THE PRICING OF MORTALITY-LINKED CONTINGENT CLAIMS: AN EQUILIBRIUM APPROACH

BY

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ABSTRACT

This study introduces an equilibrium approach to price mortality-linked securities in a discrete time economy, assuming that the mortality rate has a transformed normal distribution. This pricing method complements current studies on the valuation of mortality-linked securities, which only have discrete trading opportunities and insufficient market trading data. Like the Wang transform, the valuation relationship is still risk-neutral (preference-free) and the mortality-linked security is priced as the expected value of its terminal payoff, discounted by the risk-free rate. This study provides an example of pricing the Swiss Re mortality bond issued in 2003 and obtains an approximated closed-form solution.

KEYWORDS

Longevity risk, mortality-linked security valuation, transform normal distribution.

1. INTRODUCTION

Longevity and mortality risks have created new challenges for financial institutions, especially for life insurers, reinsurers, annuity providers and pension funds. These risks are systematic, long-trending and widespread. To avoid these risks, the insurance industry has begun to issue mortality-related securities in capital markets. These securities are called mortality-linked contingent claims (MLCCs).[1]

One major task in securitization is determining the values of MLCCs. The MLCC valuation literature contains three main methods: the Wang transform (Wang, 2000, 2002), the arbitrage-free pricing method of Cairns *et al.* (2006b) and the Sharpe ratio method of Milevsky *et al.* (2005).[2] First, Wang's approach provides a distortion operator that transforms the underlying distribution to a risk-adjusted distribution that can be used to calculate the value of MLCCs.

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Under the risk-adjusted probability, the prices of MLCCs are the expected value of cash flows discounted by the risk-free rate. Lin and Cox (2008) and Cox *et al.* (2006) used this approach to price the Swiss Re mortality bond issued in 2003. Second, Cairns *et al.* (2006b) used the arbitrage-free approach to price the European Investment Bank (EIB) longevity bond issued in 2004. The arbitrage-free approach assumes that if the market prohibits arbitrage opportunities, at least one risk-neutral measure can be derived to obtain security prices. Cairns *et al.* (2006b) estimate the market prices of longevity risk by the issue price of EIB longevity bond. Third, Milevsky *et al.* (2005) suggest that insurers bearing non-diversifiable mortality risk request a risk premium with a Sharpe ratio equal to that of a well-diversified portfolio in the capital market. Thus, they derive the Sharpe ratio valuation method for pricing mortality risk in an incomplete market. Bayraktar *et al.* (2009) and Young (2008) use this method to price MLCCs. Bauer *et al.* (2010) and Chen *et al.* (2010) compare and comment on the robustness of these approaches and provide guidance for choosing among these pricing methods.

In this paper, we propose an alternative valuation method that is distinct from above three approaches. The proposed method possesses the risk-neutral valuation relationship (RNVR) property of Rubinstein (1976) and Brennan (1979). The classical literature of asset pricing, Rubinstein (1976) and Brennan (1979), presents a risk-neutral valuation approach in which asset prices are equal to the expected cash flow of contingent claims discounted by the risk-free rate. Their option pricing formula is identical to that of Black and Scholes (1973), when the representative agent has a constant relative risk aversion (CRRA)/constant absolute risk aversion (CARA) preference, the aggregate wealth and the underlying assets have a joint lognormal/normal distribution. Their risk-neutral valuation approach depends on the triple economic assumptions: preference, wealth and underlying distributions, and do not require transaction data or perfect hedge.

One major problem in applying their risk-neutral valuation approach to pricing MLCCs is that the underlying mortality distribution may not be lognormal or normal, especially when the mortality rate is accompanied by longevity or catastrophe risk. For example, when mortality processes accompany a jump effect, the terminal distribution tends toward a positive skew (Lin and Cox, 2008). To take this risk into consideration, this study introduces a transformed normal distribution (Johnson et al., 1994) to accommodate high-order moments of mortality risk in the MLCC pricing. The transformed normal distribution includes the lognormal, four-parameter lognormal, and S_U distribution as special cases. This distribution can have negative, zero, or positive skewness and can be more leptokurtic than the standard lognormal distribution. Based on this distributional generalization, we derive a risk-neutral valuation approach for MLCCs and the result still satisfies the RNVR property of Rubinstein (1976), Brennan (1979) and Camara (2003).[3] The meaning of this transformed normal distribution differs from that of Wang's transform; it means the distribution in a general form that can be transformed into a normal distribution without distorting the underlying distribution.

Based on the triple assumptions and the distributional generalization, the prices of MLCCs are the expected end-of-period payoffs discounted at risk-free rate, taken with respect to a risk-neutral transformed normal density. This result is valuable for current MLCCs studies in three aspects. First, unlike the three valuation approaches mentioned above, it provides an alternative valuation approach for MLCCs in an incomplete market. This approach places much more restrictive assumptions on underlying distribution, individual wealth, and preference compared with the no-arbitrage pricing approach. However, when a replication portfolio or a perfect hedge is unavailable for MLCCs, this approach can still provide valuable price information in the securitization. Second, we use the proposed approach to price the Swiss Re mortality bond issued in 2003 as an example, obtaining approximated closed-form solutions for the mortality bond under some specific distributions of transformed normal distribution. The pricing results can be regarded as a referring price for the MLCC issuer. Third, our approach requires no market transaction data. Most MLCCs are traded in the over-the-counter market, and the trading data is unavailable. In the spirit of direct pricing, this approach can evaluate the MLCCs payoff without referring to other asset prices.

The remainder of this paper is organized as follows. Section 2 sets the valuation methods in a discrete time economy; and explains how to obtain the risk-neutral valuation relationship. Section 3 presents the decomposition of the Swiss Re mortality bond and the approximation method, and derives the closed-form formulas under three specific underlying distributions. Section 4 uses the Lin and Cox (2008) model and the Chen and Cox (2009) model to project the mortality distributions and fit them by the transformed normal distributions to obtain pricing parameters. This section also determines the price of the Swiss Re mortality bond and calculates its premium spread. Section 5 offers a conclusion and implications.

2. Equilibrium pricing model

Following Brennan (1979) and Camara (2003), this section reviews the pricing process of contingent claims with uncertain payoffs. Let $E^P[\cdot]$ be the expected value operator under the actual probability measure. U is the utility function of the representative investor. The current consumption is X_0, and the initial wealth and the end-of-period wealth are W_0 and W_T. $P_{j0}(q)$ and $P_{jT}(q)$ are the prices of security j written on the mortality underlying q at $t = 0$ and T, where $j = 1, \ldots, J$, denotes jth security in the market. The demand for security j is y_j. The representative agent is non-satiated, risk-averse, and attempts to maximize the expected utility by choosing current consumption and future payoff:

$$\underset{X_o,\, y_j}{\text{Max}}\; U(X_0) + E^P\left\{U\left[(W_0 - X_0)\, e^{rT} + \sum_{j=1}^{J} y_j (P_{j1}(q) - P_{j0}(q)e^{rT})\right]\right\},$$

where r is risk-free rate. Replacing $(W_0 - X_0)\, e^{rT} + \sum_{j=1}^{J} y_j (P_{j1}(q) - P_{j0}e^{rT})$ by W_T and following the equilibrium condition yields the familiar economic pricing rule:

$$P_{j0}(q) = e^{-rT} \frac{E^P\left[U'(W_T)\, P_{jT}(q)\right]}{E^P\left[U'(W_T)\right]} = e^{-rT} E^P\left[\xi(W_T)\, P_{jT}(q)\right], \qquad (1)$$

where

$$\xi(W_T) = \frac{U'(W_T)}{E^P\left[U'(W_T)\right]}$$

is the pricing kernel. Conditioning $\xi(W_T)$ with respect to the underlying q produces the asset-specific pricing kernel, defined as

$$\phi(q) = E^P\left[\xi(W_T)\,|q\right]. \qquad (2)$$

This asset-specific pricing kernel reduces the integral dimension of the pricing problem such that the valuation only involves a single integral. The rest of this study focuses on one underlying and one contingent claim written on it; thus, the subscripts j in (1) can be suppressed as

$$P_0 = e^{-rT} E^P\left[\phi(q)\, P_T(q)\right]. \qquad (3)$$

Equation (3) states that the price of security can be expressed as the expected payoff multiplied by its marginal rate of substitution, discounted by the risk-free rate. Here, the pricing kernel still depends on the marginal utility and the pricing relationship is preference-dependent.

