □

3.2. *The centralizer $\mathcal{Z}^r(f)$ is virtually isomorphic to its linearization.* In this part, we discuss the relationship between $\mathcal{Z}^r(f)$ and its linearization $\{L_g : g \in \mathcal{Z}^r(f)\}$, and we prove that the centralizer $\mathcal{Z}^r(f)$ is virtually isomorphic to $\{L_g : g \in \mathcal{Z}^r(f)\}$.

THEOREM 3.4. *Let f be a C^r $(r > 1)$ partially hyperbolic diffeomorphism on $\mathbb{T}^3$ whose linear part L_f is Anosov. Then one has that:*
- *for each $g \in \mathcal{Z}^r(f)$ which is homotopic to the identity, there exists an integer $l = l_g$ which is a factor of $|\det(L_f - \mathrm{Id}_{\mathbb{R}^3})|$ such that $g^l = \mathrm{Id}_{\mathbb{T}^3}$;*
- *$\#\{g \in \mathcal{Z}^r(f) : g \text{ is homotopic to } \mathrm{Id}_{\mathbb{T}^3}\} \leq |\det(L_f - \mathrm{Id}_{\mathbb{R}^3})|$.*

Before giving the proof of Theorem 3.4, we need to make some preparations. We first show that if $g \in \mathcal{Z}^r(f)$ is homotopic to the identity and a lift of g admits a fixed point on $\mathbb{R}^3$, then g is the identity.

PROPOSITION 3.5. *Let $f \in \mathrm{Diff}^r(\mathbb{T}^3)$ $(r > 1)$ be a partially hyperbolic diffeomorphism and $g \in \mathcal{Z}^r(f)$. Make the following assumptions.*
- *The linear part L_f of f is Anosov.*
- *There exists a lift $G : \mathbb{R}^3 \to \mathbb{R}^3$ of g with the following properties:*
  - *$G(x + n) = G(x) + n$ for any $x \in \mathbb{R}^3$ and $n \in \mathbb{Z}^3$;*
  - *G admits a fixed point $q \in \mathbb{R}^3$.*

Then $G = \mathrm{Id}_{\mathbb{R}^3}$.

Proof. By Corollary 2.6 and the fact that $G - \mathrm{Id}_{\mathbb{R}^3}$ is $\mathbb{Z}^3$-periodic, one has

$$G(\tilde{\mathcal{F}}^c(q + n)) = \tilde{\mathcal{F}}^c(q + n) = \tilde{\mathcal{F}}^c(q) + n \quad \text{for any } n \in \mathbb{Z}^3.$$

By the third item of Remark 2.10, the set $\{\tilde{\mathcal{F}}^c(q + n)\}_{n \in \mathbb{Z}^3}$ is dense in $\mathbb{R}^3$.

The following claim tells us that G is center fixing.

CLAIM 3.6. *For any $x \in \mathbb{R}^3$, the center leaf $\tilde{\mathcal{F}}^c(x)$ is fixed by G.*

Proof of the claim. As the set $\{\tilde{\mathcal{F}}^c(q+n)\}_{n \in \mathbb{Z}^3}$ is dense in $\mathbb{R}^3$, for any $x \in \mathbb{R}^3$, there exists a sequence of points $x_k \in \{\tilde{\mathcal{F}}^c(q+n)\}_{n \in \mathbb{Z}^3}$ such that x_k converges to x. Since $G - \mathrm{Id}_{\mathbb{R}^3}$ is $\mathbb{Z}^3$-periodic, there exists $\ell_0 > 0$ such that $\mathrm{d}(G(y), y) \leq \ell_0$ for any $y \in \mathbb{R}^3$. Since the center foliation $\tilde{\mathcal{F}}^c$ is quasi-isometric, there exist $a, b > 0$ such that for any $x, y \in \mathbb{R}^3$ with $x \in \tilde{\mathcal{F}}^c(y)$, one has $\mathrm{d}_{\tilde{\mathcal{F}}^c}(x, y) \leq a \cdot \mathrm{d}(x, y) + b$. Since the center leaf $\tilde{\mathcal{F}}^c(x_k)$ is G-invariant, one has

$$\mathrm{d}_{\tilde{\mathcal{F}}^c}(G(x_k), x_k) \leq a\, \mathrm{d}(G(x_k), x_k) + b \leq a\ell_0 + b.$$

Let $\ell_1 = a\ell_0 + b$. By the continuity of the center foliation, $\tilde{\mathcal{F}}^c_{\ell_1}(x_k)$ converges to $\tilde{\mathcal{F}}^c_{\ell_1}(x)$. By the continuity of G and the fact that $G(x_k) \in \tilde{\mathcal{F}}^c_{\ell_1}(x_k)$, one has $G(x) \in \tilde{\mathcal{F}}^c_{\ell_1}(x)$, proving that the center leaf $\tilde{\mathcal{F}}^c(x)$ is fixed by G. $\qquad\square$

CLAIM 3.7. *For any fixed point x_0 of G, one has*

$$G|_{\tilde{\mathcal{F}}^s(x_0) \cup \tilde{\mathcal{F}}^u(x_0)} = \mathrm{Id}\,|_{\tilde{\mathcal{F}}^s(x_0) \cup \tilde{\mathcal{F}}^u(x_0)}.$$

Proof of the claim. By Theorem 2.2, the foliations $\tilde{\mathcal{F}}^{cu}$ and $\tilde{\mathcal{F}}^s$ have global product structure, hence for any point $x \in \mathbb{R}^3$ and any point $y \in \tilde{\mathcal{F}}^{cs}(x)$, the center leaf $\tilde{\mathcal{F}}^c(y)$ intersects $\tilde{\mathcal{F}}^s(x)$ at a unique point.

