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Completion versus removal of redundancy by perturbation

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Abstract. A sequence $\{g_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ has the expansion property if each $f \in \overline{\text{span}}\,\{g_k\}_{k=1}^{\infty}$ has a representation $f = \sum_{k=1}^{\infty} c_k g_k$ for some scalar coefficients c_k. In this paper, we analyze the question whether there exist small norm-perturbations of $\{g_k\}_{k=1}^{\infty}$ which allow to represent all $f \in \mathcal{H}$; the answer turns out to be yes for frame sequences and Riesz sequences, but no for general basic sequences. The insight gained from the analysis is used to address a somewhat dual question, namely, whether it is possible to remove redundancy from a sequence with the expansion property via small norm-perturbations; we prove that the answer is yes for frames $\{g_k\}_{k=1}^{\infty}$ such that $g_k \to 0$ as $k \to \infty$, as well as for frames with finite excess. This particular question is motivated by recent progress in dynamical sampling.

1 Introduction

Let $\mathcal{H}$ denote a separable infinite-dimensional Hilbert space, and suppose that a given sequence $\{g_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ has the *expansion property*, i.e., that each $f \in \overline{\text{span}}\,\{g_k\}_{k=1}^{\infty}$ has a representation

$$f = \sum_{k=1}^{\infty} c_k g_k \tag{1.1}$$

for certain coefficients $c_k \in \mathbb{C}$. Our goal is to address the following question: when and how can we perform small norm-perturbations on the sequence $\{g_k\}_{k=1}^{\infty}$ and hereby obtain a sequence $\{\psi_k\}_{k=1}^{\infty}$ such that *arbitrary* elements $f \in \mathcal{H}$ have an expansion $f = \sum_{k=1}^{\infty} c_k \psi_k$ for certain coefficients $c_k \in \mathbb{C}$?

Formulated as above, the question is clearly a *completion problem*. We will show that the completion problem has an affirmative answer for the so-called Riesz sequences and frame sequences, but not for general basic sequences; along the way, we also consider a number of other completion problems. Interestingly, the insight gained from the above analysis can be used to address a somewhat dual question: when and how can a *redundant* system $\{g_k\}_{k=1}^{\infty}$ be turned into a complete but nonredundant system $\{\psi_k\}_{k=1}^{\infty}$ by small norm-perturbations? We will provide a positive answer to this question for a number of frames, in particular, for the so-called *near-Riesz bases* introduced by Holub in [13]. Additional motivation for this particular question will be provided at the end of the paper.

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The paper is organized as follows. In the rest of the introduction, we set the stage by providing a number of definitions and results from the literature. In Section 2, we present the results about the completion problem; the dual problem concerning removal of redundancy is considered in Section 3.

A sequence $\{g_k\}_{k=1}^{\infty}$ in the Hilbert space $\mathcal{H}$ is called a *frame for* $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{k=1}^{\infty} |\langle f, g_k\rangle|^2 \le B\|f\|^2, \ \forall f \in \mathcal{H}; \tag{1.2}$$

suitable numbers A, B are called *lower, resp. upper frame bounds.* The sequence $\{g_k\}_{k=1}^{\infty}$ is called a *Bessel sequence* if at least the right-hand inequality in (1.2) holds. A frame which is at the same time a basis is called a *Riesz basis.* Note that several other characterizations of frames and Riesz bases exist, e.g., in terms of operator theory. For example, if $\{e_k\}_{k=1}^{\infty}$ is a given orthonormal basis for $\mathcal{H}$, frames for $\mathcal{H}$ are precisely the sequences $\{Ve_k\}_{k=1}^{\infty}$ where $V : \mathcal{H} \to \mathcal{H}$ is a bounded surjective operator; Riesz bases correspond precisely to the case where the operator V also is injective. Finally, a sequence $\{g_k\}_{k=1}^{\infty}$ which is a frame for the (sub)space $\mathcal{K} := \overline{\text{span}}\,\{g_k\}_{k=1}^{\infty}$ is called a *frame sequence;* Riesz sequences are defined in the analogue way.

