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THE WADGE ORDER ON THE SCOTT DOMAIN IS NOT A WELL-QUASI-ORDER

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Abstract. We prove that the Wadge order on the Borel subsets of the Scott domain is not a well-quasi-order, and that this feature even occurs among the sets of Borel rank at most 2. For this purpose, a specific class of countable 2-colored posets $\mathbb{P}_{\text{emb}}$ equipped with the order induced by homomorphisms is embedded into the Wadge order on the $\mathbf{\Delta}^0_2$-degrees of the Scott domain. We then show that $\mathbb{P}_{\text{emb}}$ admits both infinite strictly decreasing chains and infinite antichains with respect to this notion of comparison, which therefore transfers to the Wadge order on the $\mathbf{\Delta}^0_2$-degrees of the Scott domain.

With the exception of Section 5, all the results presented in this article—including the main ones—are due to the sole second author.

§**1. Introduction.** The *Wadge order* $\leq_w$—named after Wadge [24]—on the subsets of a topological space X is the quasi-order induced by reductions via continuous functions. More precisely, if $A, B \subseteq X$, then $A \leq_w B$ if there exists a continuous function $f : X \to X$ such that $f^{-1}[B] = A$, i.e., $x \in A \Leftrightarrow f(x) \in B$ for all $x \in X$. The Wadge order measures the topological complexity of the subsets of X. Indeed, $A \leq_w B$ means that the membership problem for A can be reduced, via some continuous function, to the membership problem for B; or, in other words, A is topologically less complicated than B.

The Wadge order is a refinement of both the classical Borel and Hausdorff–Kuratowski difference hierarchies since when B is located strictly higher than A in one of these hierarchies, then $A \leq_w B$ holds. Over the last 50 years, this quasi-order has been extensively studied in the context of *Polish spaces*—i.e., the separable completely metrizable spaces [1, 2, 5, 9–11, 13, 14, 17, 21, 23–25].

Over the last decades, some slightly different classes of topological spaces rose interest for their involvement in computer science [6, 7, 18–20, 26]. This has been the case, in particular, of nonmetrizable—hence non-Polish—spaces occurring as domains of the semantic of programming languages. Building on a prior work of Selivanov—that extensively studied a generalized version of the Borel hierarchy to nonmetrizable spaces [19, 20]—de Brecht introduced in [4] the class of *quasi-Polish spaces*—i.e., the second countable quasi-metrizable spaces, where a quasi-metric is a

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metric whose symmetry condition has been dropped. In particular, de Brecht proved that some of the major results of descriptive set theory extend to quasi-Polish spaces (see Theorems 19, 23, 58, and 70 in [4]). He also exhibited the *Scott domain*[1] $\mathcal{P}\omega$ as a universal quasi-Polish space. More precisely, de Brecht proved that the quasi-Polish spaces are—up to homeomorphism—exactly the $\mathbf{\Pi}^0_2$-subsets of $\mathcal{P}\omega$ (Theorem 24 in [4]), where $\mathcal{P}\omega$ is the power set of the integers equipped with the topology where a basic open set is composed of all the sets that contain a fixed finite subset of the integers (Definition 2.1).

More results by de Brecht suggest that a reasonable descriptive set theory still holds in the quasi-Polish setting. Unfortunately, very few is known about the Wadge order in this context. To the contrary, the Polish spaces X whose Wadge order on the Borel subsets is well founded and contains no infinite antichain—or in other words, $\leq_w$ is a well-quasi-order on the Borel subsets of X—were recently characterized in [17] as the *zero-dimensional* ones—i.e., Polish spaces admitting a clopen basis. Whether this result generalizes to quasi-Polish spaces remains open. In a first attempt to tackle this question, we propose to study the Wadge order on the subsets of the Scott domain $\mathcal{P}\omega$.

Several results have already been obtained by Selivanov who proved the existence of $\leq_w$-antichains of size 4 for $\mathcal{P}\omega$, as well as the existence of $\leq_w$-minimal sets at each level of the difference hierarchy of open sets [19]; and by Becher and Grigorieff who exhibited, for each infinite level α of the difference hierarchy of open sets, some strictly $\leq_w$-increasing chains of sets of length α, and also described the $\leq_w$-maximal sets for each such level for a large number of quasi-Polish spaces including $\mathcal{P}\omega$ [3]. In this article, we show both that the Wadge order on the subsets of $\mathcal{P}\omega$ is ill-founded and that it admits infinite antichains. Moreover, we show that these properties occur already within the differences of ω open sets, i.e., at a very low level of topological complexity:

THEOREM 6.1. $\left(D_\omega(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega), \leq_w\right)$ *is ill-founded.*

THEOREM 7.1. $\left(D_\omega(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega), \leq_w\right)$ *has infinite antichains.*

These results are obtained through a generalization of a construction introduced by Selivanov in [19]. More precisely, we define an order-embedding from a class of 2-colored countable posets $\mathbb{P}_{\text{emb}}$ (Definition 3.4) endowed with the usual notion of comparison by homomorphisms (as done in [12, 27]) into the Wadge order on the $\mathbf{\Delta}^0_2$-degrees of $\mathcal{P}\omega$, where a degree is an equivalence class induced by $\leq_w$ (see Definition 2.6):

THEOREM 4.9. *There exists an order-embedding*:

$$(\mathbb{P}_{\text{emb}}, \preccurlyeq_c)/\equiv_c \;\longrightarrow\; (\mathbf{\Delta}^0_2(\mathcal{P}\omega), \leq_w)/\equiv_w.$$

Different approaches have already been considered for tackling the problem of classifying subsets of non-Polish spaces according to their topological complexity. For instance, Pequignot and Selivanov studied the quasi-order obtained from reductions via admissible representations independently [16, 22], and Motto Ros, Schlicht,

[1] The Scott domain was first introduced by Scott as a denotational semantic for the λ-calculus [18].

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and Selivanov investigated the quasi-order obtained from classes of reductions that are larger than the continuous ones [15].

The article is organized as follows. We fix notations and general definitions in Section 2, where we also recall results such as the characterizations of some of the topological classes obtained by Selivanov in [19]. In Section 3, we define the class of posets $\mathbb{P}_{\mathrm{emb}}$ (Definition 3.4) that we embed into the Wadge order of $\mathcal{P}\omega$ (Theorem 4.9) in Section 4. This order-embedding is the main construction of this article. A game characterization of reductions between 2-colored posets is introduced in Section 5 (Definition 5.1) in order to show, in Sections 6 and 7, that the Wadge order of $\mathcal{P}\omega$ is ill-founded (Theorem 6.1) and that it has infinite antichains (Theorem 7.1). We conclude in Section 8 with open questions.

§2. Preliminaries.

2.1. General notations. As usual, we denote by ω or $\mathbb{N}$ the set of all integers and by $\aleph_0$ its cardinality. We also write ω^+ for $\omega \setminus \{0\}$ and ω_1 for the first uncountable ordinal. We use the letters i, j, k, l, m, n for integers and α, β, γ for arbitrary ordinals. Since every ordinal is regarded as the set of its predecessors, if $n \in \omega$, the notation $x \cap n$ stands for $x \cap \{0, 1, \dots, n-1\}$.

Given any sets X, Y, if $f : X \to Y$ is a function, $A \subseteq X$, and $B \subseteq Y$, then we write $f[A] = \{f(x) \mid x \in A\}$ and $f^{-1}[B] = \{x \mid f(x) \in B\}$. If f is injective, we write $f^{-1}(y)$ for the unique element $x \in X$ such that $f(x) = y$.

An X-sequence—or simply a sequence—is a function $s : \alpha \to X$—denoted by $(s_\beta)_{\beta<\alpha}$—from some ordinal $\alpha = \mathrm{lh}(s)$ called the length of the sequence to X. In this article, we will mainly consider sequences such that $\alpha \in \omega + 1 = \omega \cup \{\omega\}$. We use the letters s, t to denote sequences. The only sequence of length 0—the empty sequence—is denoted by $\varnothing$. If s, t are sequences, then t is a prefix of s, written $t \sqsubseteq s$, if $\mathrm{lh}(t) \leq \mathrm{lh}(s)$ and $s_k = t_k$ for all $k < \mathrm{lh}(t)$. If $t \sqsubseteq s$ but $s \not\sqsubseteq t$, we write $t \sqsubset s$. If s, t are X-sequences, the concatenation of s and t is defined by $s^\frown t = (s_0, \dots, s_{\mathrm{lh}(s)-1}, t_0, \dots, t_{\mathrm{lh}(t)-1})$. The set of all X-sequences of finite length is denoted by $X^{<\omega}$.

A tree $T \subseteq X^{<\omega}$ is a set of finite X-sequences closed under the prefix relation.[2] It is well founded if it has no infinite branch,[3] in which case the rank of any $t \in T$ is (well) defined by $\sqsupseteq$-induction: $\mathrm{rk}_T(t) = 0$ if t is $\sqsubseteq$-maximal and $\mathrm{rk}_T(t) = \sup\{\mathrm{rk}_T(s) + 1 \mid t \sqsubset s\}$ otherwise. The rank $\mathrm{rk}(T)$ of a nonempty well-founded tree T is the ordinal $\mathrm{rk}_T(\varnothing)$.

2.2. Order-theoretic notations. A quasi-order on a set Q is any reflexive and transitive relation[4] $\leq_q \subseteq Q \times Q$. Whenever $\leq_q$ is clear from the context, we write Q for the couple $(Q, \leq_q)$. We will use the letters P, Q for quasi-orders and $p \in P, q \in Q$ for their elements. As usual, $q_0 \leq_q q_1$ stands for $(q_0, q_1) \in \leq_q$, and $q_0 <_q q_1$ for $q_0 \leq_q q_1$ but $q_1 \not\leq_q q_0$. If $q_0 \not\leq_q q_1$ and $q_1 \not\leq_q q_0$, then q_0 and q_1 are said to be incomparable which is denoted by $q_0 \perp_q q_1$. If Q is a quasi-order and $P \subseteq Q$, then

[2] If $t \in T$ and $s \sqsubseteq t$, then $s \in T$.

[3] An infinite branch is a function $f : \omega \to T$ such that, if $n < m$, then $f(n) \sqsubset f(m)$.

[4] A binary relation $\leq_q$ on Q is reflexive if, for all $q \in Q$, $(q, q) \in \leq_q$, and transitive if, for any $q_0, q_1, q_2 \in Q$, $(q_0, q_1), (q_1, q_2) \in \leq_q$ implies $(q_0, q_2) \in \leq_q$.

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P equipped with the induced relation is a quasi-order. An infinite antichain in Q is a sequence $(q_n)_{n<\omega}$ of pairwise incomparable elements, and a strictly $\leq_q$-increasing (respectively, a strictly $\leq_q$-decreasing) sequence is a sequence $(q_n)_{n<\omega}$ such that $q_n <_q q_{n+1}$ (respectively, $q_{n+1} <_q q_n$) for all $n \in \omega$. A well-quasi-order is a quasi-order Q that has no infinite antichain and no strictly $\leq_q$-decreasing sequence. We denote by $\mathrm{Pred}(q) = \{q' \in Q \mid q' \leq_q q\}$ the set of predecessors of $q \in Q$, and by $\mathrm{Pred}_{\mathrm{im}}(q) = \{q' \in Q \mid (q' <_q q) \wedge \neg\exists q'' \in Q\ (q' <_q q'' \wedge q'' <_q q)\}$ the set of its immediate predecessors.

We use homomorphisms in order to compare structures. A homomorphism[5] between two quasi-orders is an order-preserving function. If there exists an injective homomorphism $\varphi : P \to Q$, then we write $P \xrightarrow{\text{1-1 h.}} Q$; if it is injective and preserves immediate predecessors,[6] then we write $P \rightarrowtail Q$. Notice that $P \rightarrowtail Q$ is more rigid than $P \xrightarrow{\text{1-1 h.}} Q$; hence, it describes more local behaviors.

If q and q' are elements of a quasi-order Q such that $q \leq_q q'$ and $q' \leq_q q$, then we write $q \equiv_q q'$. The relation $\equiv_q$ is an equivalence relation whose equivalence classes are denoted by $[q] = \{q' \in Q \mid q \equiv_q q'\}$. The quotient set $Q/\equiv_q = \{[q] \mid q \in Q\}$ inherits the quasi-order $\leq_q$. More precisely, we set $[q] \leq_q [q']$ if and only if $q \leq q'$. The set $Q/\equiv_q$ equipped with $\leq_q$ is a poset, i.e., a quasi-order whose order-relation is a partial order.[7]

We denote the class of countable posets by $\mathbb{P}$. If $P \in \mathbb{P}$, then we can always consider $\leq_p \subseteq \alpha \times \alpha$ where $\alpha \in \omega \cup \{\omega\}$ via any bijection $P \leftrightarrow \alpha$; so that all the posets we consider are posets on $P \in \omega \cup \{\omega\}$. An order-embedding is a homomorphism between two posets $\varphi : P \to Q$ such that for any $p_0, p_1 \in P$, $p_0 \leq_p p_1$ if and only if $\varphi(p_0) \leq_q \varphi(p_1)$. Thus, order-embeddings are injective. The main poset studied in this article will be the set of *finite* subsets of the integers ordered by inclusion $(\mathcal{P}_{<\omega}(\omega), \subseteq)$.

