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Dual ground state solutions for the critical nonlinear Helmholtz equation

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Using a dual variational approach, we obtain nontrivial real-valued solutions of the critical nonlinear Helmholtz equation

$$-\Delta u - k^2 u = Q(x)|u|^{2^*-2}u, \quad u \in W^{2,2^*}(\mathbb{R}^N)$$

for $N \geqslant 4$, where $2^* := 2N/(N-2)$. The coefficient $Q \in L^\infty(\mathbb{R}^N)\backslash\{0\}$ is assumed to be nonnegative, asymptotically periodic and to satisfy a flatness condition at one of its maximum points. The solutions obtained are so-called *dual ground states*, that is, solutions arising from critical points of the dual functional with the property of having minimal energy among all nontrivial critical points. Moreover, we show that no dual ground state exists for $N = 3$.

Keywords: Nonlinear Helmholtz equation; critical exponent; dual variational method; nonvanishing; nonexistence results

2010 Mathematics subject classification: 35J20; 35J05

1. Introduction

In this paper, we focus our attention on the existence of nontrivial real-valued solutions of the critical nonlinear Helmholtz equation

$$-\Delta u - k^2 u = Q(x)|u|^{p-2}u, \quad u \in W^{2,p}(\mathbb{R}^N) \tag{1.1}$$

for $N \geqslant 3$, $k \neq 0$, and where $Q \in L^\infty(\mathbb{R}^N)\backslash\{0\}$ is a nonnegative weight function and $p = 2^* := 2N/(N-2)$ is the critical Sobolev exponent. Recently [**11**], the existence of solutions to (1.1) has been proven for all p in the noncritical interval $(2(N+1)/(N-1), 2N/(N-2))$. A direct variational approach leads to some difficulties. Indeed, by classical results of Rellich [**20**] and Kato [**13**], solutions of the Helmholtz equation decay at most like

$$u(x) = O(|x|^{-(N-1)/2}), \quad \text{as } |x| \to \infty.$$

Therefore, solutions of (1.1) can only be expected to lie in $L^p(\mathbb{R}^N)$ or $W^{2,p}(\mathbb{R}^N)$ for $p > 2N/(N-1)$. In particular, H^1-solutions will not exist in general, as would be required when using the natural energy functional associated with (1.1).

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The authors of [11] considered instead the integral equation

$$u = \mathbf{R}_k(Q|u|^{p-2}u), \quad u \in L^p(\mathbb{R}^N), \tag{1.2}$$

where $\mathbf{R}_k$ denotes the real part of the resolvent operator $\mathcal{R}_k : f \mapsto \Phi_k * f$ of $-\Delta - k^2$. Here, Φ_k is the (complex valued) radial outgoing fundamental solution of the Helmholtz equation, that is, the convolution $\Phi_k * f$ solves the inhomogeneous Helmholtz equation $-\Delta u - k^2 u = f$ and satisfies the Sommerfeld outgoing radiation condition

$$\partial_r u(x) - \mathrm{i}ku(x) = o(|x|^{(1-N)/2}), \quad \text{as } |x| \to \infty.$$

For $p \in [2(N+1)/(N-1), 2N/(N-2)]$, solutions of the integral equation (1.2) by [11, lemma 4.3] belong to $W^{2,q}(\mathbb{R}^N)$ with $p \leqslant q < \infty$, so that, by Sobolev embeddings u is indeed a strong solution of (1.1).

This dual variational approach, based on the dual energy functional $J_{Q,p}$ given by

$$J_{Q,p}(v) = \frac{1}{p'} \int_{\mathbb{R}^N} |v|^{p'} \,\mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^N} v\mathbf{A}_{Q,p}v \,\mathrm{d}x, \quad v \in L^{p'}(\mathbb{R}^N),$$

where $p' = p/(p-1)$ and $\mathbf{A}_{Q,p}v = Q^{1/p}\mathbf{R}_k(Q^{1/p}v)$, admits a better behaved structure. The functional $J_{Q,p}$ is of class C^1 and has the mountain pass geometry. Therefore, the properties of Q determine whether it satisfies the Palais–Smale condition and this, in turn, is linked in an essential way to compactness properties of the Birman–Schwinger type operator $\mathbf{A}_{Q,p}$. For noncritical p, the operator $\mathbf{A}_{Q,p}$ is compact if Q vanishes at infinity, and then $J_{Q,p}$ satisfies the Palais–Smale condition. When Q is periodic, this is not the case anymore, but $\mathbf{A}_{Q,p}$ still has some local compactness. In combination with a crucial nonvanishing property [11, theorem 3.1] of the quadratic form associated with $\mathbf{R}_k$, a nontrivial critical point can then be obtained as a weak limit after translation of a Palais–Smale sequence at the mountain pass level. The problem (1.1) becomes more delicate in the critical case $p = 2^*$. Applying to the differential equation (1.1) the rescalings

$$u \mapsto u_{r,x_0}, \quad \text{where } u_{r,x_0}(x) = r^{(N-2)/2}u(r(x-x_0)), \tag{1.3}$$

the linear term vanishes as $r \to \infty$ and, since the limit problem

$$-\Delta u = Q(x_0)|u|^{2^*-2}u \quad \text{in } \mathbb{R}^N \tag{1.4}$$

possesses nontrivial solutions, the local compactness of $\mathbf{A}_Q := \mathbf{A}_{Q,2^*}$ is lost. In the case where Q vanishes at infinity, the functional $J_Q := J_{Q,2^*}$ therefore, does not satisfy the Palais-Smale condition at every level. Indeed, if u is a nontrivial

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solution of (1.4) for some $x_0 \in \mathbb{R}^N$, consider the function v given by

$$v = Q^{1/2^+}(x_0)|u|^{2^*-2}u,$$

where $2^+ = 2N/(N+2)$ is the conjugate exponent to 2^*. Then, v satisfies the dual equation

$$|v|^{2^+-2}v = Q^{1/2^*}(x_0)\mathbf{R}_0(Q^{1/2^*}(x_0)v),$$

where $\mathbf{R}_0 = (-\Delta)^{-1}$. The sequence $(v_n)_n$ defined from v via the dual rescalings $v_n(x) := n^{(N+2)/2}v(n(x-x_0))$ is then a Palais-Smale sequence for the dual functional J_Q at level $c = N^{-1}\|v\|_{2^+}^{2^+} = N^{-1}\|\nabla u\|_2^2$, and this sequence has no converging subsequence in $L^{2^+}(\mathbb{R}^N)$.

In analogy to the study of the critical problem (1.1) on a bounded domain, starting with the celebrated work of Brézis and Nirenberg [**3**], we shall try to recover some kind of compactness by comparing the mountain pass level L_Q of the functional J_Q with the least energy level L_Q^* among all possible limiting problems (1.4) with $x_0 \in \mathbb{R}^N$. From the duality between the Sobolev and the Hardy–Littlewood–Sobolev inequalities, it follows that

$$L_Q^* = \frac{S^{N/2}}{N\|Q\|_\infty^{(N-2)/2}},$$

where S denotes the optimal constant in the Sobolev inequality (see § 3.2 for more details).

The general strategy consists roughly in two steps:

(I) show that at every level $0 < \beta < L_Q^*$, the Palais–Smale condition is satisfied, and

(II) establish the strict inequality $L_Q < L_Q^*$.

Ambrosetti and Struwe [**2**] confirmed that, for the Dirichlet problem on a bounded domain, this scheme is also adapted to the dual variational framework. However, whereas the authors in [**2**] reduce the proof of the Palais–Smale condition for the dual functional to the proof of the same property for the direct functional, we do not have for the problem (1.1) on $\mathbb{R}^N$ a direct functional at hand. In our approach towards the above steps (I) and (II), we choose instead to work directly with the resolvent operators for the original and the limit problems, via the corresponding fundamental solutions. More precisely, we start by deriving accurate upper and lower bounds on the difference of these fundamental solutions (lemma 2.1). This involves a detailed study of Bessel functions for small arguments. Based on these estimates, we then prove a new local compactness property for the difference operator $\mathbf{R}_k - \mathbf{R}_0$, where $\mathbf{R}_0 = (-\Delta)^{-1}$ (see proposition 2.4). In addition, we show that $\mathbf{R}_k$ remains locally compact in subcritical Lebesgue spaces. Combining these properties, step (I) can be completed in the case where Q vanishes at infinity.

The next step is to prove the strict inequality $L_Q < L_Q^*$. There, the lower bound on the difference of the fundamental solutions plays a key role. Indeed, it implies that in dimension $N \geqslant 4$ the quadratic form of the operator $\mathbf{R}_k - \mathbf{R}_0$ is positive

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for positive functions supported in sets of small diameter. For such functions, the energy of J_Q can thus be made smaller than that of the dual functional associated with (1.4). Since we are working with a nonconstant Q, an additional requirement (see (Q2) in theorem 1.1 below) is needed to complete the argument. The condition that we impose controls the way in which Q approaches its maximum value $\|Q\|_\infty$. The same condition also appears in several related critical problems, and it seems to go back to the work of Escobar [7]. Let us mention that Egnell [6] provided examples of critical problems on bounded domains for which this assumption is necessary. More recently, this condition was also used in a paper by Chabrowski and Szulkin [4], on a strongly indefinite critical nonlinear Schrödinger equation on $\mathbb{R}^N$ with periodic coefficients. There, the authors work in a direct variational framework and use generalized linking arguments to show the existence of a Palais–Smale sequence at some level. The condition (Q2) is used to prove that this level lies strictly below L_Q^*. A nontrivial critical point is then obtained with the help of Lions' local compactness lemma [18] (see also [22, lemma 1.21]). Our approach to treat periodic, and more generally asymptotically periodic functions Q, is inspired by [4], although our arguments differ significantly. Working within the dual framework, we can simply use the mountain pass theorem without Palais–Smale condition, but we need to show that the nonvanishing property for $\mathbf{R}_k$, proven in [11, theorem 3.1] for noncritical exponents continues to hold in the critical case $p = 2^*$.

As already pointed out by Brézis and Nirenberg [3], there is a strong contrast between the dimensions $N = 3$ and $N \geqslant 4$, for problems with the critical exponent. In the present case, the estimates on the difference of the fundamental solutions have the opposite sign for $N = 3$, so that $\mathbf{R}_k - \mathbf{R}_0$ acts negatively on positive functions. This does not permit to verify step (II) above and we show that, in fact, $L_Q = L_Q^*$ holds for any bounded $Q \geqslant 0$ in this case. Moreover, we find that the mountain pass level L_Q is not achieved.

As indicated in [11], every nontrivial critical point $v \in L^{2^+}(\mathbb{R}^N)$ of J_Q is related, via the transformation

$$u = \mathbf{R}_k(Q^{1/2^*} v),$$

to a nontrivial strong solution $u \in W^{2,2^*}(\mathbb{R}^N)$ of (1.1) (see § 3.1 for more details). The solutions we obtain in the present paper have the distinctive property that the corresponding critical point of J_Q has minimal energy among all nontrivial critical points. Following the terminology introduced in a recent paper [8], we call such solutions *dual ground states* of (1.1) (cf. § 3.1 for the precise definition). The main result in the present paper is the following.

THEOREM 1.1. *Let $N \geqslant 3$ and consider $Q \in L^\infty(\mathbb{R}^N)\backslash\{0\}$ such that $Q \geqslant 0$ a.e. in $\mathbb{R}^N$.*

(i) *If $N \geqslant 4$ and Q satisfies the following conditions,*
  (Q1) *$Q = Q_{per} + Q_0$, where $Q_{per}, Q_0 \geqslant 0$ are such that Q_{per} is periodic and $Q_0(x) \to 0$ as $|x| \to \infty$;*

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(Q2) *there exists* $x_0 \in \mathbb{R}^N$ *with* $Q(x_0) = \max_{\mathbb{R}^N} Q$ *and, as* $|x - x_0| \to 0$,

$$Q(x_0) - Q(x) = \begin{cases} o(|x - x_0|^2), & \text{if } N \geqslant 5, \\ O(|x - x_0|^2), & \text{if } N = 4, \end{cases}$$

then, the problem (1.1) *with* $p = 2^*$ *has a dual ground state.*

(ii) *If* $N = 3$, *no dual ground state exists for* (1.1) *with* $p = 2^*$.

Note that the assumption (Q1) also allows for the cases $Q = Q_0$ and $Q = Q_{\text{per}}$. Let us point out that, for a maximum point x_0 of Q, the condition $Q(x_0) - Q(x) = O(|x - x_0|^2)$ as $x \to x_0$ is satisfied as soon as Q is twice differentiable at x_0. The assumption $Q(x_0) - Q(x) = o(|x - x_0|^2)$ as $x \to x_0$ is more restrictive and requires some additional flatness of Q at x_0 (cf. [**6**]). For small $k > 0$, the condition (Q2) seems to be sharp. However, using a scaling argument, we can slightly weaken this assumption in dimensions $N \geqslant 5$ in the following sense: If Q is given and x_0 is some maximum point of Q for which $Q(x_0) - Q(x) = O(|x - x_0|^2)$ as $x \to x_0$. Then, if (Q1) is satisfied, there is some $k_0 > 0$ such that for all $k \geqslant k_0$ the equation (1.1) with $p = 2^*$ has a dual ground state. Concerning the existence of multiple solutions, the method developed recently (see [**8**, theorem 4.1]) for the high-frequency limit $k \to \infty$ can be combined with the results of the present paper to relate the number of dual bound states (i.e., solutions of (1.1) associated with critical points of the dual functional) to the topology of the set M of maximum points of Q. More precisely, for every given continuous Q vanishing at infinity and satisfying the condition $Q(x_0) - Q(x) = O(|x - x_0|^2)$ as $x \to x_0$ for some of its maximum points x_0 there is $k^* > 0$ such that for all $k \geqslant k^*$ the problem (1.1) has at least $\text{cat}_{M_\delta}(M)$ dual bound states, where M_δ is some neighbourhood of M and cat denotes the Ljusternik–Schnirelman category.

Previous results on the critical equation (1.1) had been obtained in the radial case in [**10**] and very recently in [**19**], where a broad class of nonlinearities is considered. Up to our knowledge, theorem 1.1 is the first result concerning solutions of the nonlinear Helmholtz equation with critical nonlinearity and nonradial Q. Let us also mention that the lower critical case $p = 2(N + 1)/(N - 1)$ is still open. There, we expect completely different phenomena than for $p = 2^*$. A suitable method, therefore, needs to be found and we will address this issue in a forthcoming paper.