One of the key reasons for the interest in frames is that a frame has the expansion property: in fact, given any frame $\{g_k\}_{k=1}^{\infty}$, there exists a so-called *dual frame* $\{f_k\}_{k=1}^{\infty}$ such that

$$f = \sum_{k=1}^{\infty} \langle f, f_k\rangle g_k, \ \forall f \in \mathcal{H}.$$

In general, the dual frame $\{f_k\}_{k=1}^{\infty}$ is not unique: indeed, the case where $\{g_k\}_{k=1}^{\infty}$ is a Riesz basis is characterized precisely by the existence of a unique dual. We refer to [9] for more information about frames and Riesz bases, also about their history and applications.

The following lemma collects a number of well-known results concerning norm-perturbations of various sequences with the expansion property.

Lemma 1.1 *Consider two sequences $\{g_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ in $\mathcal{H}$, satisfying that*

$$\sum_{k=1}^{\infty} \|g_k - h_k\|^2 < A,$$

for a value of A as specified below. Then, the following holds:

(i) [7] *If $\{g_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with lower bound A, then $\{h_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$.*

(ii) [8] *If $\{g_k\}_{k=1}^{\infty}$ is a Riesz sequence with lower bound A, then $\{h_k\}_{k=1}^{\infty}$ is a Riesz sequence; furthermore, codim$(\overline{\text{span}}\,\{g_k\}_{k=1}^{\infty})$ = codim$(\overline{\text{span}}\{h_k\}_{k=1}^{\infty})$.*

Alternative norm-perturbation conditions are formulated in [6]; however, they need that we have access to information about a dual frame, which is not the case in the current paper. Note also that a number of classical results about norm-perturbation (typically for orthonormal sequences) are collected in [17]. Observe that more general perturbation results are available in the literature, typically formulated in terms of certain operators rather than norm-perturbations (see, e.g., [9] and the references therein).

2 Completion via norm-perturbation

Our main interest is to consider the completion problem for sequences $\{g_k\}_{k=1}^\infty$ having the expansion property. However, we first state a number of other completion properties, some of which will be needed in latter proofs. Given any sequence $\{g_k\}_{k=1}^\infty$ in $\mathcal{H}$, we define its *excess* $\mathcal{E}(\{g_k\}_{k=1}^\infty)$ as the maximal number of elements that can be removed without changing the spanned space, i.e.,

$$\mathcal{E}(\{g_k\}_{k=1}^\infty) := \max \sharp\{J \subset \mathbb{N} \,|\, \overline{\operatorname{span}}\{g_k\}_{k\in\mathbb{N}\setminus J} = \overline{\operatorname{span}}\,\{g_k\}_{k=1}^\infty\}. \tag{2.1}$$

Furthermore, we will use the standard convention and say that a sequence $\{g_k\}_{k=1}^\infty$ in $\mathcal{H}$ is *norm-bounded below* if there exists a constant $C > 0$ such that $\|g_k\| \geq C$ for all $k \in \mathbb{N}$.

Proposition 2.1 *Let $\{g_k\}_{k=1}^\infty$ be a sequence in $\mathcal{H}$. Then, the following hold:*

(i) *If $\{g_k\}_{k=1}^\infty$ is not norm-bounded below, there exists a complete sequence $\{\psi_k\}_{k=1}^\infty$ in $\mathcal{H}$ such that*

$$\|g_k - \psi_k\| \to 0 \text{ as } k \to \infty; \tag{2.2}$$

(ii) *If $\mathcal{E}(\{g_k\}_{k=1}^\infty) \geq \operatorname{codim}(\overline{\operatorname{span}}\,\{g_k\}_{k=1}^\infty)$, there exists a complete sequence $\{\psi_k\}_{k=1}^\infty$ in $\mathcal{H}$ such that (2.2) holds.*

(iii) *If $\{g_k\}_{k=1}^\infty$ is convergent, there exists a complete sequence $\{\psi_k\}_{k=1}^\infty$ in $\mathcal{H}$ such that (2.2) holds; in particular, $\{\psi_k\}_{k=1}^\infty$ converges to the same limit as $\{g_k\}_{k=1}^\infty$.*

In all the stated cases, given any $\delta > 0$, the sequence $\{\psi_k\}_{k=1}^\infty$ can be chosen such that additionally $\|g_k - \psi_k\| \leq \delta$ for all $k \in \mathbb{N}$.