Now, let x_0 be a fixed point of G. By Corollary 2.6, one has $G(\tilde{\mathcal{F}}^s(x_0)) = \tilde{\mathcal{F}}^s(x_0)$ and $G(\tilde{\mathcal{F}}^u(x_0)) = \tilde{\mathcal{F}}^u(x_0)$. One only needs to show that G restricted to $\tilde{\mathcal{F}}^s(x_0)$ is the identity, and the case for the strong unstable manifold works analogously. By Claim 3.6, for any $z \in \tilde{\mathcal{F}}^s(x_0) \subset \tilde{\mathcal{F}}^{cs}(x_0)$, one has $G(\tilde{\mathcal{F}}^c(z)) = \tilde{\mathcal{F}}^c(z)$. As $\tilde{\mathcal{F}}^c(z)$ intersects $\tilde{\mathcal{F}}^s(x_0)$ at a unique point, by the fact that $G(\tilde{\mathcal{F}}^s(x_0)) = \tilde{\mathcal{F}}^s(x_0)$, one has $\{G(z)\} = G(\tilde{\mathcal{F}}^c(z) \cap \tilde{\mathcal{F}}^s(x_0)) = \tilde{\mathcal{F}}^c(z) \cap \tilde{\mathcal{F}}^s(x_0) = \{z\}$.

Now, we show that G is the identity. As $\mathrm{Acc}(q)$ is saturated by strong stable and strong unstable leaves and q is a fixed point of G, by Claim 3.7, the map G coincides with the identity on $\mathrm{Acc}(q)$. There are two cases to discuss according to the accessibility property.

If f is accessible, by Lemma 2.19, each lift of f to the universal cover is also accessible, hence $\mathrm{Acc}(q) = \mathbb{R}^3$ which implies $G = \mathrm{Id}_{\mathbb{R}^3}$.

If f is not accessible, by Theorem 2.20, f is Anosov. Consider a projection p of the fixed point G on $\mathbb{T}^3$. Then g coincides with the identity on the union of the strong stable and unstable manifolds of p. As f is Anosov, then the union of the strong stable and unstable manifolds of p is dense in $\mathbb{T}^3$, hence $g = \mathrm{Id}_{\mathbb{T}^3}$ which in return implies $G = \mathrm{Id}_{\mathbb{R}^3}$ since G has fixed points. Now the proof of Proposition 3.5 is completed. $\qquad\square$

As a consequence, one has the following corollary.

COROLLARY 3.8. *Let $f \in \mathrm{Diff}^r(\mathbb{T}^3)$ $(r > 1)$ be a partially hyperbolic diffeomorphism whose linear part L_f is Anosov, and let $g \in \mathcal{Z}^r(f)$. If g is homotopic to the identity, then there exists $l \in \mathbb{N}$ which is a factor of $|\det(L_f - \mathrm{Id}_{\mathbb{R}^3})|$ such that $g^l = \mathrm{Id}_{\mathbb{T}^3}$.*

Proof. Consider a lift of F of f to $\mathbb{R}^3$. Let $g \in \mathcal{Z}^r(f)$ be a diffeomorphism homotopic to the identity. By Lemma 3.1, there exist a positive integer l which is a factor of $|\det(L_f - \mathrm{Id}_{\mathbb{R}^3})|$ and a lift $\widehat{G}$ of g^l to $\mathbb{R}^3$ such that $F \circ \widehat{G} = \widehat{G} \circ F$. By Lemma 3.2, $\widehat{G}$ admits fixed points, hence $\widehat{G}$ satisfies the assumption of Proposition 3.5, which gives that $g^l = \mathrm{Id}_{\mathbb{T}^3}$. $\qquad\square$

Remark 3.9. Notice that Lemmas 3.1 and 3.2 are stated for C^1-partially hyperbolic DA diffeomorphisms. By the proof of Proposition 3.5, Corollary 3.8 holds for C^1-partially hyperbolic DA diffeomorphisms provided that they are accessible, as one only requires the regularity with $r > 1$ when dealing with the non-accessible case.

We have obtained the first item in Theorem 3.4. The following result completes the proof of Theorem 3.4.

PROPOSITION 3.10. *Let f be a C^r ($r > 1$) partially hyperbolic diffeomorphism on $\mathbb{T}^3$ whose linear part L_f is Anosov. Then*

$$\#\{g \in \mathcal{Z}^r(f) : g \text{ is homotopic to the identity}\} \leq |\det(L_f - \mathrm{Id}_{\mathbb{R}^3})|.$$

Proof. Let $p_1, \ldots, p_k$ be all the fixed points of L_f, where $k = |\det(L_f - \mathrm{Id})|$. Consider the semi-conjugacy $h : \mathbb{T}^3 \to \mathbb{T}^3$ between f and L_f which is homotopic to the identity. Then all the fixed points of f are contained in $\bigcup_{i=1}^k h^{-1}(p_i)$, and $h^{-1}(p_i)$ is an f-invariant center segment. Let I_i be the smallest connected segments containing all fixed points of f in $h^{-1}(p_i)$. If f reverses the orientation of the center foliation, then each I_i is reduced to a single point. Since $\mathcal{F}^c$ is orientable, we give it an orientation. Let $I_i = [a_i, b_i]^c$ such that the direction from a_i to b_i gives the positive orientation. Let $\mathcal{E} = \{a_i\}_{i=1}^k$.

