Glasgow Math. J. **63** (2021) 280–290. © The Author(s) 2020. Published by Cambridge University Press on behalf of Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089520000178.

HERMITIANS IN MATRIX ALGEBRAS WITH OPERATOR NORM

MICHAEL J. CRABB
School of Mathematics and Statistics, University of Glasgow, Glasgow G12 8QW, Scotland, UK

JOHN DUNCAN
Department of Mathematical Sciences, University of Arkansas, Fayetteville, AR 72701, USA
e-mail: jduncan@uark.edu

and COLIN M. McGREGOR
School of Mathematics and Statistics, University of Glasgow, Glasgow G12 8QW, Scotland, UK
e-mail: Colin.McGregor@glasgow.ac.uk

To the memory of Michael J. Crabb

(Received 1 February 2020; revised 27 March 2020; accepted 31 March 2020; first published online 27 April 2020)

Abstract. We investigate the real space H of Hermitian matrices in $M_n(\mathbb{C})$ with respect to norms on $\mathbb{C}^n$. For absolute norms, the general form of Hermitian matrices was essentially established by Schneider and Turner [Schneider and Turner, *Linear and Multilinear Algebra* (1973), 9–31]. Here, we offer a much shorter proof. For non-absolute norms, we begin an investigation of H by means of a series of examples, with particular reference to dimension and commutativity.

2010 *Mathematics Subject Classification*. Primary 15A60, 47A12

1. Introduction. The theory of numerical ranges has flourished for over half a century. Much of the early interest focused on Hermitian elements, which generalised the notion of self-adjoint elements in normed star algebras. Some intriguing examples of Hermitians were given in the finite dimensional setting, but we know of no general attack on describing all Hermitians in that setting. Surprisingly, Bauer hardly mentions the issue in his splendid lecture course at Stanford [**1**]. Michael Crabb (obit 5 December, 2019) contributed significantly to this and other areas of mathematics, particularly with his many short and ingenious proofs. This paper is dedicated to him.

We work only with the algebra $M_n(\mathbb{C})$ endowed with the operator norm $|\cdot|$ derived from a norm $\|\cdot\|$ on $\mathbb{C}^n$. We wish to describe the real space of Hermitians, H, for all possible cases. Basic facts and notations for Hermitians may be found in [**2**, **3**].

Section 2 deals with absolute norms on $\mathbb{C}^n$. The main result was essentially proved by Schneider and Turner [**7**], but without the use of the final version of the Vidav–Palmer Theorem. The proof we give is much shorter. We show that $H + iH$ is a direct sum of C^*-algebras, and we identify all the possible C^*-algebra involutions on the subalgebras.

In Section 3, the norms on $\mathbb{C}^n$ are non-absolute, and the situation becomes more complicated. Here, we begin an investigation of H, with particular reference to dimension and commutativity, illustrating various possibilities with a selection of interesting examples.

2. Absolute norms. We begin with a simple lemma. Recall that a norm on $\mathbb{C}^n$ is *absolute* if

$$\|z\| = \|(z_1, \ldots, z_n)\| = \|(|z_1|, \ldots, |z_n|)\|,$$

for all $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and that $T \in M_n(\mathbb{C})$ is Hermitian if and only if $\|\exp(itT)z\| = \|z\|$ for all $t \in \mathbb{R}$, $z \in \mathbb{C}^n$. We write E_{jk} for the usual elementary matrix.

LEMMA 2.1. *The norm $\|\cdot\|$ on $\mathbb{C}^n$ is absolute if and only if E_{jj} is Hermitian with respect to $|\cdot|$ for every j.*

Proof. For $t, \theta_1, \ldots, \theta_n \in \mathbb{R}$,

$$\theta_1 E_{11} + \cdots + \theta_n E_{nn} = \mathrm{diag}(\theta_1, \ldots, \theta_n) = \Delta \quad \text{(say)}$$

and

$$\|\exp(it\Delta)z\| = \|(\exp(it\theta_1)z_1, \ldots, \exp(it\theta_n)z_n)\|. \tag{2.1}$$

If each E_{jj} is Hermitian, then $\|z\| = \mathrm{LHS}(2.1)$ and if $\|\cdot\|$ is absolute, then $\|z\| = \mathrm{RHS}(2.1)$. The result follows. □

We give next a useful lemma on Lie products of Hermitians (the result holds in any Banach algebra).

LEMMA 2.2. *Let E, F, and T be Hermitians in $M_n(\mathbb{C})$ with E, F orthogonal idempotents. Then*

(i) $[[T, E], E] = TE - 2ETE + ET \in H$,
(ii) $i[[[T, E], E], F] = iETF - iFTE \in H$,
(iii) $[[[[T, E], E], F], F] = ETF + FTE \in H$.