Proof For the proof of (i), given $\delta > 0$, choose a frame $\{f_k\}_{k=1}^\infty$ for $\mathcal{H}$ such that $\|f_k\| \leq \delta$ for all $k \in \mathbb{N}$ and $\|f_k\| \to 0$ as $k \to \infty$; for example, letting $\{e_k\}_{k=1}^\infty$ denote any orthonormal basis, we can take

$$\{f_k\}_{k=1}^\infty = \left\{\delta\, e_1, \frac{\delta}{\sqrt{2}}\, e_2, \frac{\delta}{\sqrt{2}}\, e_2, \frac{\delta}{\sqrt{3}}\, e_3, \frac{\delta}{\sqrt{3}}\, e_3, \frac{\delta}{\sqrt{3}}\, e_3, \dots\right\}.$$

Denote the lower frame bound for the frame $\{f_k\}_{k=1}^\infty$ by A. Choose now a subsequence $\{g_{k_n}\}_{n=1}^\infty$ of $\{g_k\}_{k=1}^\infty$ such that $\|g_{k_n}\|^2 \leq \frac{3A}{\pi^2 n^2}$, $n \in \mathbb{N}$; then,

$$\sum_{n=1}^\infty \|f_n - (f_n + g_{k_n})\|^2 = \sum_{n=1}^\infty \|g_{k_n}\|^2 \leq \frac{A}{2}.$$

Using Lemma 1.1(i), this implies that $\{f_n + g_{k_n}\}_{n=1}^\infty$ is a frame for $\mathcal{H}$ and hence complete. Thus, the sequence $\{\psi_k\}_{k=1}^\infty$ formed from $\{g_k\}_{k=1}^\infty$ by replacing the elements $\{g_{k_n}\}_{n=1}^\infty$ by $\{f_n + g_{k_n}\}_{n=1}^\infty$ will satisfy the requirements.

For the proof of (ii), we first assume additionally that $M := \operatorname{codim}(\overline{\operatorname{span}}\,\{g_k\}_{k=1}^\infty)$ is finite. Without loss of generality and only for notational convenience, assume that the sequence $\{g_k\}_{k=1}^\infty$ is ordered such that $g_1, \dots, g_M \in \overline{\text{span}}\{g_k\}_{k=M+1}^\infty$, and take an orthonormal basis $\{e_k\}_{k=1}^M$ for the orthogonal complement $(\overline{\text{span}}\,\{g_k\}_{k=1}^\infty)^\perp$. Then, the sequence

$$\{\psi_k\}_{k=1}^\infty = \left\{g_1 + \delta e_1, g_2 + \frac{\delta}{2}\, e_2, \dots, g_M + \frac{\delta}{M} e_M, g_{M+1}, g_{M+2}, \dots\right\}$$

satisfies the requirements. The case where $\mathcal{E}(\{g_k\}_{k=1}^\infty) = \text{codim}(\overline{\text{span}}\,\{g_k\}_{k=1}^\infty) = \infty$ is similar and only requires minor notational modifications.

For the proof of (iii), assume that the sequence $\{g_k\}_{k=1}^\infty$ converges to $f \in \mathcal{H}$. Given $\delta > 0$, choose $K \in \mathbb{N}$ such that $\|f - g_k\| \le \delta/2$ for $k \ge K$. Let $\{e_k\}_{k=1}^\infty$ denote an orthonormal basis for $\mathcal{H}$, and define $\{\psi_k\}_{k=1}^\infty$ by

$$\psi_k := \begin{cases} g_k, & \text{if } k = 1, \dots, K-1; \\ f, & \text{if } k = K; \\ f + \frac{\delta}{2^{k-K}} e_{k-K}, & \text{if } k > K. \end{cases}$$

Then, $\text{span}\{e_k\}_{k=1}^\infty \subseteq \text{span}\{\psi_k\}_{k=1}^\infty$, so $\text{span}\{\psi_k\}_{k=1}^\infty$ is clearly complete. Furthermore, for $k \ge K$,

$$\|g_k - \psi_k\| \le \|g_k - f\| + \|f - \psi_k\| \le \delta,$$

and $\|g_k - \psi_k\| \to 0$ as $k \to \infty$. ∎

We are now ready to consider the completion problem for Riesz sequences and frame sequences. The proofs rely on an interesting result proved recently by Olevskii.

Lemma 2.2 [15, 16] *If $\{e_k\}_{k=1}^\infty$ is an orthonormal sequence in $\mathcal{H}$, there exists an orthonormal basis $\{\chi_k\}_{k=1}^\infty$ for $\mathcal{H}$ such that*

$$\|e_k - \chi_k\| \to 0 \text{ as } k \to \infty.$$

In addition, given any $\delta > 0$, the sequence $\{\chi_k\}_{k=1}^\infty$ can be chosen such that $\|e_k - \chi_k\| \le \delta$ for all $k \in \mathbb{N}$.

