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UNIVARIATE AND MULTIVARIATE LIKELIHOOD RATIO ORDERING OF GENERALIZED ORDER STATISTICS AND ASSOCIATED CONDITIONAL VARIABLES

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In this article, we establish some results concerning the univariate and multivariate likelihood ratio order of generalized order statistics and the special case of m-generalized order statistics and their associated conditional variables. These results, in addition to being new, also generalizes some of the known results in the literature. Finally, some applications of all these results are indicated.

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1. INTRODUCTION

Distributional as well as dependence properties of conditional ordered random variables have been studied with great interest in the last three decades; see, for example, Langberg, Leon, and Proschan [22], Belzunce, Franco and Ruiz [7], and Belzunce, Lillo, Ruiz, and Shaked [8]. Given a set of order statistics $X_{1:n}, \ldots, X_{n:n}$ from n independent and identically random variables, $X_1, \ldots, X_n$, these authors have discussed various properties of the conditional random variables $[X_{s:n} - t | X_{s-1:n} = t]$ for $1 < s \leq n$. Further results in this direction can be found in the recent works of Li and Zuo [26], Li and Chen [23], and Li and Zhao [24,25]. Some other forms of conditioning have also been considered recently; for example, Asadi and Bairamov [2], Asadi [1], Khaledi and Shaked [19], Li and Zhao [25], and Sadegh [30] have studied properties of the conditional random variables $[X_{s:n} - t | X_{r:n} > t]$ and $[t - X_{r:n} | X_{s:n} \leq t]$ for $1 \leq r \leq s \leq n$. In the case of independent and nonidentically distributed random variables, Zhao, Li, and Balakrishnan [37] established some results for $[X_{s:n} - t | X_{r:n} \leq t < X_{r+1:n}]$ for $1 \leq r < s \leq n$. Extensions of some of these results to the case of record values have been discussed by Khaledi and Shojaei [20] and Khaledi, Amiripour, Hu, and Shojaei [17].

Here, we seek some generalized results in the context of generalized order statistics (GOSs). For the special case of m-GOSs, Hu, Jin, and Khaledi [11], Xie and Hu [34], and Zhao and Balakrishnan [36] have discussed stochastic comparisons in the likelihood ratio order of the above conditional random variables replacing the usual order statistics by m-GOSs. The purpose of this article is twofold: first, to present new results and extend some of the known results to the case of GOSs and, second, to provide a new proof for these results in a unified and easy manner.

The rest of this article is organized as follows. In Section 2 we first recall some definitions and preliminary results that are needed for the main results to be established in Section 3. In Section 4 we describe some applications of these results. Throughout the article, the survival function associated with a distribution function $F(x)$ will be denoted by $\overline{F}(x) \equiv 1 - F(x)$.

2. DEFINITIONS AND KNOWN RESULTS

Order statistics and record values have found important applications in several fields of science and engineering, as is evident from the volumes of Balakrishnan and Rao [5,6]. Due to the close similarity among some distributional, structural, and dependence properties of order statistics and record values, Kamps [14,15] introduced the model of generalized order statistics, which includes, as particular cases, random vectors of order statistics and record values and some other models of interest such as sequential order statistics and progressively censored order statistics.

We now present the definition of generalized order statistics, due to Kamps [14,15].

Definition 2.1: *Let $n \in \mathbb{N}$, $k \geq 1$, $m_1, \ldots, m_{n-1} \in \mathbb{R}$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$, be parameters such that $\gamma_r = k + n - r + M_r \geq 1$ for all $r \in \{1, \ldots, n-1\}$, and*

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let $\tilde{m} = (m_1, \ldots, m_{n-1})$ *if* $n \geq 2$ ($\tilde{m} \in \mathbb{R}$ *arbitrary, if* $n = 1$). *The random vector* $(U_{(1,n,\tilde{m},k)}, \ldots, U_{(n,n,\tilde{m},k)})$ *with joint density function*

$$h(u_1, \ldots, u_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{j=1}^{n-1} (1 - u_j)^{m_j} \right) (1 - u_n)^{k-1},$$

defined over the cone $0 \leq u_1 \leq \cdots \leq u_n \leq 1$, *is called the uniform generalized order statistics. Now, for a given distribution function* F, *the random vector*

$$(X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)}) \equiv \left(F^{-1}(U_{(1,n,\tilde{m},k)}), \ldots, F^{-1}(U_{(n,n,\tilde{m},k)})\right)$$

is called the generalized order statistics (GOSs) based on the distribution F.

In the special case when $m_1 = \cdots = m_{n-1} = m$, the variables $(X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)})$ are called m-GOSs and are denoted by $(X_{(1,n,m,k)}, \ldots, X_{(n,n,m,k)})$.

Stochastic comparisons of GOSs have been discussed rather extensively by Franco, Ruiz, and Ruiz [10], Belzunce, Mercade, and Ruiz [9], Khaledi [16], Hu and Zhuang [12,13], Khaledi and Kochar [18], Qiu and Wu [29], and Xie and Hu [35]. The stochastic comparison of conditional generalized order statistics has also been considered as a natural extension of the corresponding results on conditional order statistics and record values. Most of these results are on the likelihood ratio order that we describe; one can refer to Shaked and Shanthikumar [32] for pertinent details.

Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors with density functions $f_{\mathbf{X}}$ and $f_{\mathbf{Y}}$, respectively. We say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the multivariate likelihood ratio order, denoted by $\mathbf{X} \leq_{\rm lr} \mathbf{Y}$, if

$$\begin{aligned} &f_{\mathbf{X}}(x_1, x_2, \ldots, x_n) f_{\mathbf{Y}}(y_1, y_2, \ldots, y_n) \\ &\quad \leq f_{\mathbf{X}}(x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n) f_{\mathbf{Y}}(x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n) \end{aligned}$$

for all $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, where $\wedge$ and $\vee$ denote the minimum and maximum operations, respectively.