Proof. When E and F are Hermitian so is $i(EF - FE)$. □

Suppose now that the norm on $\mathbb{C}^n$ is absolute. By Lemma 2.1, H contains all real diagonal matrices. It is well known that these are all the Hermitians for any ℓ^p norm, $p \neq 2$. Suppose that T is Hermitian in $M_n(\mathbb{C})$, but not diagonal. For $j < k$, apply Lemma 2.2 with $E = E_{jj}$, $F = E_{kk}$ to give

$$U_{jk} := it_{jk}E_{jk} - it_{kj}E_{kj} \in H, \qquad V_{jk} := t_{jk}E_{jk} + t_{kj}E_{kj} \in H.$$

Add all these V_{jk} to give a Hermitian which is T with all diagonal entries replaced by 0. It follows that the diagonal of T is Hermitian so that all diagonal entries of T are real. For brevity, write $\alpha = t_{jk}$, $\beta = t_{kj}$. Since all eigenvalues of V_{jk} are real, we must have $\alpha\beta \geq 0$. We cannot have $\alpha\beta = 0$ unless $\alpha = \beta = 0$ since the zero matrix is the only nilpotent Hermitian. If both are non-zero, we must have $\beta = r\overline{\alpha}$ for some $r > 0$. By taking real linear combinations of U_{jk} and V_{jk}, we may suppose that $\alpha = 1$. When $\alpha = 1$, we denote U_{jk}, V_{jk} by X_{jk}, Y_{jk}, respectively. Hence, the subalgebra consisting of all matrices with non-zero entries only in these four positions has a basis for its subspace of Hermitians given by

$$E_{jj}, E_{kk}, X_{jk}, Y_{jk}.$$

By the Vidav–Palmer Theorem, this subalgebra is a C^*-algebra.

https://doi.org/10.1017/S0017089520000178 Published online by Cambridge University Press

It is tempting to hope that we must have $r = 1$ and the usual off diagonal self-adjoints as Hermitians, but this is not the case. For example, in $M_2(\mathbb{C})$, with

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

let $S^\flat$ be defined by

$$S^\flat := \begin{bmatrix} \overline{a} & r\overline{c} \\ (1/r)\overline{b} & \overline{d} \end{bmatrix}.$$

Then $\flat$ is an involution on $M_2(\mathbb{C})$. Now define an inner product on $\mathbb{C}^2$ by

$$\langle (z_1, z_2), (w_1, w_2) \rangle = z_1\overline{w_1} + rz_2\overline{w_2}.$$

It is easy to verify that the identity representation is a $\flat$-representation of $M_2(\mathbb{C})$ on this inner product space and that each of our 2×2 Hermitians is self-adjoint with respect to $\flat$.

We ask next which collections of non-zero X_{jk}, Y_{jk} can appear for a given Hermitian T. Certainly, we cannot have an arbitrary collection of X_{jk}. For $u < v < w$, note that $i[Y_{uv}, Y_{uw}] = rX_{vw}$ and X_{vw} has ratio s/r, where r and s are the ratios for Y_{uv} and Y_{uw}, respectively. If there is no X_{uv} with $u = 1$ and $v \geq 2$, then the associated C^*-algebra is 1-dimensional and is a direct summand. Suppose that the set of all X_{1v} from all non-diagonal Hermitian T is given by $v = v_1, v_2, \dots, v_k$. (By addition, we may even suppose that this occurs for one particular T.) Then, we also have $X_{v_1 w}$ with $w = v_2, \dots, v_k$. But, we cannot have $X_{1t} \in H$ for any $t \neq v_2, \dots, v_m$ since

$$i[X_{1v_1}, X_{v_1 t}] = cX_{1t}$$

for some non-zero real c. Continue this argument and then change the basis to give a subalgebra which is $M_m(\mathbb{C})$ for some m. In fact, we can obtain a $*$-representation of this $M_m(\mathbb{C})$ by generalising the involution $\flat$ introduced above. Let $r_1 = 1$ and let $r_2, \dots, r_{m-1}$ be the ratios for $X_{12}, \dots, X_{1m}$ (and these determine the ratios for all other X_{uv}). For $S = [s_{jk}]$, let $S^\flat$ be defined by $S^\flat = [(r_k/r_j)\overline{s}_{kj}]$. It is routine to verify that $\flat$ gives an involution on $M_m(\mathbb{C})$. Now define an inner product on $\mathbb{C}^m$ by

$$\langle z, w \rangle = r_1 z_1 \overline{w_1} + \cdots + r_m z_m \overline{w_m}.$$

It is routine to verify that the identity representation is a $\flat$-representation of $M_m(\mathbb{C})$ on $\mathbb{C}^m$ with this inner product and so $M_m(\mathbb{C})$ is a C^*-algebra with the operator norm on this inner product space, with the involution $\flat$. We recall that any C^*-algebra has unique norm, unique for the given involution on the algebra. Clearly, $M_m(\mathbb{C})$ has infinitely many involutions associated with the positive parameters r_j. The maximal full matrix C^*-algebras clearly give a direct sum. We have thus proved:

THEOREM 2.3. *Let $A = M_n(\mathbb{C})$ with operator norm from an absolute norm on $\mathbb{C}^n$. Then, either $H + iH$ is all diagonal matrices or is a direct sum of C^*-algebras with involutions as above. All such possibilities can occur, for example, by taking the norm on $\mathbb{C}^n$ to be the maximum of the inner product norms on each of the corresponding subspaces of $\mathbb{C}^n$.*

The matrix subalgebra summands may take any dimensions. By choosing appropriate summands, we see that the (real) dimension of H can take many values from n up to n^2; the list of possible values depends on the partitions of n.

https://doi.org/10.1017/S0017089520000178 Published online by Cambridge University Press

We already know that, for commutative Banach algebras, much pathology can occur with Hermitians. Pathology can also occur for the case of non-commutative algebras of matrices with operator norm on $\mathbb{C}^n$ with an absolute norm, as the following remarkable example shows.