Theorem 2.3 *Let $\{g_k\}_{k=1}^\infty$ be a sequence in $\mathcal{H}$. Then, the following hold:*

(i) *If $\{g_k\}_{k=1}^\infty$ is a Riesz sequence, there exists a Riesz basis $\{\psi_k\}_{k=1}^\infty$ for $\mathcal{H}$ such that*

$$\|g_k - \psi_k\| \to 0 \text{ as } k \to \infty. \tag{2.3}$$

(ii) *If $\{g_k\}_{k=1}^\infty$ is a frame sequence, there exists a frame $\{\psi_k\}_{k=1}^\infty$ for $\mathcal{H}$ such that* (2.3) *holds.*

(iii) *If $\{g_k\}_{k=1}^\infty$ is a Bessel sequence, there exists a complete Bessel sequence $\{\psi_k\}_{k=1}^\infty$ such that* (2.3) *holds.*

In all the stated cases, given any $\delta > 0$, the sequence $\{\psi_k\}_{k=1}^\infty$ can be chosen such that $\|g_k - \psi_k\| \le \delta$ for all $k \in \mathbb{N}$.

Proof We first prove (iii). Thus, let $\{g_k\}_{k=1}^\infty$ be a Bessel sequence in $\mathcal{H}$, and let $\mathcal{K} := \overline{\text{span}}\,\{g_k\}_{k=1}^\infty$; we can assume that $\mathcal{K}^\perp \ne \{0\}$. Furthermore, if $\mathcal{K}$ is finite-dimensional, clearly $\mathcal{E}(\{g_k\}_{k=1}^\infty) = \infty$, and thus the results follow from Proposition 2.1(ii).

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Therefore, we now assume that $\mathcal{K}$ is infinite-dimensional. Now, by the standard properties of a Bessel sequence [9], choose an orthonormal basis $\{e_k\}_{k=1}^\infty$ for $\mathcal{K}$ and a bounded operator $U : \mathcal{K} \to \mathcal{K}$ such that $g_k = Ue_k$, $k \in \mathbb{N}$. Associated with the orthonormal sequence $\{e_k\}_{k=1}^\infty$, choose the orthonormal basis $\{\chi_k\}_{k=1}^\infty$ for $\mathcal{H}$ as in Lemma 2.2, and define a bounded operator $V : \mathcal{H} \to \mathcal{H}$ by

$$V = U \text{ on } \mathcal{K}, \quad V = I \text{ on } \mathcal{K}^\perp. \tag{2.4}$$

Because the range of the operator U contains the vectors $\{g_k\}_{k=1}^\infty$, it is dense in $\mathcal{K}$. Thus, the range of the operator V is dense in $\mathcal{H}$; this implies that the sequence $\{\psi_k\}_{k=1}^\infty := \{V\chi_k\}_{k=1}^\infty$ is complete in $\mathcal{H}$. A direct calculation reveals that $\{\psi_k\}_{k=1}^\infty$ is a Bessel sequence. Furthermore, for all $k \in \mathbb{N}$,

$$||g_k - \psi_k|| = ||Ue_k - V\chi_k|| = ||Ve_k - V\chi_k|| \le ||V||\,||e_k - \chi_k||.$$

Because the operator V only depends on the sequence $\{g_k\}_{k=1}^\infty$ (and the fixed choice of $\{e_k\}_{k=1}^\infty$), this proves (iii). This also gives the proof of (i) and (ii). Indeed, if $\{g_k\}_{k=1}^\infty$ is a frame sequence, the range of the operator U equals $\mathcal{K}$, which implies that the range of the operator V equals $\mathcal{H}$, and hence $\{\psi_k\}_{k=1}^\infty$ is a frame for $\mathcal{H}$; and if $\{f_k\}_{k=1}^\infty$ is a Riesz sequence, the operator $U : \mathcal{K} \to \mathcal{K}$ is bijective, implying that $V : \mathcal{H} \to \mathcal{H}$ is bijective, and hence that $\{\psi_k\}_{k=1}^\infty$ is a Riesz sequence. ■

