Differential Equation of a Loxodrome on a Sphere

Sergio Kos, Duško Vranić and Damir Zec

(*Rijeka College of Maritime Studies, Croatia*)

1. INTRODUCTION. A curve that cuts all meridians of a rotating surface at the same angle is called a loxodrome. If the shape of the Earth is approximated by a sphere, then the loxodrome is a logarithmic spiral that cuts all meridians at the same angle and asymptotically approaches the Earth's poles but never meets them. Since maritime surface navigation defines the course as the angle between the current meridian and the longitudinal direction of the ship, it may be concluded that the loxodrome is the curve of the constant course, which means that whenever navigating on an unchanging course we are navigating according to a loxodrome.

2. THE FIRST DIFFERENTIAL FORM OF THE SPHERE. The sphere is defined by the following vector equation:

$$\vec{r} = (r \sin\Theta \cos\Phi,\ r \sin\Theta \sin\Phi,\ r \cos\Phi) \tag{1}$$

where:

$\Theta = 90° - \phi$, the complement of geographical latitude,

$\phi =$ geographical latitude,

$\Phi =$ difference of geographical longitudes as the angle at the centre of Earth as depicted by a sphere ($-\pi \leqslant \Phi \leqslant \pi$), and

$r =$ radius of the Earth as a sphere.

The Gaussian basic or fundamental magnitudes of the first order for a sphere, in other words the coefficients of the first differential form, are:

$$E = r^2, \tag{2}$$

$$F = 0 - \text{prerequisite of mutual verticality of meridians and parallels}, \tag{3}$$

$$G = r^2 \sin^2\Theta. \tag{4}$$

The first differential form for the sphere from the above is:

$$d_s^2 = E\, d_u^2 + 2\, F\, d_u\, d_v + G\, d_v^2 \tag{5}$$

respectively,

$$d_s^2 = r^2\, d\Theta^2 + r^2 \sin^2 \Theta d\Phi^2 \tag{6}$$

since

$$d\Theta = d_u \quad \text{and} \quad d\Phi = d_v.$$

3. THE ANGLE BETWEEN CURVES ON THE SPHERE. The angle between two curves on the surface of a sphere is defined by the angle between their tangents at the point of intersection; hence the angle between a curve on the surface of a sphere wherein $d\Theta \neq 0$ and $d\Phi \neq 0$ and the meridian. In other words, the angle between the loxodrome and meridian can be determined by means of the following formulation of differential geometry:

$$\cos\alpha = \frac{\sqrt{r^2}\, d\Theta}{\sqrt{r^2(d\Theta^2 + \sin^2 \Theta d\Phi^2)}}. \tag{7}$$

By mathematical adjustment of formula (7), we arrive at a differential equation of the loxodrome on a sphere in its basic form:

$$\cos^2\alpha(d\Theta^2 + \sin^2 \Theta d\Phi^2) = d\Theta^2. \tag{8}$$

With the solution of the differential equation, we reach a general form for the equation of a loxodrome on a sphere:

$$\Phi = \pm \tan\alpha \ln\tan\frac{\Theta}{2} + C. \tag{9}$$

https://doi.org/10.1017/S0373463399008395 Published online by Cambridge University Press

The general equation for the loxodrome on the Earth as a sphere is shown by means of the geographical latitude $\phi = 90° - \Theta$, and acquires the following form:

$$\Phi = \pm \tan\alpha \ln\left[\tan\left(45° - \frac{\phi}{2}\right)\right]^{-1} + C, \tag{10}$$

hence:

$$\Phi = \pm \tan\alpha \ln\cot\left(45° - \frac{\phi}{2}\right) + C, \tag{11}$$

and:

$$\Phi = \pm \tan\alpha \ln\tan\left(45° + \frac{\phi}{2}\right) + C. \tag{12}$$

Using equation (12), we arrive at angle Φ expressed in radians, where C is the constant of integration. The loxodrome is a curve that winds spirally around the North pole ($\Theta = 0$, $\Phi =$ arbitrary), and respectively around the South pole ($\Theta = \pi$, $\Phi =$ arbitrary), but never reaches the poles and, in the process, cuts all the meridians at the same constant angle α. In spite of this, the loxodrome has a definite length, and this can be proven.