EXAMPLE 2.4. Let $\|\cdot\|$ be the absolute norm on $\mathbb{C}^3$ given by

$$\|(x, y, z)\| = \max\{\sqrt{|x|^2 + |y|^2}, \sqrt{|y|^2 + |z|^2}\}.$$

Then, $\mathbb{C}^3$ has two 2-dimensional subspaces each with the ℓ^2 norm and there are no non-diagonal Hermitians in $M_3(\mathbb{C})$.

We begin by identifying the dual norm $\|\cdot\|'$ for $\|\cdot\|$. Recall that, for non-negative a, b, x, y, the Cauchy–Schwartz inequality gives

$$ax + by \leq \sqrt{a^2 + b^2}\sqrt{x^2 + y^2}$$

with equality if and only if $ay = bx$.

LEMMA 2.5. *The dual norm is given by*

$$\|(\xi, \eta, \zeta)\|' = \sqrt{(|\xi| + |\zeta|)^2 + |\eta|^2}.$$

Proof. When $|x| \geq |z|$, we have

$$\begin{aligned} |\langle(\xi, \eta, \zeta), (x, y, z)\rangle| &\leq |\xi x| + |\eta y| + |\zeta z| \leq (|\xi| + |\zeta|)|x| + |\eta||y| \\ &\leq \sqrt{(|\xi| + |\zeta|)^2 + |\eta|^2}\sqrt{(|x| + |y|)^2} \leq \sqrt{(|\xi| + |\zeta|)^2 + |\eta|^2}\,\|(x, y, z)\| \end{aligned}$$

and similarly when $|x| \leq |z|$.

When $|\zeta| > 0$, let $p = |\xi|/(|\xi| + |\zeta|)$ and $q = |\xi|/|\zeta|$. Then

$$\begin{aligned} |\langle(\xi, \eta, \zeta), (\overline{\xi}, p\overline{\eta}, q\overline{\zeta})\rangle| &= (|\xi| + |\zeta|)|\xi| + |\eta|p|\eta| \\ &= \sqrt{(|\xi| + |\zeta|)^2 + |\eta|^2}\sqrt{|\xi|^2 + p^2|\eta|^2} \quad (\text{since } (|\xi| + |\zeta|)p|\eta| = |\xi||\eta|) \\ &= \sqrt{(|\xi| + |\zeta|)^2 + |\eta|^2}\,\|(\overline{\xi}, p\overline{\eta}, q\overline{\zeta})\|, \end{aligned}$$

and similarly when $|\xi| > 0$. The remaining cases are trivial. □

For v in a normed space X with $\|v\| = 1$, we write

$$D(v) = \{\psi \in X' : \psi(v) = 1 = \|\psi\|\}.$$

Proof of Example 2.4 Let

$$T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \in H.$$

For the usual bases $\{e_j\}$ and $\{\psi_j\}$ for the space and its dual, we have $\psi_j \in D(e_j)$, so $\psi_j(Te_j) \in \mathbb{R}$. Thus, $a, e, k \in \mathbb{R}$ and $\Delta = \mathrm{diag}(a, e, k) \in H$. Hence, $S = T - \Delta \in H$, where

$$S = \begin{bmatrix} 0 & b & c \\ d & 0 & f \\ g & h & 0 \end{bmatrix} \in H.$$

https://doi.org/10.1017/S0017089520000178 Published online by Cambridge University Press

For $1/\sqrt{2} = |x| = |y| \geq |z|$, we have $(\bar{x}, \bar{y}, 0) \in D(x, y, z)$ so that

$$b\bar{x}y + c\bar{x}z + d\bar{y}x + f\bar{y}z \in \mathbb{R}. \tag{2.2}$$

Taking $y = 1/\sqrt{2}$ and $z = 0$ in (2.2) leads to $\bar{b} = d$. Hence, (2.2) gives $c\bar{x}z + f\bar{y}z \in \mathbb{R}$, and taking $z = 1/\sqrt{2}$ now leads to $c = f = 0$. Since our norms 'reverse', i.e. $\|(x, y, z)\| = \|(z, y, x)\|$ and $\|(\xi, \eta, \zeta)\|' = \|(\zeta, \eta, \xi)\|'$, we similarly have $\bar{h} = f$ and $g = d = 0$. Hence, $b = c = d = f = g = h = 0$ and the proof is complete.

