ELLIPSOIDS ARE THE ONLY LOCAL MAXIMIZERS OF THE VOLUME PRODUCT

MATHIEU MEYER AND SHLOMO REISNER

Abstract.   Using previous results about shadow systems and Steiner symmetrization, we prove that the local maximizers of the volume product of convex bodies are actually the global maximizers, that is: ellipsoids.

Let $K \subset \mathbb{R}^n$ be a convex body (a compact and convex set with non-empty interior). For $z \in \text{int}(K)$, the interior of K, let K^z be the polar of K with respect to z:

$$K^z = \{y \in \mathbb{R}^n; \langle y - z, x - z \rangle \leqslant 1 \text{ for every } x \in K\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^n$. It is well-known that K^z is also a convex body, that $z \in \text{int}(K^z)$ and that $(K^z)^z = K$. The *volume product* of K, $\Pi(K)$ (or $\Pi_n(K)$ if the dimension is to be specified), is given by the following formula:

$$\Pi(K) := \min_{z \in \text{int}(K)} |K||K^z|,$$

where $|A|$ denotes the Lebesgue measure of a Borel subset A of $\mathbb{R}^n$. The unique point $z = s(K) \in K$, where this minimum is reached, is called the Santaló point of K. We denote $K^* = K^{s(K)}$. Blaschke [3] (1917) proved for dimensions $n = 2$ and $n = 3$ that

$$\Pi(K) = |K||K^*| \leqslant \Pi(B_2^n),$$

where $B_2^n = \{x \in \mathbb{R}^n; |x| \leqslant 1\}$ ($|x| = \sqrt{\langle x, x \rangle}$) is the Euclidean unit ball in $\mathbb{R}^n$. This was generalized to all dimensions by Santaló [14] (1948).

It then took some time to establish the case of equality: one has $\Pi(K) = \Pi(B_2^n)$ if and only if K is an ellipsoid. This was done by Saint-Raymond [13] (1981), when K is centrally symmetric and by Petty [12] (1982), in the general case. Another proof was given by Meyer and Pajor [9] (1990), based on Steiner symmetrization.

Campi and Gronchi [5] (2006), introduced the use of shadow systems for volume product problems. Fix a direction $u \in S^{n-1}$. A shadow system (K_t) along the direction u is a family of convex sets (K_t), $t \in [a, b]$ such that

$$K_t = \text{conv}\{x + t\alpha(x)u; x \in A\},$$

where A is a given bounded subset of $\mathbb{R}^n$ and $\alpha : A \to \mathbb{R}$ ia a given bounded function, called the *speed* of the shadow system. An example is given by the *Steiner symmetrization* of a convex body K with respect to the hyperplane $u^\perp$ orthogonal to $u \in S^{n-1}$. Let K be described as

$$K = \{y + su;\ y \in P_u K,\ s \in I(y)\},$$

where P_u is the orthogonal projection onto $u^\perp$ and $I(y)$ is some non-empty closed interval depending on $y \in P_u K$. The Steiner symmetral $St_u(K)$ is defined by

$$St_u(K) := \left\{ y + su;\ y \in P_u K,\ s \in \frac{I(y) - I(y)}{2} \right\}.$$

For $t \in [-1, 1]$, let

$$K_t = \left\{ y + su;\ y \in P_u K,\ s \in \frac{1-t}{2} I(y) - \frac{1+t}{2} I(y) \right\}.$$

The family (K_t), $t \in [-1, 1]$ forms a shadow system such that $K_{-1} = K$, K_1 is the reflection of K with respect to $u^\perp$ and K_0 is the Steiner symmetral of K with respect to $u^\perp$. As a matter of fact, setting $A = K_0$, and $I(y) = [a(y), b(y)]$ for $y \in P_u K$, one has for $t \in [-1, 1]$ that

$$K_t = \left\{ z - t\frac{a(P_u z) + b(P_u z)}{2} u;\ z \in K_0 \right\}.$$

The following theorem was proved in [11] as Theorem 1 and Proposition 7 there.