The likelihood ratio order can be used to define the MTP_2 (multivariate totally positive of order 2) dependence notion. Given a random vector $\mathbf{X}$ with density function $f_{\mathbf{X}}$, we say that $\mathbf{X}$ or $f_{\mathbf{X}}$ is MTP_2 if $\mathbf{X} \leq_{\rm lr} \mathbf{X}$.

In the univariate case, given two random variables X and Y with density functions f and g, respectively, we say that X is less than Y in the likelihood ratio order, denoted by $X \leq_{\rm lr} Y$, if $f(t)g(s) \leq f(s)g(t)$ for all $s < t \in \mathbb{R}$. The likelihood ratio order is related to the hazard rate order. Given two variables X and Y with survival functions $\overline{F}$ and $\overline{G}$, respectively, we say that X is less than Y in the hazard rate order, denoted by $X \leq_{\rm hr} Y$, if $\overline{F}(t)\overline{G}(s) \leq \overline{F}(s)\overline{G}(t)$ for all $s < t \in \mathbb{R}$. It is known that

$$X \leq_{\rm lr} Y \Rightarrow X \leq_{\rm hr} Y. \tag{2.1}$$

We now recall some properties of the likelihood ratio order that are useful in the subsequent developments. These properties include the preservation under monotone transformations, marginalization, and conditioning on sublattices. A subset $L \subseteq \mathbb{R}^n$ is called a sublattice if $\mathbf{x}, \mathbf{y} \in L$ implies $\mathbf{x} \wedge \mathbf{y} \in L$ and $\mathbf{x} \vee \mathbf{y} \in L$.

THEOREM 2.2 (Theorem 1.C.18 of Shaked and Shanthikumar [32]): *Given two random variables X and Y, if $X \le_{\rm lr} Y$ and ϕ is any increasing (decreasing) function, then $\phi(X) \le_{\rm lr} (\ge_{\rm lr})\ \phi(Y)$.*

THEOREM 2.3: *Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors such that $\mathbf{X} \le_{\rm lr} \mathbf{Y}$. Then we have the following:*

(i) *$\mathbf{X}_I \le_{\rm lr} \mathbf{Y}_I$ for all subsets $I \subseteq \{1, 2, \ldots, n\}$, where $\mathbf{X}_I$ and $\mathbf{Y}_I$ are the vectors of components of $\mathbf{X}$ and $\mathbf{Y}$ with index in I (Theorem 6.E.4 of Shaked and Shanthikumar [32]).*

(ii) *$[\mathbf{X}|\mathbf{X} \in L] \le_{\rm lr} [\mathbf{Y}|\mathbf{Y} \in L]$ for all sublattices $L \subseteq \mathbb{R}^n$ (Theorem 3.11.4 of Müller and Stoyan [27]).*

For example, given two distributions F and G and m-GOSs based on F and G, given by $(X_{(1,n,m,k)}, \ldots, X_{(n,n,m,k)})$ and $(Y_{(1,n,m,k)}, \ldots, Y_{(n,n,m,k)})$, respectively, Zhao and Balakrishnan [36] presented conditions for

$$[X_{(s,n,m,k)} - t | X_{(r,n,m,k)} > t] \le_{\rm lr} [Y_{(s,n,m,k)} - t | Y_{(r,n,m,k)} > t]$$

for $1 \le r \le s \le n, m \ge -1$ and any $t \in \mathbb{R}$. One of the goals of this article is to provide similar results on the likelihood ratio order more generally for conditional GOSs, thus generalizing those for the m-GOSs.

Some other results have also been established for conditional m-GOSs in the same vector. For example, we recall the following result.

THEOREM 2.4 (Theorem 3.2 of Hu et al. [11]): *Given a random vector $(X_{(1,n,m,k)}, \ldots, X_{(n,n,m,k)})$ of m-GOSs from an absolutely continuous distribution, we have*

$$[X_{(s,n,m,k)} - t | X_{(r,n,m,k)} > t] \le_{\rm lr} [X_{(s',n',m,k)} - t | X_{(r',n',m,k)} > t]$$

for $s > r$ and $s' - s = r' - r \ge \max\{0, n' - n\}$ and any $t \in \mathbb{R}$.

We now present some technical results that are useful for establishing the main results in the next section.

LEMMA 2.5 (Lemma 2.5 of Zhao and Balakrishnan [36]): *Given random variables X and Y with distribution functions F and G and hazard rate functions r_F and r_G, respectively, if either*

(i) $X \le_{\rm lr} Y$ and $m \ge 0$ or

(ii) $X \le_{\rm hr} Y$, $r_G(x)/r_F(x)$ is increasing in x and $-1 \le m < 0$,

then

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(a) $h_m(G(x))/h_m(F(x))$ *is increasing in* $x \in \mathbb{R}$, *and*

(b) the function

$$\phi(x,u) = \frac{h_m(G(x)) - h_m(G(u))}{h_m(F(x)) - h_m(F(u))}$$

is increasing in $(x,u) \in \mathbb{R}$, *where*

$$h_m(x) = \begin{cases} \frac{1}{m+1}(1-(1-x)^{m+1}), & m \neq -1 \\ -\log(1-x), & m = -1. \end{cases} \quad \textbf{(2.2)}$$

We end this section by presenting the expression for the joint density function of any subset of m-GOSs.