3. Non-absolute norms. Here, we begin an analysis of the case when the norm $\|\cdot\|$ on $\mathbb{C}^n$ is not absolute. We start with a useful result of Bauer [1, p. 38]. Let P be invertible in $M_n(\mathbb{C})$. Define an associated norm on $\mathbb{C}^n$ by $\|v\|_P = \|Pv\|$. We have an associated operator norm $|\cdot|_P$ for the norm $\|\cdot\|_P$. Bauer shows that the numerical range of A with respect to $|\cdot|_P$ is just the numerical range of PAP^{-1} with respect to $|\cdot|$. To see this, observe that if $\|v\|_P = 1$ and $\psi \in D_P(v)$ then $\|Pv\| = 1$ and $\psi P^{-1} \in D(Pv)$ so that $\psi(Av) = \psi P^{-1}((PAP^{-1})Pv)$. Choose a real subspace of mutually commuting Hermitians of maximal dimension, say ν. Since every Hermitian matrix is diagonable, it follows by Prasolov [6, p. 174] that we may suppose that these mutually commuting Hermitians are real diagonal matrices. So for a non-absolute norm, $\nu \leq n - 1$. It may be that any subspace X of the real diagonals can occur in this way with a norm on $\mathbb{C}^n$ for which $H = X$.

We start with an example where $n = 2$ and $\nu = 1$ and then use it to show by means of further examples that for any $n \geq 2$, ν can take any value from 1 to $n - 1$.

LEMMA 3.1. *For the norm on $\mathbb{C}^2$ defined by*

$$\|(z, w)\| = \max\{|z|, |w|, |z + w|\},$$

we have $H = \{rI : r \in \mathbb{R}\}$.

Proof. Let

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H,$$

where $a, b, c, d \in \mathbb{C}$, so that $f(Tv) \in \mathbb{R}$ whenever $\|v\| = 1$ and $f \in D(v)$. Apply this with $(1, 1) \in D(1, 0)$ and $(1, 1) \in D(0, 1)$ to give $a + c, b + d \in \mathbb{R}$. Since T has real eigenvalues, we also have $a + d \in \mathbb{R}$. With $\omega = \exp(2\pi i/3)$, we have $(1, 0) \in D(1, \omega)$ and $(1, 0) \in D(\overline{\omega}, 1)$ and so $a + b\omega, a + b\overline{\omega} \in \mathbb{R}$. By elementary algebra, we now have $a, b, c, d \in \mathbb{R}$ and $b = 0$. Interchange coordinates to give $c = 0$. If $a \neq d$, then $\mathrm{diag}(1, 0)$ and $\mathrm{diag}(0, 1)$ are in H and the norm is absolute by Lemma 2.1; but $\|(1, -1)\| = 1$ and $\|(1, 1)\| = 2$. So, $a = d$ and the proof is complete. □

THEOREM 3.2. *Let $N = \{1, \ldots, n\}$ $(n \geq 2)$ and let P be a subset of N with $|P| \geq 2$. Then for the norm on $\mathbb{C}^n$ defined by*

$$\|(z_1, \ldots, z_n)\| = \max\{|z_j|, |z_s + z_t| : j \in N,\ s, t \in P,\ s < t\},$$

H consists of all real matrices of the form $\mathrm{diag}(a_1, \ldots, a_n)$, where, for all $p \in P$, $a_p = r$ for some real r. Hence, H has real dimension $n - |P| + 1$.

Proof. Let T be Hermitian for this norm. Given $j < k$, define a norm on $\mathbb{C}^2$ by

$$|||(z_j, z_k)||| = \|(0, \ldots, z_j, \ldots, z_k, \ldots, 0)\|.$$

https://doi.org/10.1017/S0017089520000178 Published online by Cambridge University Press

Observe that $|||\cdot|||$ is either the supremum norm or the norm in Lemma 3.1, and

$$|||(z_j, z_k)||| \le \|(z_1, \ldots, z_n)\|$$

for arbitrary entries. Given $|||(z_j, z_k)||| = 1$ and $(\psi_j, \psi_k) \in D(z_j, z_k)$, we have

$$(0, \ldots, \psi_j, \ldots, \psi_k, \ldots, 0) \in D(0, \ldots, z_j, \ldots, z_k, \ldots, 0)$$

since

$$\begin{aligned}&|\langle(0, \ldots, \psi_j, \ldots, \psi_k, \ldots, 0), (z_1, \ldots, z_n)\rangle| \\ &\quad = |\psi_j z_j + \psi_k z_k| \le |||(z_j, z_k)||| \le ||(z_1, \ldots, z_n)||.\end{aligned}$$

Let U be the 2×2 matrix whose entries are the entries of T in positions (j, j), (j, k), (k, j), (k, k), respectively. It follows easily that U is Hermitian on $\mathbb{C}^2$ for $|||\cdot|||$. So, U is real diagonal, and if $j, k \in P$, then $U = cI$ for some real c. Since these restrictions apply to each such U, it follows that T has the required form. □

We look next at two norms on $\mathbb{C}^3$ and $\mathbb{C}^4$, for which $\nu = 2$. Both have higher dimensional analogues which we describe briefly in subsequent remarks.

Let $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$.

