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No steady water waves of small amplitude are supported by a shear flow with a still free surface

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The two-dimensional free-boundary problem describing steady gravity waves with vorticity on water of finite depth is considered. It is proved that no small-amplitude waves are supported by a horizontal shear flow whose free surface is still, that is, it is stagnant in a coordinate frame such that the flow is time-independent in it. The class of vorticity distributions for which such flows exist includes all positive constant distributions, as well as linear and quadratic ones with arbitrary positive coefficients.

Key words: channel flow, surface gravity waves, waves/free-surface flows

1. Introduction

We consider the two-dimensional nonlinear problem of steady waves in a horizontal open channel that has uniform rectangular cross-section and is occupied by an inviscid incompressible heavy fluid, say, water. The water motion is assumed to be rotational, which, according to observations, is the type of motion commonly occurring in nature (see e.g. Thomas 1990; Swan, Cummins & James 2001, and references therein). There are two essential features that distinguish this type of motion from the irrotational one. The first is that interior stagnation points and closed streamlines exist for some rotational flows with waves (see e.g. Wahlén 2009). Secondly, any set of stagnation points on the free surface of irrotational waves consists only of isolated points, whereas no such points occur on the surface of a uniform stream. On the contrary, there are shear flows for which these points fill up the whole free surface in the rotational case (see Kozlov & Kuznetsov 2011*b*), and the present work deals just with this case. It is also worth mentioning that properties of shear flows might be completely different when the corresponding vorticity distributions are of the same type, but have opposite signs. In particular, shear flows are unidirectional (like irrotational uniform streams) for any *negative* linear vorticity distribution; whereas if a linear vorticity distribution is *positive*, then there are shear flows having as many counter-currents as one pleases. A brief characterization of results obtained for the problem under consideration and a similar one dealing with waves on water of infinite depth is given in Kozlov & Kuznetsov (2012). Further details can be found in the survey article by Strauss (2010).

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In the present paper, our aim is to prove that no small-amplitude waves are supported by a horizontal shear flow whose free surface is still (in other words, it is stagnant in a coordinate frame in which the flow is time-independent). The reason for undertaking this study is as follows. Both versions of bifurcation theory – one used by Ehrnström, Escher & Wahlén (2011) in the case of constant vorticity, and the other one developed by Kozlov & Kuznetsov (2012) for general sufficiently smooth vorticity distributions – correctly describe the bifurcation of small-amplitude waves from any shear flow with non-stagnant free surface, but fail when it is stagnant. All steady flows with horizontal free surfaces are investigated in detail in Kozlov & Kuznetsov (2011*b*) provided their stream functions depend on the vertical coordinate only. Furthermore, the above-mentioned existence proof for Stokes waves with general vorticity is based on a dispersion equation introduced and investigated in Kozlov & Kuznetsov (2012). Thus, the results obtained here are complementary to those in the latter paper. It is also worth mentioning that the case considered here that deals with the absence of waves is essentially distinguished from that when waves do not arise on the free surface of the critical irrotational flow (see Kozlov & Kuznetsov (2008), theorem 1(i); the latter result complements the proof of the Benjamin–Lighthill conjecture for the near-critical case obtained in Kozlov & Kuznetsov (2010, 2011*a*)). Further details concerning the hydrodynamic interpretation of the present result are given in § 3.

As in Kozlov & Kuznetsov (2011*b*, 2012), no assumption is made about the absence of counter-currents in a shear flow. Moreover, we impose no restriction on the type of waves; they may be solitary, periodic with an arbitrary number of crests per period, whatever. However, the slope of the free surface profile is supposed to be bounded by a constant given *a priori*. Also, certain conditions that will be described later are imposed on the vorticity distribution.

1.1. *Statement of the problem*

Let an open channel of uniform rectangular cross-section be bounded below by a horizontal rigid bottom and let water occupying the channel be bounded above by a free surface not touching the bottom. The surface tension is neglected and the pressure is constant on the free surface. The water motion is supposed to be two-dimensional and rotational, which, combined with the incompressibility of water, allows us to seek the velocity field in the form $(\psi_y, -\psi_x)$, where $\psi(x, y)$ is referred to as the *stream function* (see e.g. the book by Lavrentiev & Shabat (1980)). It is also supposed that the vorticity distribution ω (which is a function of ψ as explained in § 1 of the cited book) is a prescribed Lipschitz function on $\mathbb{R}$ subject to some conditions (see (1.6) and (1.8) below).

We use non-dimensional variables chosen so that the constant volume rate of flow per unit span and the constant acceleration due to gravity are scaled to unity in our equations. For this purpose, lengths and velocities are scaled to $(Q^2/g)^{1/3}$ and $(Qg)^{1/3}$, respectively; here Q and g are the dimensional quantities for the rate of flow and the gravitational acceleration, respectively. We recall that $(Q^2/g)^{1/3}$ is the depth of the critical uniform stream in the irrotational case (see e.g. Benjamin 1995).

In appropriate Cartesian coordinates (x, y), the bottom coincides with the x-axis and gravity acts in the negative y-direction. We choose the frame of reference so that the velocity field is time-independent, as well as the unknown free-surface profile. The latter is assumed to be the graph of $y = \eta(x)$, $x \in \mathbb{R}$, where η is a positive C^1-function. Therefore, the longitudinal section of the water domain is $D = \{x \in \mathbb{R},\ 0 < y < \eta(x)\}$, and ψ is assumed to belong to $C^2(D) \cap C^1(\bar{D})$.