We shall now briefly describe the structure of the paper. In § 2, we study the Helmholtz resolvent operator in the Lebesgue space $L^{2^+}(\mathbb{R}^N)$. Recalling first the construction of the fundamental solution of the Helmholtz equation and its asymptotic properties, we derive in lemma 2.1 new upper and lower bounds on the difference of the latter and the fundamental solution of Laplace's equation, for small arguments. The proof consists in the precise estimation of Bessel functions and their derivatives, and the result is of crucial importance for the whole paper. As a first application, we prove in proposition 2.4 that the difference

$$\mathbf{R}_k - \mathbf{R}_0 \ : \ L^{2^+}(\mathbb{R}^N) \quad \to \quad L^{2^*}_{\text{loc}}(\mathbb{R}^N),$$

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where $\mathbf{R}_0$ denotes the Laplace resolvent operator, is compact. There, we start by decomposing the fundamental solution of the Helmholtz equation in a similar way as in [**11**] and then apply the upper bounds obtained in lemma 2.1. Another essential property of the Helmholtz resolvent, the nonvanishing property, is established in the case $p = 2^*$ in theorem 2.5. Its proof relies on improvements of previous results from [**11**] by means of the Hardy–Littlewood–Sobolev inequality. After this study of the Helmholtz resolvent, we turn in § 3 to the existence of dual ground states of (1.1) with $p = 2^*$. We start by recalling the dual variational framework set up in [**11**] and the characterization of the dual mountain pass level L_Q. Using the compactness properties of $\mathbf{R}_k$ and of $\mathbf{R}_k - \mathbf{R}_0$ established in § 2, we then analyse the behaviour of Palais–Smale sequences for J_Q at the level L_Q. Under the assumption $L_Q < L_Q^*$, we obtain in proposition 3.3 the existence of a nontrivial critical point for J_Q in the case where Q is asymptotically periodic. The nonvanishing property plays here a key role in handling the periodic part Q_{per} of the coefficient Q. Section 3.3 is then devoted to estimating the dual mountain pass level L_Q under the additional 'flatness condition' ($Q2$). There, we show that the positive lower bound on the difference of the fundamental solutions given by lemma 2.1 yields the strict inequality $L_Q < L_Q^*$, in the case $N \geqslant 4$. Combining the above results, we obtain in § 3.4 the existence of dual ground states stated in theorem 1.1. The paper concludes with the 3-dimensional case, in which we show that $L_Q = L_Q^*$ holds and, by a contradiction argument, we obtain the nonexistence of dual ground states for (1.1) with $p = 2^*$ in this case.

We close this introduction by fixing some notation. Throughout the paper, we denote by $B_r(x)$ the open ball in $\mathbb{R}^N$ with radius r and centre at x. Moreover, we set $B_r = B_r(0)$. The constant $\omega_N := 2\pi^{N/2}/(N\Gamma(N/2))$, where Γ is the gamma function, represents the volume of the unit ball B_1. By $\mathbb{1}_M$ we shall denote the characteristic function of a measurable set $M \subset \mathbb{R}^N$. We write $\mathcal{S}(\mathbb{R}^N)$ for the space of Schwartz functions and $\mathcal{S}'$ for its dual, that is, the space of tempered distributions. Furthermore, we shall indifferently denote by $\widehat{f}$ or $\mathcal{F}(f)$ the Fourier transform of a function $f \in \mathcal{S}'$. For $1 \leqslant s \leqslant \infty$, we abbreviate the norm in $L^s(\mathbb{R}^N)$ by $\|\cdot\|_s$.

2. The Helmholtz resolvent in the critical case

2.1. Fundamental solutions

Without loss of generality and to simplify formulas, we consider the problem (1.1) with $k = 1$. The general case follows by rescaling the independent variable.

For $N \geqslant 3$, the radial outgoing fundamental solution of the Helmholtz equation $-\Delta u - u = \delta_0$ in $\mathbb{R}^N$ is given by

$$\Phi(x) := \frac{\mathrm{i}}{4}(2\pi|x|)^{(2-N)/2} H^{(1)}_{(N-2)/2}(|x|), \quad \text{for } x \in \mathbb{R}^N \backslash \{0\}, \tag{2.1}$$

where $H^{(1)}_{(N-2)/2}$ denotes the Hankel function of the first kind of order $(N-2)/2$. For a function $f \in \mathcal{S}(\mathbb{R}^N)$ the convolution $u := \Phi * f \in \mathcal{C}^\infty(\mathbb{R}^N)$ is a solution of the inhomogeneous Helmholtz equation $-\Delta u - u = f$ which satisfies the Sommerfeld outgoing radiation condition $\partial_r u(x) - \mathrm{i}u(x) = o(|x|^{(1-N)/2})$, as $|x| \to \infty$. Moreover, it is known (see [**12**]) that, in the sense of tempered distributions, the Fourier

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transform of Φ is given by

$$\widehat{\Phi}(\xi) = (2\pi)^{-N/2}\frac{1}{|\xi|^2-(1+\mathrm{i}0)} := (2\pi)^{-N/2}\lim_{\varepsilon\to 0^+}\frac{1}{|\xi|^2-(1+\mathrm{i}\varepsilon)}. \tag{2.2}$$

Since we shall be considering real-valued solutions of the Helmholtz equation in the sequel, we turn our attention to

$$\Psi(x) := \mathrm{Re}(\Phi(x)) = -\frac{1}{4}(2\pi|x|)^{(2-N)/2}Y_{(N-2)/2}(|x|), \quad \text{for } x\in\mathbb{R}^N\backslash\{0\}, \tag{2.3}$$

where $Y_{(N-2)/2}$ denotes the Bessel function of the second kind of order $(N-2)/2$. Ψ should be seen as the fundamental solution of the Helmholtz equation associated with real-valued standing waves.

Let us recall some well-known facts concerning the Bessel functions of the second kind: For nonnegative orders ν and positive arguments t, the asymptotic behaviour of $Y_\nu(t)$ is given by (see [**15**, remark 5.16.2])

$$Y_\nu(t) = -\frac{2^\nu\Gamma(\nu)}{\pi t^\nu}(1+O(t)), \qquad \text{as } t\to 0, \quad \text{if } \nu>0, \tag{2.4}$$

$$Y_0(t) = -\frac{2}{\pi}\ln\frac{2}{t}+O(1), \qquad \text{as } t\to 0, \tag{2.5}$$

$$Y_\nu(t) = -\sqrt{\frac{2}{\pi t}}\cos\left(t-\frac{(2\nu-1)\pi}{4}\right)(1+O(t^{-1})), \quad \text{as } t\to\infty, \quad \text{for all } \nu\geqslant 0. \tag{2.6}$$

As a consequence, we find that

$$\Psi(x) = \begin{cases} \dfrac{1}{N(N-2)\omega_N}|x|^{2-N}(1+O(|x|)), & \text{as } |x|\to 0,\\[2ex] \dfrac{1}{2}(2\pi|x|)^{(1-N)/2}\cos\left(|x|-\dfrac{(N-3)\pi}{4}\right)(1+O(|x|^{-1})), & \text{as } |x|\to\infty. \end{cases} \tag{2.7}$$

Denoting by y_ν the first positive zero of Y_ν with $\nu\geqslant 0$, we deduce from the asymptotics (2.4) and (2.5), that $Y_\nu(t)<0$ for all $t\in(0,y_\nu)$ and therefore $\Psi(x)>0$ for all $|x|<y_\nu$.

Recalling that for $N\geqslant 3$ the fundamental solution Λ of Laplace's equation in $\mathbb{R}^N$ is given by

$$\Lambda(x) = \frac{1}{N(N-2)\omega_N}|x|^{2-N}, \quad \text{for } x\in\mathbb{R}^N\backslash\{0\}, \tag{2.8}$$

we see from (2.7) that $\Psi(x)$ behaves like $\Lambda(x)$ for small $|x|$. Our first result gives more precise estimates on the way $\Psi(x)$ approaches $\Lambda(x)$ as $|x|\to 0$. In particular, we observe a strong contrast between the dimension $N=3$ and the higher dimensions $N\geqslant 4$.

LEMMA 2.1. *Let $r>0$ be given such that $r<y_{(N-4)/2}$ if $N\geqslant 4$ and $r<\pi$ if $N=3$.*

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(i) *There exist $\kappa_1, \kappa_2 > 0$ only depending on r and N, such that for all $x \in B_r$,*

$$\begin{cases} \kappa_1 |x|^{4-N} \leqslant \Psi(x) - \Lambda(x) \leqslant \kappa_2 |x|^{4-N}, & \text{if } N \geqslant 5, \\ \kappa_1 |\ln|x|\,| \leqslant \Psi(x) - \Lambda(x) \leqslant \kappa_2 |\ln|x|\,|, & \text{if } N = 4, \\ -\kappa_1 |x| \geqslant \Psi(x) - \Lambda(x) \geqslant -\kappa_2 |x|, & \text{if } N = 3. \end{cases}$$

(ii) *For every multiindex $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \geqslant 1$, there exists $\kappa_3 > 0$ only depending on $|\alpha|$, r and N, such that*

$$|\partial^\alpha (\Psi(x) - \Lambda(x))| \leqslant \kappa_3 |x|^{4-N-|\alpha|}, \quad \text{for all } x \in B_r. \tag{2.9}$$

Proof. We start by considering for $\nu \geqslant 1$ the function $\eta_\nu \colon [0, \infty) \to \mathbb{R}$ given by

$$\eta_\nu(t) := \begin{cases} -c_\nu t^\nu Y_\nu(t), & t > 0, \\ 1, & t = 0, \end{cases} \quad \text{where } c_\nu = \frac{\pi}{2^\nu \Gamma(\nu)}.$$

Remark that η_ν is continuous, as a consequence of (2.4) and since Y_ν is analytic on $(0, \infty)$. In addition, for $t > 0$, the recursion formula $\mathrm{d}/\mathrm{d}t[t^\nu Y_\nu(t)]' = t^\nu Y_{\nu-1}(t)$ (see [**15**, p. 105]) gives

$$\eta_\nu'(t) = -c_\nu t^\nu Y_{\nu-1}(t).$$

Hence, η_ν is strictly increasing in the interval $(0, y_{\nu-1})$ and, in particular, $\eta_\nu > 1$ in this interval. Moreover, using the asymptotic expansions for small arguments (2.4) and (2.5), we see that

$$\lim_{t \to 0^+} \frac{\eta_\nu'(t)}{t} = -c_\nu \lim_{t \to 0^+} t^{\nu-1} Y_{\nu-1}(t) = \frac{1}{2(\nu-1)}, \quad \text{if } \nu > 1,$$

$$\text{and} \quad \lim_{t \to 0^+} \frac{\eta_1'(t)}{t|\ln t|} = -\frac{\pi}{2} \lim_{t \to 0^+} \frac{Y_0(t)}{-\ln t} = 1.$$

Therefore, given $0 < r < y_{\nu-1}$ and since $y_0 < 1$, there exist constants $\kappa_1' = \kappa_1'(\nu, r)$ and $\kappa_2'(\nu, r)$ such that

$$\frac{\eta_\nu'(t)}{t} \geqslant 2\kappa_1', \quad \text{if } \nu > 1, \quad \text{and} \quad \frac{\eta_1'(t)}{t|\ln t|} \geqslant 2\kappa_1', \quad \text{for all } 0 < t < r,$$

$$\left|\frac{\eta_\nu'(t)}{t}\right| \leqslant 2\kappa_2', \quad \text{if } \nu > 1, \quad \text{and} \quad \left|\frac{\eta_1'(t)}{t \ln t}\right| \leqslant 2\kappa_2', \quad \text{for all } 0 < t < r.$$

Writing

$$\frac{\eta_\nu(t) - 1}{t^2} = \int_0^1 s \frac{\eta_\nu'(st)}{st} \, \mathrm{d}s \quad \text{and} \quad \frac{\eta_1(t) - 1}{t^2 |\ln t|} = \int_0^1 s \left|\frac{\ln(st)}{\ln t}\right| \frac{\eta_1'(st)}{st|\ln(st)|} \, \mathrm{d}s,$$

we obtain the bounds

$$\eta_\nu(t) - 1 \geqslant \kappa_1' t^2, \quad \text{if } \nu > 1, \quad \text{and} \quad \eta_1(t) - 1 \geqslant \kappa_1' t^2 |\ln t|, \quad \text{for all } 0 < t < r, \tag{2.10}$$

$$|\eta_\nu(t) - 1| \leqslant \kappa_2' t^2, \quad \text{if } \nu > 1, \quad \text{and} \quad |\eta_1(t) - 1| \leqslant \kappa_2'' t^2 |\ln t|, \quad \text{for all } 0 < t < r, \tag{2.11}$$

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with some $\kappa_2'' = \kappa_2''(r) > 0$. The assertion (i) in case $N \geqslant 4$ follows from (2.10) and (2.11), since we have

$$\Psi(x) - \Lambda(x) = \Lambda(x)(\eta_{(N-2)/2}(|x|) - 1) = \frac{1}{N(N-2)\omega_N}|x|^{2-N}(\eta_{(N-2)/2}(|x|) - 1).$$

In the case $N = 3$, we have

$$\Psi(x) - \Lambda(x) = \frac{1}{4\pi|x|}(\cos|x| - 1). \tag{2.12}$$

Remark that

$$\frac{\cos t - 1}{t^2} = -\int_0^1 s\frac{\sin(st)}{st}\,\mathrm{d}s, \quad t \mapsto \frac{\sin t}{t} \text{ is decreasing in } [0,\pi], \quad \lim_{t\to 0}\frac{\sin t}{t} = 1,$$

and $|\sin t/t| \leqslant 1$ for all $t > 0$. We thus conclude that for given $0 < r_0 < \pi$ there is a constant $\kappa_1 = \kappa_1(r_0) > 0$ such that

$$\frac{\cos t - 1}{t^2} \leqslant -\kappa_1, \quad \text{for all } 0 < t < r_0, \quad \text{and} \quad \frac{\cos t - 1}{t^2} \geqslant -\frac{1}{2}, \quad \text{for all } t > 0.$$

Plugging these estimates in (2.12) yields the assertion (i) for $N = 3$ with $\kappa_2 = 1/(8\pi)$.