LEMMA 2.6: *Given a random vector* $(X_{(1,n,m,k)}, \ldots, X_{(n,n,m,k)})$ *of m-GOSs from an absolutely distribution F and density function f, the joint density of* $(X_{(r_1,n,m,k)}, X_{(r_2,n,m,k)}, \ldots, X_{(r_i,n,m,k)})$, *for* $r_1 < r_2 < \cdots < r_i$ *and* $\{r_1, r_2, \ldots, r_i\} \subset \{1,2,3,\ldots,n\}$, *is given by*

$$\begin{aligned} f_{(r_1,r_2,\ldots,r_i)}(x_{r_1}, x_{r_2}, \ldots, x_{r_i}) = {} & \frac{c_{r_i-1}}{(r_1-1)!\prod_{j=1}^{i-1}(r_{j+1}-r_j)!} \\ & \times (\overline{F}(x_{r_i}))^{k+n-r_i+M_{r_i}-1} f(x_{r_i}) h_m^{r_1-1}(F(x_{r_1})) \\ & \times \prod_{j=1}^{i-1} (\overline{F}(x_{r_j}))^m [h_m(F(x_{r_{j+1}})) - h_m(F(x_{r_j}))]^{r_{j+1}-r_j} f(x_{r_j}), \end{aligned} \quad \textbf{(2.3)}$$

for $x_{r_1} < x_{r_2} < \cdots < x_{r_i}$, *where* $c_{r_i-1} = \prod_{i=1}^{r_i} \gamma_i$ *and* h_m *is as given in* 2.2.

PROOF: The joint density of $(X_{(r_1,n,m,k)}, X_{(r_2,n,m,k)}, \ldots, X_{(r_i,n,m,k)})$, for $r_1 < r_2 < \cdots < r_i$, follows from the joint density of uniform generalized order statistics

$$(U_{(r_1,n,m,k)}, U_{(r_2,n,m,k)}, \ldots, U_{(r_i,n,m,k)}).$$

Taking into account that $U_{(1,n,m,k)} \leq U_{(2,n,m,k)} \leq \cdots \leq U_{(r_i,n,m,k)}$, and denoting by $h_{(u_1,u_2,\ldots,u_{r_i})}$ the joint density of $(U_{(1,n,m,k)}, U_{(2,n,m,k)}, \ldots, U_{(r_i,n,m,k)})$, the joint density of

$$(U_{(r_1,n,m,k)}, U_{(r_2,n,m,k)}, \ldots, U_{(r_i,n,m,k)})$$

can be derived as follows:

$$\begin{aligned} & h_{(r_1,r_2,\ldots,r_i)}(u_{r_1}, u_{r_2}, \ldots, u_{r_i}) \\ & = \int_0^{u_{r_1}} \int_{u_1}^{u_{r_1}} \cdots \int_{u_{r_1-2}}^{u_{r_1}} \int_{u_{r_1}}^{u_{r_2}} \int_{u_{r_1}+1}^{u_{r_2}} \cdots \int_{u_{r_2}-2}^{u_{r_2}} \cdots \int_{u_{r_{i-1}-2}}^{u_{r_{i-1}}} \int_{u_{r_{i-1}}}^{u_{r_i}} \int_{u_{r_{i-1}+1}}^{u_{r_i}} \cdots \end{aligned}$$

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$$\times \int_{u_{r_i}-3}^{u_{r_i}} \int_{u_{r_i-2}}^{u_{r_i}} h(u_1, u_2 \ldots, u_{r_i})\, du_{r_i-1}\, du_{r_i-2} \ldots du_{r_{i-1}+2}\, du_{r_{i-1}+1}\, du_{r_{i-1}-1} \ldots$$

$$\times du_{r_2-1} \cdots du_{r_1+2}\, du_{r_1+1}\, du_{r_1-1} \cdots du_2\, du_1$$

$$= \int_0^{u_{r_1}} \int_{u_1}^{u_{r_1}} \cdots \int_{u_{r_1-2}}^{u_{r_1}} \int_{u_{r_1}}^{u_{r_2}} \int_{u_{r_1}+1}^{u_{r_2}} \cdots \int_{u_{r_2}-2}^{u_{r_2}} \cdots \int_{u_{r_{i-1}-2}}^{u_{r_{i-1}}} \int_{u_{r_{i-1}}}^{u_{r_i}} \int_{u_{r_{i-1}+1}}^{u_{r_i}} \cdots$$

$$\times \int_{u_{r_i}-3}^{u_{r_i}} \int_{u_{r_i-2}}^{u_{r_i}} c_{r_i-1}(1-u_{r_i})^{k+n-r_i+M_{r_i}-1} \prod_{j=1}^{r_1-1} h'_m(u_j)(1-u_{r_1})^m$$

$$\times \prod_{j=r_1+1}^{r_2-1} h'_m(u_j)(1-u_{r_2})^m \prod_{j=r_2+1}^{r_3-1} h'_m(u_j) \cdots (1-u_{r_{i-1}})^m$$

$$\times \prod_{j=r_{i-1}+1}^{r_i-1} h'_m(u_j) du_{r_i-1}\, du_{r_i-2} \cdots du_{r_{i-1}+2}\, du_{r_{i-1}+1}\, du_{r_{i-1}-1} \cdots$$

$$\times du_{r_2-1} \cdots du_{r_1+2}\, du_{r_1+1}\, du_{r_1-1} \cdots du_2\, du_1$$

$$= c_{r_i-1}(1-u_{r_i})^{k+n-r_i+M_{r_i}-1} \int_0^{u_{r_1}} \int_{u_1}^{u_{r_1}} \cdots \int_{u_{r_1-2}}^{u_{r_1}} \prod_{j=1}^{r_1-1} h'_m(u_j)(1-u_{r_1})^m \cdots$$

$$\times \int_{u_{r_1}}^{u_{r_2}} \int_{u_{r_1}+1}^{u_{r_2}} \cdots \int_{u_{r_2-2}}^{u_{r_2}} \prod_{j=r_1+1}^{r_2-1} h'_m(u_j)(1-u_{r_2})^m \cdots$$