Since the surface tension is neglected, ψ and η must satisfy the following free-boundary problem:

$$\psi_{xx} + \psi_{yy} + \omega(\psi) = 0, \quad (x, y) \in D; \tag{1.1}$$

$$\psi(x, 0) = 0, \quad x \in \mathbb{R}; \tag{1.2}$$

$$\psi(x, \eta(x)) = 1, \quad x \in \mathbb{R}; \tag{1.3}$$

$$|\nabla \psi(x, \eta(x))|^2 + 2\eta(x) = 3r, \quad x \in \mathbb{R}. \tag{1.4}$$

Here r is a constant considered as the problem's parameter and referred to as the total head (see e.g. Keady & Norbury (1978)). This statement has long been known and its derivation from the governing equations and the assumptions about the boundary behaviour of water particles can be found, for example, in Constantin & Strauss (2004).

Note that the boundary condition (1.3) yields that relation (1.4) (Bernoulli's equation) can be written as follows:

$$[\partial_n \psi(x, \eta(x))]^2 + 2\eta(x) = 3r, \quad x \in \mathbb{R}. \tag{1.5}$$

Here and below ∂_n denotes the normal derivative on ∂D, and the normal $n = (n_x, n_y)$ has unit length and points out of D.

1.2. *Assumptions and the result*

We begin with the conditions that are imposed on the vorticity distribution ω in our main theorem. Let r_c denote the critical value of r for ω (see Kozlov & Kuznetsov (2011*b*, p. 386) for its definition). The role of r_c is analogous to the total head of the critical stream in the irrotational case; that is, for $r < r_c$, problem (1.1)–(1.4) has no solutions of the form $(U(y), h)$, where $h = \text{const.}$ (they are referred to as *stream solutions* and describe shear flows). First, we require that

$$\text{for some } r > r_c, \text{ problem (1.1)–(1.4) has a stream solution for which } U_y(h) = 0. \tag{1.6}$$

This implies that $r = 2h/3$ in (1.4). Thus the Bernoulli constant for which we are going to consider our problem is expressed in terms of the depth of the corresponding shear flow with stagnant free surface. In Kozlov & Kuznetsov (2011*b*), it is proved that a finite number, say, $n \geqslant 1$, of stream solutions $(U^{(j)}, h^{(j)})$, $j = 1, \ldots, n$, exists for the same r, but for them we have $h^{(j)} \neq h$ and, what is more important, $U_y^{(j)}(h^{(j)}) \neq 0$.

Note that, if some pair (ψ, η) satisfies problem (1.1)–(1.4) for the same r as (U, h), then the last equality yields that equation (1.5) for (ψ, η) takes the form

$$[\partial_n \psi(x, \eta(x))]^2 = 2[h - \eta(x)], \quad x \in \mathbb{R}. \tag{1.7}$$

Hence $h - \eta(x) \geqslant 0$, which means that, if there exists a wavy flow perturbing the shear one of the depth h, then the free surface of waves lies under the level $y = h$.

The second restriction that we impose on ω is as follows:

$$\mu = \operatorname*{ess\,sup}_{\tau \in (-\infty, \infty)} \omega'(\tau) < \frac{\pi^2}{h^2}. \tag{1.8}$$

This bound for μ is equal to the fundamental Dirichlet eigenvalue for the operator $-\mathrm{d}^2/\mathrm{d}^2 y$ on the interval $(0, h)$. As in Keady & Norbury (1978), where a similar condition was introduced, inequality (1.8) is essential for the validity of a certain version of the maximum principle. It holds for domains close to a strip of constant width and is applied in the proof of lemma 2, whereas our proof of the main result is based on this lemma.

Now we are in a position to formulate the following.

MAIN THEOREM. *Let the vorticity distribution ω satisfy (1.6) and (1.8). Then for any $B>0$ there exists $\varepsilon(\mu,h,B)>0$ such that every solution (ψ,η) of problem (1.1)–(1.4) corresponding to the same r as (U,h) coincides with the latter one if*

$$|\eta_x(x)| \leqslant B \quad \text{and} \quad h-\eta(x)<\varepsilon \quad \text{for all } x\in\mathbb{R}. \tag{1.9}$$

The first and second inequalities (1.9) mean that the wave profile η has bounded slope and sufficiently small amplitude, respectively.

2. Proof of the main theorem

Our proof is based on two lemmas. In the first, we estimate the normal derivative of a solution satisfying an auxiliary boundary value problem in the domain D. In the second, some particular perturbation of the stream function is estimated through the perturbation of the free surface profile. This requires the problem to be reformulated in terms of perturbations prior to formulating and proving lemmas.

2.1. *Reformulation of the problem*

First, we consider problem (1.1)–(1.4) as a perturbation of that for (U,h) and write the problem for

$$\phi(x,y)=\psi(x,y)-U(y) \quad \text{and} \quad \zeta(x)=h-\eta(x), \tag{2.1}$$

which is as follows:

$$\nabla^2\phi+\omega(U+\phi)-\omega(U)=0, \quad (x,y)\in D, \quad \nabla=(\partial_x,\partial_y); \tag{2.2}$$
$$\phi(x,0)=0, \quad x\in\mathbb{R}; \tag{2.3}$$
$$\phi(x,h-\zeta(x))=1-U(h-\zeta(x)), \quad x\in\mathbb{R}; \tag{2.4}$$
$$\left[\partial_n\phi+\frac{U_y(y)}{(1+\zeta_x^2)^{1/2}}\right]^2_{y=h-\zeta(x)}=2\zeta(x), \quad x\in\mathbb{R}. \tag{2.5}$$

The last condition is a consequence of (1.5) and yields that ζ is a non-negative function. Thus, our aim is to show that the ϕ and ζ that satisfy this problem vanish if the condition (1.8) is fulfilled for ω.