To prove the assertion (ii), we notice that for $\alpha \in \mathbb{N}_0^N$ and $k = |\alpha|$, an induction argument based on the recursion formula $\mathrm{d}/\mathrm{d}t[t^{-\nu}Y_\nu(t)]' = -t^{-\nu}Y_{\nu+1}(t)$ (see [**15**, p. 105]) gives

$$\partial^\alpha(\Psi(x) - \Lambda(x)) = \sum_{\ell=0}^{\lfloor k/2\rfloor} f_{k-\ell}(|x|)P_{k-2\ell}(x),$$

where for $m \in \mathbb{N}_0$, $P_m(x)$ is a homogeneous polynomial of degree m and where

$$f_m(t) = (-1)^m\frac{2^m\Gamma((N-2)/2 + m)}{\Gamma((N-2)/2)} \times \frac{t^{2-N-2m}}{N(N-2)\omega_N}[\eta_{(N-2)/2+m}(t) - 1], \quad t > 0.$$

As a consequence, given $r > 0$, there is a constant $\gamma = \gamma(N, k, r) > 0$ such that

$$|\partial^\alpha(\Psi(x) - \Lambda(x))| \leqslant \gamma|x|^{2-N-k}\sum_{\ell=0}^{\lfloor k/2\rfloor}[\eta_{(N-2)/2+k-\ell}(|x|) - 1], \quad \text{for all } |x| < r.$$

Using (2.11) and remarking that $(N-2)/2 + k - \ell > 1$ for $k \geqslant 1$ and $0 \leqslant \ell \leqslant \lfloor k/2\rfloor$, we obtain the desired assertion and the lemma is proven. □

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2.2. Compactness properties

Here, and in the next section, we discuss the properties of the resolvent Helmholtz operator $\mathbf{R} := \mathbf{R}_1$ given by the convolution $f \mapsto \Psi * f$ for $f \in \mathcal{S}(\mathbb{R}^N)$, where Ψ is given in (2.3). Let us first remark that as a consequence of an estimate of Kenig, Ruiz and Sogge [**14**, theorem 2.3], this mapping extends as a continuous linear operator

$$\mathbf{R} \ : \ L^{2^+}(\mathbb{R}^N) \ \to \ L^{2^*}(\mathbb{R}^N).$$

In particular, there exists a constant $C_0 > 0$ only depending on N such that

$$\|\mathbf{R}v\|_{2^*} \leqslant C_0\|v\|_{2^+}, \quad \text{for all } v \in L^{2^+}(\mathbb{R}^N). \tag{2.13}$$

Let us denote by

$$\mathbf{R}_0 \ : \ L^{2^+}(\mathbb{R}^N) \ \to \ L^{2^*}(\mathbb{R}^N)$$

the linear operator given by the convolution with the fundamental solution of Laplace's equation

$$\mathbf{R}_0 v := \Lambda * v, \quad v \in L^{2^+}(\mathbb{R}^N).$$

Notice that $\mathbf{R}_0$ is well defined and continuous, as a consequence of the weak Young inequality [**17**, p. 107].

Remark 2.2. The results in this and the next sections are stated and proven for the real part $\mathbf{R}$ of the resolvent, but they remain valid for the full resolvent $\mathcal{R}$: $L^{2^+}(\mathbb{R}^N) \to L^{2^*}(\mathbb{R}^N)$ which is the extension of the convolution map $f \mapsto \Phi * f$, $f \in \mathcal{S}(\mathbb{R}^N, \mathbb{C})$.

Lemma 2.3. *For all $1 \leqslant t < 2^*$ and all $r > 0$ the operator $\mathbb{1}_{B_r}\mathbf{R} : L^{2^+}(\mathbb{R}^N) \to L^t(\mathbb{R}^N)$ is compact.*

Proof. By elliptic estimates (see [**11**, proposition A.1]), we can find for every $r > 0$ a constant $D_r > 0$ such that $\|\mathbf{R}v\|_{W^{2,2^+}(B_r)} \leqslant D_r\|v\|_{2^+}$ for all $v \in L^{2^+}(\mathbb{R}^N)$. Since the embedding $W^{2,2^+}(B_r) \hookrightarrow L^t(B_r)$ is compact for all $1 \leqslant t < 2^*$, and all $r > 0$, we deduce that the operator $\mathbb{1}_{B_r}\mathbf{R} : L^{2^+}(\mathbb{R}^N) \to L^t(\mathbb{R}^N)$ is compact for all $1 \leqslant t < 2^*$ and all $r > 0$. □

Proposition 2.4.

(i) *The difference $\mathbf{R} - \mathbf{R}_0$ is a continuous linear mapping from $L^{2^+}(\mathbb{R}^N)$ into $W^{2,2^*}(\mathbb{R}^N)$*

(ii) *For all $r > 0$, the operator $\mathbb{1}_{B_r}(\mathbf{R} - \mathbf{R}_0)$: $L^{2^+}(\mathbb{R}^N) \to L^{2^*}(\mathbb{R}^N)$ is compact.*

Proof. In the sequel, for $\mu \in \mathbb{R}$, C and C_μ shall denote constants depending on N and on N, μ respectively, but which may change from line to line.

To prove (i) we shall use a decomposition of Ψ, similar to the one introduced in [**11**, § 3] for Φ. We fix a radial $\psi \in \mathcal{S}(\mathbb{R}^N)$ such that $\widehat{\psi} \in \mathcal{C}_c^\infty(\mathbb{R}^N)$, $0 \leqslant \widehat{\psi} \leqslant 1$,

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$\widehat{\psi}(\xi) = 1$ for $||\xi| - 1| \leqslant 1/6$ and $\widehat{\psi}(\xi) = 0$ for $||\xi| - 1| \geqslant 1/4$. Write $\Psi = \Psi_1 + \Psi_2$ with

$$\Psi_1 := (2\pi)^{-N/2}(\Psi * \psi), \quad \Psi_2 = \Psi - \Psi_1. \tag{2.14}$$

Then, for every $f \in \mathcal{S}(\mathbb{R}^N)$ and $\alpha \in \mathbb{N}_0^N$, the properties of the convolution of Schwartz functions with a tempered distribution (see [**21**, theorem 7.19]) allow to write

$$(\partial^\alpha \Psi_1) * f = (2\pi)^{-N/2}[\Psi * (\partial^\alpha \psi)] * f = (2\pi)^{-N/2}\Psi * [(\partial^\alpha \psi) * f],$$

where $\partial^\alpha \psi \in \mathcal{S}(\mathbb{R}^N)$. Hence, from (2.13) and Young's inequality for the convolution, we obtain the estimate

$$\|(\partial^\alpha \Psi_1) * f\|_{2^*} = (2\pi)^{-N/2}\|\Psi * [(\partial^\alpha \psi) * f]\|_{2^*} \leqslant (2\pi)^{-N/2} C_0 \|\partial^\alpha \psi\|_1 \|f\|_{2^+},$$

for all $f \in \mathcal{S}(\mathbb{R}^N)$. As a consequence, the convolution $f \mapsto (\partial^\alpha \Psi_1) * f$, $f \in \mathcal{S}(\mathbb{R}^N)$, extends as a continuous map from $L^{2^+}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$ for every $\alpha \in \mathbb{N}_0^N$.

Turning to Ψ_2, we have by definition $\widehat{\Psi_2} = (1 - \widehat{\psi})\widehat{\Psi}$ and, since taking real parts in (2.2) yields

$$\widehat{\Psi_2}(\xi) = (2\pi)^{-N/2} \lim_{\varepsilon \to 0^+} \frac{|\xi|^2 - 1}{(|\xi|^2 - 1)^2 + \varepsilon^2}(1 - \widehat{\psi}(\xi)) = (2\pi)^{-N/2} \frac{1 - \widehat{\psi}(\xi)}{|\xi|^2 - 1},$$

we get $\widehat{\Psi_2} \in \mathcal{C}^\infty(\mathbb{R}^N)$ and $\widehat{\Psi_2}(\xi) = (2\pi)^{-N/2}(|\xi|^2 - 1)^{-1}$ for $|\xi| \geqslant 5/4$. This gives $\partial^\beta \widehat{\Psi_2} \in L^1(\mathbb{R}^N)$ for all $\beta \in \mathbb{N}_0^N$ such that $2 + |\beta| > N$. Therefore, using standard differentiation properties of the Fourier transform, the fact that $\widehat{\Psi_2}$ (and so Ψ_2) is radial and that $\mathcal{F}(f)(\xi) = \mathcal{F}^{-1}(f)(-\xi)$, we obtain

$$\||\cdot|^{|\beta|}\Psi_2\|_\infty = \left\|\mathcal{F}\left(\partial^\beta \widehat{\Psi_2}\right)\right\|_\infty \leqslant \left\|\partial^\beta \widehat{\Psi_2}\right\|_{L^1(\mathbb{R}^N)} \leqslant C_{|\beta|}, \quad \text{for all } |\beta| > N - 2.$$

Choosing $\beta \in \mathbb{N}_0^N$ with $|\beta| = N$, we obtain that

$$|\Psi_2(x)| \leqslant C|x|^{-N}, \quad \text{for all } x \in \mathbb{R}^N. \tag{2.15}$$

Using the same argument with $\partial^\alpha \Psi_2$ in place of Ψ_2, for every $\alpha \in \mathbb{N}_0^N$, we get

$$|\partial^\alpha \Psi_2(x)| \leqslant C_{|\alpha|}|x|^{-N-|\alpha|}, \quad \text{for all } x \in \mathbb{R}^N \text{ and all } \alpha \in \mathbb{N}_0^N. \tag{2.16}$$

From lemma 2.1, we obtain estimates on $\partial^\alpha(\Psi_2 - \Lambda)(x)$ for $|x|$ small. For large values of $|x|$, we use (2.16) and $|\partial^\alpha \Lambda(x)| \leqslant C_{|\alpha|}|x|^{2-N-|\alpha|}$, which follows easily

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from (2.8). Altogether, we get for $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$,

$$|\partial^\alpha(\Psi_2 - \Lambda)(x)| \leqslant \begin{cases} C_{|\alpha|} \min\{|x|^{4-N-|\alpha|}, |x|^{2-N-|\alpha|}\}, & N=3,\ N \geqslant 5,\ \text{or} \\ & N=4 \text{ and } |\alpha| \geqslant 1, \\ C \min\{1 + |\ln|x||, |x|^{-2}\}, & N=4 \text{ and } |\alpha| = 0. \end{cases} \tag{2.17}$$

As a consequence, denoting by $L_w^{N/(N-2)}(\mathbb{R}^N)$ the weak-$L^{N/(N-2)}$ space, we infer that

$$\partial^\alpha(\Psi_2 - \Lambda) \in L_w^{N/(N-2)}(\mathbb{R}^N), \quad \text{for all } \alpha \in \mathbb{N}_0^N \quad \text{such that } |\alpha| \leqslant 2.$$

From the weak Young inequality, the convolution $f \mapsto \partial^\alpha(\Psi_2 - \Lambda) * f$, $f \in \mathcal{S}(\mathbb{R}^N)$, extends as a continuous map from $L^{2^+}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$ for such α. Summarizing and using the fact that

$$\|(\Psi - \Lambda) * f\|^2_{W^{2,2^*}} \leqslant 2 \sum_{|\alpha| \leqslant 2} \|\partial^\alpha(\Psi_2 - \Lambda) * f\|^2_{2^*} + 2 \sum_{|\alpha| \leqslant 2} \|(\partial^\alpha \Psi_1) * f\|^2_{2^*},$$

we obtain that the convolution $f \mapsto (\Psi - \Lambda) * f$ extends as a continuous map from $L^{2^+}(\mathbb{R}^N)$ into $W^{2,2^*}(\mathbb{R}^N)$. Therefore, the operator

$$\mathbf{R} - \mathbf{R}_0 : L^{2^+}(\mathbb{R}^N) \to W^{2,2^*}(\mathbb{R}^N)$$

is continuous and (i) is proven.

By the Rellich–Kondrachov theorem, the embedding $W^{2,2^*}_{\text{loc}}(\mathbb{R}^N) \hookrightarrow L^t_{\text{loc}}(\mathbb{R}^N)$ is compact for all $1 \leqslant t < 2N/(N-6)_+$. Thus, we obtain the compactness of $\mathbb{1}_{B_r}(\mathbf{R} - \mathbf{R}_0) : L^{2^+}(\mathbb{R}^N) \to L^{2^*}(\mathbb{R}^N)$ for all $r > 0$, which proves (ii). □

2.3. Nonvanishing property and related estimates

As a key ingredient for the existence result in § 3 below, we prove that the nonvanishing property of the quadratic form associated with the Helmholtz resolvent holds true in the space $L^{p'}(\mathbb{R}^N)$ with $p = 2^*$. This property has been proved in [**11**, theorem 3.1] in the noncritical range $2(N+1)/(N-1) < p < 2^*$.

THEOREM 2.5. *Consider a bounded sequence $(v_n)_n \subset L^{2^+}(\mathbb{R}^N)$ satisfying*

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} v_n \mathbf{R} v_n \, \mathrm{d}x \right| > 0. \tag{2.18}$$

Then there exists $R > 0$, $\zeta > 0$ and a sequence $(x_n)_n \subset \mathbb{R}^N$ such that, up to a subsequence,

$$\int_{B_R(x_n)} |v_n|^{2^+} \, \mathrm{d}x \geqslant \zeta, \quad \textit{for all } n. \tag{2.19}$$

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Proof. Let us assume by contradiction that

$$\lim_{n\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int_{B_\rho(y)}|v_n|^{2^+}\,\mathrm{d}x\right)=0,\quad \text{for all } \rho>0. \tag{2.20}$$

Consider the decomposition $\Psi=\Psi_1+\Psi_2$ introduced in (2.14), and denote by $\mathbf{R}_1$: $L^{2^+}(\mathbb{R}^N)\to L^{2^*}(\mathbb{R}^N)$ the continuous extension of the convolution map $f\mapsto\Psi_1*f$, $f\in\mathcal{S}(\mathbb{R}^N)$. Since lemma 3.4 in [**11**] holds for the critical exponent $p=2^*$, we obtain by density of $\mathcal{S}(\mathbb{R}^N)$ in $L^{2^+}(\mathbb{R}^N)$ that

$$\int_{\mathbb{R}^N} v_n\mathbf{R}_1 v_n\,\mathrm{d}x\to 0,\quad \text{as } n\to\infty, \tag{2.21}$$

taking real parts. Turning to Ψ_2, we note that the estimate (2.15) and the behaviour of Ψ close to $x=0$ given by (2.7) yield the existence of some constant $C'=C'(N)>0$ such that

$$|\Psi_2(x)|\leqslant C'\min\{|x|^{2-N},|x|^{-N}\},\quad \text{for all } x\neq 0. \tag{2.22}$$

Setting $M_R:=\mathbb{R}^N\backslash B_R$ for $R>1$, we deduce from (2.22) that

$$\|\Psi_2\|_{L^{N/(N-2)}(M_R)}\leqslant C'\left(\int_{|x|\geqslant R}|x|^{-N^2/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N}\to 0,\quad \text{as } R\to\infty.$$