A 2-colored poset is a triple $\mathsf{P} = (P, \leq_p, \mathrm{c}_p)$ where $\leq_p$ is a partial order on P and $\mathrm{c}_p : P \to 2$ is a 2-coloring. We usually use the letters P, Q for 2-colored posets. As done in [12, 27], we compare them via homomorphisms.[8] If there exists a homomorphism from P to Q, then we write $\mathsf{P} \preccurlyeq_c \mathsf{Q}$; if this homomorphism is injective, then we write $\mathsf{P} \xrightarrow{\text{1-1 h.}}_c \mathsf{Q}$; if it is injective and preserves immediate predecessors, then we write $\mathsf{P} \rightarrowtail_c \mathsf{Q}$. Notice that $\preccurlyeq_c$ is a quasi-order on 2-colored posets. We will denote by $\equiv_c$ the induced equivalence relation.

2.3. Topological notations. This article focuses on the study of a particular topological space first introduced by Scott as a universal model of the semantic of λ-calculus [18].

[5]A homomorphism between two quasi-orders P and Q is a function $\varphi : P \to Q$ such that for any $p_0, p_1 \in P$, if $p_0 \leq_p p_1$, then $\varphi(p_0) \leq_q \varphi(p_1)$.

[6]A function $\varphi : P \to Q$ preserves immediate predecessors if, for any $p_0, p_1 \in P$, whenever $p_0 \in \mathrm{Pred}_{\mathrm{im}}(p_1)$, then $\varphi(p_0) \in \mathrm{Pred}_{\mathrm{im}}(\varphi(p_1))$.

[7]A quasi-order $(P, \leq_p)$ is a partial order if $\leq_p$ is antisymmetric, i.e., for any $p_0, p_1 \in P$, $p_0 \leq_p p_1$ and $p_1 \leq_p p_0$ implies $p_0 = p_1$.

[8]A homomorphism between P, Q two 2-colored posets is a quasi-order homomorphism $\varphi : P \to Q$ such that for all $p \in P$, $\mathrm{c}_p(p) = \mathrm{c}_q(\varphi(p))$.

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Definition 2.1. The *Scott domain* is the power set of the integers $\mathcal{P}(\omega)$ equipped with the topology generated by the basis

$$\big\{\mathcal{O}_F \mid F \in \mathcal{P}_{<\omega}(\omega)\big\}, \text{ where } \mathcal{O}_F = \{x \subseteq \omega \mid F \subseteq x\}.$$

The Scott domain is a nonmetrizable—in fact non-Hausdorff (T_2), and even non-Fréchet (T_1)—compact space which is connected and Kolmogorov (T_0).

From now on and throughout this article, we use the notation $\mathcal{P}\omega$ for the Scott domain; F, G, H for finite subsets of ω; x, y, z for arbitrary subsets of ω; and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ for subsets of $\mathcal{P}\omega$.

Our ultimate goal is to study the topological complexity of subsets of $\mathcal{P}\omega$. In metrizable spaces, this study begins with the definition of the Borel hierarchy (Section 11.B in [10]). However, the same construction would not work with $\mathcal{P}\omega$ for it is not metrizable. To overcome this obstacle, Selivanov introduced a new version of the Borel hierarchy for arbitrary spaces [19, 20]. This generalization extends the original one and induces a well-behaved hierarchy (see [4] for more details). In the rest of this section, $\mathcal{T}$ denotes a topology on a set X. As usual, we denote by X both the topological space and the underlying set without any risk of confusion.

Definition 2.2. We define $\boldsymbol{\Sigma}^0_1(X) = \mathcal{T}$, and for $1 < \alpha < \omega_1$,

$$\boldsymbol{\Sigma}^0_\alpha(X) = \left\{ \bigcup_{n\in\omega} (B_n \setminus B'_n) \;\middle|\; B_n, B'_n \in \boldsymbol{\Sigma}^0_{\beta_n}(X),\ \beta_n < \alpha \right\}.$$

We also define $\boldsymbol{\Pi}^0_\alpha(X) = \{A \subseteq X \mid X \setminus A \in \boldsymbol{\Sigma}^0_\alpha(X)\}$, $\boldsymbol{\Delta}^0_\alpha(X) = \boldsymbol{\Sigma}^0_\alpha(X) \cap \boldsymbol{\Pi}^0_\alpha(X)$ for $\alpha < \omega_1$. Finally, we define the *Borel sets* as $\boldsymbol{\mathcal{B}}(X) = \bigcup_{\alpha\in\omega_1} \boldsymbol{\Sigma}^0_\alpha(X)$.
The *Borel hierarchy on* X is the quasi-order

$$\Big(\big\{\boldsymbol{\Sigma}^0_\alpha(X), \boldsymbol{\Pi}^0_\alpha(X)\big\}_{\alpha\in\omega_1}, \subseteq\Big).$$

As customary in descriptive set theory, we consider the Hausdorff–Kuratowski difference hierarchy as a first refinement of the Borel hierarchy (see Section 22.E in [10]). Its definition relies on the difference operation.

Definition 2.3. If $0 < \alpha < \omega_1$ and $(A_\beta)_{\beta<\alpha}$ is a sequence of subsets of X, then

$$D_\alpha\big((A_\beta)_{\beta<\alpha}\big) = \bigcup \left\{ A_\beta \setminus \cup_{\gamma<\beta} A_\gamma \;\middle|\; \begin{array}{l} \beta < \alpha, \text{ and} \\ \alpha \text{ and } \beta \text{ have different parities} \end{array} \right\} \subseteq X.$$

If $0 < \alpha, \beta < \omega_1$, then

$$D_\alpha\big(\boldsymbol{\Sigma}^0_\beta\big)(X) = \Big\{ D_\alpha\big((A_\gamma)_{\gamma<\alpha}\big) \mid (A_\gamma)_{\gamma<\alpha} \subseteq \boldsymbol{\Sigma}^0_\beta(X) \Big\} \subseteq \mathcal{P}(X).$$

Finally, we set $\check{D}_\alpha\big(\boldsymbol{\Sigma}^0_\beta\big)(X) = \Big\{ A \subseteq X \mid X \setminus A \in D_\alpha\big(\boldsymbol{\Sigma}^0_\beta\big)(X) \Big\}$.
The *Hausdorff–Kuratowski difference hierarchy on* X is the quasi-order

$$\Big(\big\{D_\alpha\big(\boldsymbol{\Sigma}^0_\beta\big)(X), \check{D}_\alpha\big(\boldsymbol{\Sigma}^0_\beta\big)(X)\big\}_{\alpha,\beta\in\omega_1}, \subseteq\Big).$$

All Borel and Hausdorff–Kuratowski classes previously defined are closed under continuous preimages.[9] This suggests a natural further investigation of topological

[9] A class of subsets $\Gamma(X) \subseteq \mathcal{P}(X)$ is closed under continuous preimages if for any $A \in \Gamma(X)$, $f : X \to X$ continuous, then $f^{-1}[A] \in \Gamma(X)$.

complexity through the lens of Wadge reducibility, a notion of comparison first studied thoroughly by Wadge in his Ph.D. thesis [24].

Definition 2.4. Let $A, B \subseteq X$. The set A is *Wadge reducible* to B, written $A \leq_w B$, if there exists a continuous function $f : X \to X$ such that for all $x \in X$,

$$x \in A \iff f(x) \in B,$$

i.e., $f^{-1}[B] = A$.
A is *Wadge equivalent* to B, written $A \equiv_w B$, if $A \leq_w B$ and $B \leq_w A$ hold.

Since both the identity and the composition of continuous functions are continuous, $\leq_w$ induces a quasi-order on the subset of X, and thus the binary relations $\nleq_w$, $<_w$ and $\perp_w$ are well defined.

Definition 2.5. Let X be any topological space and $\Gamma(X) \subseteq \mathcal{P}(X)$ be any class closed under continuous preimages. The *Wadge order on the* Γ*-subsets of* X is the quasi-order $(\Gamma(X), \leq_w)$.

For the equivalence relation $\equiv_w$, we have a special terminology:

Definition 2.6. Let X be any topological space, $A \subseteq X$ and $\Gamma(X) \subseteq \mathcal{P}(X)$ be any class closed under continuous preimages.
The *Wadge degree of* A is its $\equiv_w$-equivalence class $[A] = \{B \subseteq A \mid A \equiv_w B\}$.
The *Wadge order on the* Γ*-degrees of* X is the poset $(\Gamma(X), \leq_w)/\equiv_w$.

Notice that the Wadge order on the Γ-subsets of X admits an infinite antichain (respectively, a strictly $\leq_w$-decreasing sequence) if and only if the Wadge order on the Γ-degrees of X also admits an infinite antichains (respectively, a strictly $\leq_w$-decreasing sequence).

2.4. Selivanov's toolbox. We will restrict ourselves to the study of the quasi-order $(\mathbf{\Delta}^0_2(\mathcal{P}\omega), \leq_w)$. As mentioned in the Introduction, some results have already been obtained on this quasi-order in [19] and [3]. The main result of this article (Theorem 4.9) comes as a generalization of a construction introduced by Selivanov in [19] that we recall here.

Definition 2.7 (p. 56 in [19]). Let T_α be any well-founded tree of rank $\omega \leq \alpha < \omega_1$, $\xi : \omega^{<\omega} \to \omega$ be any injective mapping such that $\xi(\varnothing) = 0$, and $e : T_\alpha \to \mathcal{P}_{<\omega}(\omega)$ be defined as $e(s) = \{\xi(t) \mid t \sqsubseteq s\}$. The sets Y_α and Z_α are defined by:

1. $Y_\alpha = e[T^1_\alpha]$, where $T^1_\alpha = \{s \in T_\alpha \mid \text{lh}(s) \text{ is odd}\}$,
2. $Z_\alpha = B(T_\alpha) \cup Y_\alpha$, where $B(T_\alpha) = \{x \subseteq \omega \mid \forall s \in T_\alpha\ x \nsubseteq e(s)\}$.

In [19], it is shown that, given any $\omega \leq \alpha < \omega_1$, Y_α and Z_α are differences of α open sets, Wadge incomparable, and $\leq_w$-minimal among true differences of α open sets. More precisely,

Theorem 2.8 (Propositions 5.9 and 6.4 in [19]). *For $n \in \omega$, $\omega \leq \alpha, \beta < \omega_1$ and $\mathcal{A} \in \mathbf{\Delta}^0_2(\mathcal{P}\omega) \setminus \check{D}_\alpha(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega)$, we have:*

1. *$D_n(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega) \setminus \check{D}_n(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega)$ and $\check{D}_n(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega) \setminus D_n(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega)$ form two incomparable Wadge degrees,*
2. *$Y_\alpha, Z_\alpha \in D_\alpha(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega) \setminus \check{D}_\alpha(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega)$,*

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3. $Y_\alpha \perp_w Z_\beta$,
4. *if* $\omega \in \mathcal{A}$, *then* $Z_\alpha \leq_w \mathcal{A}$,
5. *if* $\omega \notin \mathcal{A}$, *then* $Y_\alpha \leq_w \mathcal{A}$.

The proof of Theorem 2.8 makes use of Selivanov's characterizations of the $\mathbf{\Delta}^0_2$-subsets and of the $D_\alpha(\mathbf{\Sigma}^0_1)$-subsets of $\mathcal{P}\omega$. Since our proof will also require these characterizations, we first recall them. For this purpose, if $x, y \in \mathcal{P}\omega$ are such that $x \subseteq y$, we introduce the notation

$$[x, y] = \{z \in \mathcal{P}\omega \mid x \subseteq z \subseteq y\}.$$

Definition 2.9 (Definition 2.4 in [19]). $\mathcal{A} \subseteq \mathcal{P}\omega$ is *approximable* if, for all $x \in \mathcal{A}$, there exists $F \in \mathcal{P}_{<\omega}(\omega)$ such that $F \subseteq x$ and $[F, x] \subseteq \mathcal{A}$.