In order to simplify the boundary condition (2.4), we put

$$v(x,y)=\phi(x,y)-u(x,y) \quad \text{where } u(x,y)=[1-U(h-\zeta(x))]\frac{y}{h-\zeta(x)}, \tag{2.6}$$

thus obtaining the following problem for v and ζ:

$$\nabla^2 v+\omega(U+u+v)=\omega(U)-\nabla^2 u, \quad (x,y)\in D; \tag{2.7}$$
$$v(x,0)=0, \quad x\in\mathbb{R}; \tag{2.8}$$
$$v(x,h-\zeta(x))=0, \quad x\in\mathbb{R}; \tag{2.9}$$
$$\left[\frac{\partial_n v}{(1+\zeta_x^2)^{1/2}}+\frac{1-U(y)}{y}+\frac{\zeta_x^2\,U_y(y)}{1+\zeta_x^2}\right]^2_{y=h-\zeta(x)}=\frac{2\,\zeta(x)}{1+\zeta_x^2}, \quad x\in\mathbb{R}. \tag{2.10}$$

Now we list a couple of properties that will be used below. If ζ is small enough, then Taylor's theorem and the chain rule of differentiation immediately yield the inequalities

$$|u(x,y)| \leqslant C\,[\zeta(x)]^2, \quad |u_y(x,y)| \leqslant C\,[\zeta(x)]^2 \quad \text{and} \quad |u_x(x,y)| \leqslant C\,|\zeta_x(x)|\,\zeta(x). \tag{2.11}$$

In the first and second of these, the constant C depends on the stream solution (U,h), whereas the constant is absolute in the last inequality. Hence the conditions imposed on ω yield that $|v|$ is bounded on $\bar{D}$.

2.2. *Two lemmas*

For the convenience of the reader we recall the definitions of functional spaces used in what follows. (An elementary description of their properties can be found, for example, in Michlin (1978), part 1.) Let (a,b) be a finite subinterval of $\mathbb{R}$, then $\|f\|^p_{L^p(a,b)} = \int_a^b |f(x)|^p \,\mathrm{d}x$ for $p \geqslant 1$ and $\|f\|^2_{W^{1,2}(a,b)} = \int_a^b (|f(x)|^2 + |f'(x)|^2)\,\mathrm{d}x$, whereas $f \in W^{1,2}_{loc}(\mathbb{R})$ provided $f \in W^{1,2}(a,b)$ for any (a,b). Furthermore, $\|f\|^2_{L^2(D_t)} = \int_{D_t} |f(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y$, where

$$D_t = \{(x,y) : x \in (t-1, t+2),\ y \in (0, \eta(x)\} \tag{2.12}$$

and $t \in \mathbb{R}$ is arbitrary; $f \in L^2_{loc}(D)$ provided $f \in L^2(K)$ for any domain $K \subset D$ with a compact closure. Finally, $\|f\|^2_{W^{1,2}(D_t)} = \|f\|^2_{L^2(D_t)} + \int_{D_t} |\nabla f|^2 \,\mathrm{d}x\,\mathrm{d}y$.

LEMMA 1. *Let $y = \eta(x)$ be a fixed curve such that the first condition (1.9) is fulfilled. Let also $\eta(x) \geqslant h_-$ for all x, where h_- is some positive constant. If w is a solution of the problem*

$$\nabla^2 w = f,\ (x,y) \in D, \quad w(x,0) = 0,\ x \in \mathbb{R}, \quad w(x,\eta(x)) = H,\ x \in \mathbb{R}, \tag{2.13}$$

with $f \in L^2_{loc}(D)$ and $H \in W^{1,2}_{loc}(\mathbb{R})$, then for every $t \in \mathbb{R}$ the following estimate holds:

$$\left\| \partial_n w|_{y=\eta(x)} \right\|_{L^2(t,t+1)} \leqslant C\left[\|f\|_{L^2(D_t)} + \|H\|_{W^{1,2}(t-1,t+2)} + \|w\|_{W^{1,2}(D_t)}\right], \tag{2.14}$$

where the constant C does not depend on f, H and t.