Example 3.3. For the norm on $\mathbb{C}^3$ given by

$$\|(x, y, z)\| = \max\{|w^{-1}x + y + wz| : w \in \mathbb{T}\},$$

the Hermitians matrices are the real linear combinations of I and $\mathrm{diag}(-1, 0, 1)$.

We first prove a lemma which is useful when investigating this and similar norms.

Lemma 3.4. *For $a_n \in \mathbb{C}$ $(n = 0, \pm1, \ldots, \pm N)$, let*

$$\sum_{n=-N}^{N} a_n w^n \in \mathbb{R} \quad (w \in \mathbb{T}).$$

Then, $a_0 \in \mathbb{R}$ and $a_{-n} = \overline{a_n}$ $(n = 1, \ldots, N)$.

Proof. Let $A = \sum_{n=-N}^{N} a_n w^n$ and $B = \sum_{n=-N}^{N} a_n \overline{w}^n$. Then $A, B \in \mathbb{R}$ for all $w \in \mathbb{T}$. Write $w = \exp(i\theta)$ and consider $A \pm B$ to give

$$\mathrm{Im}\, a_0 + \sum_{n=1}^{N} \mathrm{Im}(a_n + a_{-n}) \cos n\theta = 0 \quad \text{and} \quad \sum_{n=1}^{N} \mathrm{Re}(a_n - a_{-n}) \sin n\theta = 0.$$

The Fourier coefficients all vanish and the result follows. □

Proof of Example 3.3 It follows from [**2**, Example 6.1] that $U = \mathrm{diag}(-1, 0, 1)$ is Hermitian. The norm is not absolute, so if all Hermitian matrices are real diagonal, then I, U will form a basis for H since its dimension cannot exceed 2.

Let

$$T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \in H.$$

https://doi.org/10.1017/S0017089520000178 Published online by Cambridge University Press

For $w \in \mathbb{T}$, let ϕ_w be the functional $(w^{-1}, 1, w)$. Then, $w^{-n}\phi_w \in D(e_n)$, and hence $w^{-n}\phi(Te_n) \in \mathbb{R}$, for each of the usual basis vectors $e_{-1} = (1, 0, 0)$, $e_0 = (0, 1, 0)$, $e_1 = (0, 0, 1)$. Thus,

$$a + wd + w^2 g \in \mathbb{R}, \quad w^{-1}b + e + wh \in \mathbb{R}, \quad w^{-2}c + w^{-1}f + k \in \mathbb{R},$$

and applying Lemma 3.4 with $N = 3$, we obtain, in turn,

$$a \in \mathbb{R} \text{ and } d = g = 0, \quad e \in \mathbb{R} \text{ and } b = \overline{h}, \quad k \in \mathbb{R} \text{ and } c = f = 0.$$

Hence

$$T = \begin{bmatrix} a & \overline{h} & 0 \\ 0 & e & 0 \\ 0 & h & k \end{bmatrix} \quad \text{and} \quad i[T, U] = \begin{bmatrix} 0 & i\overline{h} & 0 \\ 0 & 0 & 0 \\ 0 & -ih & 0 \end{bmatrix}.$$

Since $i[T, U]$ is both Hermitian and nilpotent, $h = 0$. So $T = \operatorname{diag}(a, e, k)$ with $a, e, k \in \mathbb{R}$ and the proof is complete.

EXAMPLE 3.5. For the norm on $\mathbb{C}^4$ given by

$$\|(x, y, u, v)\| = \max\{|w^{-2}x + w^{-1}y + wu + w^2 v| : w \in \mathbb{T}\},$$

the Hermitians matrices are the real linear combinations of I and $\operatorname{diag}(-2, -1, 1, 2)$.

Proof. This is similar to the proof of Example 3.3. Let $T = (t_{jk}) \in H$. For $w \in \mathbb{T}$, let ϕ_w be the functional (w^{-2}, w^{-1}, w, w^2). Then, for example, $w\phi_w \in D(e_2)$, where $e_2 = (0, 1, 0, 0)$ and hence

$$w\phi_w(Te_2) = w^{-1}t_{12} + t_{22} + w^2 t_{32} + w^3 t_{42} \in \mathbb{R}.$$

Thus

$$w^{-3}0 + w^{-2}0 + w^{-1}t_{12} + t_{22} + w0 + w^2 t_{32} + w^3 t_{42} \in \mathbb{R},$$

and applying Lemma 3.4 with $N = 3$, we obtain

$$t_{22} \in \mathbb{R} \quad \text{and} \quad t_{12} = t_{32} = t_{42} = 0.$$

Similar arguments applied to the other three usual basis vectors establish that T is a real diagonal matrix.