Remark 2.4 Despite the fact that $\delta > 0$ can be chosen arbitrarily small in Theorem 2.3, there is a restriction on how "close" the sequence $\{\psi_k\}_{k=1}^\infty$ can be to the sequence $\{g_k\}_{k=1}^\infty$. Indeed, if $\{g_k\}_{k=1}^\infty$ is a (noncomplete) Riesz sequence with lower bound A, then the sequence $\{\psi_k\}_{k=1}^\infty$ in Theorem 2.3(i) must satisfy that

$$\sum_{k=1}^\infty ||g_k - \psi_k||^2 \ge A; \tag{2.5}$$

otherwise, Lemma 1.1(ii) would imply that $\{\psi_k\}_{k=1}^\infty$ is noncomplete as well. A similar result holds for frame sequences, although the lower bound on the infinite sum in (2.5) will involve the gap between two particular subspaces of $\mathcal{H}$ (see [8, 10] for more detailed information).

Theorem 2.3 makes it natural to ask whether a basic sequence (i.e., a Schauder basis for a subspace) also can be extended to a Schauder basis for $\mathcal{H}$ by small norm-perturbations of the elements. The following example shows that the answer is no, in general, unless additional assumptions are added.

Example 2.5 Let $\{e_k\}_{k=1}^\infty$ denote an orthonormal basis for $\mathcal{H}$, and consider the sequence

$$\{g_k\}_{k=1}^\infty = \{2e_2, 4e_4, 6e_6, \dots\} = \{2ke_{2k}\}_{k=1}^\infty.$$

Clearly, $\{g_k\}_{k=1}^\infty$ is a basic sequence. Now, given any $\delta \in]0, 2\sqrt{6}\pi^{-1}[$, consider a sequence $\{\psi_k\}_{k=1}^\infty$ in $\mathcal{H}$ such that $||g_k - \psi_k|| \le \delta$ for all $k \in \mathbb{N}$. Then,

$$\sum_{k=1}^{\infty} \|e_{2k} - \frac{1}{2k}\psi_k\|^2 = \sum_{k=1}^{\infty} \frac{1}{4k^2}\|g_k - \psi_k\|^2 \le \frac{\pi^2\delta^2}{24} < 1.$$

Because $\{e_{2k}\}_{k=1}^{\infty}$ forms a Riesz sequence with lower bound $A = 1$, Lemma 1.1(ii) implies that $\{(2k)^{-1}\psi_k\}_{k=1}^{\infty}$ also forms a Riesz sequence, spanning a space of the same codimension as $\{e_{2k}\}_{k=1}^{\infty}$; in particular, $\{\psi_k\}_{k=1}^{\infty}$ cannot be complete in $\mathcal{H}$, and hence is not a Schauder basis for $\mathcal{H}$.

3 Removal of redundancy via norm-perturbations

In this section, the focus is on sequences $\{g_k\}_{k=1}^{\infty}$ having the expansion property on the entire underlying Hilbert space $\mathcal{H}$. Such expansions might be redundant, i.e., a given $f \in \mathcal{H}$ might have expansions $f = \sum_{k=1}^{\infty} c_k g_k$ for more than one choice of the scalar coefficients $\{c_k\}_{k=1}^{\infty}$. A typical example of a redundant sequence is a frame $\{g_k\}_{k=1}^{\infty}$ which is not a Riesz basis. Our goal is to show that for certain frames $\{g_k\}_{k=1}^{\infty}$, the redundancy can be removed via small norm-perturbations of the vectors g_k.

Our first observation, stated next, does not even need the frame assumption or any other expansion property.

Theorem 3.1 *Consider any sequence $\{g_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ such that $g_k \to 0$ as $k \to \infty$. Then, given any $\delta > 0$, there exists a Riesz basis $\{\psi_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ such that*