For any $g \in \mathcal{Z}^r(f)$ which is homotopic to the identity, by Corollary 2.6, the center foliation $\mathcal{F}^c$ is g-invariant and g preserves the orientation of $\mathcal{F}^c$. Therefore for each $i \in \{1, \ldots, k\}$, there exists $j \in \{1, \ldots, k\}$ such that $g(I_i) = I_j$ and $g(a_i) = a_j$. $\qquad\square$

CLAIM 3.11. *For any $g \in \mathcal{Z}^r(f)$ which is homotopic to the identity, if there exists some $i \in \{1, \ldots, k\}$ such that $g(a_i) = a_i$, then $g = \mathrm{Id}_{\mathbb{T}^3}$.*

Proof of the claim. Let $\tilde{a}_i \in \mathbb{R}^3$ be lift of a_i. As a_i is a fixed point for g, there exists a lift G of g such that $G(\tilde{a}_i) = \tilde{a}_i$. Since $g \in \mathcal{Z}^r(f)$ is homotopic to the identity, by Proposition 3.5, one has $G = \mathrm{Id}_{\mathbb{R}^3}$, hence $g = \mathrm{Id}_{\mathbb{T}^3}$. $\qquad\square$

CLAIM 3.12. *For any $i, j \in \{1, \ldots, k\}$, there exists at most one $g \in \mathcal{Z}^r(f)$ such that:*

- $g(a_i) = a_j$;
- *g is homotopic to the identity.*

Proof of the claim. Assume that there exist $i, j \in \{1, \ldots, k\}$ and two diffeomorphisms $g_1, g_2 \in \mathcal{Z}^r(f)$ such that:

- $g_1(a_i) = g_2(a_i) = a_j$;
- g_1 and g_2 are homotopic to the identity.

Let $g = g_1 \circ g_2^{-1}$. Then $g \in \mathcal{Z}^r(f)$ is homotopic to the identity and has a_j as a fixed point. By Claim 3.11, $g = \mathrm{Id}_{\mathbb{T}^3}$, hence $g_1 = g_2$. $\qquad\square$

By Claims 3.11 and 3.12, and the fact that $g \in \mathcal{Z}^r(f)$ which is homotopic to the identity must send a_1 to some a_j, one has

$$\#\{g \in \mathcal{Z}^r(f) : g \text{ is homotopic to the identity}\} \leq k = |\det(L_f - \mathrm{Id}_{\mathbb{R}^3})|. \qquad\square$$

3.3. *The linearization of the centralizer is virtually $\mathbb{Z}^2$: rigidity case.* In this section we discuss the case where the linear part of the centralizer

$$\mathcal{L}(\mathcal{Z}^r(f)) = \{L_g : g \in \mathcal{Z}^r(f)\}$$

is virtually $\mathbb{Z}^2$. The following theorem is the main result of this section.

THEOREM 3.13. *Let f be a C^r ($r > 1$) partially hyperbolic diffeomorphism on $\mathbb{T}^3$ whose linear part L_f is Anosov. If there exists $g \in \mathcal{Z}^r(f)$ such that $L_g^m \notin \{L_f^n\}_{n \in \mathbb{Z}}$ for any $m \neq 0$, then f is $C^{r-\varepsilon}$-conjugate to L_f for every $\varepsilon > 0$.*

Remark 3.14. It is clear that Theorem 1.3 is a direct consequence of Theorems 3.4 and 3.13.

The proof of the following result is a standard fact for Cartan $\mathbb{Z}^2$-linear action on $\mathbb{T}^3$, and for completeness we give a sketch of the its proof.

COROLLARY 3.15. *Let f be a C^1 partially hyperbolic diffeomorphism on $\mathbb{T}^3$ whose linear part L_f is Anosov. If there exists $g \in \mathcal{Z}^1(f)$ such that $L_g^m \notin \{L_f^n\}_{n \in \mathbb{Z}}$ for any $m \neq 0$, then there exists $\tilde{g} \in \mathcal{Z}^1(f)$ such that its linear part $L_{\tilde{g}}$ satisfies the following properties:*

- *$L_{\tilde{g}}$ is contracting along $E_{L_f}^u$ and $E_{L_f}^s$;*
- *$L_{\tilde{g}}$ is uniformly expanding along $E_{L_f}^c$;*
- *the splitting $E_{L_f}^u \oplus_\prec E_{L_f}^s \oplus_\prec E_{L_f}^c$ is dominated for $L_{\tilde{g}}$.*

Sketch of the proof. By assumption, the rank of $\mathcal{L}(\mathcal{Z}^1(f))$ is 2. By Lemma 2.14, each element (except Id and $-$ Id) in $\mathcal{L}(\mathcal{Z}^1(f))$ is a hyperbolic automorphism on $\mathbb{T}^3$. Hence $\mathcal{L}(\mathcal{Z}^1(f))$ induces a $\mathbb{Z}^2$-linear action on $\mathbb{T}^3$ which is a Cartan action.

By Proposition 2.1, the $\mathbb{Z}^2$-linear action on $\mathbb{T}^3$ given by $\mathcal{L}(\mathcal{Z}^1(f))$ leaves $E_{L_f}^u$, $E_{L_f}^c$ and $E_{L_f}^s$ invariant. Then the Lyapunov exponents along $E_{L_f}^u$, $E_{L_f}^c$ and $E_{L_f}^s$ can be seen as linear functionals defined on $\mathbb{Z}^2$, and we denote them by λ^u, λ^c and λ^s respectively. By Corollary 2.2.14 in [**KN**], the Lyapunov hyperplanes (given by $\ker \lambda^* = 0 \subset \mathbb{R}^2$ with $* = s, c, u$) are in general position and are completely irrational. Since every element in the action has determinant ± 1, one has $\lambda^s \oplus \lambda^c \oplus \lambda^u = 0$. Then using linear algebra one can find $\tilde{g} \in \mathcal{Z}^1(f)$ satisfying the desired properties. $\qquad\square$

A priori, one does not know if the diffeomorphism g obtained in Corollary 3.15 is partially hyperbolic. To conclude, one needs further discussion.