Let $\Delta = \operatorname{diag}(-2, -1, 1, 2)$. A routine calculation gives $\| \exp(it\Delta)\| = 1$ so that $\Delta \in H$. Since T is Hermitian and real diagonal, we can write $T = pI + q\Delta + rE_{11} + sE_{44}$, where $p, q, r, s \in \mathbb{R}$. Then, $U = rE_{11} + sE_{44} \in H$. Considering $\| \exp(itU)z\| = \|z\|$ for small positive t and $z = (1, 1, 1, 1)$ leads to $r = s = 0$. Hence, T is a real linear combination of I and Δ, as required. □

REMARKS. In [4, Proposition 2.3], the norm of Example 3.3 is generalised to higher dimensional $\mathbb{C}^{2n+1}$ by defining

$$\|(z_{-n}, \ldots, z_{-1}, z_0, z_1, \ldots, z_n)\| = \max\left\{\left|z_0 + \sum_{j=1}^{n}(w^{-j}z_{-j} + w^j z_j)\right| : w \in \mathbb{T}\right\}, \tag{3.3}$$

and it follows from the proposition in [4] that every diagonal Hermitian is a real linear combinations of I and $T = \operatorname{diag}(-n, \ldots, -1, 0, 1, \ldots, n)$. We conjecture that all Hermitians

https://doi.org/10.1017/S0017089520000178 Published online by Cambridge University Press

for this norm are diagonal so that H is the real linear span of I and T. A lengthy argument does establish this for $\mathbb{C}^5$, but a different technique may be required for higher dimensions.

A similar generalisation of the norm of Example 3.5 is obtained for higher dimensional $\mathbb{C}^{2n}$ by omitting the z_0 terms in (3.3). And correspondingly, we conjecture that H is the real linear span of I and $\mathrm{diag}(-n, \ldots, -1, 1, \ldots, n)$.

For the above examples, the real linear space H has always been a commutative set. Can there be non-absolute norms for which H is not commutative? We show that the answer is yes, and we begin with such a norm on $\mathbb{C}^3$. We lead into this norm via two self-adjoint matrices. Let $A = i(E_{12} - E_{21})$ and $B = i(E_{13} - E_{31})$. We seek a non-absolute norm on $\mathbb{C}^3$ for which A and B are Hermitian. We must also have I Hermitian and also $i[A, B] = i(E_{23} - E_{32})$. We show in fact that for our norm these four matrices give a basis for H. The norm must lead to

$$|\exp(itA)| = |\exp(itB)| = 1 \quad (t \in \mathbb{R}).$$

An easy computation gives

$$\exp(itA) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \exp(itB) = \begin{bmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{bmatrix}.$$

Our first construction of a norm was as follows. Let G be the closed group generated by all $\exp(isA)$, $\exp(itB)$ with $s, t \in \mathbb{R}$. Note that G is a compact subgroup of the unitary matrices. Let $\|\cdot\|_G$ be the norm on $\mathbb{C}^3$ defined by

$$\|v\|_G = \sup_{g \in G} \|gv\|_\infty,$$

where $\|\cdot\|_\infty$ is the supremum norm on $\mathbb{C}^3$. It is routine to verify that $|\exp(itA)| = |\exp(itB)| = 1$ for this norm so that $A, B \in H$ with $AB \neq BA$. We were able to prove that $\|\cdot\|_G$ is not an absolute norm, but it is difficult to do computations with this norm. We found a related norm for which computations are much easier. We define the norm $\|\cdot\|_0$ on $\mathbb{C}^3$ by

$$\|(x, y, z)\|_0 = \max\{|\alpha x + \beta y + \gamma z| : (\alpha, \beta, \gamma) \in S\},$$

where

$$S = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha^2 + \beta^2 + \gamma^2 = 1\}.$$

We have

$$\exp(itA)(x, y, z) = (x \cos t - y \sin t, x \sin t + y \cos t, z),$$

and so

$$\|\exp(itA)(x, y, z)\|_0 = \max_{(\alpha,\beta,\gamma)\in S} |(\alpha \cos t + \beta \sin t)x + (\beta \cos t - \alpha \sin t)y + \gamma z|.$$

Since the map $(\alpha, \beta, \gamma) \to (\alpha \cos t + \beta \sin t, \beta \cos t - \alpha \sin t, \gamma)$ is a rotation of $\mathbb{R}^3$, it follows that $|\exp(itA)| = 1$ for $\|\cdot\|_0$ and so A is Hermitian. Similarly, B is Hermitian. The link between $\|\cdot\|_G$ and $\|\cdot\|_0$ is remarkable.

https://doi.org/10.1017/S0017089520000178 Published online by Cambridge University Press

PROPOSITION 3.6. *We have* $\|(x, y, z)\|_G = \|(x, y, z)\|_0$.

Proof. ($\leq$) By compactness, we may choose $g \in G$ such that

$$g(x, y, z) = (p, q, r) \quad \text{and} \quad \|(x, y, z)\|_G = \|(p, q, r)\|_\infty.$$

By replacing g, if necessary, with $\exp(itA)g$ or $\exp(itB)g$, where $t = \pi/2$, we may suppose without loss that $\|(x, y, z)\|_G = |p|$. All matrices in G have real entries and so $p = ax + by + cz$, where $a, b, c \in \mathbb{R}$ and $[a, b, c]$ is the first row of g. Then, $|p| = |ax + by + cz|$, and since g is unitary, we have $a^2 + b^2 + c^2 = 1$. Hence, $\|(x, y, z)\|_G \leq \|(x, y, z)\|_0$.