Proof. By χ we denote a smooth cut-off function such that $\chi(x) = 1$ for $x \in (t, t+1)$, $\chi(x) = 0$ for $x \in (-\infty, t-1/2) \cup (t+3/2, +\infty)$ and $0 \leqslant \chi(x) \leqslant 1$ for all x. Let us multiply the equality

$$\nabla^2(\chi w) = \chi f + w\nabla^2\chi + 2\,\nabla w \cdot \nabla\chi \tag{2.15}$$

by $(\chi w)_y$ and integrate over D, thus obtaining

$$-\frac{1}{2}\int_D (|\nabla(\chi w)|^2)_y \,\mathrm{d}x\,\mathrm{d}y + \int_{\partial D} (\chi w)_y\ \partial_n(\chi w)\,\mathrm{d}s$$
$$= \int_D (\chi f + w\nabla^2\chi + 2\,\nabla w \cdot \nabla\chi)\,(\chi w)_y\ \mathrm{d}x\,\mathrm{d}y. \tag{2.16}$$

The expression on the left-hand side arises after applying the first Green's formula; ds stands for an element of the arclength. Introducing ∂_t so that $\nabla = (\partial_t, \partial_n)$ on $y = \eta(x)$, we transform the left-hand side as follows:

$$
\begin{aligned}
&-\frac{1}{2}\int_{-\infty}^{+\infty}\left[|\nabla(\chi w)|^2\right]_{y=0}^{y=\eta(x)}\,\mathrm{d}x\\
&\quad+\int_{-\infty}^{+\infty}\left[(n_y\partial_n-n_x\partial_t)(\chi w)\,\partial_n(\chi w)\right]_{y=\eta(x)}\sqrt{1+\eta_x^2}\,\mathrm{d}x-\int_{-\infty}^{+\infty}\left[(\chi w)_y^2\right]_{y=0}\mathrm{d}x\\
&=\int_{-\infty}^{+\infty}\left(n_y\sqrt{1+\eta_x^2}-\frac{1}{2}\right)[\partial_n(\chi w)]^2_{y=\eta(x)}\mathrm{d}x\\
&\quad-\int_{-\infty}^{+\infty}\left[n_x\sqrt{1+\eta_x^2}\,\partial_t(\chi w)\,\partial_n(\chi w)+\frac{1}{2}\,|\partial_t(\chi w)|^2\right]_{y=\eta(x)}\mathrm{d}x\\
&\quad-\int_{-\infty}^{+\infty}\left[(\chi w)_y^2\right]_{y=0}\mathrm{d}x.
\end{aligned}
\tag{2.17}
$$

We substitute the last expression into (2.16) and take into account that

$$
n_x\sqrt{1+\eta_x^2}=-\eta_x, \tag{2.18}
$$

whereas the first factor in the first integrand is equal to $1/2$. Then we arrive, after rearranging terms and multiplying by 2, at the following equality:

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}[\partial_n(\chi w)]^2_{y=\eta(x)}\mathrm{d}x\\
&\quad=\int_{-\infty}^{+\infty}\left[|\partial_t(\chi w)|^2-2\,\eta_x\,\partial_t(\chi w)\,\partial_n(\chi w)\right]_{y=\eta(x)}\mathrm{d}x\\
&\qquad+\int_{-\infty}^{+\infty}\left[(\chi w)_y^2\right]_{y=0}\mathrm{d}x+2\int_D(\chi f+w\nabla^2\chi+2\,\nabla w\cdot\nabla\chi)\,(\chi w)_y\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
\tag{2.19}
$$

Since the left-hand side in (2.14) is less than that in the last equality, it is sufficient to estimate each term on the right-hand side with proper constants in order to complete the proof of the required inequality (2.14).

First, we have that

$$
\begin{aligned}
&\left|\int_{-\infty}^{+\infty}[\eta_x\,\partial_t(\chi w)\,\partial_n(\chi w)]_{y=\eta(x)}\mathrm{d}x\right|\\
&\qquad\leqslant\frac{1}{4}\int_{-\infty}^{+\infty}[\partial_n(\chi w)]^2_{y=\eta(x)}\mathrm{d}x+4\,B^2\int_{-\infty}^{+\infty}[\partial_t(\chi w)]^2_{y=\eta(x)}\mathrm{d}x,
\end{aligned}
\tag{2.20}
$$

because $y=\eta(x)$ satisfies the first condition (1.9). Furthermore, the assumption that $\eta(x)\geqslant h_-$ for all x, where the constant $h_->0$, allows us to apply the general theory of elliptic boundary value problems (see e.g. Agmon, Douglis & Nirenberg 1959), from which it follows that

$$
\int_{-\infty}^{+\infty}\left[(\chi w)_y^2\right]_{y=0}\mathrm{d}x\leqslant C\left[\|f\|_{L^2(D_t)}+\|H\|_{W^{1,2}(t-1,t+2)}+\|w\|_{W^{1,2}(D_t)}\right], \tag{2.21}
$$

where C depends only on h_-. Finally, using the Schwarz and Cauchy inequalities, one readily obtains that the absolute value of the integral over D is estimated by the right-hand side in the last inequality. □

Applying lemma 1 to problem (2.2)–(2.4) (we are able to do this because ω is a Lipschitz function), we obtain the following corollary.

Corollary 1. *If ϕ is defined by the first formula (2.1), then the estimate (2.14) for ϕ takes the form:*

$$\left\| \partial_n \phi|_{y=h-\zeta(x)} \right\|_{L^2(t,t+1)} \leqslant C \left[\|\phi\|_{W^{1,2}(D_t)} + \|1 - U(h-\zeta)\|_{W^{1,2}(t-1,t+2)} \right]. \tag{2.22}$$

Moreover, the last term in the square brackets does not exceed $C(\varepsilon + B)\, \|\zeta\|_{L^2(t-1,t+2)}$ provided conditions (1.9) are fulfilled.