To analyze the expected value of (3), we have to specify the explicit form of the terminal wealth, underlying assets, and representative agent's preference. We propose a transformed-normal distribution to describe them.

DEFINITION. *The transformed normal distribution is defined by the transformation of a random variable q such that*

$$f\left(\frac{q-\alpha}{\beta}\right) = x \sim N(\mu, \sigma^2), \qquad (4)$$

where α, β, μ and σ are parameters ($\beta, \sigma > 0$) and f is a strictly monotonic differentiable function. $N(\mu, \sigma^2)$ is a normal distribution with mean μ and variance σ^2.

Appendix A provides details about transformed normal distribution. This definition is identical to the four-parameter transformed normal distribution of Johnson (1949) and Johnson *et al.* (1994), but differs slightly from the three-parameter transformed normal distribution of Camara (2003). This study adapts Johnson's definition because it fits the mortality data better than Camara's definition. The transformed normal distributions are much more general than normal or lognormal distributions. Some well-known distributions can be included as special cases of the transformed normal distributions. For example, if $\alpha = 0$, $\beta = 1$ and f is log function, then q has a lognormal distribution.

To the risk-neutral valuation, first, assume that the terminal wealth W_T and the underlying q have a joint transformed normal distribution:

$$(f(W_T),\ f_1(q)) \sim \mathbf{N}(\mu_w, \mu, \sigma_w, \sigma, \rho), \tag{5}$$

where $f()$ and $f_1()$ are strictly monotonic differentiable functions defined in (4); $\mathbf{N}$ denotes the bivariate normal distribution with means μ_w and μ, standard deviations σ_w and σ, and a correlation coefficient ρ. The subscripts w denote the parameters with respect to wealth.

Second, specify the representative agent's preference. Assume the marginal utility of the representative agent has the form

$$U'(W_T) = \exp(\delta f(W_T)), \tag{6}$$

where δ is a constant and $f()$ is identical to the one in (5). This utility specification is quite general and includes several types of utility, such as HARA utility. For example, we can choose

$$f(W_T) = \ln\left[a^{\frac{1}{\gamma-1}} \times \left(\frac{aW_T}{1-\gamma} + b\right)\right] \text{ and } \delta = \gamma - 1,$$

where $\gamma \neq 1$, $a > 0$, $\frac{aW_T}{1-\gamma} + b > 0$, and $b = 1$ if $\gamma = -\infty$, then we obtain the marginal utility of HARA utility.[4] This transform function satisfies the strictly monotonic requirement in the definition.

Following these triple assumptions in (5) and (6), the security price can be derived as

$$P_0 = e^{-rT} E^Q[P_T(q)], \tag{7}$$

where $E^Q[\cdot]$ is the expected value operator under the Q probability measure with respect to a risk-neutral transformed normal density and a new location parameter μ^Q.

Proof. See Appendix B.

This risk-neutral density has a shifted location of the underlying density μ^Q, which is unrelated to the preference parameter. Thus, we obtain a risk-neutral valuation relationship for MLCC pricing.

3. Mortality-linked contingent claim valuation

This section first describes the payoffs of the Swiss Re mortality bond, which can be decomposed into three bull spreads. Next, we price these bull spreads and obtain the mortality bond price in closed-form solutions.

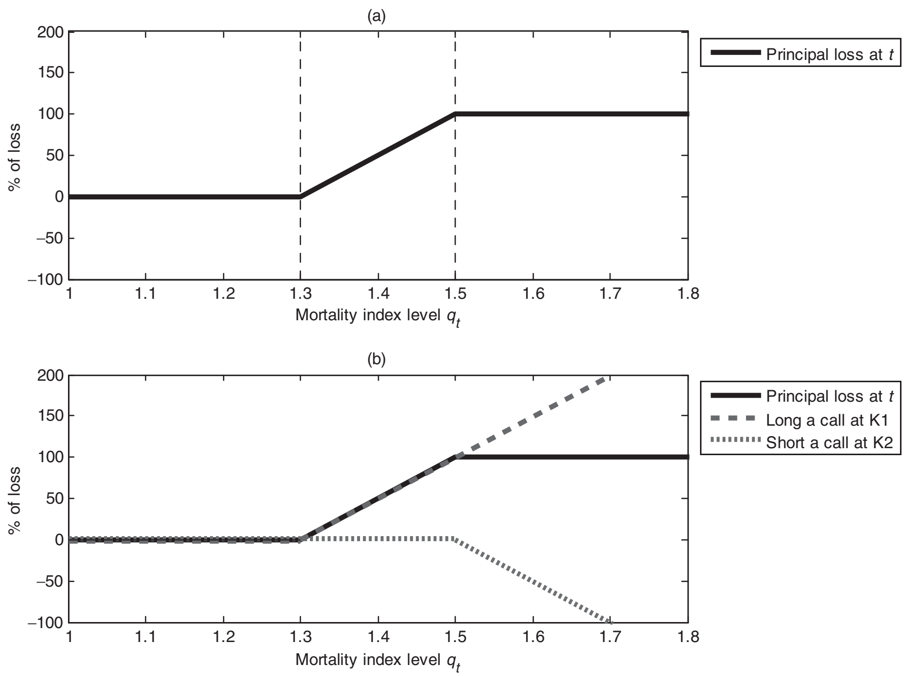

FIGURE 1: Loss ratio L_t and a bull spread payoff with exercise price K_1 and K_2.

3.1. The decomposition of Swiss Re mortality bond

The Swiss Re Insurance Company issued a three-year mortality bond in December 2003 through a special-purpose vehicle, Vita Capital.[5] The total amount was \$400 million, and the bondholders receive coupons quarterly at a rate of three-month U.S. dollar LIBOR plus 135 basis points. The principal is not fully protected and depends on the mortality index weighted by five countries' mortality experiences.[6] If the mortality index q_t in year t exceeds 130% of the 2002 level q_0, the principal decreases by 5% for every 1% increase in the index. If q_t exceeds 150% of q_0, the principal is exhausted. The principal loss ratio at time t, L_t, can be written as

$$L_t = \begin{cases} 0 & \text{if } q_t < 1.3q_0 \\ (q_t - 1.3q_0)/0.2q_0 & \text{if } 1.3q_0 \leq q_t \leq 1.5q_0 \\ 1 & \text{if } q_t > 1.5q_0 \end{cases} \quad (8)$$

where t = 1, 2 and 3 for years 2004, 2005 and 2006, respectively.

Figure 1(a) shows that L_t is in the form of a bull spread depending on the mortality level q_t. A bull spread payoff can be replicated by buying a call option with one strike price and selling another call option with a different strike price. Figure 1(b) indicates that the loss ratio in Figure 1(a) can be replicated by buying a call option at a strike price $K_1 = 1.3q_0$ and simultaneously selling the other call option at strike price $K_2 = 1.5q_0$. The aggregate payoff in Figure 1(b) is identical to the payoff in Figure 1(a).

Thus, the loss ratio can be written as a bull spread:

$$L_t = \text{Max}\left(\frac{q_t - K_1}{K_2 - K_1}, 0\right) - \text{Max}\left(\frac{q_t - K_2}{K_2 - K_1}, 0\right). \quad (9)$$

The aggregate loss ratio at time T is $\sum_t L_t = L_1 + L_2 + L_3$ and the principal return to the bondholders is

$$B_T = \text{Max}\left(1 - \sum_t L_t,\ 0\right). \quad (10)$$

Following the risk-neutral valuation approach in Section 2, the mortality bond value at time 0 is

$$B_0 = e^{-rT} E^Q\left[\text{Max}\left(1 - \sum_t L_t,\ 0\right)\right] \times \text{FaceValue}, \quad (11)$$

where Q is the probability measure with respect to the risk-adjusted transformed normal density defined in Section 2. Equation (11) is ready to evaluate the mortality bond price B_0.