($\geq$) Let $(\alpha, \beta, \gamma) \in S$ with $\|(x, y, z)\|_0 = |\alpha x + \beta y + \gamma z|$. Let $(1, \theta, \phi)$ be the polar coordinates of (α, β, γ) so that

$$\alpha = \cos\theta \cos\phi, \quad \beta = \sin\theta \cos\phi, \quad \gamma = \sin\phi.$$

Then, with $g = \exp(-i\phi B)\exp(-i\theta A)$, we have $g(x, y, z) = (p, q, r)$, where

$$p = \cos\theta \cos\phi\, x + \sin\theta \cos\phi\, y + \sin\phi\, z = \alpha x + \beta y + \gamma z.$$

Hence, $\|(x, y, z)\|_G \geq \|(x, y, z)\|_0$ and the proof is complete. □

Next we show that, for the norm $\|(x, y, z)\|_0$, the space of Hermitians H is the real linear span of I, A, B and $i[A, B]$. First, we obtain information on values of $\||(x, y, z)\|_0$ and its dual norm

$$\|(\xi, \eta, \zeta)\|_0' = \max\{|\xi x + \eta y + \zeta z| : \|(x, y, z)\|_0 \leq 1\}.$$

LEMMA 3.7. *We have*

(i) $\|(x, y, z)\|_0$ *is invariant under any permutation of the three entries,*
(ii) *for* $x, y, z \in \mathbb{R}$, $\|(x, y, z)\|_0 = \sqrt{x^2 + y^2 + z^2}$, $\quad \|(ix, y, z)\|_0 = \max\{|x|, \sqrt{y^2 + z^2}\}$,
(iii) $\|(\xi, \eta, \zeta)\|_0'$ *is invariant under any permutation of the three entries,*
(iv) *for* $\xi, \eta, \zeta \in \mathbb{R}$, $\|(\xi, \eta, \zeta)\|' \leq \sqrt{\xi^2 + \eta^2 + \zeta^2}$, $\quad \|(i\xi, \eta, \zeta)\|_0' \leq |\xi| + \sqrt{\eta^2 + \zeta^2}$.

Proof. (i) and (ii) These are clear except for the last assertion. For $x, y, z \in \mathbb{R}$,

$$\begin{aligned} \|(ix, y, z)\|_0^2 &= \max\{|\alpha ix + \beta y + \gamma c|^2 : (\alpha, \beta, \gamma) \in S\} \\ &= \max\{(\alpha x)^2 + (\beta y + \gamma z)^2 : (\alpha, \beta, \gamma) \in S\} \\ &= \max\{\alpha^2 x^2 + (1 - \alpha^2)(y^2 + z^2) : \alpha^2 \in [0, 1]\} \\ &= \max\{x^2, y^2 + z^2\} \end{aligned}$$

and the result follows.

(iii) This follows from (i).

(iv) For $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha^2 + \beta^2 + \gamma^2 = 1$, and $(x, y, z) \in \mathbb{C}^3$,

$$|\alpha x + \beta y + \gamma c| \leq \|(x, y, z)\|_0$$

and the first part of (iv) follows. Then using this, we have

$$\begin{aligned} |i\alpha x + \beta y + \gamma c| &\leq |i(\alpha x + 0y + 0c| + |0x + \beta y + \gamma c| \\ &\leq (|\alpha| + \sqrt{\beta^2 + \gamma^2})\|(x, y, z)\|_0, \end{aligned}$$

which gives the second part of (iv). □

https://doi.org/10.1017/S0017089520000178 Published online by Cambridge University Press

LEMMA 3.8. *Let*

$$T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}.$$

be Hermitian with respect to $\|\cdot\|_0$ *on* $\mathbb{C}^3$. *Then*

(i) $a, e, k \in \mathbb{R}$ *and* $b, c, d, f, g, h \in i\mathbb{R}$,
(ii) $b = -d$, $c = -g$, $f = -h$.

Proof. We consider

$$\psi(Tv) = \xi(ax + by + cz) + \eta(dx + ey + fz) + \zeta(gx + hy + kz) \in \mathbb{R},$$

for various $v = (x, y, z)$ and $\psi = (\xi, \eta, \zeta)$, where $||v|| = 1$ and $\psi \in D(v)$. Throughout we have $x, y, z, \xi, \eta, \zeta \in \mathbb{R}$.

(1) Let $v = (-i, y, z)$, $\psi = (i, 0, 0)$ with $y^2 + z^2 \leq 1$. With $y = z = 0$, this gives $a \in \mathbb{R}$. Then $y = 1, z = 0$ and $y = 0, z = 1$ give, respectively, $a + ib \in \mathbb{R}$ and $a + ic \in \mathbb{R}$, and so $b, c \in i\mathbb{R}$.
(2) Similar reasoning with $v = (x, -i, z)$, $\psi = (0, i, 0)$ and $v = (x, y, -i)$, $\psi = (0, 0, i)$ gives $e \in \mathbb{R}$, $d, f \in i\mathbb{R}$ and $k \in \mathbb{R}$, $g, h \in i\mathbb{R}$.
(3) With $v = (1/\sqrt{2}, 1/\sqrt{2}, 0)$, $\psi = (1/\sqrt{2}, 1/\sqrt{2}, 0)$, we find that $b + d \in \mathbb{R} \cap i\mathbb{R}$ and so $b = -d$. Similarly, we find $c = -g$ and $f = -h$. □

THEOREM 3.9. *For* $\mathbb{C}^3$ *with norm* $\|\cdot\|_0$, *the Hermitians in* $M_3(\mathbb{C})$ *are the real linear span of the linearly independent matrices* I, A, B *and* $i[A, B]$.