Hence, by Young's inequality,

$$\begin{aligned}&\sup_{n\in\mathbb{N}}\left|\int_{\mathbb{R}^N}v_n[(\mathbb{1}_{M_R}\Psi_2)*v_n]\,\mathrm{d}x\right|\\ &\quad\leqslant\|\Psi_2\|_{L^{N/(N-2)}(M_R)}\sup_{n\in\mathbb{N}}\|v_n\|^2_{L^{2^+}(\mathbb{R}^N)}\to 0,\quad \text{as } R\to\infty.\end{aligned} \tag{2.23}$$

Consider a decomposition of $\mathbb{R}^N$ into disjoint N-cubes $\{Q_\ell\}_{\ell\in\mathbb{N}}$ of side length R, and let for each ℓ the N-cube Q'_ℓ have the same centre as Q_ℓ but side length $3R$. From the estimate (2.22) and the Hardy–Littlewood–Sobolev inequality [**17**, theorem 4.3], there is a constant $C''=C''(N)$ such that

$$\begin{aligned}\left|\int_{\mathbb{R}^N}v_n[(\mathbb{1}_{B_R}\Psi_2)*v_n]\,\mathrm{d}x\right|&\leqslant\sum_{\ell=1}^{\infty}\int_{Q_\ell}\left(\int_{|x-y|<R}|\Psi_2(x-y)\,||v_n(x)|\,|v_n(y)|\,\mathrm{d}y\right)\mathrm{d}x\\ &\leqslant C'\sum_{\ell=1}^{\infty}\int_{Q_\ell}\left(\int_{Q'_\ell}\frac{|v_n(x)|\,|v_n(y)|}{|x-y|^{N-2}}\,\mathrm{d}y\right)\mathrm{d}x\\ &\leqslant C'\sum_{\ell=1}^{\infty}\left(\int_{Q'_\ell}|v_n(x)|^{2^+}\,\mathrm{d}x\right)^{(N+2)/N}\end{aligned}$$

$$\leqslant C' \left[\sup_{\ell \in \mathbb{N}} \int_{Q'_\ell} |v_n(x)|^{2^+} \, \mathrm{d}x \right]^{2/N} \sum_{\ell=1}^{\infty} \int_{Q'_\ell} |v_n(x)|^{2^+} \, \mathrm{d}x$$

$$\leqslant C'' \left[\sup_{y \in \mathbb{R}^N} \int_{B_{3R\sqrt{N}}(y)} |v_n(x)|^{2^+} \, \mathrm{d}x \right]^{2/N} 3^N \|v_n\|_{2^+}^{2^+},$$

for all n. Therefore, the boundedness of $(v_n)_n$ and the assumption (2.20) give

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} v_n [(\mathbb{1}_{B_R} \Psi_2) * v_n] \, \mathrm{d}x = 0, \quad \text{for every } R > 0. \tag{2.24}$$

Combining (2.21), (2.23) and (2.24), we obtain

$$\int_{\mathbb{R}^N} v_n \mathbf{R} v_n \, \mathrm{d}x = \int_{\mathbb{R}^N} v_n \mathbf{R}_1 v_n \, \mathrm{d}x + \int_{\mathbb{R}^N} v_n [\Psi_2 * v_n] \, \mathrm{d}x \to 0, \quad \text{as } n \to \infty,$$

contradicting the assumption (2.18). The theorem follows. □

Let us recall a result obtained recently [**8**, lemma 2.4], on the bilinear form associated with the operator $\mathbf{R}$ for functions having disjoint support.

Lemma 2.6. *Let $p > 2(N+1)/(N-1)$. There exists a constant $D = D(N,p) > 0$ such that for any $R > 0$, $r \geqslant 1$ and $u, v \in L^{p'}(\mathbb{R}^N)$ with supp$(u) \subset B_R$ and supp$(v) \subset \mathbb{R}^N \backslash B_{R+r}$,*

$$\left| \int_{\mathbb{R}^N} u \mathbf{R} v \, \mathrm{d}x \right| \leqslant D r^{-\lambda_p} \|u\|_{p'} \|v\|_{p'}, \quad \textit{where } \lambda_p = \frac{N-1}{2} - \frac{N+1}{p}.$$

The proof (see [**8**]) uses the decomposition $\Psi = \Psi_1 + \Psi_2$ introduced in (2.14) for the fundamental solution of the Helmholtz equation. The dominant term in the estimate comes from the convolution with Ψ_1 and is obtained as follows. First remark that the Fourier transforms $\widehat{\Psi_1}$ and $\widehat{\Psi_1}\widehat{\varphi}$ coincide, for any φ satisfying $\widehat{\varphi} \equiv 1$ on the set $\{||\xi| - 1| \leqslant 1/4\}$. Choosing such a φ for which, in addition, supp$(\widehat{\varphi})$ is contained in $\{||\xi| - 1| \leqslant 1/2\}$, one can replace v with $\varphi * v$ and apply the result in [**11**, proposition 3.3] to the convolution $(1_{M_r} \Psi_1) * (\varphi * v)$ giving the asserted decay rate. The remaining convolution terms are estimated using Young's inequality and these only give lower order contributions.

Based on this estimate, we prove a technical result which will be used in § 3.2 below to deal with a remainder term in an estimate derived from the Hardy–Littlewood–Sobolev inequality.

Lemma 2.7. *Let $(z_n)_n \subset L^{2^+}(\mathbb{R}^N)$ be a bounded sequence. Then, for every $\varepsilon > 0$, there exists $\rho_\varepsilon > 0$ such that*

$$\liminf_{n \to \infty} \left| \int_{\mathbb{R}^N} \mathbb{1}_{B_\rho} z_n \mathbf{R} \big(\mathbb{1}_{M_\rho} z_n \big) \, \mathrm{d}x \right| < \varepsilon, \quad \textit{for all } \rho \geqslant \rho_\varepsilon.$$

Here, $M_\rho := \mathbb{R}^N \backslash B_\rho$.

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Proof. Let $\zeta := \sup\{\|z_n\|_{2^+} : n \in \mathbb{N}\}$. We first see that by lemma 2.6 there is a constant $D = D(N) > 0$ such that

$$\left| \int_{\mathbb{R}^N} \mathbb{1}_{B_\rho} z_n \mathbf{R}(\mathbb{1}_{M_{2\rho}} z_n)\,\mathrm{d}x \right| \leqslant D\zeta^2 \rho^{-1/N}, \quad \text{for all } n \in \mathbb{N} \quad \text{and every } \rho \geqslant 1.$$

Hence, setting $\rho_0 := \max\{1, (2D\zeta^2/\varepsilon)^N\}$, we find

$$\left| \int_{\mathbb{R}^N} \mathbb{1}_{B_\rho} z_n \mathbf{R}\big(\mathbb{1}_{M_{2\rho}} z_n\big)\,\mathrm{d}x \right| \leqslant \frac{\varepsilon}{2}, \quad \text{for all } n \in \mathbb{N} \quad \text{and every } \rho \geqslant \rho_0.$$

Next, we choose $\eta > 0$ such that $\eta < (\varepsilon/(2C_0\zeta))^{2^+}$, where $C_0 > 0$ is such that (2.13) holds, and we claim that

$$\exists \rho_1 > 0 \quad \text{such that } \liminf_{n\to\infty} \int_{B_{2\rho}\setminus B_\rho} |z_n|^{2^+}\,\mathrm{d}x < \eta, \quad \text{for all } \rho \geqslant \rho_1. \tag{2.25}$$

Suppose this is not the case. Then, for every $k \in \mathbb{N}$ we can find a radius $\rho_k \geqslant k$ and an index $n_0(k) \in \mathbb{N}$ for which

$$\int_{B_{2\rho_k}\setminus B_{\rho_k}} |z_n|^{2^+}\,\mathrm{d}x \geqslant \eta, \quad \text{for all } n \geqslant n_0(k).$$

Moreover, we can assume without loss of generality that $n_0(k+1) \geqslant n_0(k)$ and $\rho_{k+1} \geqslant 2\rho_k$. For each $\ell \in \mathbb{N}$, it follows that

$$\zeta^{2^+} \geqslant \int_{\mathbb{R}^N} |z_n|^{2^+}\,\mathrm{d}x \geqslant \sum_{k=1}^{\ell} \int_{B_{2\rho_k}\setminus B_{\rho_k}} |z_n|^{2^+}\,\mathrm{d}x \geqslant \ell\eta, \quad \text{for all } n \geqslant n_0(\ell).$$

For ℓ large enough, we obtain a contradiction, and the claim is proven.

As a consequence of the above results, we can write for $\rho \geqslant \rho_\varepsilon := \max\{\rho_0, \rho_1\}$,

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} \mathbb{1}_{B_\rho} z_n \mathbf{R}(\mathbb{1}_{M_\rho} z_n)\,\mathrm{d}x \right| \\ &\quad \leqslant \left| \int_{\mathbb{R}^N} \mathbb{1}_{B_\rho} z_n \mathbf{R}(\mathbb{1}_{M_{2\rho}} z_n)\,\mathrm{d}x \right| + \left| \int_{\mathbb{R}^N} \mathbb{1}_{B_\rho} z_n \mathbf{R}(\mathbb{1}_{B_{2\rho}\setminus B_\rho} z_n)\,\mathrm{d}x \right| \\ &\quad \leqslant \frac{\varepsilon}{2} + C_0\zeta \|\mathbb{1}_{B_{2\rho}\setminus B_\rho} z_n\|_{2^+}, \end{aligned}$$

using Hölder's inequality and the estimate (2.13). The conclusion then follows from the claim (2.25). □

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3. Existence via the dual variational method

3.1. The dual energy functional

We follow the path established in [11] and use a dual variational framework to find nontrivial solutions for the problem

$$-\Delta u - u = Q(x)|u|^{2^*-2}u, \quad u \in W^{2,2^*}(\mathbb{R}^N), \tag{3.1}$$

where $Q \in L^\infty(\mathbb{R}^N)$ is a nonnegative function which is not identically zero. Setting $v = Q^{1/2^+}|u|^{2^*-2}u$, we shall study the fixed-point problem

$$|v|^{2^+-2}v = Q^{1/2^*}\mathbf{R}(Q^{1/2^*}v), \quad v \in L^{2^+}(\mathbb{R}^N), \tag{3.2}$$

where $\mathbf{R}$ denotes the resolvent Helmholtz operator defined in § 2.2. For the Birman-Schwinger type operators associated with the Helmholtz and Laplace resolvents respectively, we introduce the notation

$$\mathbf{A}_Q v := Q^{1/2^*}\mathbf{R}(Q^{1/2^*}v) \quad \text{and} \quad \mathbf{G}_Q v := Q^{1/2^*}\mathbf{R}_0(Q^{1/2^*}v), \quad v \in L^{2^+}(\mathbb{R}^N). \tag{3.3}$$

We consider the functional

$$J_Q(v) := \frac{1}{2^+}\int_{\mathbb{R}^N} |v|^{2^+}\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} v\mathbf{A}_Q v\,\mathrm{d}x, \quad \text{for } v \in L^{2^+}(\mathbb{R}^N). \tag{3.4}$$

It is known that $J \in \mathcal{C}^1(L^{2^+}(\mathbb{R}^N), \mathbb{R})$ and from the symmetry of $\mathbf{A}_Q$ (cf. [11, lemma 4.1]), we have

$$J_Q'(v)w = \int_{\mathbb{R}^N} \left(|v|^{2^+-2}v - \mathbf{A}_Q v\right) w\,\mathrm{d}x, \quad \text{for all } v, w \in L^{2^+}(\mathbb{R}^N).$$

We detect solutions of (3.1) by finding critical points of the functional J_Q. Indeed, for $v \in L^{2^+}(\mathbb{R}^N)$, we have $J_Q'(v) = 0$ if and only if v satisfies (3.2). Setting $u = \mathbf{R}(Q^{1/2^*}v)$, we see that u solves $u = \mathbf{R}(Q|u|^{2^*-2}u)$. Any solution of this integral equation has the following properties:

Lemma 3.1 (Special case of [11, lemma 4.3]). *Let $Q \in L^\infty(\mathbb{R}^N)$ and consider a solution $u \in L^{2^*}(\mathbb{R}^N)$ of $u = \mathbf{R}(Q|u|^{2^*-2}u)$. Then, $u \in W^{2,q}(\mathbb{R}^N)$ for all $2^* \leqslant q < \infty$ and u is a strong solution of (3.1). Moreover, u is the real part of a function $\widetilde{u}$ which satisfies Sommerfeld's outgoing radiation condition in the form*

$$\lim_{R\to\infty} \frac{1}{R}\int_{B_R} \left|\nabla\widetilde{u}(x) - \mathrm{i}\widetilde{u}(x)\frac{x}{|x|}\right|^2 \mathrm{d}x = 0.$$

In addition, u satisfies the following asymptotic relation

$$\lim_{R\to\infty} \frac{1}{R}\int_{B_R} \left|u(x) - \sqrt{\frac{\pi}{2}}\,\mathrm{Re}\left[\frac{\mathrm{e}^{\mathrm{i}|x|-\mathrm{i}(N-3)\pi/4}}{|x|^{(N-1)/2}}\mathcal{F}(Q|u|^{2^*-2}u)\left(\frac{x}{|x|}\right)\right]\right|^2 \mathrm{d}x = 0.$$

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As shown in [**11**, lemma 4.2], the functional J_Q has the mountain pass geometry, that is,

(MP1) there exists $\delta > 0$ and $\rho > 0$ such that $J_Q(v) \geqslant \delta > 0$,

for all $v \in L^{2^+}(\mathbb{R}^N)$ with $\|v\|_{2^+} = \rho$;

(MP2) there exists $v_0 \in L^{2^+}(\mathbb{R}^N)$ such that $\|v_0\|_{2^+} > \rho$ and $J_Q(v_0) < 0$.