$$\|g_k - \psi_k\| \le \delta, \; \forall k \in \mathbb{N}.$$

Proof First, given any $\delta > 0$, choose $K \in \mathbb{N}$ such that $\|g_k\| < \delta/2$ for $k \ge K$. We will now construct $\{\psi_k\}_{k=1}^{\infty}$ recursively, of the form $\psi_k := g_k + \varphi_k$ with the vectors φ_k chosen as described next. First, take $\varphi_1 \in \mathcal{H}$ such that $\|\varphi_1\| \le \delta$ and $\psi_1 \ne 0$. Then, choose $\varphi_2 \in \mathcal{H}$ such that $\|\varphi_2\| \le \delta$ and $\{\psi_1, \psi_2\}$ is linearly independent. Continuing recursively, we finally choose $\varphi_K \in \mathcal{H}$ such that $\|\varphi_K\| \le \delta$ and $\{\psi_1, \psi_2, \ldots, \psi_K\}$ is linearly independent. Then, $\{\psi_1, \psi_2, \ldots, \psi_K\}$ is a Riesz basis for the subspace $V := \text{span}\{\psi_1, \psi_2, \ldots, \psi_K\}$. Now, choose an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ for $V^{\perp}$ and define ψ_k for $k > K$ by $\psi_k := \frac{\delta}{2} e_k$. Then, $\{\psi_k\}_{k=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$ and $\|g_k - \psi_k\| \le \delta$ for all $k \in \mathbb{N}$. ∎

The result in Theorem 3.1 immediately applies to a number of well-known frames in the literature.

Example 3.2 We state a number of examples of frames $\{g_k\}_{k=1}^{\infty}$ such that $g_k \to 0$ as $k \to \infty$:

(i) Given any orthonormal basis $\{e_k\}_{k=1}^{\infty}$ for $\mathcal{H}$, the family

$$\{g_k\}_{k=1}^{\infty} := \left\{ e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \ldots \right\}$$

is a frame for $\mathcal{H}$. Clearly, $g_k \to 0$ as $k \to \infty$. Note that this particular frame was used in the proof of Proposition 2.1.

(ii) Let again $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$, and fix any $\alpha \in]0,1[$. Let $\lambda_\ell := 1 - \alpha^{-\ell}$ for $\ell \in \mathbb{N}$, and define the vectors

$$g_k := \sum_{\ell=1}^{\infty} \lambda_\ell^k \sqrt{1-\lambda_\ell^2}\, e_\ell,\ k \in \mathbb{N}.$$

Then, $\{g_k\}_{k=1}^{\infty}$ is a frame (the so-called Carleson frame), a result proved by Aldroubi et al. in [1, 2]. It is easy to see that $g_k \to 0$ as $k \to \infty$. Note that $\{g_k\}_{k=1}^{\infty}$ is heavily redundant: it can be proved that for any $N \in \mathbb{N}$, any subfamily $\{g_{Nk}\}_{k\in\mathbb{N}}$ of $\{g_k\}_{k=1}^{\infty}$ is a redundant frame as well. From this point of view, it is surprising that $\{g_k\}_{k=1}^{\infty}$ can be approximated by a Riesz basis, as stated in Theorem 3.1.

(iii) More generally than (ii), it was proved in [12] that any redundant frame that can be represented as an operator orbit $\{g_k\}_{k=1}^{\infty} = \{T^k\varphi\}_{k=1}^{\infty}$ for a bounded operator $T : \mathcal{H} \to \mathcal{H}$ and some $\varphi \in \mathcal{H}$ will have the property that $g_k \to 0$ as $k \to \infty$.

In order to reach the next result, we need the following lemma. Recall that the *deficit* of a sequence $\{g_k\}_{k=1}^{\infty}$ is defined as the codimension of the vector space $\overline{\text{span}}\,\{g_k\}_{k=1}^{\infty}$.

Lemma 3.3 *Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Given any $\delta > 0$ and any $N \in \mathbb{N}$, there exists an orthonormal system $\{\varepsilon_k\}_{k=1}^{\infty}$ with deficit N such that $\|e_k - \varepsilon_k\| \le \delta$ for all $k \in \mathbb{N}$.*

Proof Take any orthonormal system $\{\varphi_k\}_{k=1}^{\infty}$ with deficit N, and choose via Lemma 2.2 an orthonormal basis $\{\chi_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ such that $\|\varphi_k - \chi_k\| \le \delta$ for all $k \in \mathbb{N}$. Then, choose the unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that $e_k = U\chi_k$, and let $\varepsilon_k := U\varphi_k$, $k \in \mathbb{N}$. Then, $\{\varepsilon_k\}_{k=1}^{\infty}$ is an orthonormal system with deficit N, and $\|e_k - \varepsilon_k\| = \|U\chi_k - U\varphi_k\| = \|\chi_k - \varphi_k\| \le \delta$ for all $k \in \mathbb{N}$, as desired. ■