However, the mortality bond payoff in (11) is a form of options on options or compound options that is complex to be analyzed in the framework. We impose an additional assumption to simplify the payoff function. If (i) the probability of sequential catastrophes occurring in three years is small (Lin and Cox, 2008), and (ii) not every sequential mortality jump erodes the principal completely, the payoff function can be approximated by

$$E^Q\left[\text{Max}\left(1 - \sum L_t,\ 0\right)\right] \simeq \text{Max}\left(1 - E^Q\left[\sum L_t\right],\ 0\right). \quad (12)$$

Proof. See Appendix C.

Based on this approximation, the mortality bond price can be written as

$$\begin{aligned} B_0 &= e^{-rT} E^Q\left[\text{Max}\left(1 - \sum L_t,\ 0\right)\right] \times \text{FaceValue} \\ &\simeq e^{-rT}\text{Max}\left(1 - E^Q\left[\sum L_t\right],\ 0\right) \times \text{FaceValue} \\ &= e^{-rT}\text{Max}\left(\text{FaceValue} - \sum_{t=1}^{3}\left(C_t^1 - C_t^2\right),\ 0\right), \end{aligned} \quad (13)$$

where

$$\begin{aligned} C_t^1 &= E^Q[\text{Max}(q_t - K_1, 0)/(K_2 - K_1)] \times \text{FaceValue and} \\ C_t^2 &= E^Q[\text{Max}(q_t - K_2, 0)/(K_2 - K_1)] \times \text{FaceValue} \end{aligned} \quad (14)$$

are call payoffs at maturity date T with respect to underlying q_t and strike prices K_1 and K_2. If the values of C_t^1 and C_t^2 are known, we obtain B_0. In the next section, we use this approximation method to value the mortality bond price, which equals its face value minus the prices of three bull spreads.

The low-correlation assumption is restricted, but greatly simplifies the payoff of the mortality bond. An alternative simplification is the approximation method of Lin and Cox (2008):

$$\sum L_t = \text{Max}\left(\frac{q_{\max} - K_1}{K_2 - K_1}, 0\right) - \text{Max}\left(\frac{q_{\max} - K_2}{K_2 - K_1}, 0\right), \tag{15}$$

where $q_{\max} = \text{Max}(q_1, q_2, q_3)$.

These two simplifications provide a snapshot of the multi-period valuation as a single-period one. In Section 4, we use both methods to price the Swiss Re mortality bond.

3.2. Specification and valuation

Equation (13) shows that the terminal payoff of the Swiss Re mortality bond is related to the three bull spreads. Dropping subscript t of q_t for conciseness, the call value depending on the mortality rate q is

$$C = e^{-rT} E^{Q} [\text{Max}(q - K, 0)] . \tag{16}$$

Assuming that the underlying q has a specific transformed normal distribution that leads to the option price solution possessing a Black–Scholes closed-form type. We discuss three cases of underlying q following (a) the S_U distribution, (b) the four-parameter lognormal distribution and (c) the lognormal distribution, and derive their analytic pricing formulas.

(a) *The S_U distribution*. Assuming the mortality rate q has a S_U distribution, the terminal wealth W_T has a transform normal distribution, and their transformed variables $\sinh^{-1}(\frac{q-\alpha}{\beta}) = \ln(\frac{q-\alpha}{\beta} + \sqrt{1 + (\frac{q-\alpha}{\beta})^2})$ and $f(W_T)$ have a bivariate normal distribution:

$$\left(f(W_T),\ \sinh^{-1}\left(\frac{q-\alpha}{\beta}\right)\right) \sim \mathbf{N}(\mu_w, \mu, \sigma_w, \sigma, \rho),$$

where β and σ are positive constants and ρ is the correlation coefficient. The representative agent's utility function is

$$U'(W_T) = \exp(\delta f(W_T)),$$

including HARA utility. Using the RNVR derived in Section 2, the option price in (16) is

$$C = \frac{\beta}{2} e^{-rT + \mu^Q + \frac{1}{2}\sigma^2} \cdot \Phi(d_1) - \frac{\beta}{2} e^{-rT - \mu^Q + \frac{1}{2}\sigma^2} \cdot \Phi(d_2) + (\alpha - K) e^{-rT} \cdot \Phi(d_3) , \tag{17}$$

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where

$$d_1 = \frac{-\sinh^{-1}\left(\frac{K-\alpha}{\beta}\right)+\mu^Q}{\sigma} + \sigma,$$

$$d_2 = \frac{-\sinh^{-1}\left(\frac{K-\alpha}{\beta}\right)+\mu^Q}{\sigma} - \sigma,$$

$$d_3 = \frac{-\sinh^{-1}\left(\frac{K-\alpha}{\beta}\right)+\mu^Q}{\sigma} \text{ and } \mu^Q = \sinh^{-1}\left(\frac{1}{\beta}e^{-\frac{1}{2}\sigma^2}(q_0 e^{rT} - \alpha)\right). \tag{18}$$

The term $\Phi(\cdot)$ is the cumulative standard normal distribution.

Proof. See Appendix D.

(*b*) *The four-parameter lognormal distribution*. Assume that the terminal wealth has a transform normal distribution and the mortality rate q has a four-parameter lognormal distribution, i.e., $f_1(q) = \ln(\frac{q-\alpha}{\beta})$, and they have a jointly normal distribution:

$$\left(f(W_T),\ \ln(\frac{q-\alpha}{\beta})\right) \sim \mathbf{N}(\mu_w, \mu, \sigma_w, \sigma, \rho).$$

The representative agent's utility function is $U'(W_T) = \exp(\delta f(W_T))$ which includes a HARA utility. The option price is

$$C = \beta e^{-rT+\mu^Q+\frac{1}{2}\sigma^2 T} \cdot \Phi(d_1) + (\alpha - K)e^{-rT} \cdot \Phi(d_2), \tag{19}$$

where

$$d_1 = \frac{-\ln\left(\frac{K-\alpha}{\beta}\right)+\mu^Q}{\sigma} + \sigma,$$

$$d_2 = \frac{-\ln\left(\frac{K-\alpha}{\beta}\right)+\mu^Q}{\sigma} \text{ and } \mu^Q = \ln\left(\frac{1}{\beta}(q_0 e^{rT} - \alpha)\right) - \frac{1}{2}\sigma^2.$$

Proof. See Appendix E.

(*c*) *Lognormal distribution.* The lognormal distribution can be included in (*b*) by choosing $\alpha = 0$ and $\beta = 1$. For paper's completeness, we discuss it briefly. Assuming that the mortality rate q and terminal wealth have a joint lognormal/normal distribution and the agent has a utility function displaying a CRRA/CARA preference, the option formulas are the same as those of Black and Scholes (1973). This result is the same as those of Rubinstein (1976) and Brennan (1979). We omit the formula and proof here.

4. Parameter estimation and valuation

This section applies two stochastic mortality models as data generating processes (DGPs) to generate the future distribution of q for each time t. The models are the mortality catastrophe model of Lin and Cox (2008) and the Lee–Carter model with jumps of Chen and Cox (2009). The transformed normal distributions are employed to fit these mortality distributions. The Swiss Re mortality bond price is calculated by substituting the obtained parameters into the closed-form formulas.

The mortality data are obtained from the National Center for Health Statistics (NCHS).[7] NCHS reports the U.S. age-adjusted death rate per 100,000 standard million people for selected causes of death. The observation period starts from 1900 to 2002.

We first show the quantile–quantile plot (Q–Q plot) of the mortality distributions and the fitting distributions of S_U, four-parameter lognormal, and lognormal distributions. They are plotted in Figure 2. Figure 2 shows that the S_U distribution has better-fitting results than the four-parameter lognormal distribution and the lognormal distribution in both the Lin–Cox and Chen–Cox mortality distributions. Most dots plotted in the S_U-fitting fall on the 45° line, and the dots in the four-parameter-lognormal and lognormal fitting tend to deviate from the 45° line in tails. This shows that the four-parameter-lognormal and lognormal fittings perform worse than the S_U distribution in the extreme or catastrophe events. The fitting results of the mortality data in the years 2003 and 2004 are similar to those of the year 2005, and are not shown here. Therefore, throughout this section, we demonstrate our results based on the S_U distribution.