Now, we show that the strong stable and unstable bundles are jointly integrable.

THEOREM 3.16. *Let f be a C^1 partially hyperbolic diffeomorphism on $\mathbb{T}^3$ whose linear part L_f is Anosov. If there exists $g \in \mathcal{Z}^1(f)$ such that $L_g^m \notin \{L_f^n\}_{n \in \mathbb{Z}}$ for any $m \neq 0$, then f is not accessible.*

Proof. Recall that we always assume L_f has two positive Lyapunov exponents. By Corollary 3.15, one can assume that $g \in \mathcal{Z}^1(f)$ satisfies the following properties:

- L_g is contracting along $E_{L_f}^u$ and $E_{L_f}^s$;
- L_g is uniformly expanding along $E_{L_f}^c$;
- the splitting $E_{L_f}^u \oplus_\prec E_{L_f}^s \oplus_\prec E_{L_f}^c$ is dominated for L_g.

Up to replacing f and g by f^2 and g^2, one can assume that f and g preserve the orientation of $E_{L_f}^s, E_{L_f}^c, E_{L_f}^u$. Let $\mathcal{F}^u, \mathcal{F}^s, \mathcal{F}^c$ be the strong unstable, strong stable and center foliations of f, respectively. Let $\mathcal{W}^u, \mathcal{W}^s, \mathcal{W}^c$ be the strong unstable, strong stable and center foliations of L_f, respectively. Their corresponding center stable and center unstable foliations would be denoted by $\mathcal{F}^{cs}, \mathcal{F}^{cu}, \mathcal{W}^{cs}, \mathcal{W}^{cu}$.

Consider a lift F of f to $\mathbb{R}^3$. By Lemma 3.1, there exists $0 < l \le |\det(L_f - \mathrm{Id}_{\mathbb{R}^3})|$ such that g^l admits a lift to $\mathbb{R}^3$ which commutes with F. For simplicity, we will assume that $l = 1$. Let G be the lift of g such that $F \circ G = G \circ F$. By Lemma 3.2, F and G have a common fixed point $p \in \mathbb{R}^3$. Let $H : \mathbb{R}^3 \to \mathbb{R}^3$ be the continuous surjective map given by Corollary 2.8 such that:

- $H \circ F = L_f \circ H$ and $H \circ G = L_g \circ H$;
- $H - \mathrm{Id}_{\mathbb{R}^3}$ is $\mathbb{Z}^3$-periodic.

Then $H(p) = 0$. □

CLAIM 3.17. *G is topologically contracting along the foliation $\tilde{\mathcal{F}}^s$, that is, for any two points x, y on the same $\tilde{\mathcal{F}}^s$-leaf, one has $\lim_{n \to +\infty} d(G^n(x), G^n(y)) = 0$.*

Proof of the claim. By Theorem 2.9, the map H is injective along each leaf of $\tilde{\mathcal{F}}^s$ and sends a leaf of $\tilde{\mathcal{F}}^s$ to a leaf of $\tilde{\mathcal{W}}^s$. Recalling that L_g is uniformly contracting along $\tilde{\mathcal{W}}^s$ and $H \circ G = L_g \circ H$, one deduces that G is topologically contracting along $\tilde{\mathcal{F}}^s$. □

CLAIM 3.18. *There exists $K > 0$ such that for any $x \in \mathbb{R}^3$ and $y \in \tilde{\mathcal{F}}^u(x)$, one has*

$$\limsup_{n \to +\infty} d_{\tilde{\mathcal{F}}^u}(G^n(x), G^n(y)) \le K.$$

Proof of the claim. Recall that $p \in \mathbb{R}^3$ is a fixed point of G and $H(p) = 0$. Since the foliations $\tilde{\mathcal{F}}^{cs}$ and $\tilde{\mathcal{F}}^u$ have the global product structure, the space of $\tilde{\mathcal{F}}^{cs}$-leaves can be identified as $\tilde{\mathcal{F}}^u(p) \cong \mathbb{R}$. Similarly, the space of $\tilde{\mathcal{W}}^{cs}$-leaves of L_f can be identified as $\tilde{\mathcal{W}}^u(0): \mathbb{R}^3/\tilde{\mathcal{F}}^{cs} \cong \tilde{\mathcal{W}}^u(0) \cong \mathbb{R}$.

Thanks to Corollary 2.6, one can consider the action G^{cs}, induced by G, on the space of $\tilde{\mathcal{F}}^{cs}$-leaves. Then one has the following commuting diagram, and G^{cs} can be identified

as the diffeomorphism $G : \tilde{\mathcal{F}}^u(p) \to \tilde{\mathcal{F}}^u(p)$:

$$
\begin{array}{ccc}
\mathbb{R}^3 & \xrightarrow{\;G\;} & \mathbb{R}^3 \\
{\scriptstyle \mathrm{proj}_{cs}}\downarrow & & \downarrow{\scriptstyle \mathrm{proj}_{cs}} \\
\mathbb{R}^3/\tilde{\mathcal{F}}^{cs} & \xrightarrow{\;G^{cs}\;} & \mathbb{R}^3/\tilde{\mathcal{F}}^{cs} \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\
\tilde{\mathcal{F}}^u(p) & \xrightarrow{\;G\;} & \tilde{\mathcal{F}}^u(p)
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{R}^3/\tilde{\mathcal{F}}^{cs} & \xrightarrow{\;F\;} & \mathbb{R}^3/\tilde{\mathcal{F}}^{cs} \\
{\scriptstyle H^{cs}}\downarrow & & \downarrow{\scriptstyle H^{cs}} \\
\mathbb{R}^3/\tilde{\mathcal{W}}^{cs} & \xrightarrow{\;L_f\;} & \mathbb{R}^3/\tilde{\mathcal{W}}^{cs} \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\
\tilde{\mathcal{W}}^u(0) & \xrightarrow{\;L_f\;} & \tilde{\mathcal{W}}^u(0)
\end{array}
$$

By Theorem 2.9, the map H sends a center stable leaf of F to a center stable leaf of L_f which induces a map H^{cs} from the space of $\tilde{\mathcal{F}}^{cs}$-leaves to the space of $\tilde{\mathcal{W}}^{cs}$-leaves.