$$\times \int_{u_{r_{i-1}-2}}^{u_{r_{i-1}}} \int_{u_{r_{i-1}}}^{u_{r_i}} \int_{u_{r_{i-1}+1}}^{u_{r_i}} \cdots \int_{u_{r_i-3}}^{u_{r_i}} \int_{u_{r_i-2}}^{u_{r_i}} \prod_{j=r_{i-1}+1}^{r_i-1} h'_m(u_j)(1-u_{r_{i-1}})^m$$

$$\times du_{r_i-1}\, du_{r_i-2} \cdots du_{r_{i-1}+2}\, du_{r_{i-1}+1}$$

$$\times du_{r_{i-1}-1} \cdots du_{r_2-1} \cdots du_{r_1+2}\, du_{r_1+1}$$

$$\times du_{r_1-1} \cdots du_2\, du_1$$

$$= \frac{c_{r_i-1}}{(r_i - r_{i-1} - 1)!(r_{i-1} - r_{i-2} - 1)! \cdots (r_2 - r_1 - 1)!(r_1 - 1)!}$$

$$\times (1-u_{r_i})^{k+n-r_i+M_{r_i}-1}(1-u_{r_{i-1}})^m \cdots (1-u_{r_1})^m$$

$$\times [h_m(u_{r_i}) - h_m(u_{r_{i-1}})]^{r_i-r_{i-1}-1}[h_m(u_{r_{i-1}}) - h_m(u_{r_{i-2}})]^{r_{i-1}-r_{i-2}-1} \ldots$$

$$[h_m(u_{r_2}) - h_m(u_{r_1})]^{r_2-r_1-1} g_m^{r_1-1}(u_{r_1})$$

$$= \frac{c_{r_i-1}}{(r_1-1)! \prod_{j=1}^{i-1}(r_{j+1}-r_j)!}(1-u_{r_i})^{k+n-r_i+M_{r_i}-1} g_m^{r_1-1}(u_{r_1})$$

$$\times \prod_{j=1}^{i-1}(1-u_{r_j}))^m [h_m(u_{r_{j+1}}) - h_m(u_{r_j})]^{r_{j+1}-r_j}. \quad \textbf{(2.4)}$$

The result in (2.3) follows readily from the expression in (2.4). ■

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3. NEW RESULTS AND GENERALIZATIONS

First, we present a result on the likelihood ratio order of conditional GOSs under different forms of conditioning.

THEOREM 3.7: *Let X and Y be absolutely continuous variables with distribution functions F and G and hazard rates r_F and r_G, respectively. Let $\mathbf{X} = (X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)})$ and $\mathbf{Y} = (Y_{(1,n,\tilde{m},k)}, \ldots, Y_{(n,n,\tilde{m},k)})$ be random vectors of generalized order statistics based on F and G, respectively. If either*

(i) $X \leq_{\text{lr}} Y$ *and* $m_i \geq 0$ *for all* i, *or*

(ii) $X \leq_{\text{hr}} Y$, $r_G(x)/r_F(x)$ *is increasing, and* $m_i \geq -1$ *for all* i,

then

$$[X_{(s,n,\tilde{m},k)} | \mathbf{X} \in L] \leq_{\text{lr}} [Y_{(s,n,\tilde{m},k)} | \mathbf{Y} \in L]$$

for all sublattices $L \subseteq \mathbb{R}^n$.

PROOF: First, we mention that under conditions (i) and (ii), Belzunce et al. [9] proved that $(X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)}) \leq_{\text{lr}} (Y_{(1,n,\tilde{m},k)}, \ldots, Y_{(n,n,\tilde{m},k)})$. Now, upon applying parts (ii) and (i) in Theorem 2.3, we obtain the result. ■

Although this result is easy to prove from properties of the likelihood ratio order, there are numerous applications of this result, given the different sublattices that we can choose. We now list several examples of this result.

Example 3.8: Let $(x_{(1)} \leq \cdots \leq x_{(n)})$ denote a nondecreasing arrangement of the components of a vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Now, let us consider the sublattice $L = [(x_{(1)}, \ldots, x_{(n)}) \in \mathbb{R}^n | x_{(r)} > t]$. Then, from Theorems 3.7 and 2.2, we have, for $r \leq s$ and any $t \in \mathbb{R}$, and under conditions (i) and (ii) in Theorem 3.7, that

$$[X_{(s,n,\tilde{m},k)} - t | X_{(r,n,\tilde{m},k)} > t] \leq_{\text{lr}} [Y_{(s,n,\tilde{m},k)} - t | Y_{(r,n,\tilde{m},k)} > t].$$

This result extends Theorem 3.1 of Zhao and Balakrishnan [36] and Theorem 2.2 of Khaledi et al. [17] to the case of GOSs.

Example 3.9: Let $(x_{(1)} \leq \cdots \leq x_{(n)})$ denote a nondecreasing arrangement of the components of a vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Now, let us consider the sublattice $L = [(x_{(1)}, \ldots, x_{(n)}) \in \mathbb{R}^n | x_{(s)} \leq t]$. Then, from Theorems 3.7 and 2.2, we have, for $r < s$ and any $t \in \mathbb{R}$, and under conditions (i) and (ii) in Theorem 3.7, that

$$[t - X_{(r,n,\tilde{m},k)} | X_{(s,n,\tilde{m},k)} \leq t] \geq_{\text{lr}} [t - Y_{(r,n,\tilde{m},k)} | Y_{(s,n,\tilde{m},k)} \leq t].$$

This result extends Theorem 3.2 in Zhao and Balakrishnan [36] and Theorem 3.3 (b) in Khaledi et al. [17] to the case of GOSs.