The mountain pass level

$$L_Q := \inf_{P\in\mathcal{P}} \max_{t\in[0,1]} J_Q(P(t)),$$

where

$$\mathcal{P} = \left\{P \in \mathcal{C}([0,1], L^{2^+}(\mathbb{R}^N)) \,:\, P(0) = 0 \text{ and } J_Q(P(1)) < 0\right\},$$

is therefore, well defined, $0 < L_Q < \infty$, and by the same arguments as in [**9**, lemma 4.1], it can be characterized as the following infimum

$$\begin{aligned} L_Q &= \inf\left\{J_Q(t_v v) \,:\, v \in L^{2^+}(\mathbb{R}^N) \quad \text{with} \int_{\mathbb{R}^N} v\mathbf{A}_Q v \,\mathrm{d}x > 0\right\} \\ &= \inf\left\{\frac{1}{N}\left(\frac{\|v\|_{2^+}^2}{\int_{\mathbb{R}^N} v\mathbf{A}_Q v\,\mathrm{d}x}\right)^{N/2} \,:\, v \in L^{2^+}(\mathbb{R}^N) \quad \text{with} \int_{\mathbb{R}^N} v\mathbf{A}_Q v\,\mathrm{d}x > 0\right\}. \end{aligned} \tag{3.5}$$

Here, for $v \in L^{2^+}(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} v\mathbf{A}_Q v\,\mathrm{d}x > 0$,

$$t_v = \left(\frac{\int_{\mathbb{R}^N} |v|^{2^+}\,\mathrm{d}x}{\int_{\mathbb{R}^N} v\mathbf{A}_Q v\,\mathrm{d}x}\right)^{1/(2-2^+)} \tag{3.6}$$

denotes the unique $t > 0$ with the property $J_Q(t_v v) = \max_{t>0} J_Q(tv)$. Remarking that for every such v, $J_Q'(t_v v)v = 0$, we see that if L_Q is achieved by some critical point of J_Q, then L_Q coincides with the least-energy level, that is,

$$L_Q = \inf\{J_Q(v) \,:\, v \in L^{2^+}(\mathbb{R}^N)\backslash\{0\} \quad \text{with} \;\; J_Q'(v) = 0\}.$$

Following the terminology introduced in [**8**], we will call a solution u of the nonlinear Helmholtz equation (3.1) a *dual ground state*, if $u = \mathbf{R}(Q^{1/2^*}v)$ and $v \in L^{2^+}(\mathbb{R}^N)$ is a critical point of the functional J_Q at the mountain pass level, that is, $J_Q'(v) = 0$ and $J_Q(v) = L_Q$. As a consequence of the discussion at the beginning of the section, every dual ground state u has the properties stated in lemma 3.1.

3.2. Palais–Smale sequences

In this section, we investigate the properties of Palais–Smale sequences for the functional J_Q. Recall that a sequence $(v_n)_n \subset L^{2^+}(\mathbb{R}^N)$ is called a *Palais–Smale*

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sequence for J_Q if $(J_Q(v_n))_n$ is bounded and $\|J'_Q(v_n)\|_* \to 0$ as $n \to \infty$. Here, $\|\cdot\|_*$ denotes the dual norm to $\|\cdot\|_{2^+}$. If, in addition, $J_Q(v_n) \to \beta$ as $n \to \infty$ for some $\beta \in \mathbb{R}$, $(v_n)_n$ is called a *(PS)$_\beta$-sequence* for J_Q. We start by considering sequences which satisfy a localized version of the above property. For this purpose, we introduce the following piece of notation: If $v \in L^{2^+}(\mathbb{R}^N)$, we let $J'_Q(v)\mathbb{1}_{B_r}$ denote the continuous linear form $w \mapsto J'_Q(v)(\mathbb{1}_{B_r}w)$ on $L^{2^+}(\mathbb{R}^N)$.

Lemma 3.2. *Let $(v_n)_n \subset L^{2^+}(\mathbb{R}^N)$ be a bounded sequence such that, for all $r > 0$, $\|J'_Q(v_n)\mathbb{1}_{B_r}\|_* \to 0$ as $n \to \infty$. Then, up to a subsequence,*

(i) *$(v_n)_n$ has a weak limit $v \in L^{2^+}(\mathbb{R}^N)$.*

(ii) *For all $1 \leqslant q < 2^+$ and all $r > 0$, we have $\mathbb{1}_{B_r}v_n \to \mathbb{1}_{B_r}v$, strongly in $L^q(\mathbb{R}^N)$ and $v_n \to v$ a.e. on $\mathbb{R}^N$, as $n \to \infty$.*

(iii) *$J'_Q(v) = 0$.*

(iv) *As $n \to \infty$, we have for all $r > 0$,*

$$\|\mathbb{1}_{B_r}(v_n - v)\|_{2^+}^{2^+} = \int_{\mathbb{R}^N} \mathbb{1}_{B_r}(v_n - v)\boldsymbol{A}_Q(v_n - v)\,\mathrm{d}x + o(1). \tag{3.7}$$

Proof. Since $(v_n)_n$ is bounded in $L^{2^+}(\mathbb{R}^N)$, there exists $v \in L^{2^+}(\mathbb{R}^N)$ and a subsequence which we still denote by $(v_n)_n$ such that $v_n \rightharpoonup v$ weakly. This proves (i). From now on, we restrict to this particular subsequence. To prove (ii), let $r > 0$, $1 \leqslant t < 2^*$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$. For $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N}(\mathbb{1}_{B_r}|v_n|^{2^+-2}v_n - \mathbb{1}_{B_r}|v_m|^{2^+-2}v_m)\varphi\,\mathrm{d}x\right| \\
&\quad= \left|[J'_Q(v_n) - J'_Q(v_m)](\mathbb{1}_{B_r}\varphi) + \int_{\mathbb{R}^N}\mathbb{1}_{B_r}\varphi\mathbf{A}_Q(v_n - v_m)\,\mathrm{d}x\right| \\
&\quad\leqslant \|J'_Q(v_n)\mathbb{1}_{B_r} - J'_Q(v_m)\mathbb{1}_{B_r}\|_*\|\mathbb{1}_{B_r}\varphi\|_{2^+} + \|\mathbb{1}_{B_r}\mathbf{A}_Q(v_n - v_m)\|_t\|\varphi\|_{t'} \\
&\quad\leqslant C[\|J'_Q(v_n)\mathbb{1}_{B_r}\|_* + \|J'_Q(v_m)\mathbb{1}_{B_r}\|_*]\|\varphi\|_{t'} \\
&\qquad+ \|Q\|_\infty^{1/2^*}\|\mathbb{1}_{B_r}\mathbf{R}(Q^{1/2^*}(v_n - v_m))\|_t\|\varphi\|_{t'},
\end{aligned}$$

where the constant $C > 0$ depends on N and r. The first expression in the last line vanishes as $n, m \to \infty$, by assumption, and the second one also vanishes due to lemma 2.3. Therefore, arguing by density, we find that $(\mathbb{1}_{B_r}|v_n|^{2^+-2}v_n)_n$ is a Cauchy sequence in $L^t(\mathbb{R}^N)$. Since the mapping $N : L^t(\mathbb{R}^N) \to L^{t/(2^*-1)}(\mathbb{R}^N)$ given by $N(u) := |u|^{2^*-2}u$ is well defined and Lipschitz continuous, it follows that $(\mathbb{1}_{B_r}v_n)_n = (N(\mathbb{1}_{B_r}|v_n|^{2^+-2}v_n))_n$ is a Cauchy sequence in $L^q(\mathbb{R}^N)$ for all $1 \leqslant q < 2^+$. Since these spaces are complete, and since $\mathbb{1}_{B_r}v_n \rightharpoonup \mathbb{1}_{B_r}v$ in each of these spaces, we obtain the desired strong convergence $\mathbb{1}_{B_r}v_n \to \mathbb{1}_{B_r}v$ in $L^q(\mathbb{R}^N)$ for all $1 \leqslant q < 2^+$. Going to a subsequence, we also have the pointwise convergence $v_n(x) \to v(x)$ as $n \to \infty$, for almost every $x \in \mathbb{R}^N$.

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Assertion (iii) now follows from (ii), since for $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ and $r > 0$ such that $\operatorname{supp}(\varphi) \subset B_r$, we have

$$\begin{aligned} J_Q'(v)\varphi &= \int_{\mathbb{R}^N} \mathbb{1}_{B_r}|v|^{2^+-2}v\varphi \,\mathrm{d}x - \int_{\mathbb{R}^N} \mathbb{1}_{B_r}\varphi\mathbf{A}_Q v\,\mathrm{d}x \\ &= \lim_{n\to\infty}\left[\int_{\mathbb{R}^N} \mathbb{1}_{B_r}|v_n|^{2^+-2}v_n\varphi\,\mathrm{d}x - \int_{\mathbb{R}^N}\mathbb{1}_{B_r}\varphi\mathbf{A}_Q v_n\,\mathrm{d}x\right] \\ &= \lim_{n\to\infty} J_Q'(v_n)\mathbb{1}_{B_r}\varphi = 0. \end{aligned}$$

To prove assertion (iv), we use the assumption $\|J_Q'(v_n)\mathbb{1}_{B_r}\|_* \to 0$ as $n\to\infty$ and write

$$\begin{aligned} o(1) = J_Q'(v_n)\mathbb{1}_{B_r}v_n &= \|\mathbb{1}_{B_r}v_n\|_{2^+}^{2^+} - \int_{\mathbb{R}^N}\mathbb{1}_{B_r}v_n\mathbf{A}_Q v_n\,\mathrm{d}x \\ &= \|\mathbb{1}_{B_r}v_n\|_{2^+}^{2^+} - \int_{\mathbb{R}^N}\mathbb{1}_{B_r}(v_n-v)\mathbf{A}_Q(v_n-v)\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N}\mathbb{1}_{B_r}v_n\mathbf{A}_Q v\,\mathrm{d}x - \int_{\mathbb{R}^N}\mathbb{1}_{B_r}v\mathbf{A}_Q(v_n-v)\,\mathrm{d}x. \end{aligned} \tag{3.8}$$

The last expression vanishes as $n\to\infty$, since $v_n \rightharpoonup v$ in $L^{2^+}(\mathbb{R}^N)$ and since $\mathbf{A}_Q$: $L^{2^+}(\mathbb{R}^N) \to L^{2^*}(\mathbb{R}^N)$ is continuous. Furthermore, using (iii), the weak convergence $v_n \rightharpoonup v$ and the Brézis–Lieb lemma [**22**, lemma 1.32], which applies due to (ii), we obtain

$$\begin{aligned} &\|\mathbb{1}_{B_r}v_n\|_{2^+}^{2^+} - \int_{\mathbb{R}^N}\mathbb{1}_{B_r}v_n\mathbf{A}_Q v\,\mathrm{d}x \\ &= \|\mathbb{1}_{B_r}v_n\|_{2^+}^{2^+} - \left(\|\mathbb{1}_{B_r}v\|_{2^+}^{2^+} - \int_{\mathbb{R}^N}\mathbb{1}_{B_r}v\mathbf{A}_Q v\,\mathrm{d}x\right) - \int_{\mathbb{R}^N}\mathbb{1}_{B_r}v_n\mathbf{A}_Q v\,\mathrm{d}x \\ &= \|\mathbb{1}_{B_r}(v_n-v)\|_{2^+}^{2^+} + o(1), \quad \text{as } n\to\infty. \end{aligned}$$

Substituting in (3.8), the desired conclusion follows. □

In the above proof, the fact that the operator $\mathbb{1}_{B_r}\mathbf{R}$: $L^{2^+}(\mathbb{R}^N) \to L^t(\mathbb{R}^N)$ is compact for $1 \leqslant t < 2^*$ was essential. For $t = 2^*$, the compactness does not hold anymore and therefore the assertion (ii) is false in this case. However, in view of the proposition 2.4, this is only caused by the noncompactness of the operator $\mathbb{1}_{B_r}\mathbf{R}_0$: $L^{2^+}(\mathbb{R}^N) \to L^{2^*}(\mathbb{R}^N)$ associated with the fundamental solution of Laplace's equation. In the next result, we prove that local strong convergence can be restored, provided the mountain pass level L_Q lies below the threshold value given by the least-energy level L_Q^* of the functional

$$J_Q^*(v) := \frac{1}{2^+}\int_{\mathbb{R}^N}|v|^{2^+}\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} v\mathbf{G}_q v\,\mathrm{d}x,$$

where, using the notation (3.3), $\mathbf{G}_q v = q^{1/2^*}\mathbf{R}_0(q^{1/2^*}v)$ for $q = \|Q\|_\infty$. As mentioned in the introduction, this functional arises from the limit of suitable rescaling

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of J_Q. The least-energy level L_Q^* can be characterized by a formula similar to (3.5), namely

$$L_Q^* = \inf\left\{\frac{1}{N}\left(\frac{\|v\|_{2^+}^2}{\|Q\|_\infty^{2/2^*}\int_{\mathbb{R}^N} v\mathbf{R}_0 v\,\mathrm{d}x}\right)^{N/2} : v \in L^{2^+}(\mathbb{R}^N)\backslash\{0\}\right\}. \tag{3.9}$$

We note incidentally that it can be expressed in terms of the optimal constant S for the Sobolev inequality in $\mathbb{R}^N$,

$$\|\nabla u\|_2^2 \geqslant S\|u\|_{2^*}^2, \quad \text{for all } u \in L^{2^*}(\mathbb{R}^N) \quad \text{with } \nabla u \in L^2(\mathbb{R}^N). \tag{3.10}$$

Indeed, it is known (see [**16**] and also [**5**]) that the Sobolev inequality is dual to the Hardy–Littlewood–Sobolev inequality

$$\int_{\mathbb{R}^N} v\mathbf{R}_0 v\,\mathrm{d}x \leqslant S^{-1}\|v\|_{2^+}^2 \tag{3.11}$$

and that the optimal constants are inverse to each other. Hence, we obtain

$$L_Q^* = \frac{S^{N/2}}{N\|Q\|_\infty^{(N-2)/2}}. \tag{3.12}$$

Proposition 3.3. *Let $Q \in L^\infty(\mathbb{R}^N)\backslash\{0\}$ have the form $Q = Q_{\mathrm{per}} + Q_0$, for some $Q_{\mathrm{per}}, Q_0 \geqslant 0$ such that Q_{per} is $\mathbb{Z}^N$-periodic and $Q_0(x) \to 0$ as $|x| \to \infty$.*

If $(v_n)_n \subset L^{2^+}(\mathbb{R}^N)$ is a (PS)$_\beta$–sequence for J_Q such that $\beta = L_Q < L_Q^*$, then there exists $w \in L^{2^+}(\mathbb{R}^N)$, $w \neq 0$, such that $J_Q'(w) = 0$ and $J_Q(w) = L_Q$.

Proof. Since $(v_n)_n$ is a (PS)$_\beta$-sequence for J_Q with $\beta > 0$, it is bounded (see [**11**, lemma 4.2]). Using lemma 3.2, we find

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N} Q^{1/2^*}v_n\mathbf{R}(Q^{1/2^*}v_n)\,\mathrm{d}x &= \left(\frac{1}{2^+}-\frac{1}{2}\right)^{-1}\lim_{n\to\infty}\left[J_Q(v_n) - \frac{1}{2^+}J_Q'(v_n)v_n\right]\\ &= N\beta > 0.\end{aligned}$$

Hence, the nonvanishing theorem 2.5 gives the existence of a sequence $(x_n)_n \subset \mathbb{R}^N$ and of constants $R, \zeta > 0$ such that, up to a subsequence,

$$\begin{aligned}&\int_{B_R(x_n)} |v_n|^{2^+}\,\mathrm{d}x \geqslant \|Q\|_\infty^{-2^+/2^*}\\ &\int_{B_R(x_n)} |Q^{1/2^*}v_n|^{2^+}\,\mathrm{d}x \geqslant \|Q\|_\infty^{-2^+/2^*}\zeta > 0, \quad \text{for all } n.\end{aligned} \tag{3.13}$$

Moreover, we may assume that $(x_n)_n \subset \mathbb{Z}^N$ by making R larger, if necessary. We now distinguish the two cases.