LEMMA 2. *Let the conditions imposed on ω in the main theorem be fulfilled. If ζ is sufficiently small and $|\zeta_x| \leqslant B$ for some $B > 0$, then there exist $\delta > 0$, depending on $(\pi/h)^2 - \mu$, h and B, and $C_\delta > 0$ such that the inequality*

$$\int_D \mathrm{e}^{-\delta|t-x|} (v^2 + |\nabla v|^2)\, \mathrm{d}x\, \mathrm{d}y \leqslant C_\delta \int_{-\infty}^{+\infty} \mathrm{e}^{-\delta|t-x|} \zeta^2 \left(\zeta^2 + \zeta_x^2\right) \mathrm{d}x \tag{2.23}$$

holds for every function v satisfying relations (2.7)–(2.9) and all $t \in \mathbb{R}$.

Proof. Let $\chi_1(x)$ denote a cut-off function equal to unity on $(-1, 1)$ and vanishing for $|x| > 2$, whereas $\chi_N(x) = \chi_1(x/N)$. We write (2.2) in the form

$$\nabla^2 v + \omega(U + u + v) - \omega(U + u) = \omega(U) - \omega(U + u) - \nabla^2 u, \tag{2.24}$$

multiply it by $-v(x)\chi_N(x-t)/\cosh\delta(x-t)$ with some $\delta > 0$, and integrate over D. After applying the first Green's formula and integrating by parts on the left-hand side, we arrive at the following equality:

$$\begin{aligned} &\int_D \left\{ \frac{\chi_N(x-t)}{\cosh\delta(x-t)} \left(|\nabla v|^2 - v \int_0^v \omega'(U + u + \tau)\, \mathrm{d}\tau \right) - \frac{v^2}{2} \left[\frac{\chi_N(x-t)}{\cosh\delta(x-t)} \right]_{xx} \right\} \mathrm{d}x\, \mathrm{d}y \\ &\quad = \int_D \frac{\chi_N(x-t)}{\cosh\delta(x-t)}\, v \left[\nabla^2 u + \omega(U + u + v) - \omega(U) \right] \mathrm{d}x\, \mathrm{d}y. \end{aligned} \tag{2.25}$$

Here the boundary conditions (2.8) and (2.9) are also taken into account.

Using assumption (1.8), we get that the absolute value of the left-hand side is greater than or equal to

$$\begin{aligned} &\int_D \left\{ \frac{\chi_N(x-t)}{\cosh\delta(x-t)} \left[|\nabla v|^2 - (\mu + 3\,\delta^2)\, v^2 \right] \right. \\ &\quad \left. - \frac{v^2}{2} \left| \frac{\chi_N''(x-t)}{\cosh\delta(x-t)} + 2\, \chi_N'(x-t)\, [1/\cosh\delta(x-t)]' \right| \right\} \mathrm{d}x\, \mathrm{d}y, \end{aligned} \tag{2.26}$$

because $|\,(1/\cosh\delta x)''\,| \leqslant 3\,\delta^2/\cosh\delta x$. Furthermore, we have that

$$\int_0^{h-\zeta} v_y^2\, \mathrm{d}y \geqslant \delta^2 \int_0^{h-\zeta} v_y^2\, \mathrm{d}y + (1 - \delta^2)\, (\pi/h)^2 \int_0^{h-\zeta} v^2\, \mathrm{d}y, \tag{2.27}$$

which gives that the integral in the first line of (2.26) is estimated from below by the following expression:

$$\int_D \frac{\chi_N(x-t)}{\cosh\delta(x-t)} \left\{ \left(v_x^2 + \delta^2 v_y^2 \right) + \left[(1 - \delta^2) \left(\frac{\pi}{h} \right)^2 - \mu - 3\delta^2 \right] v^2 \right\} \mathrm{d}x\, \mathrm{d}y. \tag{2.28}$$

In view of assumption (1.8), the number in the square brackets is positive provided δ is chosen sufficiently small.

Now we turn to estimating from above the absolute value of the right-hand side in (2.25). First, the Cauchy inequality yields that

$$\begin{aligned}\left|\int_D \frac{\chi_N(x-t)}{\cosh\delta(x-t)}\,[\omega(U+u+v)-\omega(U)]\,\mathrm{d}x\,\mathrm{d}y\right| \\ \leqslant C_\omega \int_D \frac{\chi_N(x-t)}{\cosh\delta(x-t)}\,|u+v|\,\mathrm{d}x\,\mathrm{d}y \\ \leqslant \delta^2 \int_D \frac{\chi_N(x-t)}{\cosh\delta(x-t)}\,v^2\,\mathrm{d}x\,\mathrm{d}y + \frac{C_\omega^2}{4\delta^2}\int_D \frac{\chi_N(x-t)}{\cosh\delta(x-t)}\,u^2\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{2.29}$$

where C_ω is the Lipschitz constant of ω. Second, we apply the first Green's formula to the other term and get, in view of the boundary conditions (2.8) and (2.9), that its absolute value can be written as follows:

$$\left|\int_D \left\{\frac{\chi_N(x-t)}{\cosh\delta(x-t)}\nabla u\cdot\nabla v + v u_x\left[\frac{\chi_N'(x-t)}{\cosh\delta(x-t)} + \chi_N(x-t)\left[\frac{1}{\cosh\delta(x-t)}\right]'\right]\right\}\mathrm{d}x\,\mathrm{d}y\right|. \tag{2.30}$$

Here the first and third terms do not exceed

$$\frac{\delta^2}{2}\int_D \frac{\chi_N(x-t)}{\cosh\delta(x-t)}\,|\nabla v|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2\,\delta^2}\int_D \frac{\chi_N(x-t)}{\cosh\delta(x-t)}\,|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}y \tag{2.31}$$

and

$$\delta^2\int_D \frac{\chi_N(x-t)}{\cosh\delta(x-t)}\,v^2\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{4}\int_D \frac{\chi_N(x-t)}{\cosh\delta(x-t)}\,u_x^2\,\mathrm{d}x\,\mathrm{d}y, \tag{2.32}$$

respectively, whereas we simply take the absolute value of the integrand in the second term.