Example 3.10: Let $(x_{(1)} \leq \cdots \leq x_{(n)})$ denote a nondecreasing arrangement of the components of a vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Now, let us consider the sublattice $L = [(x_{(1)}, \ldots, x_{(n)}) \in \mathbb{R}^n | t_1 < x_{(r)} \leq t_2]$. Then, from Theorems 3.7 and 2.2, we have, for $r \leq s$ and any $t_1 < t_2 \in \mathbb{R}$, and under conditions (i) and (ii) in Theorem 3.7, that

$$[X_{(s,n,\tilde{m},k)} - t_1 | t_1 < X_{(r,n,\tilde{m},k)} \leq t_2] \leq_{\text{lr}} [Y_{(s,n,\tilde{m},k)} - t_1 | t_1 < Y_{(r,n,\tilde{m},k)} \leq t_2].$$

Example 3.11: Let $(x_{(1)} \leq \cdots \leq x_{(n)})$ denote a nondecreasing arrangement of the components of a vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Now, let us consider the sublattice $L = [(x_{(1)}, \ldots, x_{(n)}) \in \mathbb{R}^n | x_{(r)} \leq t < x_{(r+1)}]$. Then, from Theorems 3.7 and 2.2, we have, for $r < s$ and any $t \in \mathbb{R}$, and under conditions (i) and (ii) in Theorem 3.7, that

$$\begin{aligned}&[X_{(s,n,\tilde{m},k)} - t | X_{(r,n,\tilde{m},k)} \leq t < X_{(r+1,n,\tilde{m},k)}] \\ &\quad \leq_{\text{lr}} [Y_{(s,n,\tilde{m},k)} - t | Y_{(r,n,\tilde{m},k)} \leq t < Y_{(r+1,n,\tilde{m},k)}].\end{aligned}$$

Example 3.12: Let $(x_{(1)} \leq \cdots \leq x_{(n)})$ denote a nondecreasing arrangement of the components of a vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Now, let us consider the sublattice $L = [(x_{(1)}, \ldots, x_{(n)}) \in \mathbb{R}^n | x_{(r)} = t]$. Then, from Theorems 3.7 and 2.2, we have, for $r < s$ and any $t \in \mathbb{R}$, and under conditions (i) and (ii) in Theorem 3.7, that

$$[X_{(s,n,\tilde{m},k)} - t | X_{(r,n,\tilde{m},k)} = t] \leq_{\text{lr}} [Y_{(s,n,\tilde{m},k)} - t | Y_{(r,n,\tilde{m},k)} = t].$$

Example 3.13: Let $(x_{(1)} \leq \cdots \leq x_{(n)})$ denote a nondecreasing arrangement of the components of a vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Now, let us consider the sublattice $L = [(x_{(1)}, \ldots, x_{(n)}) \in \mathbb{R}^n | t_1 < x_{(r)}, x_{(s)} \leq t_2]$. Then, from Theorems 3.7 and 2.2, we have, that for $r \leq p \leq s$ and any $t_1 < t_2 \in \mathbb{R}$, and under conditions (i) and (ii) in Theorem 3.7, that

$$\begin{aligned}&[X_{(p,n,\tilde{m},k)} - t_1 | t_1 < X_{(r,n,\tilde{m},k)}, X_{(s,n,\tilde{m},k)} \leq t_2] \\ &\quad \leq_{\text{lr}} [Y_{(p,n,\tilde{m},k)} - t_1 | t_1 < Y_{(r,n,\tilde{m},k)}, Y_{(s,n,\tilde{m},k)} \leq t_2].\end{aligned}$$

Example 3.14: Let $(x_{(1)} \leq \cdots \leq x_{(n)})$ denote a nondecreasing arrangement of the components of a vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Now, let us consider the sublattice $L = [(x_{(1)}, \ldots, x_{(n)}) \in \mathbb{R}^n | x_{(r)} \leq t_1, x_{(s)} > t_2]$. Then, from Theorems 3.7 and 2.2, we have, that for $r < p < s$ and any $t_1 < t_2 \in \mathbb{R}$, and under conditions (i) and (ii) in Theorem 3.7, that

$$\begin{aligned}&[X_{(p,n,\tilde{m},k)} - t_1 | X_{(r,n,\tilde{m},k)} \leq t_1, X_{(s,n,\tilde{m},k)} > t_2] \leq_{\text{lr}} [Y_{(p,n,\tilde{m},k)} - t_1 | Y_{(r,n,\tilde{m},k)} \\ &\quad \leq t_1, Y_{(s,n,\tilde{m},k)} > t_2].\end{aligned}$$

Example 3.15: Let $(x_{(1)} \leq \cdots \leq x_{(n)})$ denote a nondecreasing arrangement of the components of a vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Now, let us consider the sublattice $L = [(x_{(1)}, \ldots, x_{(n)}) \in \mathbb{R}^n | x_{(r)} < t_1 \leq x_{(r+1)}, t_2 \leq x_s]$. Then, from Theorems 3.7 and

2.2, we have, for $r < s$ and any $t_1 < t_2 \in \mathbb{R}$, and under conditions (i) and (ii) in Theorem 3.7, that

$$[X_{(s,n,\tilde{m},k)} - t_2 | X_{(r,n,\tilde{m},k)} < t_1 < X_{(r+1,n,\tilde{m},k)}, t_2 \le X_{(s,n,\tilde{m},k)}] \\ \le_{\rm lr} [Y_{(s,n,\tilde{m},k)} - t_2 | Y_{(r,n,\tilde{m},k)} < t_1 \le Y_{(r+1,n,\tilde{m},k)}, t_2 \le Y_{(s,n,\tilde{m},k)}].$$

Observe that Theorem 3.7 compares conditional GOSs based on two different distributions but with the same set of parameters. Next, we establish a similar result for GOSs based on the same distribution but with different parameters.