A subset $\mathcal{A}$ of $\mathcal{P}\omega$ is $\mathbf{\Delta}^0_2$ if the membership of any subset $x \subseteq \omega$ to $\mathcal{A}$ can be approximated by a finite subset of x. More precisely:

Theorem 2.10 (Theorem 3.12 in [19]). *Let* $\mathcal{A} \subseteq \mathcal{P}\omega$.

$$\mathcal{A} \in \mathbf{\Delta}^0_2(\mathcal{P}\omega) \iff \mathcal{A} \text{ and } \mathcal{P}\omega \setminus \mathcal{A} \text{ are approximable.}$$

The characterization of $D_\alpha(\mathbf{\Sigma}^0_1)$-subsets of $\mathcal{P}\omega$ is a stratification of the previous result using the notion of a 1-alternating tree.

Definition 2.11 (Definition 3.5 in [19]). Let $\mathcal{A} \subseteq \mathcal{P}\omega$ and $0 < \alpha < \omega_1$. A *1-alternating tree for* $\mathcal{A}$ *of rank* α is a homomorphism of quasi-orders

$$f : (T, \sqsubseteq) \to (\mathcal{P}_{<\omega}(\omega), \subseteq)$$

from a well-founded tree $T \subseteq \omega^{<\omega}$ of rank α such that:

1. $f(\varnothing) \in \mathcal{A}$, and
2. for all $s^\frown\langle n\rangle \in T$, we have $\big(f(s) \in \mathcal{A} \leftrightarrow f(s^\frown\langle n\rangle) \notin \mathcal{A}\big)$.

Corollary 2.12 (Corollary 3.11 in [19]). *Let* $\mathcal{A} \subseteq \mathcal{P}\omega$ *and* $0 < \alpha < \omega_1$.

$$\mathcal{A} \in D_\alpha(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega) \iff \begin{cases} \mathcal{A} \in \mathbf{\Delta}^0_2(\mathcal{P}\omega) \text{ and} \\ \text{there is no 1-alternating tree for } A \text{ of rank } \alpha. \end{cases}$$

§**3. The class $\mathbb{P}_{\mathbf{emb}}$.** We define a class—called $\mathbb{P}_{\text{emb}}$—of countable 2-colored posets (Definition 3.4) that will be mapped into the Wadge order on the subsets of the Scott domain in the next section. The definition of $\mathbb{P}_{\text{emb}}$ will first be independent of $\mathcal{P}\omega$. Afterwards, we will give an order theoretic characterization of the elements of $\mathbb{P}_{\text{emb}}$ that link them to $\mathcal{P}\omega$ (Proposition 3.3).

We begin with the naming of several posets that are useful for the definition of a subclass of $\mathbb{P}$ denoted by $\mathbb{P}_{\text{shr}}$. In Figure 1, we represent each poset $(P, \leq_p)$ with its Hasse diagram $G = (P, \to)$. More precisely, if $p, q \in P$, then $p \leq_p q$ if and only if there exists a finite sequence $(p_k)_{k \leq l}$ such that $p_0 = p$, $p_l = q$ and for all $k < l$, we have $p_k \to p_{k+1}$.

In [19], Selivanov worked with well-founded trees in order to construct subsets of $\mathcal{P}\omega$. We will generalize this construction to a larger class of posets that we call *shrubs* and that share some of the properties of well-founded trees. For this purpose, we make use of the classical notion of *bounded completeness* that occurs in domain theory [6].

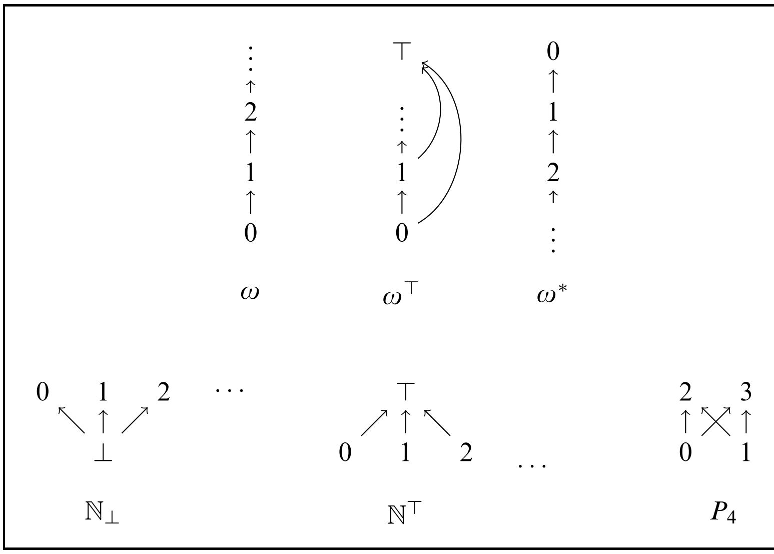

FIGURE 1. Samples of useful countable posets.

DEFINITION 3.1. Let $(P, \leq_p)$ be any poset.

1. A subset $B \subseteq P$ is *bounded* if there exists an element $u \in P$ such that $b \leq_p u$ holds for every $b \in B$. Such a $u \in P$ is called an *upper bound of* B. The set of all upper bounds of B is denoted by $\mathcal{U}_B$.
2. If the set $\mathcal{U}_B$ of all upper bounds of B has a—necessarily unique—$\leq_p$-minimal element $s_B \in \mathcal{U}_B$—i.e., $\forall u \in \mathcal{U}_B\ s_B \leq_p u$—it is called *the supremum* of B in P.
3. The poset $(P, \leq_p)$ is *bounded complete* if every bounded $S \subseteq P$ has a supremum.

A typical example of a poset which is not bounded complete is P_4 as shown in Figure 1. All other posets shown in Figure 1, as well as $(\mathcal{P}_{<\omega}(\omega), \subseteq)$ and $(\mathcal{P}\omega, \subseteq)$ are bounded complete. Notice that every bounded complete poset P has a unique $\leq_p$-minimal element, namely the supremum of the empty set, usually denoted by $\perp$.

DEFINITION 3.2. The class of all *shrubs* $\mathbb{P}_{\text{shr}} \subseteq \mathbb{P}$ is the class of all countable posets $P \in \mathbb{P}$ that satisfy:

1. $\omega \overset{1\text{-}1\ h.}{\not\longrightarrow} P$,
2. for all $p \in P$, $\text{Card}(\text{Pred}(p)) < \aleph_0$,
3. P is bounded complete.

Well-founded trees, and in particular $\mathbb{N}_\perp$, are typical examples of shrubs. More involved ones will be constructed in the proof of Theorem 6.1 (Figure 5) and of Theorem 7.1 (Figure 6). To the contrary, ω, $\omega^\top$, ω^*, $\mathbb{N}^\top$, and P_4 are typical examples of posets that are not shrubs.

In the next proposition, we give alternative characterizations to the second item of the previous definition. In particular, we show that the posets we just defined

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can be embedded into $\mathcal{P}_{<\omega}(\omega)$. We also give an alternative characterization of this second item that exclusively depends on morphisms between posets.

Proposition 3.3. *If $P \in \mathbb{P}$, then the following are equivalent*:

1. *for all* $p \in P$, $\mathrm{Card}(\mathrm{Pred}(p)) < \aleph_0$,
2. $P \xrightarrow{\textit{1-1 h.}} \mathcal{P}_{<\omega}(\omega)$,
3. $(\omega^\top \overset{\textit{1-1 h.}}{\not\longrightarrow} P)$, $(\omega^* \overset{\textit{1-1 h.}}{\not\longrightarrow} P)$ *and* $(\mathbb{N}^\top \overset{\textit{1-1 h.}}{\not\longrightarrow} P)$.

Proof. (1. $\Rightarrow$ 2.): We consider $P \in \omega \cup \{\omega\}$ and define a function:

$$\begin{aligned} e : P &\to \mathcal{P}_{<\omega}(\omega) \\ k &\mapsto \{n \mid n \leq_p k\}. \end{aligned}$$

If $k \leq_p l$, then by transitivity of $\leq_p$, we get $e(k) \subseteq e(l)$. If $k \neq l$, we consider the two cases $k <_p l$ and $k \perp_p l$ (the third case $l <_p k$ is the same as the case $k <_p l$). In both cases, $l \in e(l) \setminus e(k)$. Therefore, we obtain that e is an injective homomorphism that witnesses $P \xrightarrow{\textit{1-1 h.}} \mathcal{P}_{<\omega}(\omega)$.

(2. $\Rightarrow$ 3.): If $\varphi : Q \xrightarrow{\textit{1-1 h.}} P$, then for all $q \in Q$, the injectivity of φ implies $\mathrm{Card}\big(\mathrm{Pred}(q)\big) \leq \mathrm{Card}\big(\mathrm{Pred}(\varphi(q))\big)$. Since $\mathrm{Card}\big(\mathrm{Pred}(F)\big) < \aleph_0$ for any $F \in \mathcal{P}_{<\omega}(\omega)$, we get the result by contradiction.

(3. $\Rightarrow$ 1.): Towards a contradiction, we pick $p \in P$ such that $\mathrm{Card}(\mathrm{Pred}(p)) = \aleph_0$. We consider three different cases.

(a) Suppose there exists $q_0 <_p p$ such that there exists no immediate predecessor p' of p satisfying $q_0 \leq_p p'$. Hence, there exists $q_1 <_p p$ such that $q_0 <_p q_1$. We continue the process to construct a sequence $(q_n)_{n\in\omega}$ witnessing $\omega^\top \xrightarrow{\textit{1-1 h.}} P$ via the mapping: $\top \mapsto p$, and $n \mapsto q_n$ for any $n \in \omega$.

(b) Suppose there exist infinitely many immediate predecessors $(q_n)_{n\in\omega}$ of $p \in P$, then the mapping: $\top \mapsto p$, and $n \mapsto q_n$ for any $n \in \omega$, witnesses $\mathbb{N}^\top \xrightarrow{\textit{1-1 h.}} P$.

(c) Suppose that we are not in the situations (a) and (b); then, by the pigeonhole principle, there exists q_0 an immediate predecessor of p such that $\mathrm{Card}(\mathrm{Pred}(q_0)) = \aleph_0$. If we replace p with q_0 and start the proof again, either we get a contradiction from (a) or (b) or we exhibit q_1 an immediate predecessor of q_0 such that $\mathrm{Card}(\mathrm{Pred}(q_1)) = \aleph_0$. By an infinite iteration of this process, we obtain a sequence $(q_n)_{n\in\omega}$ witnessing $\omega^* \xrightarrow{\textit{1-1 h.}} P$ via the mapping: $0 \mapsto p$, and $n \mapsto q_{n-1}$ for any $n \in \omega^+$. ⊣

In Figure 2, we give a name to some specific 2-colored posets that are useful for the next definition: the nodes of the form • and ∘ correspond to color 1 and color 0, respectively.

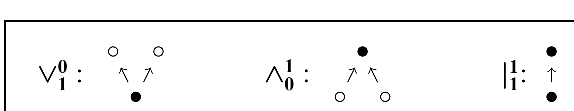

Figure 2. Samples of useful 2-colored countable posets.

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The next definition introduces the class of *embeddable posets* $\mathbb{P}_{\text{emb}}$. We will later associate a subset $\mathcal{A}_{\mathsf{P}}$ of $\mathcal{P}\omega$ to each such 2-colored poset $\mathsf{P} \in \mathbb{P}_{\text{emb}}$, where the color 1 will correspond to elements inside $\mathcal{A}_{\mathsf{P}}$.

Definition 3.4. The class of *embeddable posets* $\mathbb{P}_{\text{emb}}$ is the class of countable 2-colored posets $\mathsf{P} = (P, \leq_p, \mathrm{c}_p)$ such that $(P, \leq_p) \in \mathbb{P}_{\text{shr}}$ and whose coloring satisfies:

1. $\mathrm{c}_p(\bot) = 0$,
2. for all $k \in P$ $\leq_p$-maximal, $\mathrm{c}_p(k) = 1$,
3. $(\vee^{\mathbf{0}}_{\mathbf{1}} \not\preccurlyeq_c \mathsf{P})$, $(\wedge^{\mathbf{1}}_{\mathbf{0}} \not\preccurlyeq_c \mathsf{P})$ and $(|^{\mathbf{1}}_{\mathbf{1}} \not\preccurlyeq_c \mathsf{P})$.

If P is an embeddable poset, then the nodes of color 1 are isolated. Indeed, if $\mathsf{P} \in \mathbb{P}_{\text{emb}}$, $p \in P$ and $\mathrm{c}_p(p) = 1$, then p has a unique immediate predecessor; and p has at most one immediate successor,[10] depending on whether p is $\leq_p$-maximal or not. Moreover, if they exist, they both have color 0. Thus, we introduce the following notations.