Since $H^{-1}(x)$ is contained in a single center leaf for any $x \in \mathbb{R}^3$, it follows that H^{cs} is a homeomorphism from the space of $\tilde{\mathcal{F}}^{cs}$-leaves to the space of $\tilde{\mathcal{W}}^{cs}$-leaves. Combining with the fact that $H \circ G = L_g \circ H$, one gets that the homeomorphism $G^{cs} : \tilde{\mathcal{F}}^u(p) \to \tilde{\mathcal{F}}^u(p)$ is conjugate to $L_g : \tilde{\mathcal{W}}^u(0) \to \tilde{\mathcal{W}}^u(0)$:

$$
\begin{array}{ccc}
\mathbb{R}^3/\tilde{\mathcal{F}}^{cs} \cong \tilde{\mathcal{F}}^u(p) & \xrightarrow{\;G^{cs}\;} & \mathbb{R}^3/\tilde{\mathcal{F}}^{cs} \cong \tilde{\mathcal{F}}^u(p) \\
{\scriptstyle H^{cs}}\downarrow & & \downarrow{\scriptstyle H^{cs}} \\
\mathbb{R}^3/\tilde{\mathcal{W}}^{cs} \cong \mathcal{W}^u(0) & \xrightarrow{\;L_g\;} & \mathbb{R}^3/\tilde{\mathcal{W}}^{cs} \cong \mathcal{W}^u(0)
\end{array}
$$

By the choice of g, the linear map L_g is a contracting along $\tilde{\mathcal{W}}^u(0)$; therefore G^{cs} is topologically contracting, as is $G : \tilde{\mathcal{F}}^u(p) \to \tilde{\mathcal{F}}^u(p)$.

Let $x \in \mathbb{R}^3$ and $y \in \tilde{\mathcal{F}}^u(x)$. By the global product structure, the leaves $\tilde{\mathcal{F}}^{cs}(x)$ and $\tilde{\mathcal{F}}^{cs}(y)$ intersect $\tilde{\mathcal{F}}^u(p)$ at unique points $\widehat{x}$ and $\widehat{y}$, respectively. Since $\mathrm{d}(G^n(\widehat{x}), G^n(\widehat{y}))$ tends to 0 when n tends to infinity and the center stable foliation and strong unstable foliations are invariant under G, by Corollary 2.5, there exists a constant $K > 0$ such that

$$
\mathrm{d}_{\tilde{\mathcal{F}}^u}(G^n(x), G^n(y)) \le K + \mathrm{d}(G^n(\widehat{x}), G^n(\widehat{y})).
$$

Letting n tend to $+\infty$, one gets the posited property. $\qquad\square$

Assume, to the contrary, that f is accessible. Lemma 2.19 shows that the lift $F : \mathbb{R}^3 \to \mathbb{R}^3$ is also accessible. For every point $x \in \mathbb{R}^3$, we choose $y \in \mathcal{F}^c(x)$ such that $H(x) \ne H(y)$. Let I^c denote the segment between x and y contained in $\mathcal{F}^c(x)$.

Since F is accessible, there exists a sequence of segments $I_1, I_2, \ldots, I_k$ such that for every $j = 1, \ldots, k$, one has that:

- I_j is contained in a leaf of $\tilde{\mathcal{F}}^s$ or $\tilde{\mathcal{F}}^u$;
- the endpoints of I_j are x_{j-1} and x_j, where $x_0 = x$ and $x_k = y$.

By Claims 3.17 and 3.18,

$$
\max_{j=1,\ldots,k}\{\limsup_{n\to+\infty} \ell(G^n(I_j))\} \le K,
$$

where $\ell(\cdot)$ denotes the length of a C^1 curve. This implies the two endpoints of $G^n(I^c)$ are at uniformly bounded distance.

On the other hand, since $H(x) \neq H(y)$, by the choice of g, one has

$$\lim_{n \to +\infty} \ell(L_g^n \circ H(I^c)) = \lim_{n \to +\infty} \ell(H \circ G^n(I^c)) = +\infty.$$

Since $H - \mathrm{Id}_{\mathbb{R}^3}$ is uniformly bounded on $\mathbb{R}^3$, one has $\lim_{n \to +\infty} \ell(G^n(I^c)) = +\infty$. This contradicts to the quasi-isometric property of the center foliation $\tilde{\mathcal{F}}^c$ given by Theorem 2.2.

Now, we are ready to give the proof of Theorem 3.13.

Proof of Theorem 3.13. By Theorems 2.20 and 3.16, the strong stable and unstable bundles of f are jointly integrable, and f is Anosov. Let h be the homeomorphism such that $h \circ f = L_f \circ h$.