4. DETERMINING THE LOXODROME LENGTH ON THE SPHERE. The arc length element of any curve on the sphere is established with the following formula:

$$d_L = r\sqrt{\sin^2\Theta d\Phi^2 + d\Theta^2}. \tag{13}$$

With the solution of the above differential equation, we get:

$$L = r\int_{\Theta_1}^{\Theta_2} \sqrt{1 + \sin^2\Theta\left(\frac{d\Phi}{d\Theta}\right)^2}\, d\Theta. \tag{14}$$

Having taken the differential equation of the loxodrome into consideration, the arc length of the loxodrome is defined by the following equation:

$$L = r\int_{\Theta_1}^{\Theta_2} \frac{d\Theta}{\cos\alpha} = \frac{r}{\cos\alpha}\int_{\Theta_1}^{\Theta_2} d\Theta,$$

$$L = r\frac{\Theta_2 - \Theta_1}{\cos\alpha}, \tag{15}$$

in which:

L = length of the arc of loxodrome on the sphere,
r = radius of the Earth as a sphere,
Θ = complement of geographical latitude,
α = angle at which loxodrome cuts the meridians of the sphere.

Since radius r in angular minutes for the Earth as a sphere amounts to:

$$r = \frac{21600}{2\pi} = \frac{10800}{\pi}, \quad [r] = ', 1' = 1852 \text{ m} = 1 \text{ nautical mile},$$

equation (15) takes on the following form:

$$L = \frac{10800}{\pi}\frac{\Theta_2 - \Theta_1}{\cos\alpha} \tag{16}$$

In equations (15) and (16), angle $\Theta_2 - \Theta_1$ should be designated in radians.

4.1. *An Illustration.* Make a calculus of angle Φ at the centre of the Earth as a sphere, and of the length of the loxodrome that cuts all the meridians of the Earth's sphere under the same angle $\alpha = 109°25'$, and passes through the geographical latitudes $35°26'00''$ and $00°00'00''$.

https://doi.org/10.1017/S0373463399008395 Published online by Cambridge University Press

According to equation (12):

$$\Phi = \pm \tan\alpha \ln\tan\left(45° + \frac{\phi}{2}\right) + C,$$

$C = 0 \rightarrow$ since it is held that axis y is located on a level with the equator and, in that case, $x = 0$ and $y = 0$.

$$\Phi = \pm \tan 109°25' \ln\tan 62°43' = -1{\cdot}878373743 \text{ radians},$$
$$\Phi = -107{\cdot}6228878° = -107°37'22{\cdot}396''.$$

According to equations (15) and (16) respectively:

$$L = \frac{10800}{\pi}\,\frac{0{\cdot}6184283316}{\cos\alpha} = 6395{\cdot}2257287 \text{ nautical miles.}$$

5. CONCLUSION. It is apparent from equations (15) and (16), that the length of the loxodrome between two parallels only depends on the difference $\Theta_2 - \Theta_1$ along with the specified constant angle α. If $\Theta_1 = 0$, $\Theta_2 = \pi$, then the length of the loxodrome between the poles is:

$$L = \frac{r\pi}{\cos\alpha} \tag{17}$$

In other words, inserting r in angular minutes for the Earth as a sphere, we get:

$$L = \frac{10800}{\cos\alpha} \tag{18}$$

For $\alpha = 0°$ or $360°$, the loxodrome coincides with the meridian; hence the length of the loxodrome between poles for the Earth as a sphere amounts to 10800 nautical miles. In general, it may be concluded that the length of the loxodrome is a definite magnitude for the angles $\alpha \neq \pm\pi/2$ (where α is the angle at which the loxodrome cuts the meridians of the sphere). For $\alpha = \pm\pi/2$, the loxodrome coincides with the parallels, which are infinite in number.

REFERENCES

[1] Lipschutz, M. M. (1969). *Differential Geometry*. McGraw–Hill Book Company, New York.
[2] O'Neil, B. (1969). *Elementary Differential Geometry*. Acad. Press, New York, London.
[3] Bowditch, N. (1981, 1984). *American Practical Navigator*. Vols. I–II, DMAHTC, Washington.

https://doi.org/10.1017/S0373463399008395 Published online by Cambridge University Press