Using (2.26)–(2.32) in equality (2.25) and letting $N\to\infty$, we arrive at the following inequality:

$$\begin{aligned}\int_D \left[\left(1-\frac{\delta^2}{2}\right)v_x^2 + \frac{\delta^2}{2}v_y^2 + \left\{\left(\frac{\pi}{h}\right)^2 - \mu - \delta^2\left[5+\left(\frac{\pi}{h}\right)^2\right]\right\}v^2\right]\frac{\mathrm{d}x\,\mathrm{d}y}{\cosh\delta(x-t)} \\ \leqslant \int_D \left[\left(\frac{1}{4}+\frac{1}{2\,\delta^2}\right)|\nabla u|^2 + \frac{C_\omega^2}{4\,\delta^2}u^2\right]\frac{\mathrm{d}x\,\mathrm{d}y}{\cosh\delta(x-t)},\end{aligned} \tag{2.33}$$

because χ_N goes to unity, whereas χ_N' and χ_N'' go to zero. Now (2.23) follows from assumption (1.8) and inequalities (2.11). □

The following corollary is a consequence of lemma 2.

Corollary 2. *Let the assumptions of lemma 2 be fulfilled, and let $\zeta(x) < h$ for all $x\in\mathbb{R}$. Then*

$$\|v\|_{W^{1,2}(D_t)} \leqslant C(\delta, h, B)\sup_{\tau\in\mathbb{R}}\|\zeta\|_{L^2(\tau,\tau+1)} \quad \textit{for all } t\in\mathbb{R}. \tag{2.34}$$

Proof. It is clear that the left-hand side of (2.23) is greater than or equal to

$$\int_{t-1}^{t+2} \mathrm{e}^{-\delta|t-x|}\,\mathrm{d}x \int_0^{h-\zeta} (v^2+|\nabla v|^2)\mathrm{d}y \geqslant \mathrm{e}^{-2\delta}\,\|v\|^2_{W^{1,2}(D_t)}, \tag{2.35}$$

because $\mathrm{e}^{-2\delta} = \min_{x\in[t-1,t+2]} \mathrm{e}^{-\delta|t-x|}$. Since $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{d}\tau \int_{\tau}^{\tau+1} f(x)\,\mathrm{d}x$ for any f, we write the right-hand side of (2.23) as follows:

$$C_\delta \int_{-\infty}^{+\infty} \mathrm{d}\tau \int_{\tau}^{\tau+1} \mathrm{e}^{-\delta|t-x|}\, \zeta^2(\zeta^2 + \zeta_x^2)\mathrm{d}x. \tag{2.36}$$

This, in view of the assumptions made about ζ and ζ_x, is less than or equal to

$$C_\delta\, \mathrm{e}^{\delta}(h^2 + B^2) \int_{-\infty}^{+\infty} \mathrm{e}^{-\delta|t-\tau|}\, \|\zeta\|^2_{L^2(\tau,\tau+1)}\,\mathrm{d}\tau, \tag{2.37}$$

because $\mathrm{e}^{-\delta|t-x|} \leqslant \mathrm{e}^{\delta}\mathrm{e}^{-\delta|t-\tau|}$ provided $\tau \leqslant x \leqslant \tau + 1$. Taking the supremum of the norm, we arrive at the required inequality, because the integral of $\mathrm{e}^{-\delta|t-\tau|}$ is equal to $2/\delta$. $\square$

2.3. *Proof of the main theorem*

The assumptions made about η and η_x allow us to apply inequalities (2.11) for estimating u and Corollary 2 for estimating v. Since $\phi = u + v$, we get

$$\|\phi\|_{W^{1,2}(D_t)} \leqslant C\,(B^2 + h^2)^{1/2}\, \|\zeta\|_{L^2(t-1,t+2)} + C(\delta, h, B) \sup_{\tau\in\mathbb{R}} \|\zeta\|_{L^2(\tau,\tau+1)}, \tag{2.38}$$

and so the right-hand side does not exceed $C_1(\delta, h, B) \sup_{\tau\in\mathbb{R}} \|\zeta\|_{L^2(\tau,\tau+1)}$. Combining this fact and Corollary 1, we obtain that

$$\begin{aligned} \left\|\partial_n\phi|_{y=h-\zeta(x)}\right\|_{L^2(t,t+1)} &\leqslant C_2(\delta, h, B) \sup_{\tau\in\mathbb{R}} \|\zeta\|_{L^2(\tau,\tau+1)} \\ &\leqslant \varepsilon^{1/2} C_2(\delta, h, B) \sup_{\tau\in\mathbb{R}} \|\zeta\|^{1/2}_{L^1(\tau,\tau+1)}, \end{aligned} \tag{2.39}$$

where the last inequality is a consequence of the second assumption (1.9).