Theorem 3.4 *Consider a frame of the form $\{g_k\}_{k=1}^{\infty} = \{g_k\}_{k=1}^{N} \cup \{g_k\}_{k=N+1}^{\infty}$, where $N \in \mathbb{N}$ and $\{g_k\}_{k=N+1}^{\infty}$ is a Riesz basis for $\mathcal{H}$. Then, given any $\delta > 0$, there exists a Riesz basis $\{\psi_k\}_{k=1}^{\infty}$ such that $\|g_k - \psi_k\| \le \delta$ for all $k \in \mathbb{N}$.*

Proof First, consider an orthonormal basis for $\mathcal{H}$ indexed as $\{e_k\}_{k=N+1}^{\infty}$, and choose the bounded bijective operator $V : \mathcal{H} \to \mathcal{H}$ such that $g_k = Ve_k$ for $k = N+1, N+2, \ldots$. Using Lemma 3.3, choose an orthonormal system $\{\varepsilon_k\}_{k=N+1}^{\infty}$ with deficit N such that $\|e_k - \varepsilon_k\| \le \delta/\|V\|$ for $k = N+1, N+2, \ldots$. Then, letting $\psi_k := V\varepsilon_k$, $k = N+1, N+2, \ldots$, the family $\{\psi_k\}_{k=N+1}^{\infty}$ is a Riesz sequence with deficit N, and $\|g_k - \psi_k\| = \|Ve_k - V\varepsilon_k\| \le \delta$ for $k = N+1, N+2, \ldots$.

Now, consider the vector g_N. If $g_N \notin \overline{\text{span}}\{\psi_k\}_{k=N+1}^{\infty}$, let $\psi_N := g_N$; then, $\{\psi_k\}_{k=N}^{\infty}$ is a Riesz sequence with deficit $N-1$. On the other hand, if $g_N \in \overline{\text{span}}\{\psi_k\}_{k=N+1}^{\infty}$, choose any normalized vector $\varphi_N \notin \overline{\text{span}}\{\psi_k\}_{k=N+1}^{\infty}$, and let $\psi_N := g_N + \delta\varphi_N$; then, again $\{\psi_k\}_{k=N}^{\infty}$ is a Riesz sequence with deficit $N-1$, and $\|g_k - \psi_k\| \le \delta$ for $k = N, N+1, N+2, \ldots$. Applying now the same procedure on $g_{N-1}, g_{N-2}, \ldots, g_1$, we arrive at the desired Riesz basis $\{\psi_k\}_{k=1}^{\infty}$ in a finite number of steps. ■

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Interestingly, frames of the type considered in Theorem 3.4 were called *near-Riesz bases* by Holub in the paper [13]; the above result provides an additional reason for this name being very appropriate.

Remark 3.5 Despite the fact that $\delta > 0$ can be chosen arbitrarily small in Theorem 3.4, the Riesz basis $\{\psi_k\}_{k=1}^{\infty}$ must satisfy that $\sum_{k=1}^{\infty} ||g_k - \psi_k||^2 \geq A$, where A is the lower frame bound for $\{g_k\}_{k=1}^{\infty}$; otherwise, the results in [5] show that $\{\psi_k\}_{k=1}^{\infty}$ would be a frame with the same excess as $\{g_k\}_{k=1}^{\infty}$.

We want to point out that the proof of Theorem 3.4 somewhat hides the fact that it is highly nontrivial to get direct access to the Riesz basis $\{\psi_k\}_{k=1}^{\infty}$, especially due to the intriguing and deep construction by Olevskii playing a key role in the argument. The next example illustrates this by a concrete construction.

Example 3.6 Let again $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$, and consider the frame

$$\{g_k\}_{k=1}^{\infty} := \{e_1, e_1, e_2, e_3, e_4, \dots\},$$

consisting of the orthonormal basis and a single extra element. A natural way to try to remove the redundancy would be to fix a small $\varepsilon > 0$ and let $\psi_1 := e_1$ and for $k > 1, \psi_k := \frac{1}{2}e_{k-1} + (\frac{1}{2} + \varepsilon)e_k$. Then, for any finite sequence $\{c_k\}_{k=2}^{\infty}$,

$$\left|\left|\sum_{k=2}^{\infty} c_k \left(\left(\frac{1}{2} + \varepsilon\right) e_k - \psi_k\right)\right|\right|^2 = \frac{1}{4}\left|\left|\sum_{k=2}^{\infty} c_k e_{k-1}\right|\right|^2 = \frac{1}{4}\sum_{k=2}^{\infty} |c_k|^2.$$