Case 1: $|x_n| \to \infty$, for a subsequence.

Let us restrict to this subsequence, setting $w_n := v_n(\cdot + x_n)$ for all n. We shall use lemma 3.2 for the sequence $(w_n)_n$ and with Q replaced with Q_{per}. We, therefore,

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need to check that $\|J'_{Q_{\rm per}}(w_n)\mathbb{1}_{B_r}\|_* \to 0$ as $n \to \infty$, for all $r > 0$. For this, observe that for $r > 0$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} J'_{Q_{\rm per}}(w_n)\mathbb{1}_{B_r}\varphi &= J'_Q(v_n)\mathbb{1}_{B_r(x_n)}\varphi(\cdot - x_n) \\ &\quad + \int_{\mathbb{R}^N} \mathbb{1}_{B_r}\varphi(\mathbf{A}_{Q(\cdot+x_n)} - \mathbf{A}_{Q_{\rm per}(\cdot+x_n)})w_n \,\mathrm{d}x, \end{aligned} \tag{3.14}$$

using the fact that $Q_{\rm per}$ is invariant under $\mathbb{Z}^N$-translations. Since $(v_n)_n$ is a Palais–Smale sequence, the first term in the right-hand side goes to zero uniformly for $\|\varphi\|_{2^+} \leqslant 1$. The second term can be estimated as follows.

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} \mathbb{1}_{B_r}\varphi(\mathbf{A}_{Q(\cdot+x_n)} - \mathbf{A}_{Q_{\rm per}(\cdot+x_n)})w_n \,\mathrm{d}x \right| \\ &\quad = \left| \int_{\mathbb{R}^N} \mathbb{1}_{B_r}\varphi(Q^{1/2^*}(\cdot + x_n) - Q_{\rm per}^{1/2^*}(\cdot + x_n)) \right. \\ &\qquad\qquad \left. \mathbf{R}[(Q^{1/2^*}(\cdot + x_n) + Q_{\rm per}^{1/2^*}(\cdot + x_n))w_n] \,\mathrm{d}x \right| \\ &\quad \leqslant 2C_0\|Q\|_\infty^{1/2^*}\|\varphi\|_{2^+}\|w_n\|_{2^+}\|\mathbb{1}_{B_r}(Q^{1/2^*}(\cdot + x_n) - Q_{\rm per}^{1/2^*}(\cdot + x_n))\|_\infty, \end{aligned} \tag{3.15}$$

where $C_0 > 0$ is given by (2.13). Moreover, by assumption, $Q(x) - Q_{\rm per}(x) = Q_0(x) \to 0$ as $|x| \to \infty$. Thus, since $|x_n| \to \infty$ as $n \to \infty$ and $Q, Q_{\rm per} \geqslant 0$ are bounded functions, it follows that

$$\|\mathbb{1}_{B_r}(Q^{1/2^*}(\cdot + x_n) - Q_{\rm per}^{1/2^*}(\cdot + x_n))\|_\infty \to 0, \quad \text{as } n \to \infty, \quad \text{for all } r > 0. \tag{3.16}$$

Combining (3.14), (3.15) and (3.16), we find that $\|J'_{Q_{\rm per}}(w_n)\mathbb{1}_{B_r}\|_* \to 0$, as $n \to \infty$. Therefore, the conditions of lemma 3.2 are fulfilled and, going to a subsequence, we obtain $w_n \rightharpoonup w$ in $L^{2^+}(\mathbb{R}^N)$ and $w_n \to w$ a.e. on $\mathbb{R}^N$, for some $w \in L^{2^+}(\mathbb{R}^N)$ which satisfies $J'_{Q_{\rm per}}(w) = 0$.

Furthermore, from (3.7), we infer that, as $n \to \infty$,

$$\begin{aligned} \|\mathbb{1}_{B_r}(w_n - w)\|_{2^+}^{2^+} &= \int_{\mathbb{R}^N} \mathbb{1}_{B_r}(w_n - w)\mathbf{A}_{Q_{\rm per}}(w_n - w) \,\mathrm{d}x + o(1) \\ &= \int_{\mathbb{R}^N} \mathbb{1}_{B_r}(w_n - w)\mathbf{A}_{Q_{\rm per}}[\mathbb{1}_{B_r}(w_n - w)] \,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} \mathbb{1}_{B_r}(w_n - w)\mathbf{A}_{Q_{\rm per}}[\mathbb{1}_{M_r}(w_n - w)] \,\mathrm{d}x + o(1), \end{aligned} \tag{3.17}$$

for all $r > 0$, where $M_r := \mathbb{R}^N \backslash B_r$.

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For the first integral, we obtain with the proposition 2.4 and the characterization (3.9) of L_Q^*,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \mathbb{1}_{B_r}(w_n - w)\mathbf{A}_{Q_{\mathrm{per}}}[\mathbb{1}_{B_r}(w_n - w)]\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} \mathbb{1}_{B_r}(w_n - w)\mathbf{G}_{Q_{\mathrm{per}}}[\mathbb{1}_{B_r}(w_n - w)]\,\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^N} \mathbb{1}_{B_r}(w_n - w)(\mathbf{A}_{Q_{\mathrm{per}}} - \mathbf{G}_{Q_{\mathrm{per}}})[\mathbb{1}_{B_r}(w_n - w)]\,\mathrm{d}x \\
&\quad \leqslant \|Q\|_\infty^{2/2^*} \int_{\mathbb{R}^N} \mathbb{1}_{B_r}|w_n - w|\mathbf{R}_0[\mathbb{1}_{B_r}|w_n - w|]\,\mathrm{d}x + o(1) \\
&\quad \leqslant (NL_Q^*)^{-2/N}\|\mathbb{1}_{B_r}(w_n - w)\|_{2^+}^2 + o(1), \quad \text{as } n \to \infty,
\end{aligned}$$

where $\mathbf{G}_{Q_{\mathrm{per}}}$ is given by (3.3) with Q_{per} in place of Q. In addition, the Brézis–Lieb lemma implies

$$\begin{aligned}
\|\mathbb{1}_{B_r}(w_n - w)\|_{2^+}^{2^+} &\leqslant \|w_n - w\|_{2^+}^{2^+} = \|w_n\|_{2^+}^{2^+} - \|w\|_{2^+}^{2^+} + o(1) \\
&\leqslant \|v_n\|_{2^+}^{2^+} + o(1) = \left(\frac{1}{2^+} - \frac{1}{2}\right)^{-1}\left(J_Q(v_n) - \frac{1}{2}J_Q'(v_n)v_n\right) + o(1) \\
&= N\beta + o(1), \quad \text{as } n \to \infty,
\end{aligned}$$

since $(v_n)_n$ is a $(PS)_\beta$-sequence by assumption. Combining these two estimates, we obtain

$$\begin{aligned}
&\left[1 - \left(\frac{\beta}{L_Q^*}\right)^{2/N}\right]\|\mathbb{1}_{B_r}(w_n - w)\|_{2^+}^{2^+} \\
&\quad \leqslant \|\mathbb{1}_{B_r}(w_n - w)\|_{2^+}^{2^+} - (NL_Q^*)^{-2/N}\|\mathbb{1}_{B_r}(w_n - w)\|_{2^+}^2 + o(1) \\
&\quad \leqslant \|\mathbb{1}_{B_r}(w_n - w)\|_{2^+}^{2^+} - \int_{\mathbb{R}^N} \mathbb{1}_{B_r}(w_n - w)\mathbf{A}_{Q_{\mathrm{per}}}[\mathbb{1}_{B_r}(w_n - w)]\,\mathrm{d}x + o(1),
\end{aligned}$$

as $n \to \infty$, and (3.17) gives for all $r > 0$,

$$\begin{aligned}
&\left[1 - \left(\frac{\beta}{L_Q^*}\right)^{2/N}\right]\|\mathbb{1}_{B_r}(w_n - w)\|_{2^+}^{2^+} \\
&\quad \leqslant \int_{\mathbb{R}^N} \mathbb{1}_{B_r}(w_n - w)\mathbf{A}_{Q_{\mathrm{per}}}[\mathbb{1}_{M_r}(w_n - w)]\,\mathrm{d}x + o(1),
\end{aligned} \tag{3.18}$$

as $n \to \infty$, where the first factor on the left-hand side is strictly positive since we are assuming $\beta < L_Q^*$.

Let us now suppose by contradiction that $(\mathbb{1}_{B_r}w_n)_n$ does not converge strongly to $\mathbb{1}_{B_r}w$ in $L^{2^+}(\mathbb{R}^N)$, for some fixed $r > 0$. Then, passing to a subsequence, there

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exists $\varepsilon > 0$ such that

$$\left[1 - \left(\frac{\beta}{L_Q^*}\right)^{2/N}\right] \|\mathbb{1}_{B_r}(w_n - w)\|_{2^+}^{2^+} > \varepsilon \quad \text{for all } n.$$

Lemma 2.7 applied to the sequence $(Q_{\mathrm{per}}^{1/2^*}(w_n - w))_n$ gives $\rho_\varepsilon > 0$ such that for all $\rho \geqslant \max\{\rho_\varepsilon, r\}$, we have

$$\begin{aligned}
\varepsilon &> \liminf_{n\to\infty} \left| \int_{\mathbb{R}^N} \mathbb{1}_{B_\rho}(w_n - w)\mathbf{A}_{Q_{\mathrm{per}}}[\mathbb{1}_{M_\rho}(w_n - w)]\,\mathrm{d}x \right| \\
&\geqslant \liminf_{n\to\infty} \left[1 - \left(\frac{\beta}{L_Q^*}\right)^{2/N}\right] \|\mathbb{1}_{B_\rho}(w_n - w)\|_{2^+}^{2^+}, \quad \text{using (3.18)} \\
&\geqslant \liminf_{n\to\infty} \left[1 - \left(\frac{\beta}{L_Q^*}\right)^{2/N}\right] \|\mathbb{1}_{B_r}(w_n - w)\|_{2^+}^{2^+} \\
&\geqslant \varepsilon.
\end{aligned}$$

This contradiction proves the strong convergence $\mathbb{1}_{B_r} w_n \to \mathbb{1}_{B_r} w$ in $L^{2^+}(\mathbb{R}^N)$ as $n \to \infty$, for all $r > 0$. Using (3.13), we immediately deduce that $w \neq 0$. Moreover,

$$\begin{aligned}
J_{Q_{\mathrm{per}}}(w) &= J_{Q_{\mathrm{per}}}(w) - \frac{1}{2}J'_{Q_{\mathrm{per}}}(w)w = \frac{1}{N}\|w\|_{2^+}^{2^+} \\
&\leqslant \liminf_{n\to\infty} \frac{1}{N}\|w_n\|_{2^+}^{2^+} = \liminf_{n\to\infty} \frac{1}{N}\|v_n\|_{2^+}^{2^+} \\
&= \liminf_{n\to\infty} [J_Q(v_n) - \frac{1}{2}J'_Q(v_n)v_n] = \beta = L_Q.
\end{aligned} \tag{3.19}$$

Now, consider the function

$$\widetilde{w} := \left(\frac{Q_{\mathrm{per}}}{Q}\right)^{1/2^*} w.$$

Since $Q = Q_{\mathrm{per}} + Q_0$ with $Q_0 \geqslant 0$, we find that $|\widetilde{w}| \leqslant |w|$. In particular, we have $\widetilde{w} \in L^{2^+}(\mathbb{R}^N)$ and by definition,

$$\int_{\mathbb{R}^N} \widetilde{w}\mathbf{A}_Q\widetilde{w}\,\mathrm{d}x = \int_{\mathbb{R}^N} w\mathbf{A}_{Q_{\mathrm{per}}}w\,\mathrm{d}x = \|w\|_{2^+}^{2^+} > 0,$$

since w is a nontrivial critical point of $J_{Q_{\mathrm{per}}}$. Hence, $\widetilde{w} \neq 0$ and, setting

$$\tau := \left(\frac{\int_{\mathbb{R}^N} |\widetilde{w}|^{2^+}\,\mathrm{d}x}{\int_{\mathbb{R}^N} \widetilde{w}\mathbf{A}_Q\widetilde{w}\,\mathrm{d}x}\right)^{1/(2-2^+)},$$

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we find that $0 < \tau \leqslant 1$ and $J_Q'(\tau\widetilde{w})\widetilde{w} = 0$. In addition, since $|\widetilde{w}| \leqslant |w|$, we have

$$J_Q(\tau\widetilde{w}) = \frac{1}{N}\left(\frac{\|\widetilde{w}\|_{2^+}^2}{\displaystyle\int_{\mathbb{R}^N} \widetilde{w}\mathbf{A}_Q\widetilde{w}\,\mathrm{d}x}\right)^{N/2} \leqslant \frac{1}{N}\left(\frac{\|w\|_{2^+}^2}{\displaystyle\int_{\mathbb{R}^N} w\mathbf{A}_{Q_{\mathrm{per}}}w\,\mathrm{d}x}\right)^{N/2} = J_{Q_{\mathrm{per}}}(w).$$

Therefore, (3.5) and (3.19) yield $J_Q(\tau\widetilde{w}) = L_Q = J_{Q_{\mathrm{per}}}(w)$ and we deduce that $\tau = 1$. We now claim that $\tau\widetilde{w} = \widetilde{w}$ is a critical point for J_Q. To prove this, let $\varphi \in L^{2^+}(\mathbb{R}^N)$ be arbitrarily given and choose $\delta > 0$ such that

$$\int_{\mathbb{R}^N} (\widetilde{w} + s\varphi)\mathbf{A}_Q(\widetilde{w} + s\varphi)\,\mathrm{d}x > 0, \quad \text{for all } s \in [-\delta, \delta],$$

and, for $s \in [-\delta, \delta]$, set

$$t_s := \left(\frac{\displaystyle\int_{\mathbb{R}^N} |\widetilde{w} + s\varphi|^{2^+}\,\mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} (\widetilde{w} + s\varphi)\mathbf{A}_Q(\widetilde{w} + s\varphi)\,\mathrm{d}x}\right)^{1/(2-2^+)}.$$

Then we can write, using (3.5), the property $J_Q(\widetilde{w}) = J_Q(\tau\widetilde{w}) = \max_{t>0} J_Q(t\widetilde{w})$ and the mean-value theorem,

$$\begin{aligned}0 \leqslant J_Q(t_s(\widetilde{w} + s\varphi)) - J_Q(\widetilde{w}) &\leqslant J_Q(t_s(\widetilde{w} + s\varphi)) - J_Q(t_s\widetilde{w})\\ &= J_Q'(t_s(\widetilde{w} + s\sigma\varphi))t_s s\varphi,\end{aligned}$$

for some $\sigma \in [-1, 1]$. Dividing by $s \neq 0$ and letting $s \to 0^{\pm}$, we obtain $J_Q'(\widetilde{w})\varphi = 0$, since $t_s \to 1$, as $s \to 0$. The proposition is proven in this case.