4.1. The S_U fitting

The estimation for the S_U distribution using a maximum likelihood method sometimes is unstable. This study proposes a quantile-estimation method adapted from Slifker and Shapiro (1980) to facilitate parameter estimation. The quantile-based estimation method of Slifker and Shapiro (1980) provides two advantages. One advantage is an increase in the accuracy when the number of observations of data is sufficiently large.[8] The other advantage is that the estimators have explicit formulations. Appendix F describes the steps of estimating these parameters.

The mortality projections are simulated by 100,000 times from 2004 to 2006 and are regarded as three discrete-time distributions. Tables 1 and 2 show the basic statistics of the simulated distributions. Table 1 shows the parameters of simulated distributions from the Lin and Cox (2008) model. The basic statistics are shown for q_{2004}, q_{2005}, q_{2006} and $q_{\max}$, respectively, where $q_{\max}$ is the maximum value of the three years, defined in the approximate equation (15). In the basic statistics, the means have a decreasing trend from 0.008691 to 0.008524 from year 2004 to 2006. $q_{\max}$ has a greatest mean value of 0.008826. The standard deviations show an increasing trend from 0.000310 to 0.000483 from year 2004 to 2006. This is because the Lin and Cox model assumes that the mortality rate follows a geometric Brownian motion and that its volatility increases over time.[9] The skewness, ranging from 1.3585 to 0.52016, shows a positive skew of data for each year. This is consistent with the mortality jumps in the processes

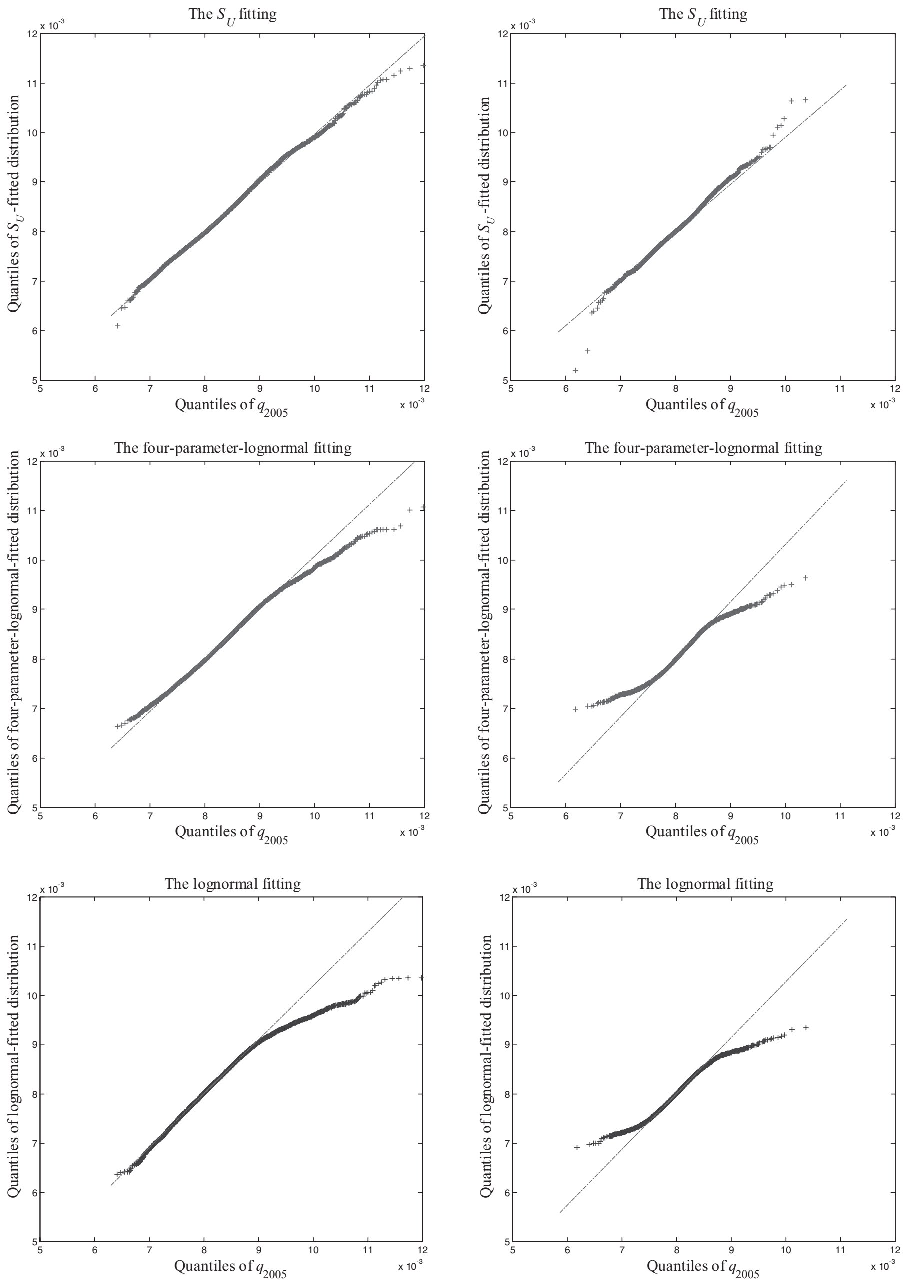

FIGURE 2: Q–Q plot of the fitted distributions for the Lin–Cox and Chen–Cox models (left half: 2005 Lin–Cox mortality distribution; right half: 2005 Chen–Cox mortality distribution).

TABLE 1

THE PARAMETERS OF THE 100,000-TIMES SIMULATED DISTRIBUTION GENERATED BY THE LIN AND COX MODEL.

		q_{2004}	q_{2005}	q_{2006}	q_{max}
Basic statistics	Mean	0.008691	0.008607	0.008524	0.008826
	Std	0.000310	0.000406	0.000483	0.000401
	Skewness	1.3585	0.67229	0.52016	1.4969
	Kurtosis	10.106	5.3958	4.4473	8.0794
Parameters fitted by S_U	α	0.008399	0.008169	0.007905	0.008403
	β	0.000298	0.000613	0.000904	0.000392
	μ	0.70780	0.58728	0.58743	0.79031
	σ	0.67281	0.50654	0.42218	0.60124
	Criteria	3.014	2.030	1.648	2.297

TABLE 2

THE PARAMETERS OF THE 100,000-TIMES SIMULATED DISTRIBUTION GENERATED BY THE CHEN AND COX MODEL.

		q_{2004}	q_{2005}	q_{2006}	q_{max}
Basic statistics	Mean	0.008167	0.008092	0.008017	0.008229
	Std	0.000255	0.000286	0.000311	0.000296
	Skewness	0.4269	0.4111	0.3178	1.0487
	Kurtosis	10.108	7.8531	6.2193	9.5241
Parameters fitted by S_U distribution	α	0.008129	0.008038	0.007946	0.008155
	β	0.000213	0.000308	0.000392	0.000242
	μ	0.12210	0.13146	0.14309	0.19654
	σ	0.84453	0.72561	0.65284	0.82233
	Criteria	7.736	4.998	3.863	6.981

leading to a skewed distribution. q_{max} shows the greatest degree of skewness because we chose the maximum value of q_t in each simulation. The simulated distributions have positive excess kurtosis, meaning that the mortality distributions are leptokurtic and fat-tailed.

The fitted transformed parameters of the mortality distribution are shown in the bottom half of Table 1. The location parameter of transformed parameters, α, shows a decreasing trend from 2004 to 2006, and q_{max} has a greatest location value. The scale parameter, β, shows an increasing trend from 2004 to 2006. After the transform, the distribution is a normal distribution with mean μ and standard deviation σ. All the criterion values are larger than one indicating that the estimation using the S_U fitting is appropriate (see Appendix F).

Table 2 shows the parameters of the simulated mortality distribution generated from the Chen and Cox (2009) model. The trends of the means and α are decreasing, but with a slower mortality improvement compared with the trends shown in Table 1. The standard deviations and β in Table 2 are smaller, implying that the dispersion risk in the Chen and Cox model is smaller than that in the Lin and Cox model. The row of criteria also shows that the S_U estimation is suitable.