Notation 3.5. *For* $\mathsf{P} \in \mathbb{P}_{\text{emb}}$, $p \in P$ *and* $\mathrm{c}_p(p) = 1$, *we denote by* p^- *its unique immediate predecessor; and, if it exists, by* p^+ *its unique immediate successor. We have* $\mathrm{c}_p(p^-) = \mathrm{c}_p(p^+) = 0$.

This means that the direct neighborhood—composed of all immediate predecessors and all immediate successors—of every node of color 1 is of one of the form given in Figure 3, depending on whether it is $\leq_p$-maximal or not.

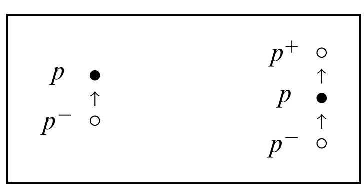

Figure 3. The two possible direct neighborhoods of any $p \in P$, where $\mathsf{P} \in \mathbb{P}_{\text{emb}}$ and $\mathrm{c}_p(p) = 1$. The first case occurs when p is $\leq_p$-maximal, and the second one when p is not.

§**4. An order-embedding into the Wadge order.** In this section, we associate a subset $\mathcal{A}_{\mathsf{P}} \in \mathbf{\Delta}^0_2(\mathcal{P}\omega)$ to each embeddable poset $\mathsf{P} \in \mathbb{P}_{\text{emb}}$, and show that this association is such that, for any $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\text{emb}}$, $\mathsf{P} \preccurlyeq_c \mathsf{Q}$ if and only if $\mathcal{A}_{\mathsf{P}} \leq_w \mathcal{A}_{\mathsf{Q}}$ (Lemma 4.5). As a consequence, we get our main result that there exists an order-embedding $(\mathbb{P}_{\text{emb}}, \preccurlyeq_c)/\equiv_c \;\rightarrow (\mathbf{\Delta}^0_2(\mathcal{P}\omega), \leq_w)/\equiv_w$ (Theorem 4.9).

We first need to label the elements of any embeddable poset.

Definition 4.1. Let $\mathsf{P} \in \mathbb{P}_{\text{emb}}$ so that $P \in \omega \cup \{\omega\}$ has a $\leq_p$-minimal element $m = \bot$ for some $m \in \omega$. The labeling l_p on P is defined by:

[10]If P is an embeddable poset, $p \in P$ is an immediate successor of $p' \in P$ if $p' \in \mathrm{Pred}_{\mathrm{im}}(p)$.

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$$\begin{aligned} l_p : P &\to \mathcal{P}_{<\omega}(\omega) \\ \bot &\mapsto \emptyset, \\ n &\mapsto \bigcup_{k \leq_p n} \{k\}. \end{aligned}$$

We notice that l_p is an injective homomorphism of posets. Therefore, for every $F \in \mathcal{P}_{<\omega}(\omega)$ in the range of l_p, $l_p^{-1}(F)$ is well defined.

We then associate a subset of the Scott domain to any embeddable poset through the labeling given by Definition 4.1.

Definition 4.2. Let $\mathsf{P} \in \mathbb{P}_{\mathrm{emb}}$, we define the subset $\mathcal{A}_{\mathsf{P}} \subseteq \mathcal{P}\omega$ as:

$$\begin{aligned} \mathcal{A}_{\mathsf{P}} &= l_p\left[\mathrm{c}_p^{-1}[\{1\}]\right] \\ &= \left\{x \subseteq \omega \mid \exists p \in P\ (\mathrm{c}_p(p) = 1 \wedge l_p(p) = x)\right\}. \end{aligned}$$

We also denote by $\mathcal{C}(\mathcal{A}_{\mathsf{P}})$ the set of all finite sets of integers contained in the labeling of an element of P:

$$\mathcal{C}(\mathcal{A}_{\mathsf{P}}) = \left\{F \subseteq \omega \mid \exists p \in P\ F \subseteq l_p(p)\right\}.$$

The next lemma gathers two crucial observations that arise from the construction given by Definition 4.2.

Lemma 4.3. *Let* $\mathsf{P} \in \mathbb{P}_{\mathrm{emb}}$ *and* $F \in \mathcal{P}_{<\omega}(\omega)$.

1. *If* $F \in \mathcal{C}(\mathcal{A}_{\mathsf{P}})$, *then* $\{p \in P \mid l_p(p) \subseteq F\}$ *has an upper bound in* P. *By Definition* 3.2, *it has a unique supremum denoted by* $s_F \in P$.
2. $F \in \mathcal{A}_{\mathsf{P}} \Leftrightarrow \left(\mathrm{c}_p(s_F) = 1 \wedge l_p(s_F) = F\right)$.

Proof. 1. Since $F \in \mathcal{C}(\mathcal{A}_{\mathsf{P}}) = \left\{F \subseteq \omega \mid \exists p \in P\ F \subseteq l_p(p)\right\}$, there exists $p_0 \in P$ such that $F \subseteq l_p(p_0)$. Thus, $p_0 \in P$ is an upper bound of $\{p \in P \mid l_p(p) \subseteq F\}$.

2. Assume first that $F \in \mathcal{A}_{\mathsf{P}} \subseteq \mathcal{C}(\mathcal{A}_{\mathsf{P}})$. Then, there exists $p_0 \in P$ such that $\mathrm{c}_p(p_0) = 1$ and $l_p(p_0) = F$. It implies that p_0 is the supremum of $\{p \in P \mid l_p(p) \subseteq F\}$, hence $p_0 = s_F$. We then get $l_p(s_F) = F$ and $\mathrm{c}_p(s_F) = 1$. Conversely, from the very definition of $\mathcal{A}_{\mathsf{P}}$, we have $\mathrm{c}_p(s_F) = 1$ and $l_p(s_F) = F$, which implies that $F \in \mathcal{A}_{\mathsf{P}}$. ⊣

The rest of this section consists in proving that the correspondence $\mathsf{P} \mapsto \mathcal{A}_{\mathsf{P}}$ satisfies that $\mathcal{A}_{\mathsf{P}} \in \mathbf{\Delta}_2^0(\mathcal{P}\omega)$ and for any $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\mathrm{emb}}$, $\mathsf{P} \preccurlyeq_c \mathsf{Q}$ if and only if $\mathcal{A}_{\mathsf{P}} \leq_w \mathcal{A}_{\mathsf{Q}}$. For this, we make use of the well-known result that a continuous mapping from $\mathcal{P}\omega$ to itself is completely determined by its behavior on $\mathcal{P}_{<\omega}(\omega)$. The proof can be safely left to the reader.

Lemma 4.4 (Exercice 5.1.62 in [7]). *Given any homomorphism of posets* $f : \mathcal{P}_{<\omega}(\omega) \to \mathcal{P}\omega$, *there exists a* unique *continuous extension of* f *to the whole Scott domain. This extension is given by*

$$\begin{aligned} \hat{f} : \mathcal{P}\omega &\to \mathcal{P}\omega \\ x &\mapsto \bigcup_{n \in \omega} f(x \cap n). \end{aligned}$$

We are now ready for our main proof.

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LEMMA 4.5. *The following mapping*

$$H : (\mathbb{P}_{\text{emb}}, \preccurlyeq_c) \to (\mathbf{\Delta}^0_2(\mathcal{P}\omega), \leq_w)$$
$$\mathsf{P} \mapsto \mathcal{A}_{\mathsf{P}}$$

satisfies that for any $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\text{emb}}$, *we have*

$$\mathsf{P} \preccurlyeq_c \mathsf{Q} \textit{ if and only if } \mathcal{A}_{\mathsf{P}} \leq_w \mathcal{A}_{\mathsf{Q}}.$$

PROOF. The proof is divided into the three Claims 4.6, 4.7, and 4.8. The first two claims show that H is a well-defined homomorphism, whereas the third one completes the proof.

CLAIM 4.6. *If* $\mathsf{P} \in \mathbb{P}_{\text{emb}}$, *then* $\mathcal{A}_{\mathsf{P}} \in \mathbf{\Delta}^0_2(\mathcal{P}\omega)$.

PROOF OF THE CLAIM. We show that $\mathcal{A}_{\mathsf{P}}$ is both approximable and co-approximable, i.e., $\mathcal{P}\omega \setminus \mathcal{A}_{\mathsf{P}}$ is approximable. $\mathcal{A}_{\mathsf{P}}$ is approximable because $\mathcal{A}_{\mathsf{P}} \subseteq \mathcal{P}_{<\omega}(\omega)$. For co-approximability, we proceed by contradiction and suppose that $\mathcal{A}_{\mathsf{P}}$ is not co-approximable for some $x \in \mathcal{P}\omega \setminus \mathcal{A}_{\mathsf{P}}$ infinite. So, we fix $F_0 \in [\emptyset, x] \cap \mathcal{A}_{\mathsf{P}}$ and set $p_0 = l_p^{-1}(F_0)$. Assume F_n and p_n are already constructed. Since $\mathcal{A}_{\mathsf{P}}$ is not co-approximable, there exists $F_{n+1} \in \big([F_n, x] \setminus \{F_n\}\big) \cap \mathcal{A}_{\mathsf{P}}$. We set $p_{n+1} = l_p^{-1}(F_{n+1})$. It follows that the function

$$\varphi : \omega \to P$$
$$n \mapsto p_n$$

witnesses $\omega \xrightarrow{\text{1-1 h.}} P$, a contradiction. $\dashv_{\text{Claim}}$

CLAIM 4.7. *If* $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\text{emb}}$ *and* $\mathsf{P} \preccurlyeq_c \mathsf{Q}$, *then* $\mathcal{A}_{\mathsf{P}} \leq_w \mathcal{A}_{\mathsf{Q}}$.

PROOF OF THE CLAIM. Suppose that $\mathsf{P} \preccurlyeq_c \mathsf{Q}$ is witnessed by $\varphi : P \to Q$. Consider the function:

$$f_\varphi : \mathcal{P}_{<\omega}(\omega) \to \mathcal{P}\omega$$
$$F \mapsto \begin{cases} l_q\big(\varphi(s_F)\big) & \text{if } F \in \mathcal{C}(\mathcal{A}_{\mathsf{P}}) \wedge \mathrm{c}_p(s_F) = 0, \\ l_q\big(\varphi(s_F)\big) & \text{if } F \in \mathcal{C}(\mathcal{A}_{\mathsf{P}}) \wedge \mathrm{c}_p(s_F) = 1 \wedge F = l_p(s_F), \\ l_q\big(\varphi(s_F^-)\big) & \text{if } F \in \mathcal{C}(\mathcal{A}_{\mathsf{P}}) \wedge \mathrm{c}_p(s_F) = 1 \wedge F \subsetneq l_p(s_F), \\ l_q\big(\varphi(s_F^+)\big) & \text{if } F \in \mathcal{C}(\mathcal{A}_{\mathsf{P}}) \wedge \mathrm{c}_p(s_F) = 1 \wedge F \nsubseteq l_p(s_F), \\ \omega & \text{otherwise,} \end{cases}$$

where s_F is defined as in Lemma 4.3; s_F^- and s_F^+ are defined as in Notation 3.5; and s_F^+ is replaced by ω whenever s_F is a maximal element in $(P, \leq_p)$.

We show that the function $\hat{f}_\varphi$ given by Lemma 4.4 satisfies $\hat{f}_\varphi^{-1}\big[\mathcal{A}_{\mathsf{Q}}\big] = \mathcal{A}_{\mathsf{P}}$. First, for $\hat{f}_\varphi$ to exist, we need f_φ to be order-preserving. Let $F, G \in \mathcal{P}_{<\omega}(\omega)$ be such that $F \subseteq G$. We have several cases to check:

1. if $G \notin \mathcal{C}(\mathcal{A}_{\mathsf{P}})$, then $f_\varphi(F) \subseteq f_\varphi(G) = \omega$.

Since $G \in \mathcal{C}(\mathcal{A}_{\mathsf{P}})$ implies $F \in \mathcal{C}(\mathcal{A}_{\mathsf{P}})$, we now suppose $F, G \in \mathcal{C}(\mathcal{A}_{\mathsf{P}})$ and thus $s_F \leq_p s_G$.