Case 2: $(x_n)_n$ is bounded. In this case, making R again larger if necessary, we can assume that (3.13) holds with $x_n = 0$ for all n.

Since $(v_n)_n$ is a $(\mathrm{PS})_\beta$-sequence, the assumptions of lemma 3.2 are satisfied. Thus, going to a subsequence, we obtain $v_n \rightharpoonup v$ in $L^{2^+}(\mathbb{R}^N)$ and $v_n \to v$ a.e. on $\mathbb{R}^N$, for some $v \in L^{2^+}(\mathbb{R}^N)$ which satisfies $J_Q'(v) = 0$. Replacing in (3.17) and the subsequent computations Q_{per} with Q, w_n with v_n and w with v, we obtain for all $r > 0$,

$$\left[1 - \left(\frac{\beta}{L_Q^*}\right)^{2/N}\right] \|\mathbb{1}_{B_r}(v_n - v)\|_{2^+}^{2^+} \leqslant \int_{\mathbb{R}^N} \mathbb{1}_{B_r}(v_n - v)\mathbf{A}_Q[\mathbb{1}_{M_r}(v_n - v)]\,\mathrm{d}x + o(1),$$

as $n \to \infty$, and the same contradiction argument as in Case 1 yields the strong convergence $\mathbb{1}_{B_r}v_n \to \mathbb{1}_{B_r}v$ as $n \to \infty$ in $L^{2^+}(\mathbb{R}^N)$, for all $r > 0$. In particular, v is a nontrivial critical point of J_Q and the characterization (3.5) of the mountain-pass

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level L_Q yields

$$L_Q \leqslant J_Q(v) = J_Q(v) - \frac{1}{2} J_Q'(v)v = \frac{1}{N}\|v\|_{2^+}^{2^+}$$
$$\leqslant \liminf_{n\to\infty} \frac{1}{N}\|v_n\|_{2^+}^{2^+} = \liminf_{n\to\infty}\big[J_Q(v_n) - \frac{1}{2}J_Q'(v_n)v_n\big] = \beta = L_Q.$$

Hence, $J_Q(v) = L_Q$ and this concludes the proof. □

REMARK 3.4.

(i) When $Q \equiv Q_0$, Case 1 does not occur in the proof above. Moreover, the operator $\mathbf{A}_Q - \mathbf{G}_Q$ is itself compact so that all arguments in the proof hold globally on $\mathbb{R}^N$. As a consequence, J_Q satisfies the Palais–Smale condition at every level $0 < \beta < L_Q^*$.

(ii) In the case where $Q \equiv Q_{\text{per}}$, we have $\widetilde{w} = w$ in the above proof and the proposition is valid for any $0 < \beta < L_Q^*$, except for the last assertion which should be replaced with $L_Q \leqslant J_Q(w) \leqslant \beta$.

3.3. Estimating the dual mountain-pass level

Our next result shows that in dimension $N \geqslant 4$, the mountain-pass level L_Q lies below the critical threshold L_Q^* if the coefficient Q satisfies some flatness condition (see condition (Q) below). This additional condition seems to go back to the works of Escobar [7] and Egnell [6] (see also [4, remark 1.2]).

To estimate L_Q, we shall use the functions

$$v_\varepsilon(x) := (N(N-2)\varepsilon)^{(N+2)/4}\left(\frac{1}{\varepsilon + |x|^2}\right)^{(N+2)/2}, \quad \varepsilon > 0. \tag{3.20}$$

It was shown by Lieb [16] (see also [17, theorem 4.3]) that, up to translation and multiplication by a constant, v_ε, $\varepsilon > 0$ are the only optimizers of the Hardy–Littlewood–Sobolev inequality (3.11), that is,

$$\int_{\mathbb{R}^N} v_\varepsilon \mathbf{R}_0 v_\varepsilon \,\mathrm{d}x = S^{-1}\|v_\varepsilon\|_{2^+}^2. \tag{3.21}$$

In addition, we have $v_\varepsilon = u_\varepsilon^{2^*-1}$, where

$$u_\varepsilon(x) := (N(N-2)\varepsilon)^{(N-2)/4}\left(\frac{1}{\varepsilon + |x|^2}\right)^{(N-2)/2}, \quad \varepsilon > 0,$$

are the Aubin–Talenti instantons (see, e.g., [3]) that optimize the Sobolev inequality (3.10) and for which the following holds:

$$\|\nabla u_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 = \|u_\varepsilon\|_{2^*}^{2^*} = S^{N/2}, \quad \text{for all } \varepsilon > 0.$$

In particular, we deduce that

$$\|v_\varepsilon\|_{2^+} = \|u_\varepsilon\|_{2^*}^{2^*-1} = S^{(N+2)/4}, \quad \text{for all } \varepsilon > 0. \tag{3.22}$$

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Proposition 3.5. *Let $N \geqslant 4$ and consider $Q \in L^\infty(\mathbb{R}^N)\backslash\{0\}$ nonnegative. Assume further, that there exists $x_0 \in \mathbb{R}^N$ with $Q(x_0) = \|Q\|_\infty$ and that*

$$Q(x_0) - Q(x) = o(|x - x_0|^2), \quad \text{as } |x - x_0| \to 0. \tag{Q}$$

Then we have

$$L_Q < L_Q^*.$$

Proof. Let us assume – without loss of generality – that $x_0 = 0$ and set $q := \|Q\|_\infty$.

We consider for $\varepsilon > 0$ the dual instanton v_ε given by (3.20) and put $v := v_1$. Fix a cut-off function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ with $0 \leqslant \varphi \leqslant 1$ on $\mathbb{R}^N$, $\varphi \equiv 1$ on $B_1(0)$ and $\varphi \equiv 0$ outside of $B_2(0)$. Setting for $\varepsilon > 0$, $\alpha > 0$,

$$v_{\varepsilon,\alpha} := \varphi_\alpha v_\varepsilon, \quad \text{where } \varphi_\alpha(x) := \varphi\left(\frac{x}{\alpha}\right),$$

we shall estimate the ratio

$$\frac{\|v_{\varepsilon,\alpha}\|_{2^+}^2}{\displaystyle\int_{\mathbb{R}^N} v_{\varepsilon,\alpha} \mathbf{A}_Q v_{\varepsilon,\alpha}\, \mathrm{d}x},$$

and we first look at the quadratic form $\int_{\mathbb{R}^N} v_{\varepsilon,\alpha} \mathbf{A}_Q v_{\varepsilon,\alpha}\, \mathrm{d}x$. Consider the decomposition

$$\begin{aligned}\int_{\mathbb{R}^N} v_{\varepsilon,\alpha} \mathbf{A}_Q v_{\varepsilon,\alpha}\, \mathrm{d}x &= \int_{\mathbb{R}^N} v_\varepsilon \mathbf{G}_q v_\varepsilon\, \mathrm{d}x - \int_{\mathbb{R}^N} (1+\varphi_\alpha) v_\varepsilon \mathbf{G}_q((1-\varphi_\alpha)v_\varepsilon)\, \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} v_{\varepsilon,\alpha}(\mathbf{A}_q - \mathbf{G}_q) v_{\varepsilon,\alpha}\, \mathrm{d}x - \int_{\mathbb{R}^N} v_{\varepsilon,\alpha}(\mathbf{A}_q - \mathbf{A}_Q) v_{\varepsilon,\alpha}\, \mathrm{d}x,\end{aligned} \tag{3.23}$$

with $\mathbf{G}_q$ as in (3.3) where Q is replaced by the constant function q, that is, $\mathbf{G}_q = q^{1/2^*}\mathbf{R}_0 q^{1/2^*}$. Starting with the first integral in the right-hand side of (3.23), we remark that (3.21) and (3.22) together with the definition of $\mathbf{G}_q$ give

$$\int_{\mathbb{R}^N} v_\varepsilon \mathbf{G}_q v_\varepsilon\, \mathrm{d}x = q^{2/2^*} \int_{\mathbb{R}^N} v_\varepsilon \mathbf{R}_0 v_\varepsilon\, \mathrm{d}x = q^{2/2^*} S^{N/2}. \tag{3.24}$$

Using the Hardy–Littlewood–Sobolev inequality, the second integral in (3.23) can be estimated as follows

$$\int_{\mathbb{R}^N} (1+\varphi_\alpha) v_\varepsilon \mathbf{G}_q((1-\varphi_\alpha)v_\varepsilon)\, \mathrm{d}x \leqslant q^{2/2^*} S^{-1} \|(1+\varphi_\alpha)v_\varepsilon\|_{2^+} \|(1-\varphi_\alpha)v_\varepsilon\|_{2^+}.$$

Moreover, since $1 - \varphi_\alpha = 0$ in $B_\alpha(0)$, we obtain

$$\begin{aligned}\|(1-\varphi_\alpha)v_\varepsilon\|_{2^+}^{2^+} &\leqslant N\omega_N (N(N-2))^{N/2} \int_{\alpha/\sqrt{\varepsilon}}^\infty r^{-(N+1)}\, \mathrm{d}r \\ &= \omega_N (N(N-2))^{N/2} \alpha^{-N} \varepsilon^{N/2}.\end{aligned}$$

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Thus, from (3.22) and since $0 \leqslant \varphi_\alpha \leqslant 1$, it follows that

$$\int_{\mathbb{R}^N} (1+\varphi_\alpha)v_\varepsilon \mathbf{G}_q((1-\varphi_\alpha)v_\varepsilon)\,\mathrm{d}x \\ \leqslant 2q^{2/2^*}(\omega_N)^{1/2^+} S^{(N-2)/4}(N(N-2))^{(N+2)/4}\alpha^{-(N+2)/2}\varepsilon^{(N+2)/4}. \tag{3.25}$$

The third integral in (3.23) can be rewritten as

$$\begin{aligned}\int_{\mathbb{R}^N} v_{\varepsilon,\alpha}(\mathbf{A}_q-\mathbf{G}_q)v_{\varepsilon,\alpha}\,\mathrm{d}x &= q^{2/2^*}\int_{\mathbb{R}^N} v_{\varepsilon,\alpha}[(\Psi-\Lambda)*v_{\varepsilon,\alpha}]\,\mathrm{d}x \\ &= q^{2/2^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} v_\varepsilon(x)v_\varepsilon(y)\varphi_\alpha(x)\varphi_\alpha(y)[\Psi(x-y)-\Lambda(x-y)]\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Since $\varphi_\alpha(x)=0$ for all $|x|\geqslant 2\alpha$, it is enough to estimate the difference $\Psi-\Lambda$ inside $B_{4\alpha}(0)$. Fixing $\alpha_0\in(0,y_0/4)$ and observing that $y_\nu<y_{\nu+1}$ for $\nu\geqslant 0$, we obtain from lemma 2.1 a constant $\kappa_0>0$ such that

$$\Psi(z)-\Lambda(z)\geqslant\begin{cases}\kappa_0|z|^{4-N}, & \text{if } N\geqslant 5,\\ \kappa_0|\ln|z||, & \text{if } N=4,\end{cases} \qquad \text{for all } 0<|z|\leqslant 4\alpha_0.$$

As a consequence, and since $\varphi_\alpha\equiv 1$ in B_α, we can write for all $0<\alpha\leqslant\alpha_0$ and $0<\varepsilon\leqslant\alpha^2$:

$$\begin{aligned}\int_{\mathbb{R}^N} v_{\varepsilon,\alpha}(\mathbf{A}_q-\mathbf{G}_q)v_{\varepsilon,\alpha}\,\mathrm{d}x &\geqslant \kappa_0 q^{2/2^*}\int_{B_\alpha}\int_{B_\alpha} v_\varepsilon(x)v_\varepsilon(y)|x-y|^{4-N}\,\mathrm{d}y\,\mathrm{d}x \\ &= \varepsilon\kappa_0 q^{2/2^*}\int_{B_{\alpha/\sqrt{\varepsilon}}}\int_{B_{\alpha/\sqrt{\varepsilon}}} v(x)v(y)|x-y|^{4-N}\,\mathrm{d}y\,\mathrm{d}x \\ &\geqslant \varepsilon 2^{4-N}\kappa_0 q^{2/2^*}\left(\int_{B_1} v(x)\,\mathrm{d}x\right)^2,\end{aligned}$$

in the case where $N\geqslant 5$. In a similar way, we obtain for $N=4$,

$$\int_{\mathbb{R}^N} v_{\varepsilon,\alpha}(\mathbf{A}_q-\mathbf{G}_q)v_{\varepsilon,\alpha}\,\mathrm{d}x \geqslant \varepsilon|\ln(2\sqrt{\varepsilon})|\kappa_0 q^{2/2^*}\left(\int_{B_1} v(x)\,\mathrm{d}x\right)^2. \tag{3.26}$$

Setting $\gamma := 2^{4-N}\kappa_0 q^{2/2^*}\left(\int_{B_1} v(x)\,\mathrm{d}x\right)^2$, the above computations yield

$$\int_{\mathbb{R}^N} v_{\varepsilon,\alpha}(\mathbf{A}_q-\mathbf{G}_q)v_{\varepsilon,\alpha}\,\mathrm{d}x \geqslant \gamma\varepsilon, \quad \text{for all } 0<\alpha\leqslant\alpha_0 \quad \text{and} \\ 0<\varepsilon\leqslant\min\left\{\alpha^2, \frac{\mathrm{e}^{-2}}{4}\right\}. \tag{3.27}$$

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To estimate the remaining integral in (3.23), we first note that since $0 \leqslant Q(x)/q \leqslant 1$, we have

$$0 \leqslant q^{1/2^*} - Q^{1/2^*}(x) \leqslant q^{1/2^*-1}(q - Q(x)), \quad \text{for all } x.$$

Thus, the assumption (Q) gives for each $\delta > 0$ a constant $\alpha_\delta > 0$ such that

$$0 \leqslant q^{1/2^*} - Q^{1/2^*}(x) \leqslant \frac{\delta}{2}|x|^2, \quad \text{for all } |x| \leqslant 2\alpha_\delta.$$