THEOREM 1. *Let K_t, $t \in [a, b]$, be a shadow system in $\mathbb{R}^n$. Then $t \to |K_t^*|^{-1}$ is a convex function on $[a, b]$. If $t \to |K_t|$ and $t \to |K_t^*|^{-1}$ are both affine functions in $[a, b]$ then, for all $t \in [a, b]$, K_t is an affine image of K_a, $K_t = A_{u,t}(K_a)$, where $A_{u,t}$ is an affine transformation that satisfies $P_u A_{u,t} = P_u$. More precisely: for some $v \in \mathbb{R}^n$ and some $c \in \mathbb{R}$, one has for all $t \in [-1, 1]$ and all $x \in \mathbb{R}^n$ that*

$$A_{u,t}(x) = x + (t - a)(\langle x, v \rangle + c)u.$$

This theorem extended and strengthened a result of Campi and Gronchi [5], who proved the first part of it when the shadow system (K_t) is composed of bodies that are centrally symmetric with respect to the same center of symmetry.

As a consequence of Theorem 1, one gets the main result of this paper.

THEOREM 2. *The convex bodies K in $\mathbb{R}^n$ which are local maximizers (with respect to the Hausdorff distance or to the Banach–Mazur distance) of the volume product in $\mathbb{R}^n$ are the ellipsoids.*

Remark. A partial result in this direction was proved by Alexander, Fradelizi and Zvavich [1] who observed that no polytope can be a local maximizer for the volume product.

Proof of Theorem 2. Suppose that K is a local maximizer. Let $u \in S^{n-1}$ and $St_u(K)$ be the Steiner symmetral of K with respect to $u^\perp$.

With the above notations we describe the Steiner symmetral of K as K_0 of a shadow system K_t, $t \in [-1, 1]$, with $K_{-1} = K$ and K_1 being the mirror reflection of K about $u^\perp$. It follows from the definition of this shadow system that it preserves the volume of K: one has $|K_t| = |K|$ for all $t \in [-1, 1]$.

By construction, for all t, K_t is the mirror reflection of K_{-t} with respect to $u^\perp$. It follows that $(K_t)^*$ is also the mirror reflection of $(K_{-t})^*$ with respect to $u^\perp$. Let

$$f(t) = (|K||(K_t)^*|)^{-1} = \frac{1}{\Pi_n(K_t)}.$$

It is clear that the function $t \to K_t$ is continuous for both the Hausdorff and the Banach–Mazur distances. Thus such is also the function $t \to (K_t)^*$. It follows that f is continuous on $[-1, 1]$.

By Theorem 1, f is convex on $[-1, 1]$ and by construction, it is even. Thus $f(t) \leqslant f(-1) = f(1)$ for all $t \in [-1, 1]$ and f has its absolute minimum at 0. Since K is a local maximum of the volume product (i.e, a local minimum of f), one has for some $-1 < c \leqslant 0$, $f(t) \geqslant f(-1)$ for all $t \in [-1, c]$. Thus f is constant on $[-1, c]$. It now follows from its convexity and the preceding observations, that f is actually constant on $[-1, 1]$ and $|(K_t)^*| = |K^*|$ for $t \in [-1, 1]$.

From the second part of Theorem 1 we conclude now that $K_0 = St_u(K)$ is an image of $K_{-1} = K$ under an affine transformation having special properties. Since this fact is true for any $u \in S^{n-1}$, application of the next lemma completes the proof.

LEMMA 3. *Let K be a convex body such that, for all $u \in S^{n-1}$, $St_u(K)$ is an image of K, $St_u(K) = A_u(K)$ where A_u is an affine transformation that satisfies $P_u A_u = P_u$. Then (and only then) K is an ellipsoid.*

Remark. Lemma 3 can be formulated in an equivalent form as: *Let K be a convex body such that, for all $u \in S^{n-1}$, the centers of the chords of K that are parallel to u are located on a hyperplane. Then (and only then) K is an ellipsoid.* With this formulation the result, in dimension two, was declared by Bertrand [2] (1842). However, his proof does not seem complete. The result was proved by Brunn [4] (1889). Gruber [8] (1974) proved the result under strongly relaxed assumptions. A number of proofs of the result appear in the literature. See, for example, Danzer, Laugwitz and Lenz [6] (1957), which uses the Löwner ellipsoid of K, or Grinberg [7] (1991) which uses an infinite sequence of symmetrizations. We introduce here, for the sake of completeness, a proof that uses the uniqueness of the John ellipsoid of K.

We also point out [**10**] for a generalization, replacing the location of midpoints of chords by the location of centroids of sections of any fixed dimension k, $1 \leqslant k \leqslant n - 1$.

Proof of Lemma 3. We notice that the property of K presented in the lemma is preserved under affine transformations (this is easy to see from the equivalent form of this property presented in the Remark above). Thus, using an affine transformation, we may assume that the John ellipsoid of K (the ellipsoid of maximal volume contained in K) is the Euclidean unit ball B_2^n. We then want to show that K is a homothetic Euclidean ball.