2. if $\mathrm{c}_p(s_F) = \mathrm{c}_p(s_G) = 0$, then $f_\varphi(F) = l_q\big(\varphi(s_F)\big) \subseteq l_q\big(\varphi(s_G)\big) = f_\varphi(G)$,
3. if $\mathrm{c}_p(s_F) = 0$ and $\mathrm{c}_p(s_G) = 1$, then $f_\varphi(F) = l_q\big(\varphi(s_F)\big) \subseteq l_q\big(\varphi(s_G^-)\big) \subseteq f_\varphi(G)$,

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4. if $c_p(s_F) = 1$ and $c_p(s_G) = 0$, then $f_\varphi(F) \subseteq l_q(\varphi(s_F^+)) \subseteq l_q(\varphi(s_G)) = f_\varphi(G)$,
5. if $c_p(s_F) = c_p(s_G) = 1$ and $s_F \neq s_G$, then there exists $p \in P$ such that $s_F <_p p <_p s_G$ holds, because there exist no two consecutive nodes colored by 1. Therefore $f_\varphi(F) \subseteq l_q(\varphi(s_F^+)) \subseteq l_q(\varphi(s_G^-)) = f_\varphi(G)$.

It only remains to consider the cases where $c_p(s_F) = c_p(s_G) = 1$, and $s_F = s_G$:

6. if $F, G \in \mathcal{A}_\mathsf{P}$, then $f_\varphi(F) = l_q(\varphi(s_F)) = l_q(\varphi(s_G)) = f_\varphi(G)$,
7. if $F \in \mathcal{A}_\mathsf{P}$ and $G \notin \mathcal{A}_\mathsf{P}$, then $f_\varphi(F) = l_q(\varphi(s_F)) \subseteq l_q(\varphi(s_F^+)) = f_\varphi(G)$,
8. if $F \notin \mathcal{A}_\mathsf{P}$ and $G \in \mathcal{A}_\mathsf{P}$, then $f_\varphi(F) = l_q(\varphi(s_F^-)) \subseteq l_q(\varphi(s_F)) = f_\varphi(G)$,
9. if $F, G \notin \mathcal{A}_\mathsf{P}$ and $F \subsetneq l_p(s_F)$, then $f_\varphi(F) = l_q(\varphi(s_F^-)) \subseteq f_\varphi(G)$,
10. if $F, G \notin \mathcal{A}_\mathsf{P}$ and $F \nsubseteq l_p(s_F)$, then $f_\varphi(F) = l_q(\varphi(s_F^+)) = f_\varphi(G)$.

This finishes the proof that $f_\varphi : \mathcal{P}_{<\omega}(\omega) \to \mathcal{P}\omega$ is order-preserving. It follows from Lemma 4.4, that f_φ has a continuous extension $\hat{f}_\varphi : \mathcal{P}\omega \to \mathcal{P}\omega$. We distinguish between three different cases to show that $\hat{f}_\varphi^{-1}[\mathcal{A}_\mathsf{Q}] = \mathcal{A}_\mathsf{P}$.

$\boldsymbol{x \in \mathcal{P}_\omega(\omega)}$: because $\mathcal{A}_\mathsf{P} \subseteq \mathcal{P}_{<\omega}(\omega)$, we have $x \notin \mathcal{A}_\mathsf{P}$. Suppose, towards a contradiction, that $\hat{f}_\varphi(x) \in \mathcal{A}_\mathsf{Q}$. Since $\mathcal{A}_\mathsf{Q} \subseteq \mathcal{P}_{<\omega}(\omega)$, there exist $F \in \mathcal{P}_{<\omega}(\omega)$ and $n \in \omega$, such that $\hat{f}_\varphi(x) = F \in \mathcal{A}_\mathsf{Q}$ and $f_\varphi(x \cap m) = F$ both hold for all $m \geq n$. We then notice that, for any $G \in \mathcal{P}_{<\omega}(\omega)$,

$$\begin{aligned} f_\varphi(G) \in \mathcal{A}_\mathsf{Q} &\Rightarrow G \in \mathcal{C}(\mathcal{A}_\mathsf{P}) \wedge c_p(s_G) = 1 \wedge G = l_p(s_G) \\ &\Rightarrow G \in \mathcal{A}_\mathsf{P}. \end{aligned}$$

Where the first implication comes from the definition of f_φ and the second from Lemma 4.3. We obtain that $x \cap m \in \mathcal{A}_\mathsf{P}$ holds for all $m \geq n$, this implies $c_p\left(l_p^{-1}(x \cap m)\right) = 1$. Since x is infinite and l_p injective, we can extract a subsequence of $\left(l_p^{-1}(x \cap m)\right)_{m\in\omega}$ witnessing $\omega \xrightarrow{\text{1-1 h.}} P$, a contradiction.

$\boldsymbol{F \in \mathcal{P}_{<\omega}(\omega) \setminus \mathcal{C}(\mathcal{A}_\mathsf{P})}$: $F \notin \mathcal{A}_\mathsf{P}$ holds by the very definition of $\mathcal{C}(\mathcal{A}_\mathsf{P})$. Hence, we have $\omega = f_\varphi(F) = \hat{f}_\varphi(F) \notin \mathcal{A}_\mathsf{Q}$.

$\boldsymbol{F \in \mathcal{C}(\mathcal{A}_\mathsf{P})}$: Suppose first that $F \in \mathcal{A}_\mathsf{P}$. By Lemma 4.3, $\hat{f}_\varphi(F) = l_q(\varphi(s_F))$ is satisfied. Moreover, from $c_q\left(\varphi(s_F)\right) = 1$, we get $\hat{f}_\varphi(F) \in \mathcal{A}_\mathsf{Q}$.
Suppose now that $F \notin \mathcal{A}_\mathsf{P}$. By Lemma 4.3, there are three cases:
1. if $c_p(s_F) = 0$, then $c_q\left(\varphi(s_F)\right) = 0$ which implies $\hat{f}_\varphi(F) \notin \mathcal{A}_\mathsf{Q}$,
2. if $c_p(s_F) = 1$ and $F \subsetneq l_p(s_F)$, then $c_q\left(\varphi(s_F^-)\right) = 0$ which implies $\hat{f}_\varphi(F) \notin \mathcal{A}_\mathsf{Q}$,
3. if $c_p(s_F) = 1$ and $F \nsubseteq l_p(s_F)$, then $c_q\left(\varphi(s_F^+)\right) = 0$ which implies $\hat{f}_\varphi(F) \notin \mathcal{A}_\mathsf{Q}$. $\dashv_{\text{Claim}}$

Claim 4.8. *If* $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\text{emb}}$ *and* $\mathcal{A}_\mathsf{P} \leq_w \mathcal{A}_\mathsf{Q}$, *then* $\mathsf{P} \preccurlyeq_c \mathsf{Q}$.

Proof of the claim. We assume that $\mathcal{A}_\mathsf{P} \leq_w \mathcal{A}_\mathsf{Q}$ is witnessed by some continuous function $f : \mathcal{P}\omega \to \mathcal{P}\omega$. We describe a reduction which witnesses $\mathsf{P} \preccurlyeq_c \mathsf{Q}$. First, we need a few observations. Let $p \in P$. Since $\omega \overset{\text{1-1 h.}}{\not\longrightarrow} P$ and all $\leq_p$-maximal elements have color 1, there exists $p' \in P$ such that both $p \leq_p p'$ and $c_p(p') = 1$ hold. Therefore, $f\left(l_p(p')\right) \in \mathcal{A}_\mathsf{Q}$. Hence, for all $p \in P$, we have $f\left(l_p(p)\right) \in \mathcal{C}(\mathcal{A}_\mathsf{Q})$. We also define, for all $p \in P$, the set

$$Q_p = \{q \in Q \mid l_q(q) \subseteq f\left(l_p(p)\right)\}.$$

Since $f(l_p(p)) \in \mathcal{C}(\mathcal{A}_\mathsf{Q})$ holds, Lemma 4.3 yields the existence of a unique supremum t_p of Q_p in Q.
We define a mapping:

$$\varphi : P \to Q$$
$$p \mapsto \begin{cases} t_p & \text{if } f(l_p(p)) \in \mathcal{A}_\mathsf{Q}, \\ t_p & \text{if } f(l_p(p)) \notin \mathcal{A}_\mathsf{Q} \wedge \mathrm{c}_q(t_p) = 0, \\ t_p^- & \text{if } f(l_p(p)) \notin \mathcal{A}_\mathsf{Q} \wedge \mathrm{c}_q(t_p) = 1 \wedge l_q(t_p) \subsetneq f(l_p(p)), \\ t_p^+ & \text{if } f(l_p(p)) \notin \mathcal{A}_\mathsf{Q} \wedge \mathrm{c}_q(t_p) = 1 \wedge l_q(t_p) \nsubseteq f(l_p(p)), \end{cases}$$

where t_p^- and t_p^+ are defined as in Notation 3.5.

For φ to be well defined, we need t_p^+ not to occur whenever t_p is a $\leq_q$-maximal element. So, suppose t_p is a $\leq_q$-maximal element. Since $\mathrm{c}_q(t_p) = 1$, then $t_p \in Q_p$ for it has a unique immediate predecessor. Thus, $l_q(t_p) \subseteq f(l_p(p))$ holds, which shows that t_p^+ does not occur in this case.
Since for every $p \in P$ we have

$$\mathrm{c}_p(p) = 1 \Leftrightarrow l_p(p) \in \mathcal{A}_\mathsf{P} \Leftrightarrow f(l_p(p)) \in \mathcal{A}_\mathsf{Q},$$

it follows from the definition of φ, that for all $p \in P$ we also have $\mathrm{c}_p(p) = \mathrm{c}_q(\varphi(p))$. Therefore, it only remains to show that φ is order-preserving. Suppose $p \leq_P p'$, we get $t_p \leq_q t_{p'}$. We proceed with cases:

1. if $\mathrm{c}_q(t_p) = \mathrm{c}_q(t_{p'}) = 0$, then $\varphi(p) = t_p \leq_q t_{p'} = \varphi(p')$,
2. if $\mathrm{c}_q(t_p) = 0$ and $\mathrm{c}_q(t_{p'}) = 1$, then $\varphi(p) = t_p \leq_q t_{p'}^- \leq_q \varphi(p')$,
3. if $\mathrm{c}_q(t_p) = 1$ and $\mathrm{c}_q(t_{p'}) = 0$, then $\varphi(p) \leq_q t_p^+ \leq_q t_{p'} = \varphi(p')$,
4. if $\mathrm{c}_q(t_p) = \mathrm{c}_q(t_{p'}) = 1$ and $t_p \neq t_{p'}$, then there exists some $q \in Q$ that satisfies $t_p <_q q <_q t_{p'}$. This finally leads to $\varphi(p) \leq_q t_p^+ \leq_q t_{p'}^- = \varphi(p')$.

It only remains to consider the cases where $\mathrm{c}_q(t_p) = \mathrm{c}_q(t_{p'}) = 1$, and $t_p = t_{p'}$:

5. if $\mathrm{c}_p(p) = \mathrm{c}_p(p') = 1$, then $\varphi(p) = t_p = t_{p'} = \varphi(p')$,
6. if $\mathrm{c}_p(p) = 1$ and $\mathrm{c}_p(p') = 0$, then $\varphi(p) = t_p \leq_q t_p^+ = \varphi(p')$,
7. if $\mathrm{c}_p(p) = 0$ and $\mathrm{c}_p(p') = 1$, then $\varphi(p) = t_p^- \leq_q t_p = \varphi(p')$,
8. if $\mathrm{c}_p(p) = \mathrm{c}_p(p') = 0$ and $l_q(t_p) \subsetneq f(l_p(p))$, then $\varphi(p) = t_p^- \leq_q \varphi(p')$,
9. if $\mathrm{c}_p(p) = \mathrm{c}_p(p') = 0$ and $l_q(t_p) \nsubseteq f(l_p(p))$, then $\varphi(p) = t_p^+ = \varphi(p')$.