Now, let us fix some notation. Recall that the partially hyperbolic splitting for f is denoted by $E^s \oplus E^c \oplus E^u$ and the partially hyperbolic splitting for L_f is denoted by $E_{L_f}^s \oplus E_{L_f}^c \oplus E_{L_f}^u$. Let $\mathcal{F}^s, \mathcal{F}^u, \mathcal{F}^c$ be the strong stable, strong unstable and center foliations of f respectively, and by Corollary 2.6, these foliations are invariant under each element of $\mathcal{Z}^r(f)$. Let $\mathcal{W}^s, \mathcal{W}^u, \mathcal{W}^c$ be the strong stable, strong unstable and center foliations of L_f, respectively.

Since $E^s \oplus E^u$ is integrable, by Theorem 2.22, one has that:

- the conjugacy h is uniformly C^1 along the center leaves of f;
- h sends the foliation $\mathcal{F}^u$ to the foliation $\mathcal{W}^u$.

In the following, we will show that h is uniformly C^1 along the leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$.

As per the discussion in Corollary 3.15, since the rank of the linearization of whole $\mathcal{Z}^r(f)$ action is 2 (which is full rank), hence $\mathcal{Z}^r(f)$ induces a maximal Cartan affine action on the torus, and there exist diffeomorphisms $g, \hat{g} \in \mathcal{Z}^r(f)$ whose linear parts satisfy the following properties:

- L_g is uniformly expanding along $E_{L_f}^c$, and L_g is uniformly contracting along $E_{L_f}^u \oplus E_{L_f}^s$;
- $L_{\hat{g}}$ is uniformly expanding along $E_{L_f}^s \oplus E_{L_f}^c$, and $L_{\hat{g}}$ is uniformly contracting along $E_{L_f}^u$;
- the splitting $E_{L_f}^u \oplus E_{L_f}^s \oplus E_{L_f}^c$ is dominated for L_g and $L_{\hat{g}}$.

Since both $h \circ g \circ h^{-1}$ and $h \circ \hat{g} \circ h^{-1}$ belong to the centralizer of L_f, by Theorem 2.11, one has that

$$h \circ g \circ h^{-1} = L_g + w_g \quad \text{and} \quad h \circ \hat{g} \circ h^{-1} = L_{\hat{g}} + w_{\hat{g}}$$

are both affine maps on $\mathbb{T}^3$. Recall that h is uniformly C^1 along the center leaves of f. Since $h(\mathcal{F}^c) = \mathcal{W}^c$, L_g and $L_{\hat{g}}$ are uniformly expanding along $E_{L_f}^c$, it follows that

$$g = h^{-1} \circ (L_g + w_g) \circ h \quad \text{and} \quad \hat{g} = h^{-1} \circ (L_{\hat{g}} + w_{\hat{g}}) \circ h$$

are uniformly expanding along E^c.

CLAIM 3.19. *The diffeomorphisms g and $\hat{g}$ are Anosov. To be precise:*

- *g is uniformly contracting along E^u and E^s;*
- *$\hat{g}$ is uniformly contracting along E^u, and uniformly expanding along E^s.*

Proof of the claim. We only prove the case for g (the case for $\widehat{g}$ works analogously), since what we need are the conjugation through h to their linear parts and the Hölder continuity of h along the leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$.

Since g is topologically conjugate to $(L_g + w_g)$ by h, it satisfies the shadowing lemma. In particular, every ergodic measure of g can be approximated by the atomic measures supported on periodic orbits. Thus to prove g is uniformly contracting along E^u and E^s, it suffices to show that the Lyapunov exponents of periodic points of g along E^u and E^s are uniformly smaller than zero.

By the continuity of the bundle E^u, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in \mathbb{T}^3$ with $\mathrm{d}(x, y) < \delta$, one has

$$-\varepsilon \leq \log \|Dg|_{E^u(x)}\| - \log \|Dg|_{E^u(y)}\| \leq \varepsilon. \tag{1}$$

Since g is conjugate to $(L_g + w_g)$ and h sends the foliations $\mathcal{F}^u$, $\mathcal{F}^s$ to the corresponding linear foliations of $(L_g + w_g)$, it follows that g is topologically contracting along the leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$. Let p be a periodic point of period k. Then $g^k|_{\mathcal{F}^u(p)} : \mathcal{F}^u(p) \to \mathcal{F}^u(p)$ is topologically contracting and has a unique fixed point p. Let $x \in \mathcal{F}^u_\delta(p)$. Then $g^{nk}(x) \in \mathcal{F}^u_\delta(p)$ for any $n \in \mathbb{N}$. By equation (1), one has

$$\exp((\chi^u(p) - \varepsilon)nk) \cdot \mathrm{d}(x, p) \leq \mathrm{d}(g^{nk}(x), g^{nk}(p))$$
$$\leq \exp((\chi^u(p) + \varepsilon)nk) \cdot \mathrm{d}(x, p), \tag{2}$$

where $\chi^u(p)$ is the Lyapunov exponent of p for g along the direction E^u. Since $h \circ g = (L_g + w_g) \circ h$, one has

$$\mathrm{d}(h \circ g^{nk}(x), h \circ g^{nk}(p)) = \mathrm{d}((L_g + w_g)^{nk}(h(x)), (L_g + w_g)^{nk}(h(p))) \tag{3}$$
$$= \exp(\chi^u(L_g) \cdot nk) \cdot \mathrm{d}(h(x), h(p)),$$

where $\chi^u(L_g)$ is the Lyapunov exponent of L_g along $E^u_{L_f}$, which is the same as the Lyapunov exponent of $(L_g + w_g)$ along $E^u_{L_f}$.