Bernoulli's equation written as follows (cf. (2.5))

$$[\zeta(x)]^{1/2} = \frac{1}{\sqrt{2}} \left| \partial_n\phi + \frac{U_y(y)}{(1+\zeta_x^2)^{1/2}} \right|_{y=h-\zeta(x)}, \quad x \in \mathbb{R}, \tag{2.40}$$

immediately yields that

$$\sup_{\tau\in\mathbb{R}} \|\zeta\|^{1/2}_{L^1(\tau,\tau+1)} \leqslant \frac{1}{\sqrt{2}} \sup_{\tau\in\mathbb{R}} \left[\left\|\partial_n\phi|_{y=h-\zeta(x)}\right\|_{L^2(\tau,\tau+1)} + C\,\|\zeta\|_{L^2(\tau,\tau+1)} \right]. \tag{2.41}$$

Using inequalities (2.39) for estimating both terms in the square brackets, we arrive at

$$\sup_{\tau\in\mathbb{R}} \|\zeta\|^{1/2}_{L^1(\tau,\tau+1)} \leqslant \varepsilon^{1/2} C \sup_{\tau\in\mathbb{R}} \|\zeta\|^{1/2}_{L^1(\tau,\tau+1)}, \tag{2.42}$$

which is impossible for sufficiently small ε. The obtained contradiction proves the theorem.

3. Discussion

In the framework of the classical approach to steady water waves with vorticity, it is proved under assumptions (1.6) and (1.8) that no waves of small amplitude are supported by a horizontal shear flow with still free surface. Here we discuss the first of these assumptions in greater detail and consider examples when both of them are fulfilled.

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The first assumption (there exists a stream solution with still free surface) yields that

$$\left.\begin{aligned} h_0 = \int_0^1 \frac{\mathrm{d}\tau}{\sqrt{s_0^2 - 2\,\Omega(\tau)}} < \infty \\ \text{where} \quad \Omega(\tau) = \int_0^\tau \omega(t)\,\mathrm{d}t \text{ and } s_0 = \sqrt{2 \max_{\tau\in[0,1]} \Omega(\tau)} \end{aligned}\right\} \tag{3.1}$$

(see Kozlov & Kuznetsov (2011*b*), the first assertion in § 4.2). For a given vorticity distribution, h_0 is the smallest depth of a shear flow for which the free surface is stagnant. Let the maximum of Ω be attained at $\tau_0 \in [0, 1]$; then $h_0 < \infty$ if and only if $\omega(\tau_0) \neq 0$ (see Kozlov & Kuznetsov (2011*b*, p. 382)), and so τ_0 is either 0 or 1. These are the conditions of either case (ii) or case (iii), according to the classification of vorticity distributions proposed in § 4.2 of the cited paper.

It is shown in Kozlov & Kuznetsov (2011*b*, §§ 5.2 and 5.3) that for $s_0 > 0$ any stream solution (U, h) that satisfies assumption (1.6) is

$$\text{either} \quad (U(y; s_0), h_k^{(+)}) \quad \text{or} \quad (U(y; -s_0), h_k^{(-)}). \tag{3.2}$$

Here $U(y; s)$ denotes (as in the cited paper) a unique solution of the Cauchy problem:

$$U_{yy} + \omega(U) = 0, \quad U(0) = 0, \quad U_y(0) = s. \tag{3.3}$$

The restriction of $U(y; +s_0)$ $(U(y; -s_0))$ on $[0, h_k^{(+)}]$ $([0, h_k^{(-)}])$ is the stream function of a shear flow with the stagnant free surface, whose depth is equal to

$$h_k^{(+)} = h_0 + 2\,k\,[h_0 - y_-(s_0)] \quad (h_k^{(-)} = h_k^{(+)} - 2\,y_-(s_0)), \quad k = 0, 1, \ldots \tag{3.4}$$

The bottom velocity is positive (negative) for flows corresponding to the plus (minus) sign in these formulae, whereas the value $y_-(s_0) < 0$ (see its definition in Kozlov & Kuznetsov (2011*b*, § 3, in particular, pp. 378 and 379)) can be finite as well as infinite depending on the vorticity distribution; it is such that $(y_-(s_0), h_0)$ is the maximal interval containing $y = 0$ inside, on which $U(y; s_0)$ increases strictly monotonically. Thus, if $y_-(s_0) > -\infty$, then $U(y; s_0)$ is periodic and the above formulae are valid for all non-negative integers k; that is, there are infinitely many shear flows with stagnant free surfaces, but they have either different numbers of counter-currents or the opposite directions of the bottom velocity. Otherwise, only the first formula (3.2) with $k = 0$ gives a stream solution satisfying assumption (1.6), and the corresponding shear flow is unidirectional.

If $s_0 = 0$, then we have $y_-(s_0) = 0$, and so all stream solutions satisfying assumption (1.6) are given by the first formula (3.2) provided $U(y; s_0)$ is periodic.

Now we turn to examples of vorticity distributions ω for which both assumptions (1.6) and (1.8) are fulfilled.