This concludes the proof that φ witnesses $\mathsf{P} \preccurlyeq_c \mathsf{Q}$. $\dashv_{\text{Claim}}$

So, Claim 4.6 proves that the mapping $H : \mathsf{P} \mapsto \mathcal{A}_\mathsf{P}$ is a well-defined mapping from $(\mathbb{P}_{\text{emb}}, \preccurlyeq_c)$ to $(\mathbf{\Delta}^0_2(\mathcal{P}\omega), \leq_w)$, and we conclude from the Claims 4.7 and 4.8 that for any $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\text{emb}}$, $\mathsf{P} \preccurlyeq_c \mathsf{Q}$ if and only if $\mathcal{A}_\mathsf{P} \leq_w \mathcal{A}_\mathsf{Q}$. $\dashv$

The previous lemma almost immediately yields the main result:

THEOREM 4.9. *The following mapping is an order-embedding*:

$$(\mathbb{P}_{\text{emb}}, \preccurlyeq_c)/\equiv_c \;\to\; (\mathbf{\Delta}^0_2(\mathcal{P}\omega), \leq_w)/\equiv_w$$
$$[\mathsf{P}] \mapsto [\mathcal{A}_\mathsf{P}].$$

PROOF. By Lemma 4.5 and the definition of the order on quotient sets, it is clear that for any $[\mathsf{P}], [\mathsf{Q}] \in (\mathbb{P}_{\text{emb}}, \preccurlyeq_c)/\equiv_c$, we have $[\mathsf{P}] \preccurlyeq_c [\mathsf{Q}]$ if and only if $[\mathcal{A}_\mathsf{P}] \leq_w [\mathcal{A}_\mathsf{Q}]$. Moreover, if $[\mathcal{A}_\mathsf{P}] = [\mathcal{A}_\mathsf{Q}]$, then $\mathcal{A}_\mathsf{P} \equiv_w \mathcal{A}_\mathsf{Q}$, and by Lemma 4.5, we have $\mathsf{P} \equiv_c \mathsf{Q}$, hence $[\mathsf{P}] = [\mathsf{Q}]$. Thus, the mapping $[\mathsf{P}] \mapsto [\mathcal{A}_\mathsf{P}]$ is an order-embedding. $\dashv$

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§**5. A reduction game on $\mathbb{P}$.** This section introduces a game characterization of reductions on 2-colored posets. This characterization and the order-embedding given in Theorem 4.9 are the essential tools that we need in order to study the Wadge order on the Scott domain.

As pointed out by an anonymous referee, this game-theoretical approach is not entirely needed in order to obtain the main results of the article (as suggested by Proposition 5.3). However, this version of the Ehrenfeucht-Fraïssé game [8] that we use captures the dynamic viewpoint that was essential—at least for the authors—in obtaining Theorems 6.1 and 7.1. Moreover, this playful viewpoint also works as a powerful tool in analyzing the Wadge order on $\mathcal{P}\omega$, a study currently undertaken by the authors.

This game comes as a standard two-player infinite game where the players choose elements of some given posets $\mathsf{P}, \mathsf{Q} \in \mathbb{P}$.

Definition 5.1. Let $\mathsf{P}, \mathsf{Q} \in \mathbb{P}$. The game $G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q})$ is defined as a two-player (I and II) game played on ω rounds. Each round $n \in \omega$ is played as follows: first I picks an element $p_n \in P$ and then II picks an element $q_n \in Q$. We further require that there exists $n_0 \in \omega$ such that, for all $n \geq n_0$, $p_n = p_{n_0}$.

We say that II *wins the game* if and only if the two following conditions are satisfied:

1. $p_n \leq_p p_m \to q_n \leq_q q_m$ holds for all $n, m \in \omega$,
2. $\mathrm{c}_p(p_n) = \mathrm{c}_q(q_n)$ for all $n \in \omega$.

Schematically, the game goes as in Figure 4.

A *run* of the game is a sequence $(p_0, q_0, p_1, q_1, \dots) \in (P \cup Q)^\omega$.

In plain English, player I moves inside the 2-colored poset P, whereas player II moves inside the 2-colored poset Q. The goal of II is to reproduce (order-wise and color-wise) in Q the run that I is producing in P. Notice that the condition of playing ultimately constant for player I is equivalent to requiring that the game stops after finitely many rounds.

Related to this game, we introduce the notion of an *ultrapositional strategy* as a strengthening of the usual notion of a strategy.

Definition 5.2. Let $\mathsf{P}, \mathsf{Q} \in \mathbb{P}$. An *ultrapositional strategy for player* II *in the game* $G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q})$ is a function $\tau : P \to Q$.

Contrary to the usual strategies that rely on the history of the opponent's run, ultrapositional strategies only take into account the last move of the opponent. An ultrapositional strategy is winning if it ensures a win whatever the opponent does.

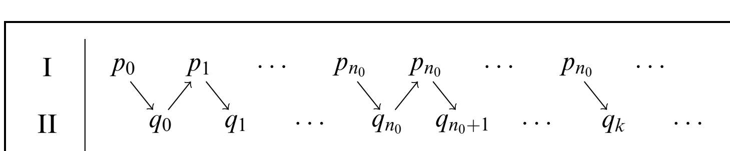

Figure 4. The game $G_{\mathbb{P}}(P, Q)$ for $\mathsf{P}, \mathsf{Q} \in \mathbb{P}$.

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Ultrapositional strategies characterize the reductions inside $\mathbb{P}$ as shown by the next proposition.

PROPOSITION 5.3. *Let* $\mathsf{P}, \mathsf{Q} \in \mathbb{P}$.

$$\mathsf{P} \preccurlyeq_c \mathsf{Q} \iff \text{II } \textit{has an ultrapositional winning strategy in } G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q}).$$

PROOF. First, suppose that $\mathsf{P} \preccurlyeq_c \mathsf{Q}$ holds and is witnessed by $\varphi : P \to Q$. Observe that φ is also an ultrapositional strategy for II in $G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q})$. From the very definition of a homomorphism between 2-colored posets, it respects the two conditions to be winning for II in $G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q})$.

Conversely, an ultrapositional winning strategy for II in $G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q})$ is a homomorphism $\varphi : P \to Q$ for it respects the two winning conditions. ⊣

We have a reduction between 2-colored posets and their subposets that are closed under the predecessor relation.

DEFINITION 5.4. Let $(Q, \leq_q)$ be a poset. A subposet $(P, \leq_p)$ is an *ideal* of $(Q, \leq_q)$ if, for all $p \in P$, we have $\{q \in Q : q \leq_q p\} \subseteq P$.

PROPOSITION 5.5. *Let* $\mathsf{P}, \mathsf{Q} \in \mathbb{P}$.

$$\textit{If } \mathsf{P} \textit{ is an ideal of } \mathsf{Q}, \textit{ then } \mathsf{P} \preccurlyeq_c \mathsf{Q}.$$

PROOF. The inclusion $i : P \to Q,\ p \mapsto p$ witnessing that $(P, \leq_p)$ is an ideal of $(Q, \leq_q)$ is an ultrapositional winning strategy for II in $G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q})$. ⊣

5.1. On the reduction game on $\mathbb{P}_{\mathbf{fin}}$. In order to simplify some later proofs, we conclude this section with some necessary conditions for an ultrapositional strategy to be winning in a subclass of the embeddable posets.

DEFINITION 5.6. A *finite branching poset* is an embeddable poset $\mathsf{P} \in \mathbb{P}_{\mathrm{emb}}$ such that every element $p \in P$ which is not $\leq_p$-minimal has finitely many successors, i.e., for all $p \in P$, if $p \neq \bot$, then:

$$\operatorname{Card}\big(\operatorname{Succ}(p)\big) = \operatorname{Card}\big(\{p' \in P \mid p \leq_p p'\}\big) < \aleph_0.$$

The class of all finite branching posets is denoted by $\mathbb{P}_{\mathrm{fin}}$.

It turns out that the image of a finitely branching poset via the order-embedding of Theorem 4.9 must be topologically reasonably simple, for we have:

PROPOSITION 5.7. *If* $\mathsf{P} \in \mathbb{P}_{\mathrm{fin}}$, *then* $\mathcal{A}_{\mathsf{P}} \in D_\omega(\boldsymbol{\Sigma}^0_1)(\mathcal{P}\omega)$.

PROOF. We use the characterization of Corollary 2.12. Since $\mathsf{P} \in \mathbb{P}_{\mathrm{emb}}$ holds, Lemma 4.5 implies that $\mathcal{A}_{\mathsf{P}} \in \boldsymbol{\Delta}^0_2(\mathcal{P}\omega)$ holds as well. Towards a contradiction, assume that $\mathcal{A}_{\mathsf{P}}$ admits a 1-alternating tree of rank ω, namely:

$$f : T_\omega \to \mathcal{P}_{<\omega}(\omega).$$

This implies that, for every $k \in \omega$, there exists a strictly $\subseteq$-increasing sequence $(F^k_m)_{m<k}$ such that $F^k_0 = f(\varnothing)$ and $F^k_m \in \mathcal{A}_{\mathsf{P}}$ both hold for all $m < k$. Thus, the sequence $\left(l_p^{-1}(F^k_m)\right)_{l<k}$ is a strictly $\leq_p$-increasing sequence of size k that satisfies

$$\mathrm{c}_p\left(l_p^{-1}(F^k_0)\right) = \mathrm{c}_p\left(l_p^{-1}(f(\varnothing))\right) = 1,$$

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for every $k \in \omega$. Therefore, we obtain

$$\mathrm{Card}\left(\mathrm{Succ}\left(l_p^{-1}\big(f(\varnothing)\big)\right)\right) = \aleph_0.$$

By definition of a finite branching poset, this implies $l_p^{-1}\big(f(\varnothing)\big) = \bot$, a contradiction for $\mathrm{c}_p(\bot) = 0$. ⊣

As a corollary, we obtain a somehow more detailed picture of Theorem 4.9.

COROLLARY 5.8. *The following mapping is an order-embedding*:

$$H : {}^{(\mathbb{P}_{\mathrm{fin}}, \preccurlyeq_c)}\big/_{\equiv_c} \longrightarrow {}^{(D_\omega(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega), \leq_w)}\big/_{\equiv_w}$$
$$[\mathsf{P}] \mapsto [\mathcal{A}_{\mathsf{P}}].$$

Now, we introduce some notations to talk about the game-theoretical strength of a given node in a finite branching poset.

Let us fix $\mathsf{P} \in \mathbb{P}_{\mathrm{fin}}$ and $p \in P$. If it exists, let $k_p \in \omega$ be the length of the largest strictly $\leq_p$-increasing sequence $(s_n)_{n<k_p}$ that satisfies $s_0 = p$ and $(\mathrm{c}_p(s_n) = \mathrm{c}_p(p) \Leftrightarrow$ n is even$)$. The *increasing strength of p in* P is

$$\mathrm{Str}_{\mathrm{incr}}(p) = \begin{cases} k_p & \text{if } k_p \in \omega \text{ exists,} \\ \omega & \text{otherwise.} \end{cases}$$

Since $\mathsf{P} \in \mathbb{P}_{\mathrm{fin}}$, the latter case can only occur when $p = \bot$. From a game-theoretical viewpoint, if $p \neq \bot$, then $\mathrm{Str}_{\mathrm{incr}}(p)$ corresponds to the length of the strongest $<_p$-increasing run that a player can take while playing in P.

In a similar manner, we define the *decreasing strength of p in* P, denoted by $\mathrm{Str}_{\mathrm{decr}}(p) = k \in \omega$, as the length of the largest strictly $\leq_p$-decreasing sequence $(s_n)_{n<k}$ that satisfies $s_0 = p$ and $(\mathrm{c}_p(s_n) = \mathrm{c}_p(p) \Leftrightarrow n$ is even$)$. It is well defined since $\mathrm{Card}(\mathrm{Pred}(p)) < \aleph_0$ holds for every $p \in P$.

The increasing and decreasing strengths of a node give a good indication of the strength it bears as a position in the game:

LEMMA 5.9. *If* $\mathsf{P}, \mathsf{Q} \in \mathbb{P}_{\mathrm{fin}}$ *and* τ *is a winning ultrapositional strategy for* II *in the game* $G_{\mathbb{P}}(\mathsf{P}, \mathsf{Q})$, *then for all* $p \in P$:

1. $\mathrm{Str}_{\mathrm{incr}}(p) \leq \mathrm{Str}_{\mathrm{incr}}\big(\tau(p)\big)$,
2. $\mathrm{Str}_{\mathrm{decr}}(p) \leq \mathrm{Str}_{\mathrm{decr}}\big(\tau(p)\big)$.

PROOF.

1. Towards a contradiction, suppose that $\mathrm{Str}_{\mathrm{incr}}(p) > \mathrm{Str}_{\mathrm{incr}}\big(\tau(p)\big)$. We proceed by cases.
   If $\mathbf{Str_{incr}}(\boldsymbol{p}) \neq \boldsymbol{\omega}$: assume that $\mathrm{Str}_{\mathrm{incr}}(p) = k$ is witnessed by a sequence $(p_n)_{n<k}$. Since τ is winning, $\big(\tau(p_n)\big)_{n<k}$ is strictly $\leq_q$-increasing and satisfies $\tau(p_0) = \tau(p)$ and $\big(\mathrm{c}_q\left(\tau(p_n)\right) = \mathrm{c}_p\left(\tau(p)\right) \Leftrightarrow n \text{ is even}\big)$. Thus $\mathrm{Str}_{\mathrm{incr}}\big(\tau(p)\big) \geq k$, a contradiction.
   If $\mathbf{Str_{incr}}(\boldsymbol{p}) = \boldsymbol{\omega}$: for all $k \in \omega$, there exists a strictly $\leq_p$-increasing sequence $(s_n)_{n<k}$ that satisfies $s_0 = p$ and $(\mathrm{c}_p(s_n) = \mathrm{c}_p(p) \Leftrightarrow n$ is even$)$. Since τ is winning, $\big(\tau(p_n)\big)_{n<k}$ is strictly $\leq_q$-increasing and satisfies $\tau(p_0) = \tau(p)$ and $\big(\mathrm{c}_q\left(\tau(p_n)\right) = \mathrm{c}_p\left(\tau(p)\right) \Leftrightarrow n \text{ is even}\big)$. Therefore, $\mathrm{Str}_{\mathrm{incr}}\big(\tau(p)\big) = \omega$, a contradiction.