By Remark 2.23, the map h is uniformly Hölder continuous along the leaves of $\mathcal{F}^u$, that is, there exist $C, \alpha > 0$ such that for any two points x_1, x_2 on the same $\mathcal{F}^u$-leaf, one has $\mathrm{d}_{\mathcal{F}^u}(x_1, x_2) \leq C \cdot (\mathrm{d}_{\mathcal{W}^u}(h(x_1), h(x_2)))^\alpha$. Hence one has

$$\mathrm{d}(hg^{nk}(x), hg^{nk}(p)) \geq C^{-1/\alpha} \cdot (\mathrm{d}(g^{nk}(x), g^{nk}(p)))^{1/\alpha}.$$

Then, combining with Equations (2) and (3), for any $n \in \mathbb{N}$, one has

$$C^{-1/\alpha} \cdot \exp((\chi^u(p) - \varepsilon)nk/\alpha) \cdot (\mathrm{d}(x, p))^{1/\alpha} \leq \exp(\chi^u(L_g)nk) \cdot \mathrm{d}(h(x), h(p)),$$

which implies that $\chi^u(p) - \varepsilon \leq \alpha \cdot \chi^u(L_g)$. The arbitrariness of ε and p gives that the Lyapunov exponents of periodic points of g along E^u are uniformly bounded away from zero. Analogous argument gives that the Lyapunov exponents of periodic points of g along E^s are uniformly bounded away from zero. Hence g is also Anosov. $\square$

By [**PiRa**], which states that the codimension-one stable (or unstable) foliation of a $C^r(r > 1)$ codimension-one Anosov diffeomorphism is C^1-smooth, one has that:

- the unstable foliation of f, which is tangent to $E^c \oplus E^u$, is C^1-smooth;
- the stable foliation of g, which is tangent to $E^u \oplus E^s$, is C^1-smooth;
- the unstable foliation of $\widehat{g}$, which is tangent to $E^s \oplus E^c$, is C^1-smooth.

As a consequence, the foliations $\mathcal{F}^s$, $\mathcal{F}^c$, $\mathcal{F}^u$ are C^1. Now, both g and $\widehat{g}$ satisfy the assumption of Theorem 2.24, hence h is uniformly C^1 along the leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$. As f, g, $\widehat{g}$ are Anosov diffeomorphisms, the leaves of $\mathcal{F}^s$, $\mathcal{F}^c$, $\mathcal{F}^u$ are C^r. By Lemma 2.17, the map h is C^r along the leaves of $\mathcal{F}^s$, $\mathcal{F}^c$, $\mathcal{F}^u$. Finally, Journé's theorem [**J**] shows that $h \in \mathrm{Diff}^{r-\varepsilon}(\mathbb{T}^3)$ for any $\varepsilon > 0$. $\qquad\square$

A. *Appendix. Proof of Theorem 2.24.*

The aim of this section is to give the proof of Theorem 2.24 which essentially follows the argument in §A of [**GS**]. See also [**Go1**].

Proof of Theorem 2.24. Assume, to the contrary, that there exist two periodic orbits whose Lyapunov exponents along the bundle E^c are different. For each periodic point p, we denote by $\chi^c(p)$ the Lyapunov exponent of p along the direction E^c. As g is Anosov, periodic measures (atomic probability measures equidistributed on a single periodic orbit) are dense among invariant measures [**Sig**]. Since g is uniformly expanding along the continuous bundle E^c, by the convexity of the set of invariant measures, there exist $0 < \chi_1 < \chi_2$ such that $\overline{\{\chi^c(p) : p \text{ is a periodic point}\}} = [\chi_1, \chi_2]$. By the shadowing lemma, for each point $x \in \mathbb{T}^3$, one has

$$\chi_1 \leq \liminf_{n \to \infty} \frac{1}{n} \log \|Dg^n|_{E^c(x)}\| \leq \limsup_{n \to \infty} \frac{1}{n} \log \|Dg^n|_{E^c(x)}\| \leq \chi_2.$$

Hence for any $\varepsilon > 0$, there exists an adapted metric $\|\cdot\|_\varepsilon$ such that

$$\chi_1 - \varepsilon \leq \log \|Dg|_{E^c(x)}\|_\varepsilon \leq \chi_2 + \varepsilon, \quad \text{for any } x \in \mathbb{T}^3.$$

By the continuity of the bundle E^c, there exists $\delta > 0$ such that for any $z_1, z_2 \in \mathbb{T}^3$ with $d(z_1, z_2) < 3\delta$, one has

$$-\varepsilon < \log \|Dg|_{E^c(z_1)}\|_\varepsilon - \log \|Dg|_{E^c(z_2)}\|_\varepsilon < \varepsilon.$$

Now, for $\varepsilon > 0$ small (which will be fixed later), one fixes periodic points p, q such that $\chi^c(p) \leq \chi_1 + \varepsilon$ and $\chi^c(q) \geq \chi_2 - \varepsilon$.

For the linear Anosov map L_g, let us denote by $\mathcal{W}^s, \mathcal{W}^c, \mathcal{W}^u, \mathcal{W}^{su}$ the foliations tangent to $E^s_{L_g}, E^c_{L_g}, E^u_{L_g}, E^s_{L_g} \oplus E^u_{L_g}$, respectively.