THEOREM 3.16: *Let* $\mathbf{X} = (X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)})$ *and* $\mathbf{X}' = (X_{(1,n,\tilde{m}',k')}, \ldots, X_{(1,n,\tilde{m}',k')})$ *be random vectors of GOSs based on the same distribution F and with parameters* k, m_i, $i = 1, \ldots, n-1$ *and* k', m_i', $i = 1, \ldots, n-1$, *respectively. If* $k \ge k'$ *and* $m_i \ge m_i'$, *for all* $i = 1, \ldots, n-1$, *then*

$$[X_{(s,n,\tilde{m},k)} | \mathbf{X} \in L] \le_{\rm lr} [X_{(s,n,\tilde{m}',k')} | \mathbf{X}' \in L]$$

for all sublattices $L \subseteq \mathbb{R}^n$.

PROOF: In this case, from a result of Belzunce et al. [9], we have $(X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)}) \le_{\rm lr} (X_{(1,n,\tilde{m}',k')}, \ldots, X_{(n,n,\tilde{m}',k')})$, and the result then follows along the same lines as in Theorem 3.7. ■

A combination of Theorems 3.7 and 3.16 will provide new results for the likelihood ratio order of conditional GOSs from different populations with different parameters. For example, we can obtain the following result.

THEOREM 3.17: *Let X and Y be absolutely continuous variables with distribution functions F and G and hazard rates* r_F *and* r_G, *respectively. Let* $\mathbf{X} = (X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)})$ *and* $\mathbf{Y} = (Y_{(1,n,\tilde{m}',k')}, \ldots, Y_{(n,n,\tilde{m}',k')})$ *be random vectors of GOSs based on F and G and with parameters k and* m_i, $i = 1, \ldots, n-1$ *and* k', m_i', $i = 1, \ldots, n-1$, *respectively. If* $k \ge k'$ *and* $m_i \ge m_i'$ *for all* $i = 1, \ldots, n-1$, *and if either*

(i) $X \le_{\rm lr} Y$ *and* $m_i \ge 0$ *for all i or* $m_i' \ge 0$ *for all i, or*

(ii) $X \le_{\rm hr} Y$, $r_G(x)/r_F(x)$ *is increasing, and* $m_i \ge -1$ *for all i or* $m_i' \ge -1$ *for all i,*

then

$$[X_{(s,n,\tilde{m},k)} - t | X_{(r,n,\tilde{m},k)} > t] \le_{\rm lr} [Y_{(s,n,\tilde{m}',k')} - t | Y_{(r,n,\tilde{m}',k')} > t]$$

for $r < s$ *and any* $t \in \mathbb{R}$.

This result extends Theorem 3.3 of Zhao and Balakrishnan [36] to the case of GOSs. We can also combine the above results with Theorem 2.4. For example, if we

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consider the sublattice in Example 3.8 and the result in Theorem 2.4, then we obtain the following result.

THEOREM 3.18: *Let X and Y be absolutely continuous variables with distribution functions F and G and hazard rates r_F and r_G, respectively. Let* $\mathbf{X} = (X_{(1,n,m,k)}, \ldots, X_{(n,n,m,k)})$ *and* $\mathbf{Y} = (Y_{(1,n,m,k)}, \ldots, Y_{(n,n,m,k)})$ *be random vectors of m-GOSs based on F and G, respectively. For* $r \leq s \leq n$, $s' - s = r' - r \geq \max\{0, n' - n\}$, *and any* $t \in \mathbb{R}$, *if either*

(i) $X \leq_{\rm lr} Y$ *and* $m \geq 0$, *or*

(ii) $X \leq_{\rm hr} Y$, $r_G(x)/r_F(x)$ *is increasing in x, and* $-1 \leq m < 0$,

then

$$[X_{(s,n,m,k)} - t | X_{(r,n,m,k)} > t] \leq_{\rm lr} [Y_{(s',n',m,k)} - t | Y_{(r',n',m,k)} > t].$$

Next, we establish a multivariate likelihood ratio ordering result for subsets of m-GOSs.

THEOREM 3.19: *Let X and Y be absolutely continuous variables with distribution functions F and G, densities f and g, and hazard rates r_F and r_g, respectively. Let* $\mathbf{X} = (X_{(1,n,m,k)}, \ldots, X_{(n,n,m,k)})$ *and* $\mathbf{Y} = (Y_{(1,n',m,k)}, \ldots, Y_{(n',n',m,k)})$ *be random vectors of m-GOSs based on distributions F and G, respectively. For* $r_1 \leq r_2 \leq \cdots \leq r_i \leq n$, $r'_1 \leq r'_2 \leq \cdots \leq r'_i \leq n'$, $r'_1 - r_1 = r'_2 - r_2 = \cdots = r'_i - r_i \geq \max\{0, n' - n\}$, *if either*

(i) $X \leq_{\rm lr} Y$ *and* $m \geq 0$, *or*

(ii) $X \leq_{\rm hr} Y$, $r_G(x)/r_F(x)$ *is increasing in x, and* $-1 \leq m < 0$,

then

$$(X_{(r_1,n,m,k)}, X_{(r_2,n,m,k)}, \ldots, X_{(r_i,n,m,k)}) \leq_{\rm lr} (Y_{(r'_1,n',m,k)}, Y_{(r'_2,n',m,k)}, \ldots, Y_{(r'_i,n',m,k)}).$$

PROOF: Let us use $f_{r_1,r_2,\ldots,r_i}(x_1, x_2, \ldots, x_i)$ and $g_{r'_1,r'_2,\ldots,r'_i}(x_1, x_2, \ldots, x_i)$ to denote the joint density of $(X_{(r_1,n,m,k)}, X_{(r_2,n,m,k)}, \ldots, X_{(r_i,n,m,k)})$ and $(Y_{(r'_1,n',m,k)}, Y_{(r'_2,n',m,k)}, \ldots, Y_{(r'_1,n',m,k)})$, respectively. Following Kochar [21] and Spizzichino [33, p. 109], if

$$\frac{g_{r'_1,r'_2,\ldots,r'_i}(x_1, x_2, \ldots, x_i)}{f_{r_1,r_2,\ldots,r_i}(x_1, x_2, \ldots, x_i)} \text{ is increasing in } (x_1 < x_2 < \cdots < x_i) \in \mathbb{R}^i \qquad \textbf{(3.1)}$$

and $(X_{(r_1,n,m,k)}, X_{(r_2,n,m,k)}, \ldots, X_{(r_i,n,m,k)})$ or $(Y_{(r'_1,n',m,k)}, Y_{(r'_2,n',m,k)}, \ldots, Y_{(r'_1,n',m,k)})$ or both are MTP_2, then

$$(X_{(r_1,n,m,k)}, X_{(r_2,n,m,k)}, \ldots, X_{(r_i,n,m,k)}) \leq_{\rm lr} (Y_{(r'_1,n',m,k)}, Y_{(r'_2,n',m,k)}, \ldots, Y_{(r'_1,n',m,k)}).$$