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2. Towards a contradiction, suppose that $\mathrm{Str}_{\mathrm{decr}}(p) > \mathrm{Str}_{\mathrm{decr}}\big(\tau(p)\big)$. We also suppose that $\mathrm{Str}_{\mathrm{decr}}(p) = k \in \omega$ is witnessed by a sequence $(p_n)_{n<k}$. Since τ is winning, $\big(\tau(p_n)\big)_{n<k}$ is strictly $\leq_q$-decreasing and satisfies $\tau(p_0) = \tau(p)$ and $\big(\mathrm{c}_q\left(\tau(p_n)\right) = \mathrm{c}_p\left(\tau(p)\right) \Leftrightarrow n \text{ is even}\big)$. Thus $\mathrm{Str}_{\mathrm{decr}}\left(\tau(p)\right) \geq k$, a contradiction. ⊣

§**6. Ill-foundedness of the Wadge order on the Scott domain.** In this section, we prove that the quasi-order $\leq_w$ is already ill-founded inside the class of ω-differences of open sets of the Scott domain.

THEOREM 6.1.

$$\big(D_\omega(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega), \leq_w\big) \textit{ is ill-founded.}$$

PROOF. The proof consists in exhibiting a strictly $\preccurlyeq_c$-decreasing sequence of posets $(\mathsf{P}_n)_{n\in\omega^+}$ in $\mathbb{P}_{\mathrm{emb}}$ and making use of the Lemma 4.5.

First, let us fix $n \in \omega^+$. We define $\mathsf{P}_n = (P_n, \leq_{p_n}, \mathrm{c}_{p_n})$ as the 2-colored countable poset with colored Hasse diagram given in Figure 5.

Formally, the set of nodes is

$$P_n = \{\bot\} \cup \{w_m, x_m, y_m\}_{m\in\omega} \\ \cup \left\{z^{2k}_m \mid k \in \omega, n \geq km\right\} \cup \left\{z^{2k+1}_m \mid k \in \omega, n \geq (k+1)m\right\},$$

the order relation is

$$\leq_{p_n} = \left\{(\bot, w_m), (w_m, x_m), (x_m, y_m), (x_{m+1}, y_m), (y_m, z^0_m)\right\}_{m\in\omega} \\ \cup \left\{(z^k_m, z^{k+1}_m) \mid k \leq \left\lfloor \frac{m}{n} \right\rfloor \cdot 2 - 1\right\},$$

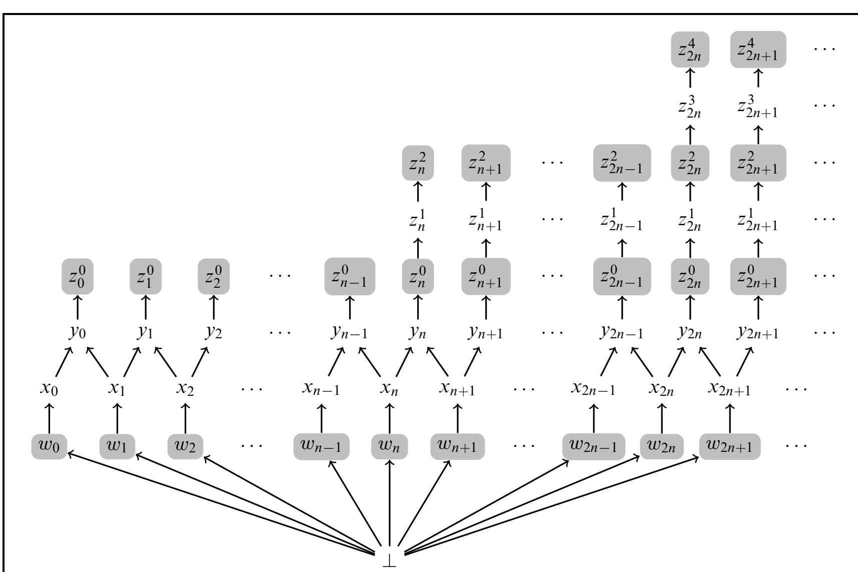

FIGURE 5. The colored Hasse diagram of $\mathsf{P}_n \in \mathbb{P}_{\mathrm{emb}}$ for $n \in \omega^+$.

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where $\lfloor \frac{m}{n} \rfloor$ denotes the integer part of $\frac{m}{n}$, and the 2-coloring is:

$$\begin{aligned} \mathrm{c}_{p_n} : P_n &\to 2 \\ p &\mapsto 0 \quad \text{if } p \in \{\perp, x_m, y_m\}_{m\in\omega} \cup \bigcup_{m\in\omega} z_m^{\text{odd}}, \\ p &\mapsto 1 \quad \text{if } p \in \{w_m\}_{m\in\omega} \cup \bigcup_{m\in\omega} z_m^{\text{even}}, \end{aligned}$$

where $z_m^{\cdot} = \{z_m^k \mid k \leq \lfloor \frac{m}{n} \rfloor \cdot 2\}$, $z_m^{\text{even}} = \{z_m^k \in z_k^{\cdot} \mid k \text{ even}\}$, and $z_m^{\text{odd}} = \{z_m^k \in z_k^{\cdot} \mid k \text{ odd}\}$.

For all $n \in \omega^+$, it is easy to check that all the requirements that are needed for P_n to belong to $\mathbb{P}_{\text{fin}}$ are fulfilled. Therefore, by Proposition 5.7, we have:

$$\mathcal{A}_{\mathsf{P}_n} \in D_\omega(\mathbf{\Sigma}_1^0)(\mathcal{P}\omega).$$

For the remainder of the proof, we need some notations. For any $k \in \omega$, we call *branch k of* P_n the set of nodes $B_k = \{w_k, x_k, y_k\} \cup z_k^{\cdot}$, and *right-shift in* P_n any sequence of moves of the form (w_k, y_k, w_{k+1}). First, we describe the behavior of an ultrapositional winning strategy facing a right-shift.

Claim 6.2. *Let $n, m \in \omega^+$ and τ be an ultrapositional strategy for* II *in $G_{\mathbb{P}}(\mathsf{P}_n, \mathsf{P}_m)$. If* I*'s moves are a right-shift (w_k, y_k, w_{k+1}) and $\tau(w_k) \in B_l$ for some $l \in \omega$, then $\tau(w_{k+1}) \in B_{l'}$ for some $l' \leq l + 1$.*

Proof of the Claim. We split the proof in two different cases.

If $\boldsymbol{l = 0}$ holds: since $w_k \leq_{p_n} y_k$, $\mathrm{c}_{p_n}(y_k) = 0$, τ is winning and $\tau(w_k) \in B_0$, we get $\tau(y_k) \in \{x_0, y_0\}$. Moreover, since $w_{k+1} \leq_{p_n} y_k$, $\mathrm{c}_{p_n}(w_{k+1}) = 1$ and τ is winning, we get:

$$\tau(w_{k+1}) \in \{w_0, w_1\} \subseteq B_0 \cup B_1.$$

If $\boldsymbol{l \in \omega^+}$ holds: once again, since $w_k \leq_{p_n} y_k$, $\mathrm{c}_{p_n}(y_k) = 0$, τ is winning and $\tau(w_k) \in B_l$, we get $\tau(y_k) \in z_{l-1}^{\text{odd}} \cup z_l^{\text{odd}} \cup \{x_l, y_l, y_{l-1}\}$. Moreover, since $w_{k+1} \leq_{p_n} y_k$, $\mathrm{c}_{p_n}(w_{k+1}) = 1$ and τ is winning, we get:

$$\tau(w_{k+1}) \in z_{l-1}^{\text{even}} \cup z_l^{\text{even}} \cup \{w_{l-1}, w_l, w_{l+1}\} \subseteq \bigcup_{l' \leq l+1} B_{l'}. \qquad \dashv_{\text{Claim}}$$

It remains to show that the sequence $(\mathsf{P}_n)_{n\in\omega_+}$ is an infinite strictly $\preccurlyeq_c$-decreasing sequence in $\mathbb{P}_{\text{emb}}$.

Claim 6.3. *If $0 < n < m < \omega$, then $\mathsf{P}_m \preccurlyeq_c \mathsf{P}_n$.*

Proof of the Claim. It suffices to observe that P_m is an ideal of P_n and use Proposition 5.5. $\dashv_{\text{Claim}}$

Claim 6.4. *If $0 < n < m < \omega$, then $\mathsf{P}_n \not\preccurlyeq_c \mathsf{P}_m$.*

Proof of the Claim. Towards a contradiction, suppose that $\mathsf{P}_n \preccurlyeq_c \mathsf{P}_m$ holds. By Proposition 5.3, player II has a winning ultrapositional strategy τ in the game $G_{\mathbb{P}}(\mathsf{P}_n, \mathsf{P}_m)$.

The idea of the proof is to construct a particular run of the game that τ cannot win. By Claim 6.2, if I plays a sequence of the form $(w_0, y_0, w_1, y_1, w_2, \dots)$ composed with right-shifts, then II's moves are limited. In particular, whenever I shifts from B_k to B_{k+1}, II can only shift from B_l to $B_{l'}$ where $l' \leq l + 1$. Because $n < m$, I can finally reach a node of greater increasing strength than the one reached by II, which leads to a contradiction.

More formally, suppose that I's first move is w_0 so that $\tau(w_0) \in B_{k_0}$ for some $k_0 \in \omega$, and that I plays a run composed with several right-shifts

$$(w_0, y_0, w_1, y_1, w_2, \dots, w_l).$$

By an iteration of Claim 6.2, we get $\tau(w_l) \in B_{l'}$ for some $l' \leq k_0 + l$. Since $n < m$, there exists $n_0 \in \omega$ such that the following inequalities work:

$$\mathrm{Str}_{\mathrm{incr}}(w_{nmn_0}) = 2mn_0 + 3 > 2nn_0 + \mathrm{Str}_{\mathrm{incr}}(w_{k_0}) \geq \mathrm{Str}_{\mathrm{incr}}\left(\tau(w_{nmn_0})\right),$$

which is a contradiction to Lemma 5.9. $\dashv_{\text{Claim}}$

So, we constructed an infinite strictly $\preccurlyeq_c$-decreasing sequence of embeddable posets, namely

$$\mathsf{P}_1 \succ_c \mathsf{P}_2 \succ_c \mathsf{P}_3 \succ_c \mathsf{P}_4 \succ_c \cdots.$$

By Lemma 4.5, we obtain an infinite strictly $\leq_w$-decreasing sequence of subsets of $\mathcal{P}\omega$, namely:

$$\mathcal{A}_{\mathsf{P}_1} >_w \mathcal{A}_{\mathsf{P}_2} >_w \mathcal{A}_{\mathsf{P}_3} >_w \mathcal{A}_{\mathsf{P}_4} >_w \cdots$$

which were also proved to be differences of ω open sets. $\dashv$

§7. Antichains in the Wadge order on the Scott domain. We prove that infinite $\leq_w$-antichains already exist within the class of ω-differences of open subsets of the Scott domain. The proof is nothing but a tailoring of the proof of Theorem 6.1.

Theorem 7.1.

$$\left(D_\omega(\boldsymbol{\Sigma}^0_1)(\mathcal{P}\omega), \leq_w\right) \textit{ has infinite antichains.}$$

Proof. We construct an infinite sequence of embeddable posets $(\mathsf{Q}_n)_{n\in\omega_+}$ that are pairwise $\preccurlyeq_c$-incomparable.

We fix $n \in \omega^+$ and define $\mathsf{Q}_n = (Q_n, \leq_{q_n}, \mathrm{c}_{q_n})$ as the 2-colored countable poset with the colored Hasse diagram given in Figure 6.