Since the factors of unit eigenvectors of L_g are algebraic, there exists $C_1 > 1$ such that for any $l > 0$ large, each strong unstable segment of L_g with length l is $C_1/\sqrt{l}$ dense in $\mathbb{T}^3$. For any $l > 0$ large, there exist $x, y \in \mathbb{T}^3$ such that

$$x \in \mathcal{W}^u_l(h(p)) \quad \text{and} \quad y \in \mathcal{W}^s_{C_1/\sqrt{l}}(x) \cap \mathcal{W}^c_{C_1/\sqrt{l}}(h(q)). \tag{4}$$

By the continuity of h, there exists $\eta > 0$ such that for any center segment for L_g of length no more than η, its pre-image under h has length no more than δ. We will use $\ell(I)$ denote the length of a C^1-curve. As $\mathcal{W}^{su}$ is a linear foliation, the holonomy map given by $\mathcal{W}^{su}$ is an isometry. Now, one chooses $\mathcal{W}^c$-center segments I_x and $I_{h(p)}$ such that:

- $\ell(I_x) = \ell(I_{h(p)}) = \eta$;
- x is an endpoint of I_x and $h(p)$ is an endpoint of $h(p)$;
- I_x is an image of $I_{h(p)}$ under the holonomy map $\mathcal{W}^{su}$.

Then $J_{\widehat{x}} = h^{-1}(I_x)$ and $J_p = h^{-1}(I_{h(p)})$ are segments tangent to E^c with length no more than δ, and J_p is the image of $J_{\widehat{x}}$ under a holonomy map of $\mathcal{F}^{su}$, where $\widehat{x} = h^{-1}(x)$.

By Remark 2.23, the homeomorphism h is uniformly Hölder continuous along the leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$, hence there exist constants $C_2 > 1$ and $\theta \in (0, \frac{1}{2})$ which is only determined by h such that $\widehat{x} \in \mathcal{F}^u_d(p)$ and $\widehat{y} \in \mathcal{F}^s_{C_2/d^\theta}(\widehat{x}) \cap \mathcal{F}^c_\delta(q)$ for d large, due to equation (4). As l can be chosen arbitrarily large, so is d.

As g is uniformly expanding and contracting along E^u and E^s respectively, let us denote

$$\tau = \sup_{x \in \mathbb{T}^3} \|Dg^{-1}|_{E^u(x)}\|_\varepsilon < 1 \quad \text{and} \quad \kappa = \sup_{x \in \mathbb{T}^3} \|Dg^{-1}|_{E^s(x)}\|_\varepsilon > 1.$$

Let N_d be the smallest integer such that $\tau^{N_d} \cdot d \leq 1$. Then $N_d \leq -(\log d/\log \tau) + 1$. Let N^1_d be the largest integer such that

$$\kappa^{N^1_d} C_2/d^\theta \leq \delta,$$

which implies that $g^{-N^1_d}(J_{\widehat{x}})$ is contained in the 3δ-neighborhood of q. Then for d large enough, one has the following estimate:

$$\begin{aligned}
\frac{N^1_d}{N_d} &\geq \frac{\theta \log d + \log \delta - \log C_2 - 1}{-(\log d/\log \tau) + 1} \cdot \frac{1}{\log \kappa} \\
&= \frac{\theta \log d + \log \delta - \log C_2 - 1}{\log d - \log \tau} \cdot \frac{-\log \tau}{\log \kappa} \\
&\geq \frac{\theta}{2} \frac{-\log \tau}{\log \kappa}.
\end{aligned}$$

As g is uniformly expanding along E^c, by the choices of J_p and δ, one has

$$\exp(-N_d(\chi^c(p) + \varepsilon)) \cdot \ell(J_p) \leq \ell(g^{-N_d}(J_p))$$

and

$$\ell(g^{-N_d}(J_x)) \leq \ell(J_x) \cdot \exp(-N^1_d(\chi^c(q) - \varepsilon)) \cdot \exp((N^1_d - N_d) \cdot (\chi_1 - \varepsilon)).$$

Then one has

$$
\begin{aligned}
\frac{\ell(g^{-N_d}(J_x))}{\ell(g^{-N_d}(J_p))} &\leq \frac{\ell(J_x)}{\ell(J_p)} \cdot \exp(-N_d^1(\chi^c(q) - \varepsilon)) \\
&\quad \cdot \exp((N_d^1 - N_d)(\chi_1 - \varepsilon)) \cdot \exp(N_d(\chi^c(p) + \varepsilon)) \\
&\leq \frac{\ell(J_x)}{\ell(J_p)} \cdot \exp(4N_d\varepsilon) \cdot \exp(N_d^1(\chi^c(p) - \chi^c(q))) \\
&\quad \cdot \exp((-N_d^1 + N_d) \cdot (\chi^c(p) - \chi_1)) \\
&\leq \frac{\ell(J_x)}{\ell(J_p)} \cdot \exp(8N_d\varepsilon) \cdot \exp(N_d^1(\chi_1 - \chi_2)) \\
&\leq \frac{\ell(J_x)}{\ell(J_p)} \cdot \exp(8\varepsilon N_d) \cdot \exp\left((\chi_1 - \chi_2) \cdot \frac{\theta}{2} \frac{(-\log \tau)}{\log \kappa} \cdot N_d \right).
\end{aligned}
$$

One only needs to choose $\varepsilon = (1/16)(\chi_2 - \chi_1) \cdot (\theta/-\log \tau)/2\log \kappa$, and one gets that $\ell(g^{-N_d}(J_x))/\ell(g^{-N_d}(J_p))$ tends to 0 when d tends to infinity. The holonomy map given by the foliation $\mathcal{F}^u$ restricted to the unstable foliation of f is uniformly C^1; therefore $\ell(g^{-N_d}(J_x))/\ell(g^{-N_d}(J_p))$ is uniformly bounded from above and below, which gives the contradiction. This proves that all the periodic points have the same Lyapunov exponent along the bundle E^c. Applying [**GS**], one gets a periodic point whose Lyapunov exponent along E^c is the same as the center Lyapunov exponent of L_f. Finally, we apply [**GS**], which shows that h is C^1 along each leaf of $\mathcal{F}^c$. $\qquad\square$

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