There are two parameters in the S_U distribution that control the scale size of the random variable q. β captures the scale effect before the transformation, and σ captures the scale size after the transformation. As Tables 1 and 2 show, β increases over time from 2004 to 2006, which is consistent with the intuition. The S_U function has a monotonic transformation on scale but has a decreasing effect on the larger variance distributions (concave). Therefore, if the scale increase effects are captured by β, σ captures the reverse effect of scale size after the transformation and has a decrease trend in Table 1 and 2.

Figure 3 shows the simulated distributions and the S_U-fitted distributions. The simulated distributions generated by the Lin and Cox (2008) model appear on the left half; the simulated distributions derived from the Chen and Cox (2009) model appear on the right half. These histograms represent the simulated frequency, and the solid line shows the S_U-fitted results. We find that the mortality distributions generated by the Lin and Cox model exhibit greater mortality improvements from 2004 to 2006 and a wider distributional risk, whereas the distributions of the Chen and Cox model are more concentrated and have a smaller mortality improvement. These figures are consistent with the parameters in Tables 1 and 2.

4.2. The mortality bond prices

Using the parameters in Tables 1 and 2, we can calculate the Swiss Re mortality bond prices and the par spreads. Some exogenous data are given: the initial mortality rate q_0 is 0.008453 from NCHS 2002 data, the risk-free interest rate is 0%,[10] the time t starts from 1 to 3 ($t = 3$ for $q_{\max}$), and the exercise prices are $K_1 = q_0 \times 1.3$ and $K_2 = q_0 \times 1.5$. For a more intuitive presentation, we assume that the face value is 1,000 instead of 400 million. Table 3 shows the prices of the call options based on the parameters in Table 1 and the option pricing equation (17).

Table 3 shows the results separately under the bull spread approximation method and the $q_{\max}$ approximation method of Lin and Cox (2008). In the bull spread method, the call prices C_t^1 have an increasing trend from 0.6539 to 1.9744. This is because the scale parameter of q_t increased from 2004 to 2006, the call prices have an increasing trend. Table 3 also shows that C_t^2 is cheaper than C_t^1. This is because the exercise price K_2 is larger than K_1, implying that C_t^2 is much more out-of-the-money than C_t^1 for all t. Substituting these call prices into the valuation equation (13) shows that loss amounts at year 2006 are $L_1 = 0.5973$, $L_2 =0.9908$, $L_3 =1.8114$, and the aggregated loss $\sum L_t =3.3995$. The mortality bond price is 996.6 and the resulting par spread is 11.4 bps.

In the results of the $q_{\max}$ approximation, the mortality bond price is 999.45 and the par spread is 1.83 bps.[11] This result is smaller than the spread of the bull spread approximation, 11.4 bps.

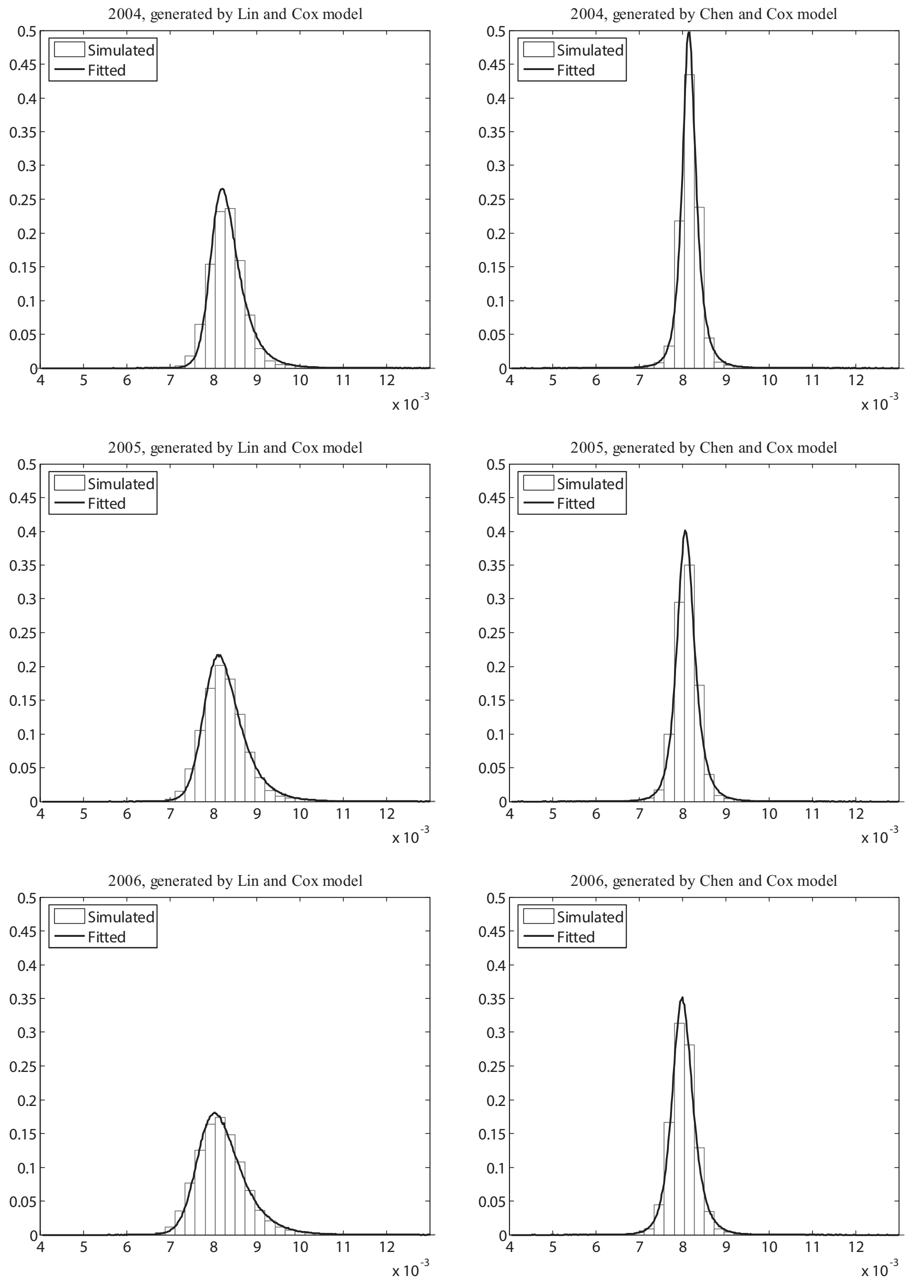

FIGURE 3: The 100,000-times simulated mortality distributions from years 2004 to 2006 and their S_U-fitted distributions (x-axis: values of q; y-axis: relative probability).

TABLE 3

THE VALUES OF CALLS AND THE MORTALITY BOND PRICE BASED ON THE LIN AND COX MODEL.

	Bull Spread Method			q_{max} Method
Call values at $t=3$	$t=1$	$t=2$	$t=3$	$t=3$
C_t^1	0.6539	1.0729	1.9744	0.5960
C_t^2	0.0566	0.0821	0.1630	0.0467
$L_t = C_t^1 - C_t^2$	0.5973	0.9908	1.8114	0.5493
$\sum L_t$			3.3995	
B_0			996.6	999.45
Par spread			11.4 bps	1.83 bps

TABLE 4

THE VALUES OF CALLS AND THE MORTALITY BOND PRICE BASED ON THE CHEN AND COX MODEL.

	Bull Spread Method			q_{max} Method
Call values at $t=3$	$t=1$	$t=2$	$t=3$	$t=3$
C_t^1	1.4043	1.1087	1.0592	1.0538
C_t^2	0.2989	0.1738	0.1353	0.2023
$L_t = C_t^1 - C_t^2$	1.1054	0.9349	0.9239	0.8515
$\sum L_t$			2.964	
B_0			997.04	999.15
Par spread			9.9 bps	2.8 bps

Table 4 shows the mortality bond prices derived from the Chen and Cox (2009) mortality model. The aggregated loss $\sum L_t$ is 2.964 dollars and the bond price is 997.04 under the bull spread approximation. The bond price is 999.15 under the q_{max} approximation. The spreads are 9.9 bps and 2.8 bps, respectively.[12] We also find that the q_{max} approximation has a lower spread than the bull spread approximation. This is because choosing q_{max} from q_1, q_2 and q_3 increases the mean but decreases the variance of the mortality distribution. If the variance of mortality risk is important to the valuation, the q_{max} approximation undervalues the mortality bond price.