Formally, the set of nodes is:

$$Q_n = \{\bot\} \cup \{x^k_m, y_m\}_{m\in\omega, k<2n} \cup \left\{z^{2k}_m \mid k \in \omega, n \geq km\right\} \cup \left\{z^{2k+1}_m \mid k \in \omega, n \geq (k+1)m\right\},$$

the order relation is:

$$\leq_{q_n} = \left\{(\bot, x^0_m), (x^k_m, x^{k+1}_m), (x^{2n-1}_m, y_m), (x^{2n-1}_{m+1}, y_m), (y_m, z^0_m)\right\}_{m\in\omega, k<2n-1} \cup \left\{(z^k_m, z^{k+1}_m) \mid k \leq \left\lfloor \frac{m}{n} \right\rfloor \cdot 2 - 1\right\},$$

and the coloring is given by the function:

$$\begin{aligned} \mathrm{c}_{p_n} : P_n &\to 2 \\ p &\mapsto 0 \quad \text{if } p \in \{\bot, x^{2k+1}_m, y_m\}_{m\in\omega, k<n} \cup \bigcup_{m\in\omega} z^{\text{odd}}_m, \\ p &\mapsto 1 \quad \text{if } p \in \{2^{2k}_m\}_{m\in\omega, k<n} \cup \bigcup_{m\in\omega} z^{\text{even}}_m. \end{aligned}$$

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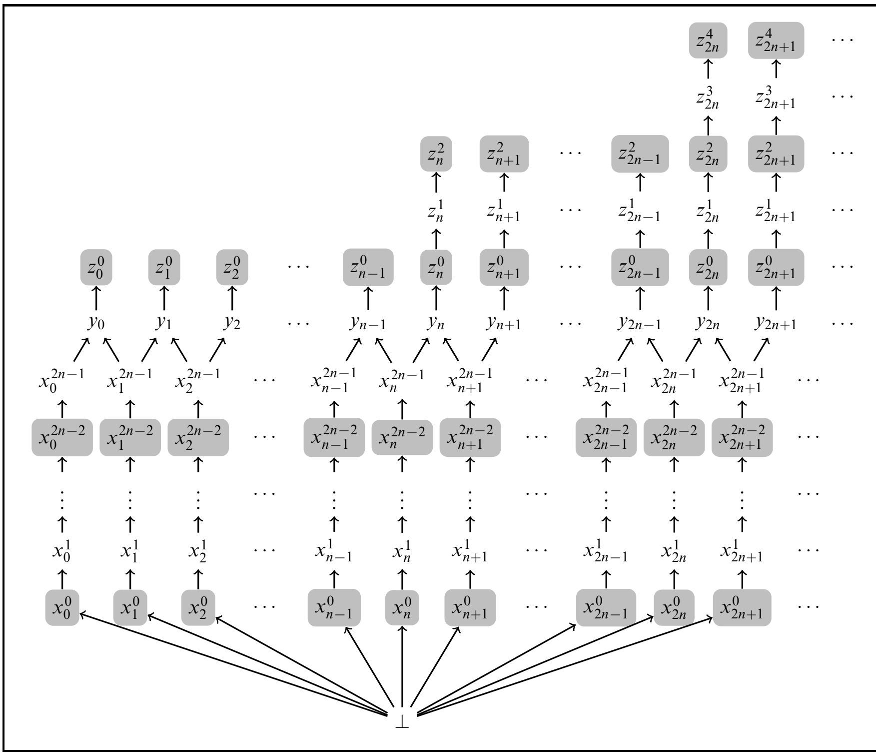

FIGURE 6. The colored Hasse diagram of $\mathsf{Q}_n \in \mathbb{P}_{\text{emb}}$ for $n \in \omega^+$.

As in the proof of Theorem 6.1, it is easy to see that $\mathsf{Q}_n \in \mathbb{P}_{\text{fin}}$, and thus $\mathcal{A}_{\mathsf{Q}_n} \in D_\omega(\mathbf{\Sigma}^0_1)(\mathcal{P}\omega)$ holds for every $n \in \omega^+$. Now, it remains to show that $(\mathsf{Q}_n)_{n\in\omega_+}$ is a sequence of pairwise $\preccurlyeq_c$-incomparable embeddable posets. For this purpose, we define a *right-shift in* Q_n as any sequence of moves of the form $(x_k^{2n-2}, y_k, x_{k+1}^{2n-2})$ for some $k \in \omega$.

CLAIM 7.2. *If* $0 < n < m < \omega$, *then* $\mathsf{Q}_m \not\preccurlyeq_c \mathsf{Q}_n$.

PROOF OF THE CLAIM. Towards a contradiction, we assume that $\mathsf{Q}_m \preccurlyeq_c \mathsf{Q}_n$ holds. By Proposition 5.3, II has an ultrapositional winning strategy τ in the game $G_{\mathbb{P}}(\mathsf{Q}_m, \mathsf{Q}_n)$.

The idea of the proof is to exhibit some specific run for I in this game that τ cannot beat. For this purpose, player I will use the fact that $n < m$ and several right-shifts to reach an element $q \in \mathsf{Q}_n$ which has a larger increasing strength than $\tau(q)$.

We consider x_0^{2m-2} as I's first move. If II's first move is x_i^{2j} for some $i \in \omega$ and $j < n$, then $\text{Str}_{\text{decr}}\left(x_0^{2m-2}\right) = 2m > 2n \geq \text{Str}_{\text{decr}}\left(x_i^{2j}\right)$, which contradicts Lemma 5.9. Since $\mathsf{c}_{q_m}(x_0^{2m-2}) = 1$, we can assume that $\tau\left(x_0^{2m-2}\right) = z_{l_0}^{2k}$ for some $k, l_0 \in \omega$.

If I's second move is y_0, then II's second move has color 0. Hence, II's second move is of the form $z_{l_0}^{2k'+1}$ for some $k' \in \omega$.

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Since $\mathrm{Str}_{\mathrm{decr}}\left(x_1^{2m-2}\right) = 2m > 2n \geq \mathrm{Str}_{\mathrm{decr}}\left(x_i^{2j}\right)$ for all $j < n$, if I's third move is x_1^{2m-2}, then Lemma 5.9 implies that II's third move cannot be of the form x_i^{2j} for some $i, j \in \omega$. So, II's third move is of the form $z_{l_0}^{2k''}$ for some $k'' \in \omega$.

Now, consider the run where I plays right-shifts:

$$\left(x_0^{2m-2}, y_0, x_1^{2m-2}, y_1, x_2^{2m-2}, y_2, \dots\right).$$

By the previous observations, II will only play in $z_{l_0}^{\cdot}$. But there exists $i_0 \in \omega$ such that

$$\mathrm{Str}_{\mathrm{incr}}(y_{i_0}) > \max\{\mathrm{Str}_{\mathrm{incr}}(q) \mid q \in z_{l_0}^{\cdot}\},$$

which contradicts Lemma 5.9. $\dashv_{\text{Claim}}$

For the last two claims, we need to introduce the notion of *branches in* Q_n. For any $k \in \omega$, we call *branch* k *of* Q_n the set of nodes $B_k = \{x_k^l, y_k\}_{l<2n} \cup z_k^{\cdot}$. The next claim, which concerns the 2-colored countable posets of the form Q_n for some $n \in \omega^+$, is a tailoring of Claim 6.2.

Claim 7.3. *Let $n, m \in \omega^+$ and τ be an ultrapositional strategy for* II *in $G_{\mathbb{P}}(\mathsf{Q}_n, \mathsf{Q}_m)$. If* I*'s moves are a right-shift $(x_k^{2n-2}, y_k, x_{k+1}^{2n-2})$ and $\tau\left(x_k^{2n-2}\right) \in B_l$ holds for some $l \in \omega$, then $\tau\left(x_{k+1}^{2n-2}\right) \in B_{l'}$ holds for some $l' \leq l+1$.*

Proof of the claim. We proceed as in the proof of Claim 6.2, except that the right-shift (w_k, y_k, w_{k+1}) in P_n is replaced by the right-shift $(x_k^{2n-2}, y_k, x_{k+1}^{2n-2})$ in Q_n. $\dashv_{\text{Claim}}$

With the help of the previous claim, we finally obtain:

Claim 7.4. *If $0 < n < m < \omega$, then $\mathsf{Q}_n \not\preccurlyeq_c \mathsf{Q}_m$.*

Proof of the claim. We proceed as in the proof of Claim 6.4. Towards a contradiction, suppose that $\mathsf{Q}_n \preccurlyeq_c \mathsf{Q}_m$ holds. By Proposition 5.3, player II has a winning ultrapositional strategy τ in the game $G_{\mathbb{P}}(\mathsf{Q}_n, \mathsf{Q}_m)$.

Suppose that I's first move is x_0^{2n-2} so that $\tau\left(x_0^{2n-2}\right) \in B_{k_0}$ for some $k_0 \in \omega$, and that I plays a run composed with several right-shifts

$$\left(x_0^{2n-2}, y_0, x_1^{2n-2}, y_1, x_2^{2n-2}, \dots, x_l^{2n-2}\right).$$

By an iteration of Claim 7.3, we get $\tau\left(x_l^{2n-2}\right) \in B_{l'}$ for some $l' \leq k_0 + l$. Since $n < m$, there exists $n_0 \in \omega$ such that the following inequalities work:

$$\mathrm{Str}_{\mathrm{incr}}\left(x_{nmn_0}^{2n-2}\right) = 2mn_0 + 3 > 2nn_0 + \mathrm{Str}_{\mathrm{incr}}\left(x_{k_0}^0\right) \geq \mathrm{Str}_{\mathrm{incr}}\left(\tau\left(x_{nmn_0}^{2n-2}\right)\right),$$

which contradicts Lemma 5.9. $\dashv_{\text{Claim}}$

So, we constructed an infinite sequence of pairwise $\preccurlyeq_c$-incomparable embeddable posets, namely $(\mathsf{Q}_n)_{n\in\omega^+}$. By Lemma 4.5, we obtain an infinite sequence of pairwise $\leq_w$-incomparable subsets of $\mathcal{P}\omega$, namely $\left(\mathcal{A}_{\mathsf{Q}_n}\right)_{n\in\omega^+}$. We also proved that all these sets are ω-differences of open sets. $\dashv$

§**8. Open questions.** We conclude with some related open questions that may serve as guidelines for future work.

In Theorem 4.9, we exhibited a partial order on a class of 2-colored countable posets which embeds in the Wadge order on the $\mathbf{\Delta}_2^0$-degrees of $\mathcal{P}\omega$. It would be desirable to find a better description of this partial order, as it was recently done in [11] for the Baire space ω^ω—the space of infinite sequence of integers endowed with the product of the discrete topology. More precisely, building on [5, 21], they showed that the Wadge order on the Borel subsets of ω^ω can be represented by countable joins of countable transfinite nests of well-founded trees labeled by 2. Although such a description seems to be out of reach for the whole Borel subsets, a reasonable question would be:

QUESTION 8.1. *Is there any already well-studied order-theoretic structure that is isomorphic to* $(\mathbf{\Delta}^0_2(\mathcal{P}\omega), \leq_w)/\equiv_w$?

We showed that some unwanted properties already occur at a very low topological complexity level in the Wadge order of $\mathcal{P}\omega$. By looking at some reductions that are more general than the continuous ones, these bad behaviors may disappear. For example, Motto Ros, Schlicht, and Selivanov consider the class of $\mathbf{\Sigma}^0_\omega$-functions $\mathcal{F}_0 = \{f : \mathcal{P}\omega \to \mathcal{P}\omega : f^{-1}(\mathcal{A}) \in \mathbf{\Sigma}^0_\omega(\mathcal{P}\omega) \text{ for any } \mathcal{A} \in \mathbf{\Sigma}^0_\omega(\mathcal{P}\omega)\}$ [15]. They show that $\leq_{\mathcal{F}_0}$[11] induces a well-quasi-order on the Borel subsets of $\mathcal{P}\omega$. Thus, the following question seems of interest:

QUESTION 8.2. *For which classes of functions* $\mathcal{F} \subseteq \mathcal{F}_0$ *containing the continuous ones is the induced order* $\leq_{\mathcal{F}}$ *on the Borel subsets of* $\mathcal{P}\omega$ *a well-quasi-order*?

Another relevant question concerns the possibility of extending our results to some other quasi-Polish spaces. We essentially focused on $\mathcal{P}\omega$ because it is universal among them. Since we showed that $\mathcal{P}\omega$ is not well behaved with respect to the Wadge order, one may ask whether there exists some characterization of the well-behaved quasi-Polish spaces, in a similar way as zero-dimensionality characterizes the well-behaved Polish spaces [17].

QUESTION 8.3. *Is there any topological criterion that singles out the quasi-Polish spaces whose Wadge order on the Borel subsets is a well-quasi-order*?

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