Let $u \in S^{n-1}$. By the assumption, $St_u(K) = A_u(K)$, A_u affine with $P_u A_u = P_u$. Hence the John ellipsoid of $St_u(K)$ is $A_u(B_2^n)$. Now $|K| = |St_u(K)| = |A_u(K)|$, so $|\det(A_u)| = 1$ and $|A_u(B_2^n)| = |B_2^n|$. By symmetry of B_2^n about $u^\perp$ and the fact that $B_2^n \subset K$, we have $B_2^n \subset St_u(K)$. By the uniqueness of the John ellipsoid we conclude that $A_u(B_2^n) = B_2^n$. Thus A_u is a linear isometry with respect to the Euclidean norm, i.e. an orthogonal transformation.

The orthogonal transformation A_u preserves $u^\perp$ by the assumption of the lemma, so it is either the identity or an orthogonal reflection by $u^\perp$. Using any of these possibilities for each $u \in S^{n-1}$, we see that K is orthogonally symmetric about any hyperplane through 0. It follows that all the points of the boundary of K have the same Euclidean norm. Thus K is a Euclidean ball centered at the origin.

This completes the proof of Lemma 3, thus also the proof of Theorem 2.   □

References

1. M. Alexander, M. Fradelizi and A. Zvavitch, Polytopes of maximal volume product. *Discrete Comput. Geom.* (to appear). *Preprint*, 2017, arXiv:1708.07914.
2. J. Bertrand, Démonstration d'un théorème de géométrie. *J. Math. Pures Appl.* **7** (1842), 215–216.
3. W. Blaschke, Über affine Geometrie VII: Neue Extremeingenschaften von Ellipse und Ellipsoid. *Ber. Verh. Sächs. Akad. Wiss. Math. Phys. Kl.* **69** (1917), 412–420.
4. H. Brunn, *Über Kurven ohne Wendepunkte*, Habilitationsschrift (Munich, 1889).
5. S. Campi and P. Gronchi, On volume product inequalities for convex sets. *Proc. Amer. Math. Soc.* **134** (2006), 2393–2402.
6. L. Danzer, D. Laugwitz and H. Lenz, Über das Löwnersche Ellipsoid und sein analogon unter den einem Eikörper einbeschriebenen Ellipsoiden. *Arch. Math.* **8** (1957), 214–219.
7. E. L. Grinberg, Isoperimetric inequalities and identities for k-dimensional cross-sections of convex bodies. *Math. Ann.* **291** (1991), 75–86.
8. P. Gruber, Über kennzeichende Eigenschaften von Euklidischen Räumen und Ellipsoiden I. *J. Reine Angew. Math.* **265** (1974), 61–83.
9. M. Meyer and A. Pajor, On the Blaschke–Santaló inequality. *Arch. Math. (Basel)* **55**(1) (1990), 82–93.
10. M. Meyer and S. Reisner, Characterizations of ellipsoids by section-centroid location. *Geom. Dedicata* **31** (1989), 345–355.
11. M. Meyer and S. Reisner, Shadow systems and volumes of polar convex bodies. *Mathematika* **53**(1) (2006), 129–148.
12. C. M. Petty, Affine isoperimetric problems. *Ann. N.Y. Acad. Sci.* **440** (1985), 113–127.
13. J. Saint Raymond, Sur le volume des corps convexes symétriques, Séminaire d'Initiation à l'Analyse: G. Choquet-M. Rogalski-J. Saint-Raymond, 20ème Année: 1980/1981, Publ. Math. Univ. Pierre et Marie Curie, vol. 46, Univ. Paris VI, Paris, 1981, Exp. No. 11, 25.
14. L. A. Santaló, Un invariante afin para los cuerpos convexos del espacio des n dimensiones. *Port. Math.* **8** (1949), 155–161.

Mathieu Meyer,
Université Paris-Est-Marne-la-Vallée,
Laboratoire d'Analyse et de Mathématiques
    Appliquées (UMR 8050),
Cité Descartes, 5 Bd Descartes,
Champs-sur-Marne,
77454 Marne-la-Vallée cedex 2,
France
E-mail: mathieu.meyer@u-pem.fr

Shlomo Reisner,
Department of Mathematics,
University of Haifa,
Haifa 31905,
Israel
E-mail: reisner@math.haifa.ac.il