First, we take the vorticity equal to an arbitrary positive constant, say, $b > 0$ (see details in Kozlov & Kuznetsov (2011*b*, § 6.1)), and obtain the simplest example of the unique stream solution satisfying (1.6) and (1.8) simultaneously. Indeed, in this case $s_0 = \sqrt{2b} > 0$, $h_0 = \sqrt{2/b} = h$ and the stream function is $U = \sqrt{2b}\,y - by^2/2$. Therefore, the corresponding shear flow has the velocity profile in the form of a straight segment which goes from $\sqrt{2b}$ on the bottom to zero on the free surface. In his study of bifurcation of waves from shear flows with constant vorticity, Wahlén (2009) also excluded the above stream solution from his considerations.

Alternatively, if the vorticity is equal to a negative constant, say, $-b < 0$, then $s_0 = 0$, and the corresponding stream solution $(U, h) = (by^2/2, \sqrt{2/b})$ gives a positive value of the flow velocity on the free surface. The existence of Stokes waves bifurcating from this shear flow is proved by Wahlén (2009), but the general results obtained by Kozlov & Kuznetsov (2012) are not applicable in this case. Presumably, the reason for this lies in the degeneration of the streamline pattern for $s_0 = 0$, which becomes clear from figures 1 and 2 in Wahlén (2009). Indeed, the velocity of flow is negative (vanishes) on the bottom for the flow shown in figure 1 (figure 2, respectively). In the middle of the flow corresponding to the negative bottom velocity (see figure 1), there is a critical layer formed by closed cat's-eye vortices. However, for $s_0 = 0$ domains with closed streamlines are attached to the bottom and separated from each other.

In the case of positive linear vorticity, that is, $\omega(\tau) = b\tau$, $b > 0$, we have that $s_0 = \sqrt{b}$ and $h_0 = \pi/(2\sqrt{b})$ (see details in Kozlov & Kuznetsov (2011*b*, § 6.3)). There are infinitely many stream solutions corresponding to s_0, and their second components are equal to $\pi k/(2\sqrt{b})$ $(k = 1, 3, 5, \ldots)$. Condition (1.8) is fulfilled only for $k = 1$, in which case the main theorem is valid, but it gives no answer for $k \geqslant 2$. However, Ehrnström *et al.* (2011) exclude from consideration all shear flows with still free surfaces in their detailed study of waves with positive linear vorticity. The reason for this is as follows: 'without this assumption the linearized operator [...] appearing in the bifurcation problem' can be shown not to be Fredholm.

The main theorem is also applicable to a shear flow with $\omega(\tau) = b\tau^2$ on $[-R, R]$ and constant $\omega(\tau)$ for $|\tau|$ outside $(-R, R)$ (the constant is taken so that ω is continuous); here $R > 1$ and b is a positive constant. For this vorticity, $s_0 = \sqrt{2b/3}$ and formula (1.6) gives that

$$h_0 = \sqrt{\frac{3}{2b}} \int_0^1 \frac{d\tau}{\sqrt{1-\tau^3}}. \tag{3.5}$$

The equation for the first component of the corresponding stream solution is as follows:

$$3U_y^2 + 2bU^3 = 2b. \tag{3.6}$$

Using elliptic functions, one can obtain its general solution (see Kamke (1959, part 3, ch. 6, § 6.5)), but this is superfluous in the present context. Of course, the smallest (if there are more than one) second component of stream solutions with still free surfaces is equal to h_0 for which, according to formula 17.4.59 in Abramowitz & Stegun (1965), we have the following expression:

$$\sqrt{\frac{3}{2b}} \frac{F(\varphi_0 \setminus \alpha_0)}{\sqrt[4]{3}} \quad \text{where} \quad \varphi_0 = \arccos \frac{\sqrt{3}-1}{\sqrt{3}+1}, \ \alpha_0 = 75^\circ, \tag{3.7}$$

and $F(\varphi \setminus \alpha)$ denotes the elliptic integral of the first kind. Then condition (1.8) is fulfilled if $\sqrt{3}\, [F(\varphi_0 \setminus \alpha_0)]^2 < \pi^2$, and this inequality is true because after simple computations one gets from table 17.5 in Abramowitz & Stegun (1965) that $F(\varphi_0 \setminus \alpha_0) < 1.9$.

Any of the shear flows described above might be called a *critical flow of the second kind*. Indeed, Stokes waves bifurcate from all shear flows whose depths are close to h for positive constant and positive linear vorticity (see Kozlov & Kuznetsov (2012, § 5)). On the other hand, the bifurcation pattern is different near a flow that is referred to as *critical* on p. 386 of Kozlov & Kuznetsov (2011*b*). We recall that this flow described by $(U(y; s_c), h(s_c))$ exists for *all* vorticity distributions. On the s-axis, the value s_c separates two intervals with different properties. On the left of s_c, there lies a finite interval, and for s belonging to it, small-amplitude Stokes waves bifurcate from the corresponding horizontal shear flows (see main theorem in Kozlov & Kuznetsov (2012)). On the right of s_c, a sufficiently small interval exists such that solitary waves are present for those s, as Hur (2008) proved. This near-critical behaviour is distinct from that outlined above, but is completely analogous to that taking place in the irrotational case when the critical uniform flow separates sub- and supercritical flows from which Stokes and solitary waves, respectively, bifurcate (see e.g. Kozlov & Kuznetsov (2010, 2011*a*)). Besides, only a uniform flow exists for the critical value of the problem's parameter in the irrotational case (see Kozlov & Kuznetsov (2008, theorem 1), for the proof). On the other hand, a similar fact for problem (1.1)–(1.4) is still an open question.

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