In Tables 3 and 4, the par spreads are smaller than the Swiss Re spread of 135 bps for both the bull spread and the q_{max} approximation. Therefore, we may conclude that the Swiss Re mortality bond (with a spread of 135 bps) is a good deal for investors. We emphasize that these numerical results do not include any transaction costs and fees. However, if the transaction costs are small or negligible, we believe that the issue price of the Swiss Re mortality bond is relatively cheap. This result is consistent with the finding of Lin and Cox (2008), and may support one of the reasons why the Swiss Re mortality bonds were so popular when they were issued.

5. Conclusion and discussion

This study provides an equilibrium pricing approach to value MLCCs and uses it to price the Swiss Re mortality bond as a numerical example. A convenient and closed-form formulation for the Swiss Re mortality bond is obtained. The proposed valuation approach has several unique features. First, relative to the no-arbitrage pricing, we make restricted assumptions on the utility, wealth and underlying distribution. The benefits are that we do not require market transaction data and replicating portfolio assumption to determine the price of MLCCs. Second, similar to the Wang transform, this valuation formulation is preference-free and the payoffs are discounted by the risk-free rate. Third, this method employs a general distribution that can be transformed into a normal distribution that is more suitable for mortality data than normal and lognormal distributions. This transformed normal distribution can integrate the high-order moments risk into pricing when the mortality jump is critical. Fourth, the utility specification covers most classical utility specifications in the mortality-linked security literature, such as the HARA utility. By applying the results of RNVR and option pricing formula, the MLCC valuation problem is simplified as a mortality forecasting problem and the individual preference does not play a crucial role in the valuation. Finally, this equilibrium valuation approach is applicable to most MLCCs, and contributes an alternative method to the existing literature to explore fair value of MLCCs.

There are several study limitations to this paper. First, we do not consider the default risk and loading fees in the valuation, which may increase the mortality bond issue price. Second, we use the U.S. mortality rate as a proxy for the five-country weighted mortality index. Because of the diversification effect, the weighted mortality index would have a lower volatility and a lower risk spread. This proxy may not change our results significantly if there are no high correlations in mortality catastrophes across these countries. Third, the valuation result depends on the mortality model selection and therefore involves the mortality model risk. However, the mortality model selection is arbitrary in this method; model builders can choose their preferred mortality projection model, such as a non-parametric model to mitigate the model risk. Thus, the remaining model risk is the transform normal distribution failing to achieve a good fitting. Finally, the proposed approximation method may undervalue the bond price if the probability of sequential mortality catastrophe is high. It may be fruitful to consider these issues in future studies.

Notes

1. See, for example, Lin and Cox (2005), Blake *et al.* (2006a, 2006b), Cairns *et al.* (2006a), Dowd *et al.* (2006), Sherris (2006), Cox and Lin (2007) and Cox *et al.* (2010).

2. For example, Denuit *et al.* (2007), Bauer *et al.* (2010), Chen and Cox (2009) and Chen *et al.* (2010) used the Wang transform; Milevsky and Promislow (2001), Biffis (2005), Biffis and Millossovich (2006), Cairns *et al.* (2006b) and Dahl and Moller (2006) used the arbitrage-free

method; Milevsky *et al.* (2005), Bayraktar and Young (2007) and Young (2008) used the Sharpe ratio method to price mortality risk.

3. Camara (2003) also obtains the RNVR property, but the definition of transformed normal distribution is different from ours.

4. The HARA utility (Ingersoll, 1987) is defined as $U(W) = \frac{1-\gamma}{\gamma}\left(\frac{aW}{1-\gamma}+b\right)^{\gamma}$, where $\gamma \neq 1$, $a > 0$, $\frac{aW}{1-\gamma}+b > 0$, and $b = 1$ if $\gamma = -\infty$. This utility displays IARA if $1 < \gamma < \infty$, CARA if $b = 0$, and DARA if $-\infty < \gamma < 1$; is IRRA if $b > 0$ (for $\gamma \neq 1$), CRRA if $b = 0$, and DRRA if $b < 0$ (for $-\infty < \gamma < 1$).

5. http://www.swissre.com/media/news_releases/

6. The weights of the five countries are the United States (70%), the United Kingdom (15%), France (7.5%), Italy (5%) and Switzerland (2.5%). The index also has weights on males (65%) and females (35%) for each country.

7. http://www.cdc.gov/nchs/nvss/mortality/

8. In the numerical study, the 103 mortality data points are used to estimate the parameters of the Lin–Cox model and the Chen–Cox model, and we regard these two mortality models as DGPs to generate future mortality projections. Then the future mortality projections are fitted by the S_U distribution for the valuation. The number of generated processes (simulation times) is 100,000, and this number is sufficiently large to achieve a good fit by using the quantile-estimation method.

9. However, compared with other years, $q_{\max}$ has a smaller standard deviation of 0.000401. We observe that the approximation of Lin and Cox (2008) creates a censor effect and $q_{\max}$ no longer has the largest standard deviation. If the standard deviation is crucial in the pricing, this approximation method may undervalue the price.

10. We discuss our results with regard to a premium spread; therefore, the assumption of risk-free rate does not change these results.

11. Comparing the results with other studies, for example, the spread of Lin and Cox (2008) is 39 bps. Because of the lack of mortality transaction data, Lin and Cox (2008) calculated their spread according to the market price of risk of property catastrophe bonds.

12. Compared with the spread derived from Chen and Cox (2009), which is 56 bps. However, the design of the mortality bonds is somewhat different. For details, please see Chen and Cox (2009).

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APPENDIX A. SUMMARY OF JOHNSON'S TRANSFORMED NORMAL DISTRIBUTION

The probability density function of the transformed normal distribution is

$$h(q) = h(q;\ f,\ \mu,\ \sigma) = \frac{1}{\sqrt{2\pi}\sigma}\left|f'(q)\right| \exp\left[-\frac{1}{2}\left(\frac{f(q)-\mu}{\sigma}\right)^2\right], \tag{20}$$

where f is the transform function. The transformed normal distribution developed by Johnson *et al.* (1994) includes three distribution families (i) the lognormal distribution S_L, (ii) the range-bounded distribution S_B and (iii) the range-unbounded distribution S_U.

Following the definition in (4), if f has a log function, $q > \alpha$ and $\beta = 0$, we obtain the family of the lognormal distribution:

$$S_L : \log(q - \alpha) = \frac{x - \mu}{\sigma}.$$

If f has a log function and q is bounded at $\alpha < q < \alpha + \beta$, then the bounded transform normal distribution is

$$S_B : \log\left(\frac{q - \alpha}{\alpha + \beta - q}\right) = \frac{x - \mu}{\sigma}.$$

If f has an inverse hyperbolic function, q is unbounded and the unbounded transform normal distribution is

$$S_U : \sinh^{-1}\left(\frac{q - \alpha}{\beta}\right) = \frac{x - \mu}{\sigma}.$$

For more details, please refer to Johnson *et al.* (1994).

APPENDIX B. DERIVE THE RISK-NEUTRAL VALUATION APPROACH IN AN EQUILIBRIUM SETTING

This proof is mathematically tedious, and we only show the main steps and concepts. For details, please refer to Brennan (1979) and Camara (2003). What we want to show is that if the representative agent's marginal utility has the form of

$U'(W) = \exp(\delta f(W))$, terminal wealth W_T and the underlying q have a jointly transformed normal distribution

$$(f(W_T),\ f_1(q)) \sim \mathbf{N}(\mu_w, \mu, \sigma_w, \sigma, \rho),$$

then the security price and payoff have an RNVR as $P_0 = e^{-rT} E^Q [P_T(q)]$.