Since $\varphi_{\varepsilon,\alpha} \equiv 0$ outside $B_{2\alpha}$, we find for $0 < \alpha \leqslant \alpha_\delta$ and $0 < \varepsilon \leqslant \alpha^2$,

$$\begin{aligned}\int_{\mathbb{R}^N} v_{\varepsilon,\alpha}(\mathbf{A}_q - \mathbf{A}_Q)v_{\varepsilon,\alpha}\,\mathrm{d}x &= \int_{\mathbb{R}^N} (q^{1/2^*} - Q^{1/2^*})v_{\varepsilon,\alpha}\mathbf{R}[(q^{1/2^*} + Q^{1/2^*})v_{\varepsilon,\alpha}]\,\mathrm{d}x \\ &\leqslant 2q^{1/2^*}C_0\|(q^{1/2^*} - Q^{1/2^*})v_{\varepsilon,\alpha}\|_{2^+}\|v_{\varepsilon,\alpha}\|_{2^+} \\ &\leqslant \delta\varepsilon q^{1/2^*}S^{(N+2)/4}C_0\left(\int_{\mathbb{R}^N} \big||x|^2 v(x)\big|^{2^+}\,\mathrm{d}x\right)^{1/2^+}.\end{aligned}$$

Choosing $\delta > 0$ such that $\delta q^{1/2^*}S^{(N+2)/4}C_0\left(\int_{\mathbb{R}^N} \big||x|^2 v(x)\big|^{2^+}\,\mathrm{d}x\right)^{1/2^+} \leqslant \gamma/2$ and setting $\alpha := \min\{\alpha_0, \alpha_\delta\}$, we obtain the estimate

$$\int_{\mathbb{R}^N} v_{\varepsilon,\alpha}(\mathbf{A}_q - \mathbf{A}_Q)v_{\varepsilon,\alpha}\,\mathrm{d}x \leqslant \frac{\gamma}{2}\varepsilon, \quad \text{for all } 0 < \varepsilon \leqslant \alpha^2. \tag{3.28}$$

With this choice of α, putting the estimates (3.24), (3.25), (3.27) and (3.28) together, the decomposition (3.23) yields

$$\begin{aligned}\int_{\mathbb{R}^N} v_{\varepsilon,\alpha}\mathbf{A}_Q v_{\varepsilon,\alpha}\,\mathrm{d}x &\geqslant q^{2/2^*}S^{N/2} + \frac{\gamma}{2}\varepsilon - \zeta\varepsilon^{(N+2)/4} \geqslant q^{2/2^*}S^{N/2} + \frac{\gamma}{4}\varepsilon \\ &> q^{2/2^*}S^{N/2}, \quad \text{for } 0 < \varepsilon \leqslant \varepsilon_0 \\ &:= \min\left\{\alpha^2, \left(\frac{\gamma}{4\zeta}\right)^{4/(N-2)}, \frac{\mathrm{e}^{-2}}{4}\right\},\end{aligned} \tag{3.29}$$

where $\zeta = 2q^{2/2^*}(\omega_N)^{1/2^*}S^{(N-2)/4}(N(N-2))^{(N+2)/4}\alpha^{-(N+2)/2}$. Hence, from (3.5), (3.12), (3.22) and (3.29), we infer that for $\alpha = \min\{\alpha_0, \alpha_\delta\}$ and $0 < \varepsilon \leqslant \varepsilon_0$,

$$\begin{aligned}L_Q &\leqslant \frac{1}{N}\left(\frac{\|v_{\varepsilon,\alpha}\|_{2^+}^2}{\displaystyle\int_{\mathbb{R}^N} v_{\varepsilon,\alpha}\mathbf{A}_Q v_{\varepsilon,\alpha}\,\mathrm{d}x}\right)^{N/2} < \frac{1}{N}\left(\frac{S^{(N+2)/2}}{q^{2/2^*}S^{N/2}}\right)^{N/2} \\ &= \frac{S^{N/2}}{Nq^{(N-2)/2}} = L_Q^*.\end{aligned}$$

This proves the desired result. □

Remark 3.6. In the case $N = 4$, using the estimate (3.26) instead of (3.27), we see that the condition (Q) can be weakened to

$$Q(x_0) - Q(x) = O(|x - x_0|^2), \quad \text{as } |x - x_0| \to 0.$$

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3.4. Existence and nonexistence of dual ground states

We are now in a position to give the proof of our main existence result for the critical nonlinear Helmholtz equation.

THEOREM 3.7. *Let $N \geqslant 4$ and consider $Q \in L^\infty(\mathbb{R}^N)\backslash\{0\}$. Assume in addition that*

(Q1) *$Q = Q_{\text{per}} + Q_0$, where $Q_{\text{per}}, Q_0 \geqslant 0$ are such that Q_{per} is $\mathbb{Z}^N$-periodic and $Q_0(x) \to 0$ as $|x| \to \infty$;*

(Q2) *there exists $x_0 \in \mathbb{R}^N$ with $Q(x_0) = \|Q\|_\infty$ and, as $|x - x_0| \to 0$,*

$$Q(x_0) - Q(x) = \begin{cases} o(|x-x_0|^2), & \text{if } N \geqslant 5, \\ O(|x-x_0|^2), & \text{if } N = 4. \end{cases}$$

Then the problem

$$-\Delta u - u = Q(x)|u|^{2^*-2}u, \quad u \in W^{2,2^*}(\mathbb{R}^N) \tag{3.30}$$

has a dual ground state.

Proof. Using the mountain pass theorem without the Palais–Smale condition (see [**1**] and [**3**, theorem 2.2]), we obtain the existence of a Palais–Smale sequence $(v_n)_n \subset L^{2^+}(\mathbb{R}^N)$ at the mountain pass level L_Q. Therefore, by propositions 3.3, 3.5 and remark 3.6, the functional J_Q possesses a critical point $w \in L^{2^+}(\mathbb{R}^N)$ of J_Q which satisfies $J_Q(w) = L_Q$. Setting $u = \mathbf{R}(Q^{1/2^*}w)$, we find that $u \in L^{2^*}(\mathbb{R}^N)$ is a dual ground state of (3.30), and this concludes the proof. □

In dimension $N = 3$, the situation completely changes. Indeed, the proof of proposition 3.5 fails, since the estimate in lemma 2.1(i) now has the opposite sign. In fact, we have the following nonexistence result.

PROPOSITION 3.8. *Let $Q \in L^\infty(\mathbb{R}^3)\backslash\{0\}$ satisfy $Q(x) \geqslant 0$ for almost every $x \in \mathbb{R}^3$. Then, there is no dual ground state for the problem*

$$-\Delta u - u = Q(x)|u|^{2^*-2}u, \quad u \in W^{2,2^*}(\mathbb{R}^3). \tag{3.31}$$

Proof. We start by proving the inequality $L_Q \leqslant L_Q^*$.

For this, let us consider the family of functions v_ε, $\varepsilon > 0$, given in (3.20). Let $0 < \delta < \|Q\|_\infty$ be arbitrary but fixed, and consider the set $\omega_\delta := \{x \in \mathbb{R}^3 : Q(x) \geqslant \|Q\|_\infty - \delta\}$. Since ω_δ has positive measure, we can choose a point $x_\delta \in \omega_\delta$ such that

$$\lim_{\alpha \to 0^+} \frac{|\omega_\delta \cap B_\alpha(x_\delta)|}{|B_\alpha(x_\delta)|} = 1, \tag{3.32}$$

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where $|\cdot|$ denotes the Lebesgue measure. With $\varphi_\delta := \mathbb{1}_{\omega_\delta \cap B_{1/2}(x_\delta)}$, we find

$$\begin{aligned}
&\int_{\mathbb{R}^3} \varphi_\delta v_\varepsilon(\cdot - x_\delta) \mathbf{A}_Q(\varphi_\delta v_\varepsilon(\cdot - x_\delta))\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \varphi_\delta(x)\varphi_\delta(y) v_\varepsilon(x-x_\delta)v_\varepsilon(y-x_\delta) Q^{1/2^*}(x)Q^{1/2^*}(y)\Psi(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \geqslant (\|Q\|_\infty - \delta)^{2/2^*} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \varphi_\delta(x)\varphi_\delta(y) v_\varepsilon(x-x_\delta)v_\varepsilon(y-x_\delta)\Psi(x-y)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

since $\Psi(x-y) = \cos|x-y|/(4\pi|x-y|) \geqslant 0$ for all $x, y \in B_{1/2}(z)$, $z \in \mathbb{R}^3$. Remarking furthermore that $v_\varepsilon(x) = \varepsilon^{-5/4} v_1(x/\sqrt{\varepsilon})$, a change of variables gives

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \varphi_\delta(x)\varphi_\delta(y) v_\varepsilon(x-x_\delta)v_\varepsilon(y-x_\delta)\Psi(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \varphi_\delta(\sqrt{\varepsilon}x + x_\delta)\varphi_\delta(\sqrt{\varepsilon}y + x_\delta) v_1(x)v_1(y) \frac{\cos(\sqrt{\varepsilon}(x-y))}{4\pi|x-y|}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \longrightarrow \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} v_1(x)v_1(y)\frac{1}{4\pi|x-y|}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^3} v_1 \mathbf{R}_0 v_1\,\mathrm{d}x, \quad \text{as } \varepsilon \to 0^+,
\end{aligned}$$

using (3.32) in the last step. Thus, we obtain

$$\begin{aligned}
\liminf_{\varepsilon\to 0^+} \int_{\mathbb{R}^3} \varphi_\delta v_\varepsilon(\cdot - x_\delta)\mathbf{A}_Q(\varphi_\delta v_\varepsilon(\cdot - x_\delta))\,\mathrm{d}x &\geqslant (\|Q\|_\infty - \delta)^{2/2^*} \int_{\mathbb{R}^3} v_1 \mathbf{R}_0 v_1\,\mathrm{d}x \\
&= (\|Q\|_\infty - \delta)^{2/2^*} S^{-1} \|v_1\|_{2^+}^2.
\end{aligned}$$

In addition, since $\|\varphi_\delta v_\varepsilon(\cdot - x_\delta)\|_{2^+} \to \|v_1\|_{2^+}$, as $\varepsilon \to 0^+$, the characterization (3.5) of L_Q yields

$$\begin{aligned}
L_Q &\leqslant \limsup_{\varepsilon\to 0^+} \frac{1}{N} \left(\frac{\|\varphi_\delta v_\varepsilon(\cdot - x_0)\|_{2^+}^2}{\int_{\mathbb{R}^3} \varphi_\delta v_\varepsilon(\cdot - x_0)\mathbf{A}_Q(\varphi_\delta v_\varepsilon(\cdot - x_0))\,\mathrm{d}x} \right)^{N/2} \\
&\leqslant \frac{S^{N/2}}{N(\|Q\|_\infty - \delta)^{(N+2)/2}}.
\end{aligned}$$

Letting now $\delta \to 0^+$, we infer from (3.12) that $L_Q \leqslant L_Q^*$.

We next assume by contradiction that L_Q is achieved. In this case, there exists $v \in L^{2^+}(\mathbb{R}^3)$ such that $\|v\|_{2^+} = 1$ and

$$\int_{\mathbb{R}^3} v\mathbf{A}_Q v\,\mathrm{d}x = (NL_Q)^{-\frac{2}{N}}.$$

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Since $L_Q \leqslant L_Q^*$ and recalling the value of L_Q^* given in (3.12), we can write

$$\begin{aligned} S^{-1}\|Q\|_\infty^{2/2^*} &= (NL_Q^*)^{-2/N} \leqslant (NL_Q)^{-2/N} = \int_{\mathbb{R}^3} v\mathbf{A}_Q v\,\mathrm{d}x \\ &\leqslant \int_{\mathbb{R}^3} Q^{1/2^*}|v|[|\Psi| * (Q^{1/2^*}|v|)]\,\mathrm{d}x \leqslant \int_{\mathbb{R}^3} Q^{1/2^*}|v|[\Lambda * (Q^{1/2^*}|v|)]\,\mathrm{d}x \\ &= \int_{\mathbb{R}^3} Q^{1/2^*}|v|\mathbf{R}_0(Q^{1/2^*}|v|)\,\mathrm{d}x \leqslant S^{-1}\|Q^{1/2^*}v\|_{2+}^2 \leqslant S^{-1}\|Q\|_\infty^{2/2^*}, \end{aligned}$$

using the fact that $|\Psi(z)| = \cos|z|/(4\pi|z|) \leqslant 1/(4\pi|z|) = \Lambda(z)$ for all $z \in \mathbb{R}^3$, and the Hardy–Littlewood–Sobolev inequality. As a consequence, all inequalities are equalities and we find $L_Q = L_Q^*$ and obtain the following identities.

$$\int_{\mathbb{R}^3} Q^{1/2^*}|v|\mathbf{R}_0\left(Q^{1/2^*}|v|\right)\,\mathrm{d}x = S^{-1}\|Q^{1/2^*}v\|_{2+}^2, \tag{3.33}$$

$$\int_{\mathbb{R}^3} Q^{1/2^*}|v|[|\Psi| * (Q^{1/2^*}|v|)]\,\mathrm{d}x = \int_{\mathbb{R}^3} Q^{1/2^*}|v|[\Lambda * (Q^{1/2^*}|v|)]\,\mathrm{d}x. \tag{3.34}$$

From (3.33) and the uniqueness of the optimizers for the Hardy–Littlewood–Sobolev inequality [**16**, **17**], we deduce that

$$Q^{1/2^*}|v| = \gamma v_\varepsilon(\cdot - x_0), \quad \text{for some } \gamma, \varepsilon > 0 \quad \text{and} \quad x_0 \in \mathbb{R}^3,$$

where v_ε is given by (3.20). In particular, $Q^{1/2^*}|v| > 0$ everywhere in $\mathbb{R}^3$, and we obtain

$$\begin{aligned} &\int_{\mathbb{R}^3} Q^{1/2^*}|v|[|\Psi| * (Q^{1/2^*}|v|)]\,\mathrm{d}x \\ &\quad = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} Q^{1/2^*}(x)|v(x)|Q^{1/2^*}(y)|v(y)|\frac{|\cos|x-y||}{4\pi|x-y|}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad < \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} Q^{1/2^*}(x)|v(x)|Q^{1/2^*}(y)|v(y)|\frac{1}{4\pi|x-y|}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad = \int_{\mathbb{R}^3} Q^{1/2^*}|v|[\Lambda * (Q^{1/2^*}|v|)]\,\mathrm{d}x. \end{aligned}$$

This contradicts (3.34) and therefore, shows that L_Q is not achieved. In particular, J_Q does not have any critical point at level L_Q, and thus no dual ground state solution of (3.31) can exist. □

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