Observe that $\{e_1\} \cup \{(\frac{1}{2} + \varepsilon)e_k\}_{k=2}^{\infty}$ is a Riesz basis with lower bound $\frac{1}{2} + \varepsilon$. Considering $\{\psi_k\}_{k=1}^{\infty}$ as a perturbation of this Riesz basis, it now follows from the results in [5] that $\{\psi_k\}_{k=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$. Note that

$$||g_k - \psi_k|| = \sqrt{\frac{1}{4} + \left(\frac{1}{2} + \varepsilon\right)^2};$$

however, this construction does not allow us to obtain $||g_k - \psi_k|| \leq \delta$ when $\delta < 2^{-1/2} \approx 0.7$. In fact, in order to obtain the result in Theorem 3.4 for smaller values of δ, it would be necessary to consider much more complicated perturbations $\{\psi_k\}_{k=1}^{\infty}$ of $\{g_k\}_{k=1}^{\infty}$, making it highly nontrivial to do this in practice.

Remark 3.7 The question of removal of redundancy is partly motivated by the research topic *dynamical sampling,* introduced in the papers [2, 3]. One of the key issues in dynamical sampling is the construction of frames as orbits $\{T^k\varphi\}_{k=0}^{\infty}$ of a bounded operator $T : \mathcal{H} \to \mathcal{H}$, for some $\varphi \in \mathcal{H}$; we encountered such frames already in Example 3.2(ii,iii). Unfortunately, it is very difficult to construct such frames, and the only concrete examples available in the literature are indeed Riesz bases [11] and the Carleson frame [2] considered in Example 3.2(ii). Furthermore, it was proved in [11] that a near-Riesz basis never has this property. This raises the natural question whether a near-Riesz basis can be approximated by a Riesz basis, and hence by an orbit of a bounded operator; Theorem 3.4 confirms that this indeed is possible. We will phrase this consequence of Theorem 3.4 as a separate result, where we index the given near-Riesz basis by $\{g_k\}_{k=0}^\infty$ for notational convenience.

Corollary 3.8 *Consider any near-Riesz basis $\{g_k\}_{k=0}^\infty$. Then, given any $\delta > 0$, there exists $\varphi \in \mathcal{H}$ and a bounded operator $T : \mathcal{H} \to \mathcal{H}$ such that*

$$||g_k - T^k\varphi|| \leq \delta, \ \forall k \in \mathbb{N}_0.$$

The results in Theorems 3.1 and 3.4 do not cover the standard (regular) redundant Gabor frames and wavelet frames: they consist of vectors with equal norm, and they have infinite excess [4]. Due to the complications discussed in Example 3.1 and the preceding text, it seems to be very difficult to answer the question whether all frames indeed can be approximated by a Riesz basis. At least for Gabor frames and wavelet frames, we can apply the following adaption of the Feichtinger theorem (finally proved in one of its equivalent formulations in [14]), showing that any frame which is norm-bounded below can be approximated by a *finite* collection of Riesz bases.

Theorem 3.9 *Let $\{g_k\}_{k=1}^\infty$ be a frame which is norm-bounded below. Then, there exists a finite partition $\{g_k\}_{k=1}^\infty = \bigcup_{j=1}^J \{g_k\}_{k\in I_j}$ with the property that for each $\delta > 0$, there exist Riesz bases $\{\psi_k\}_{k\in I_j}$, $j = 1, \ldots, J$, for $\mathcal{H}$ such that $||g_k - \psi_k|| \leq \delta$ for all $k \in \mathbb{N}$.*

Proof Choose according to the Feichtinger theorem a finite partition $\{g_k\}_{k=1}^\infty = \bigcup_{j=1}^J \{g_k\}_{k\in I_j}$ such that each sequence $\{g_k\}_{k\in I_j}$, $j = 1, \ldots, J$, is a Riesz sequence; using Theorem 2.1 in [11], we can shuffle the elements around to ensure that each of the index sets I_j is infinite. Now, the result follows directly from Theorem 2.3(i). ∎

The result in Theorem 3.9 can be formulated as an operator-theoretic result, similarly to Corollary 3.8; we leave the precise formulation to the interested reader.

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