We first show that the asset-specific pricing kernel in (2) has a lognormal distribution. Because $\delta f(W)$ and $f_1(q)$ are joint normal, we can orthogonally project $\ln [U'(W_T)]$ onto $f_1(q)$:

$$\delta f(W_T) = a + b f_1(q) + \epsilon,$$

where $f_1(q)$ is independent of ϵ. By definition, $E(\delta f(W_T)) = a + b\mu = \delta\mu_w$ and $V(\delta f(W_T)) = b^2\sigma^2 + \sigma_\epsilon^2 = \delta^2\sigma_w^2$. Using the property of linear regression and the strict monotonicity of $f_1(\cdot)$, we obtain

$$E^P\,(\delta f(W_T)|q) = a + b f_1(q) = \delta\mu_w - b\mu + b f_1(q)$$

and

$$V^P\,(\delta f(W_T)|q) = \sigma_\epsilon^2 = \delta^2\sigma_w^2 - b^2\sigma^2.$$

Since $\delta f(W_T) = \ln [U'(W_T)]$ is normal distribution, and recalling the mean for the lognormal distribution, we have

$$E^P\left[U'(W_T)\right] = \exp\left[\delta\mu_w + \frac{1}{2}\delta^2\sigma_w^2\right]$$

and

$$E^P\left[U'(W_T)\,|q\,\right] = \exp\left[\delta\mu_w - b\mu + b f_1(q) + \frac{1}{2}\left(\delta^2\sigma_w^2 - b^2\sigma^2\right)\right].$$

Therefore, the asset-specific pricing kernel is

$$\phi(q) = \frac{E^P\,[U'(W_T)\,|q\,]}{E^P\,[U'(W_T)]} = \exp\left[-b\mu + b f_1(q) - \frac{1}{2}b^2\sigma^2\right] \sim L.N.D., \quad (21)$$

with $E^P\,(\ln\phi(q)) = -\frac{1}{2}\delta^2\rho^2\sigma_w^2$ and $V^P\,(\ln\phi(q)) = \delta^2\rho^2\sigma_w^2$, where $b = \delta\rho\sigma_w/\sigma$. Note that the asset-specific pricing kernel in (21) does not change even when we revise the transform function $f(\cdot)$ of the utility setting. This means that the $\phi(q)$ is robust to other utility and wealth specifications which belong to the transformed normal distribution.

By multiplying the asset-specific pricing kernel with the payoff $P_T(q)$ in (3), the security price under true probability is

$$P_0 = e^{-rT} E^P\,[\phi(q)\,P_T(q)] = e^{-rT}\int \phi(q)\,P_T(q) h(q;\, f_1, \mu, \sigma) dq, \quad (22)$$

where $h(q;\, f_1, \mu, \sigma)$ is the transform normal density of the underlying q which is defined in (20). Collecting the terms of $\phi(q)$ and $h(q;\, f_1, \mu, \sigma)$, (22) can be written as

$$P_0 = e^{-rT} \int P_T(q) h(q;\, f_1, \mu + \delta\rho\sigma_w\sigma, \sigma) dq. \tag{23}$$

This equation has a shifted location parameter $\mu + \delta\rho\sigma_w\sigma$ and a scale parameter σ. The price in (23) is still preference dependent. This equation also has to correctly price a primary asset that pays q dollar at time T with current price q_0 (this primary asset need not really exist):

$$q_0 = e^{-rT} \int q \cdot h(q;\, f_1, \mu + \delta\rho\sigma_w\sigma, \sigma) dq. \tag{24}$$

If the preference-dependent term $\mu + \delta\rho\sigma_w\sigma$ in f_1 can be explicitly written as a function of $q_0 e^{rT}$, it can be replaced. For example, Appendices D and E set specific forms of f_1 as S_U and four-parameter lognormal distributions to derive the explicit results. Thus, we denote $\mu + \delta\rho\sigma_w\sigma$ by μ^Q and write down the preference-free pricing equation as

$$P_0 = e^{-rT} \int P_T(q) h(q;\, f_1, \mu^Q, \sigma) dq = e^{-rT} E^Q \left[P_T(q) \right], \tag{25}$$

where μ^Q has a functional form of $q_0 e^{rT}$ and represents the new location-shifted parameter under Q measure which is independent of the preference parameter.

APPENDIX C. PROOF OF THE APPROXIMATION METHOD

The loss ratios L_t have a bull spread payoff between $0 \le L_t \le 1$, for t =1, 2, 3, and the aggregated loss ratio $\sum L_t$ is bounded by $0 \le \sum L_t \le 3$. Assume that the probability of sequential mortality catastrophes is small and the events for an aggregate loss ratio larger than 1 are rare, then

$$\text{Prob}(0 \le \textstyle\sum L_t \le 3) \simeq \text{Prob}(0 \le \sum L_t \le 1) \simeq 1.$$

That is, prob$(1 < \sum L_t \le 3) \simeq 0$, then

$$E^Q \left[\text{Max} \left(1 - \textstyle\sum L_t,\ 0 \right) \right] \simeq E^Q[1 - \sum L_t] = 1 - E^Q[\sum L_t].$$

Additionally, in most of our simulation results, $0 \le E^Q \left[\sum L_t \right] \le 1$ is held, representing that

$$\text{Max} \left(1 - E^Q \left[\textstyle\sum L_t \right],\ 0 \right) \simeq 1 - E^Q \left[\sum L_t \right].$$

Therefore, $E^Q \left[\text{Max} \left(1 - \sum L_t,\ 0 \right) \right] \simeq \text{Max} \left(1 - E^Q \left[\sum L_t \right],\ 0 \right)$.

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APPENDIX D. THE CLOSED-FORM SOLUTION OF CALL OPTION UNDER S_U DISTRIBUTION

The underlying q has a S_U-type transformed normal distribution, $\sinh^{-1}\left(\frac{q-\alpha}{\beta}\right) = x \sim N(\mu, \sigma)$ where $\beta, \sigma > 0$. Following the risk-neutral valuation approach, the price of a call written on q with exercise price K is

$$C \cdot e^{rT} = E^Q[P_T(q)] = E^Q[q - K|\, q \geq K].$$

Replacing q by $\alpha + \beta \sinh x$ yields

$$\begin{aligned}
C \cdot e^{rT} &= E^Q\left[\alpha + \beta \sinh x - K|\, x \geq \sinh^{-1}\left(\tfrac{K-\alpha}{\beta}\right)\right] \\
&= \int_{\sinh^{-1}(\frac{K-\alpha}{\beta})}^{\infty} (\alpha + \beta \sinh x - K) \cdot f\left(x; \mu^Q, \sigma\right) dx \\
&= \frac{\beta}{2} \int_{\sinh^{-1}(\frac{K-\alpha}{\beta})}^{\infty} e^{x} \cdot f\left(x; \mu^Q, \sigma\right) dx \\
&\quad - \frac{\beta}{2} \int_{\sinh^{-1}(\frac{K-\alpha}{\beta})}^{\infty} e^{-x} \cdot f\left(x; \mu^Q, \sigma\right) dx \\
&\quad + (\alpha - K) \int_{\sinh^{-1}(\frac{K-\alpha}{\beta})}^{\infty} f\left(x; \mu^Q, \sigma\right) dx.
\end{aligned}$$

Because x is the normal distribution and by the property of the normal distribution, it follows that

$$\begin{aligned}
C \cdot e^{rT} &= \tfrac{\beta}{2} e^{\mu^Q + \frac{1}{2}\sigma^2} \cdot \Phi\left(\frac{-\sinh^{-1}(\frac{K-\alpha}{\beta}) + \mu^Q}{\sigma} + \sigma\right) \\
&\quad - \tfrac{\beta}{2} e^{-\mu^Q + \frac{1}{2}\sigma^2} \cdot \Phi\left(\frac{-\sinh^{-1}(\frac{K-\alpha}{\beta}) + \mu^Q}{\sigma} - \sigma\right) \\
&\quad + (\alpha - K) \cdot \Phi\left(\frac{-\sinh^{-1}(\frac{K-\alpha}{\beta}) + \mu^Q}{\sigma}\right).
\end{aligned}$$

Therefore, the option price is

$$C = \tfrac{\beta}{2} e^{-rT + \mu^Q + \frac{1}{2}\sigma^2} \cdot \Phi\left(d_1\right) - \tfrac{\beta}{2} e^{-rT - \mu^Q + \frac{1}{2}\sigma^2} \cdot \Phi\left(d_2\right) + (\alpha - K) e^{-rT} \cdot \Phi\left(d_3\right),$$