Belzunce et al. [9] showed that any vector of GOSs is MTP_2, and given that the MTP_2 property is preserved under marginalization, we have

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$(X_{(r_1,n,m,k)}, X_{(r_2,n,m,k)}, \ldots, X_{(r_i,n,m,k)})$ and $(Y_{(r_1',n',m,k)}, Y_{(r_2',n',m,k)}, \ldots, Y_{(r_1',n',m,k)})$ to be MTP_2.

Let us now prove (3.1). First, let us assume the conditions in (i). In this case, we have from (2.2), for $(x_1 < x_2 < \cdots < x_i)$, that

$$\begin{aligned} &\frac{g_{r_1',r_2',\ldots,r_i'}(x_1,x_2,\ldots,x_i)}{f_{r_1,r_2,\ldots,r_i}(x_1,x_2,\ldots,x_i)} \\ &\quad \propto \frac{g(x_i)}{f(x_i)}\left(\frac{\overline{G}(x_i)}{\overline{F}(x_i)}\right)^{k-1+(m+1)(n'-r_i')} \prod_{j=1}^{i-1}\left(\frac{\overline{G}(x_j)}{\overline{F}(x_j)}\right)^{m} \frac{g(x_j)}{f(x_j)} \\ &\quad \times \left(\frac{h_m(G(x_1))}{h_m(F(x_1)}\right)^{r_1-1} \prod_{j=1}^{i-1}\left(\frac{h_m(G(x_{j+1}))-h_m(G(x_j))}{h_m(F(x_{j+1}))-h_m(F(x_j))}\right)^{r_{j+1}'-r_j'-1} \\ &\quad \times \frac{(h_m(G(x_1))^{r_1'-r_1}}{(\overline{F}(x_i))^{(m+1)(n-r_i-n'+r_i')}}. \end{aligned}$$

Given that $m \geq 0$ and $(k-1)+(m+1)(n'-r_i') \geq 0$, we have $g(x_i)/f(x_i)$, $(\overline{G}(x_i)/\overline{F}(x_i))^{k-1+(m+1)(n'-r_i')}$, $(\overline{G}(x_j)/\overline{F}(x_j))^m$, and $g(x_j)/f(x_j)$ all to be increasing functions in $x_i, x_j \in \mathbb{R}$, respectively.

From Lemma 2.5, we also have $((h_m(G(x_{j+1}))-h_m(G(x_j)))/(h_m(F(x_{j+1}))-h_m(F(x_j))))^{r_{j+1}'-r_j'-1}$ and $(h_m(G(x_1))/h_m(F(x_1))^{r_1-1}$ to be increasing functions in $(x_j,x_{j+1}) \in \mathbb{R}^2$ and $x_1 \in \mathbb{R}$, respectively.

Finally, given that $n-r_i \geq n'-r_i'$, we have $1/(\overline{F}(x_i))^{(m+1)(n-r_i-n'+r_i')}$ to be increasing in $x_i \in \mathbb{R}$, and given that $r_1'-r_1 \geq 0$, $(h_m(G(x_1))^{r_1'-r_1}$ is also increasing in $x_1 \in \mathbb{R}$.

If we assume the conditions in (ii), then, for $(x_1 < x_2 < \cdots < x_i)$, we have

$$\begin{aligned} &\frac{g_{r_1',r_2',\ldots,r_i'}(x_1,x_2,\ldots,x_i)}{f_{r_1,r_2,\ldots,r_i}(x_1,x_2,\ldots,x_i)} \\ &\quad \propto \frac{g(x_i)}{f(x_i)}\left(\frac{\overline{G}(x_i)}{\overline{F}(x_i)}\right)^{k-1+(m+1)(n'-r_i')} \prod_{j=1}^{i-1}\left(\frac{\overline{G}(x_j)}{\overline{F}(x_j)}\right)^{m+1} \frac{r_G(x_j)}{r_F(x_j)} \\ &\quad \times \left(\frac{h_m(G(x_1))}{h_m(F(x_1)}\right)^{r_1-1} \prod_{j=1}^{i-1}\left(\frac{h_m(G(x_{j+1}))-h_m(G(x_j))}{h_m(F(x_{j+1}))-h_m(F(x_j))}\right)^{r_{j+1}'-r_j'-1} \\ &\quad \times \frac{(h_m(G(x_1))^{r_1'-r_1}}{(\overline{F}(x_i))^{(m+1)(n-r_i-n'+r_i')}}. \end{aligned}$$

and the proof in this case follows using similar arguments similar to those used above. ∎

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Remark 3.20: It is important to note that Theorem 3.19 has established, the likelihood ratio order property for joint distributions of subvectors of m-GOSs from which many results can be obtained readily upon the same arguments that in Theorem 3.7.

For example, upon using the closure property under conditioning, the following results follow immediately, and more results of these forms can be deduced in a similar manner.