Proof. It is enough to prove that for the matrix T in Lemma 3.8, $a = e = k$. Let $U = \text{diag}(a, e, k)$ so that U is Hermitian and hence

$$\|(\exp(iat)x, \exp(iet)y, \exp(ikt)z)\|_0 = \|(x, y, z)\|_0.$$

Taking $v = (i, 1, 0)$ gives $\|(i \exp(iat), \exp(iet), 0)\|_0 = 1$, and so we have

$$\|(i, \exp(i(e - a)t), 0)\|_0 = 1.$$

If $a \neq e$, we can choose t such that $\exp(i(e - a)t) = i$. Since $\|(i, i, 0)|| = \sqrt{2}$, we must have $a = e$. Similarly, we find $a = k$, completing the proof. □

REMARKS. The unit sphere of $\mathbb{C}^3$ with norm $\|\cdot\|_0$ of Proposition 3.6 is not smooth since $(0, 1, 0)$ and $(-i, 0, 0)$ are both in $D(v)$ when $v = (i, 1, 0)$. The unit sphere of the real vector subspace $\{(ix, y, z) : x, y, z \in \mathbb{R}\}$ has the shape of a cylindrical can.

In $M_2(\mathbb{C})$ with the operator norm derived from $\|\cdot\|_0$, $M_2(\mathbb{R})$ is a real C^* subalgebra of highest dimension, demonstrating that having a real C^* algebra of full real dimension does not force a large H.

We easily extend the norm $\|\cdot\|_0$ to $\mathbb{C}^n$, and we can obtain a mixture of Hermitians with real or imaginary entries by modifying the norm by a similarity matrix which is diagonal with each entry either 1 or i.

The linear independence of I, A, B and $i[A, B]$ in Theorem 3.9 holds, more generally, for Hermitian elements A and B, with $AB \neq BA$, in any complex unital Banach algebra. To see that I, A, B and $T = i[A, B]$ are linearly independent over $\mathbb{C}$ (and hence also

https://doi.org/10.1017/S0017089520000178 Published online by Cambridge University Press

over $\mathbb{R}$), consider first $U = aA + bB + cI = O$, where $a, b, c \in \mathbb{C}$. Then, $O = UB - BU = a(AB - BA)$ giving $a = 0$. Similarly, $b = 0$ and it follows that $c = 0$. Hence I, A and B are linearly independent. Now, towards a contradiction, suppose that $T = pA + qB + rI$, where $p, q, r \in \mathbb{C}$. If $p \neq 0$ then $TB - BT$ and T commute since $TB - BT = p(AB - BA) = -ipT$. By Kleinecke [5] or Širokov [9], $TB - BT$, and hence T, is quasi nilpotent. Then Sinclair [8, Proposition 2] gives $T = O$, a contradiction. So $p = 0$ and similarly $q = 0$. Then, $AB - BA = -iT = -irI$ so that $r \neq 0$. But this being so, $AB - BA$ commutes with B, and Kleinecke–Širokov–Sinclair again gives the contradiction $T = O$.

REFERENCES

1. F. L. Bauer, *Theory of norms*, infolab.stanford.edu/pub/cstr/reports/cs/tr/67/75/CS-TR-67-75.pdf (Stanford University, 1967).

2. F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras*, LMS Lecture Note Series, vol. 2 (Cambridge University Press, New York, 1971).

3. F. F. Bonsall and J. Duncan, *Numerical ranges II*, LMS Lecture Note Series, vol. 10 (Cambridge University Press, New York, 1973).

4. M. J. Crabb, J. Duncan, C. M. McGregor and T. J. Ransford, Hermitian powers: a Müntz theorem and extremal algebras, *Studia Math.* **146**(1) (2001), 83–97.

5. D. C. Kleinecke, On operator commutators, *Proc. Amer. Math. Soc.* **8** (1957), 535–536.

6. V. V. Prasolov, *Problems and theorems in linear algebra*, Mathematical Monographs, **134** (American Mathematical Society, 1996).

7. H. Schneider and R. E. Turner, Matrices Hermitian for an absolute norm, *Linear Multilinear Algebra* (1973), 9–31.

8. A. M. Sinclair, The norm of a Hermitian element in a Banach algebra, *Proc. Amer. Math. Soc.* **1** (1971), 446–450.

9. F. V. Širokov, Proof of a conjecture of Kaplansky, *Uspehi Mat. Nauk* **11** (1956), 167–168.

https://doi.org/10.1017/S0017089520000178 Published online by Cambridge University Press