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where

$$d_1 = \frac{-\sinh^{-1}(\frac{K-\alpha}{\beta})+\mu^Q}{\sigma} + \sigma$$

$$d_2 = \frac{-\sinh^{-1}(\frac{K-\alpha}{\beta})+\mu^Q}{\sigma} - \sigma$$

$$d_3 = \frac{-\sinh^{-1}(\frac{K-\alpha}{\beta})+\mu^Q}{\sigma}, \text{ and } \mu^Q = \sinh^{-1}\left(\frac{1}{\beta}e^{-\frac{1}{2}\sigma^2}(q_0 e^{rT} - \alpha)\right).$$

This is (17). The risk-adjusted mean μ^Q is obtained by the equation correctly pricing the basis asset that pays q dollars at $t = T$ with current price q_0:

$$\begin{aligned} q_0 \cdot e^{rT} &= \int_{-\infty}^{\infty} q \cdot f\left(q;\ f_1, \mu^Q, \sigma\right) dq = \int_{-\infty}^{\infty} (\alpha + \beta \sinh x) \cdot f\left(x;\ \mu^Q, \sigma\right) dx \\ &= \alpha + \beta \int_{-\infty}^{\infty} \left(\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right) \cdot f\left(x; \mu^Q, \sigma\right) dx \\ &= \alpha + \beta e^{\frac{1}{2}\sigma^2}\left(\frac{1}{2}e^{\mu^Q} - \frac{1}{2}e^{-\mu^Q}\right) = \alpha + \beta e^{\frac{1}{2}\sigma^2} \sinh \mu^Q. \end{aligned}$$

This basis asset does not need to be actually traded in the real world, as it is just a mathematical substitution. Rearrangement leads to $\mu^Q = \sinh^{-1}\left(\frac{1}{\beta}e^{-\frac{1}{2}\sigma^2}(q_0 e^{rT} - \alpha)\right)$, which is independent of preference.

APPENDIX E. THE CLOSED-FORM SOLUTION OF CALL OPTION UNDER THE FOUR-PARAMETER LOGNORMAL DISTRIBUTION

The underlying q has a four-parameter lognormal distribution, i.e., $\ln(\frac{q-\alpha}{\beta}) = x \sim N(\mu, \sigma)$ and $(q - \alpha)/\beta > 0, \sigma > 0$. Following the risk-neutral valuation, (7), the price of call option written on q with exercise price K is

$$C \cdot e^{rT} = E^Q[P_T(q)] = E^Q[q - K|\, q \geq K].$$

Substituting $q = \alpha + \beta e^x$ into the above equation yields the result

$$\begin{aligned} C \cdot e^{rT} &= E^Q\left[\alpha + \beta e^x - K \middle| \alpha + \beta e^x \geq K\right] \\ &= \int_{\ln(\frac{K-\alpha}{\beta})}^{\infty} \left(\alpha + \beta e^x - K\right) \cdot f\left(x; \mu^Q, \sigma\right) dx \\ &= \beta \int_{\ln(\frac{K-\alpha}{\beta})}^{\infty} e^x \cdot f\left(x; \mu^Q, \sigma\right) dx + (\alpha - K) \int_{\ln(\frac{K-\alpha}{\beta})}^{\infty} f\left(x; \mu^Q, \sigma\right) dx. \end{aligned}$$

Since x is normally distributed, and by the property of normal distribution, it follows that

$$C{\cdot}e^{rT} = \beta e^{\mu^Q+\frac{1}{2}\sigma^2}{\cdot}\Phi\left(\frac{-\ln(\frac{K-\alpha}{\beta})+\mu^Q}{\sigma}+\sigma\right)+(\alpha-K){\cdot}\Phi\left(\frac{-\ln(\frac{K-\alpha}{\beta})+\mu^Q}{\sigma}\right).$$

Thus, the option price is

$$C = \beta e^{-rT+\mu^Q+\frac{1}{2}\sigma^2}\cdot\Phi\left(d_1\right)+(\alpha-K)e^{-rT}\cdot\Phi\left(d_2\right),$$

where

$$d_1 = \frac{-\ln\left(\frac{K-\alpha}{\beta}\right)+\mu^Q}{\sigma\sqrt{T}}+\sigma,$$

$$d_2 = \frac{-\ln\left(\frac{K-\alpha}{\beta}\right)+\mu^Q}{\sigma} \text{ and } \mu^Q = \ln\left(\frac{1}{\beta}(q_0e^{rT}-\alpha)\right)-\frac{1}{2}\sigma^2.$$

This is (19). μ^Q is obtained from the price of the basis asset that pays q dollars at $t = T$ with current price q_0:

$$q_0\cdot e^{rT} = \int_{-\infty}^{\infty} q\cdot f\left(q;\ f_1,\ \mu^Q,\sigma\right)dq = \int_{-\infty}^{\infty}\left(\alpha+\beta e^x\right)\cdot f\left(x;\ \mu^Q,\sigma\right)dx$$
$$= \alpha+\beta\int_{-\infty}^{\infty} e^x\cdot f\left(x;\mu^Q,\sigma\right)dx = \alpha+\beta e^{\mu^Q+\frac{1}{2}\sigma^2}.$$

Rearrangement leads to $\mu^Q = \ln\left(\frac{1}{\beta}(q_0e^{rT}-\alpha)\right)-\frac{1}{2}\sigma^2$, which is independent of preference.

APPENDIX F. THE TRANSFORM NORMAL DISTRIBUTION ESTIMATORS OF SLIFKER AND SHAPIRO (1980)

Choose a fixed value $z > 0$ and use its cumulative probability of a standard normal distribution to determine the corresponding value in the raw data. For example, choosing $z = 1$ and its cumulative probability is 0.8413. Then, find the corresponding value q in the data with a cumulative probability 0.8413. Similarly, find four points $\pm z$ and $\pm 3z$ to determine the corresponding value in the data. Denote the corresponding value q_{-3z}, q_{-z}, q_z and q_{3z}. Let

$$m = q_{3z} - q_z,$$
$$n = q_{-z} - q_{-3z} \text{ and}$$
$$p = q_z - q_{-z}.$$

If the data satisfy the criterion, $mn/p^2 > 1$, the distribution can be estimated using S_U. The estimates for the four parameters are

$$\alpha = \frac{x_z + x_{-z}}{2} + \frac{n - m}{2(\frac{m}{p} + \frac{n}{p} - 2)}, \qquad \mu = \sinh^{-1}\left[\frac{\frac{m}{p} - \frac{n}{p}}{2\left(\frac{m}{p}\frac{n}{p} - 1\right)^{1/2}}\right],$$

$$\beta = \frac{2p\left(\frac{m}{p}\frac{n}{p} - 1\right)^{1/2}}{\left(\frac{m}{p} + \frac{n}{p} - 2\right)\left(\frac{m}{p} + \frac{n}{p} + 2\right)^{1/2}} \ (\beta > 0), \sigma = \frac{\cosh^{-1}\left[\frac{1}{2}\left(\frac{m}{p} + \frac{n}{p}\right)\right]}{2z} \ (\sigma > 0).$$

If the criterion mn/p^2 approaches 1, the parameters can be estimated using the four-parameter lognormal distribution:

$$\alpha = \frac{x_z + x_{-z}}{2} - \frac{p}{2}\frac{\frac{m}{p} + 1}{\frac{m}{p} - 1}, \qquad \mu = \sinh^{-1}\left[\frac{\frac{m}{p} - \frac{n}{p}}{2\left(\frac{m}{p}\frac{n}{p} - 1\right)^{1/2}}\right],$$

$$\beta = \frac{2p\left(\frac{m}{p}\frac{n}{p} - 1\right)^{1/2}}{\left(\frac{m}{p} + \frac{n}{p} - 2\right)\left(\frac{m}{p} + \frac{n}{p} + 2\right)^{1/2}} \ (\beta > 0), \sigma = \frac{\cosh^{-1}\left[\frac{1}{2}\left(\frac{m}{p} + \frac{n}{p}\right)\right]}{2z} \ (\sigma > 0).$$

If $mn/p^2 < 1$, the data cannot be estimated using the S_U or the four-parameter lognormal distribution.