COROLLARY 3.21: *Let X and Y be absolutely continuous variables with distribution functions F and G and hazard rates r_F and r_G, respectively. Let $\mathbf{X} = (X_{(1,n,m,k)}, \ldots, X_{(n,n,m,k)})$ and $\mathbf{Y} = (Y_{(1,n',m,k)}, \ldots, Y_{(n',n',m,k)})$ be random vectors of m-GOSs based on F and G, respectively. For $r \leq s \leq n$, $s' - s = r' - r \geq \max\{0, n' - n\}$, and any $t_1 < t_2 \in \mathbb{R}$, if either*

(i) $X \leq_{\text{lr}} Y$ *and* $m \geq 0$, *or*

(ii) $X \leq_{\text{hr}} Y$, $r_G(x)/r_F(x)$ *is increasing in* x, *and* $-1 \leq m < 0$,

then

$$[X_{(s,n,m,k)} - t_1 \big| t_1 < X_{(r,n,m,k)} \leq t_2] \leq_{\text{lr}} [Y_{(s',n',m,k)} - t_1 \big| t_1 < Y_{(r',n',m,k)} \leq t_2].$$

COROLLARY 3.22: *Let X and Y be absolutely continuous variables with distribution functions F and G and hazard rates r_F and r_G, respectively. Let $\mathbf{X} = (X_{(1,n,m,k)}, \ldots, X_{(n,n,m,k)})$ and $\mathbf{Y} = (Y_{(1,n',m,k)}, \ldots, Y_{(n',n',m,k)})$ be random vectors of m-GOSs based on F and G, respectively. For $r \leq p \leq s \leq n$, $s' - s = p' - p = r' - r \geq \max\{0, n' - n\}$, and any $t_1 < t_2 \in \mathbb{R}$, if either*

(i) $X \leq_{\text{lr}} Y$ *and* $m \geq 0$, *or*

(ii) $X \leq_{\text{hr}} Y$, $r_G(x)/r_F(x)$ *is increasing in* x, *and* $-1 \leq m < 0$,

then

$$\begin{aligned}[X_{(p,n,m,k)} - t_1 \big| X_{(r,n,m,k)} > t_1, X_{(s,n,m,k)} \leq t_2] \\ \leq_{\text{lr}} [Y_{(p',n',m,k)} - t_1 \big| Y_{(r',n',m,k)} > t_1, Y_{(s',n',m,k)} \leq t_2].\end{aligned}$$

4. APPLICATIONS

In this section we present some applications of the results established in the last section in some special cases of GOSs. For a detailed description of these special cases, interested readers may refer to Kamps [14,15], Balakrishnan and Aggarwala [4], Belzunce et al. [9], and Balakrishnan [3].

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4.1. Order Statistics, Record Values, k-Record Values, and Progressively Type-II Censored Orders Statistics

As mentioned earlier in Section 1, several models of ordered data are special cases of GOSs through an appropriate selection of the parameters. Order statistics from a sample of independent and identically distributed random variables are a particular case of GOSs when $k = 1$ and $m_i = 0$ for all $i = 1, \ldots, n-1$. When $k = 1$ and $m_i = -1$ for all $i = 1, \ldots, n-1$, we obtain the random vector of the first n record values or the first n epoch times of a nonhomogeneous Poisson process. A generalization of record values is the case in which $k \in \mathbb{N}$, obtaining what are called k-records. A life-testing experiment of interest in reliability studies involves N independent and identically distributed random variables placed simultaneously on test and, at the time of the mth failure, R_i surviving unites are randomly censored from the test. The progressively Type-II censored order statistics arising from such a reliability experiment can be obtained from the model of GOSs by setting $n = m$, $m_i = R_i$, and $k = R_m + 1$.

In these models, the components might represent times at which some event occurs. For example, they might be the times at which some failures occur. Hence, the results established in the last section are comparing the failure times under two different populations based on some information on previous failures or future failures. Let us consider, for example, the sublattice in Example 3.15. In this case, we are comparing the failure times of the sth component under a double monitoring scheme, as discussed recently by Poursaeed and Nematollahi [28].

Another interesting sublattice in the case of random vectors with ordered components is $h_t = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 = t_1 < \cdots < x_r = t_r < t, x_i > t \text{ for all } i = r+1, \ldots, n\}$, which describes the history of failure times of the first r failed components while the remaining components are all still alive at time t. For a description of the notion of history, one can refer to Shaked and Shanthikumar [32].

4.2. Order Statistics Under Multivariate Imperfect repair

An interesting model contained in the model of generalized order statistics is that of order statistics under multivariate imperfect repair; see Shaked and Shanthikumar [31]. Suppose n items start to function at the same time 0. Upon failure, an item undergoes a repair. If i items ($i = 0, 1, \ldots, n-1$) have already been scrapped, then, with probability p_{i+1}, the repair is unsuccessful and the item is scrapped, and with probability $1 - p_{i+1}$, the repair is successful and minimal.

Let us now consider n items with independent and identically distributed random lifetimes $X_1, \ldots, X_n$, with the same distribution F and density function f. Let $(X_{(1)}, \ldots, X_{(n)})$ be the ordered random lifetimes resulting from $X_1, \ldots, X_n$ under such a minimal repair policy. Then, the joint density function of $(X_{(1)}, \ldots, X_{(n)})$ is given by

$$f(t_1, \ldots, t_n) = n! \prod_{j=1}^{n} p_j f(t_j) (\overline{F}(t_j))^{(n-j+1)p_j - (n-j)p_{j+1} - 1} \quad \text{for } 0 \le t_1 \le \cdots \le t_n.$$

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It is evident that this is a particular case of the joint density function of generalized order statistics based on F for the choice of parameters $k = p_n$ and $m_j = (n-j+1)p_j - (n-j)p_{j+1} - 1$. The results established in the preceding section can also be applied in this case. We can also combine Theorems 3.7 and 3.16 for comparing different models, for example, conditional order statistics from independent and identically distributed random variables and conditional order statistics under multivariate imperfect repair.

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