ON POLYLOGARITHMS

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1. The nth order polylogarithm $Li_n(z)$ is defined for $|z| \leqq 1$ by

$$Li_n(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^n} \quad (n = 2, 3, \ldots)$$

([**4**, p. 169], cf. [**2**, §1.11 (14) and §1.11. 1]). The definition can be extended to all values of z in the z-plane cut along the real axis from 1 to ∞ by the formula

$$Li_n(z) = \frac{z}{(n-1)!} \int_0^{\infty} \frac{t^{n-1}}{e^t - z}\, dt \tag{1}$$

[**2**, §1.11 (3)]. Then $Li_n(z)$ is regular in the cut plane, and there is a differential recurrence relation [**4**, p. 169]

$$zLi_n'(z) = Li_{n-1}(z) \quad (n \geqq 3). \tag{2}$$

It is convenient to extend the sequence $Li_n(z)$ backwards in the manner suggested by (2) and define

$$Li_1(z) = zLi_2'(z), \quad Li_0(z) = zLi_1'(z), \ldots .$$

Then $Li_1(z) = -\log(1-z)$, and $Li_n(z)$ is a rational function of z for $n = 0, -1, -2, \ldots$. Formula (2) now holds for all integers n.

2. We now prove

THEOREM. *There is no pure recurrence relation of the form*

$$A_0(z)Li_m(z) + A_1(z)Li_{m-1}(z) + \ldots + A_r(z)Li_{m-r}(z) = 0, \tag{3}$$

where the $A_n(z)$ are algebraic functions of z, $A_0(z)$ is not identically zero, $m \geqq 1$, and $r \geqq m$ is allowed.

Suppose that there is a relation (3). Divide by $A_0(z)$, differentiate with respect to z, and use (2) for each $Li_n'(z)$. We obtain an equation of the form

$$B_0(z)Li_{m-1}(z) + \ldots + B_r(z)Li_{m-r-1}(z) = 0,$$

where $B_0(z) = \{A_1(z)/A_0(z)\}' + 1/z$. Since the $A_n(z)$ are algebraic functions of z, so are the $B_n(z)$, and $B_0(z)$ is not identically zero. We now repeat the process until we obtain

$$K_0(z)Li_1(z) + \ldots + K_r(z)Li_{-r+1}(z) = 0, \tag{4}$$

say, where the $K_n(z)$ are algebraic, and $K_0(z)$ is not identically zero. But (4) implies that $\log(1-z)$ is an algebraic function of z, which is a contradiction. Hence there is no relation of the form (3).

3. A generating function. From (1) we have

$$\sum_{n=2}^{\infty} w^{n-1}Li_n(z) = z\sum_{n=2}^{\infty}\frac{w^{n-1}}{(n-1)!}\int_0^{\infty}\frac{t^{n-1}}{e^t-z}\,dt = z\int_0^{\infty}\frac{e^{wt}-1}{e^t-z}\,dt, \tag{5}$$

on inverting the order of integration and summation, this being justified by absolute convergence if $|w| < 1$. The function on the right of (5) is thus a generating function for the $Li_n(z)$ $(n \geqq 2)$.

A special case of (5) is the known formula [**2**, §1.17 (5)]

$$\sum_{n=2}^{\infty} w^{n-1}\zeta(n) = -\psi(1-w)-\gamma \quad (|w| < 1), \tag{6}$$

where $\psi(x)$ is the logarithmic derivative of the gamma function $\Gamma(x)$ and γ is Euler's constant. This is obtained from (5) on taking $z = 1$ and using the formula [**2**, §1.7.2 (14)]

$$\psi(1+w)+\gamma = -\int_0^{\infty}\frac{e^{-wt}-1}{e^t-1}\,dt,$$

and the fact that $Li_n(1) = \zeta(n)$.

Next, let

$$\sigma(n) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^n}.$$

Then, since $\sigma(n) = (1-2^{-n+1})\zeta(n)$, (6) gives

$$\sum_{n=2}^{\infty} w^{n-1}\sigma(n) = \psi(1-\tfrac{1}{2}w)-\psi(1-w).$$

This equation was used in [**1**] to evaluate the log-sine integrals

$$\int_0^{\frac{1}{2}\pi}\{\log(2\sin\theta)\}^n\,d\theta,$$

which also have other connexions with polylogarithms [**4**, pp. 148, 151–152, 184–185, 195–198].

4. Summation of series. By means of (5) we can sum various series involving polylogarithms. When w is rational, say $w = p/q$, where p and q are integers, $q > 0$, and $|p| < q$, the substitution $u = e^{-t/q}$ reduces the integral in (5) to the integral of a rational function of u, which can be evaluated. For example, when $w = \pm\frac{1}{2}$ and z is real, we obtain the formulae

$$\sum_{n=2}^{\infty}\frac{1}{2^{n-1}}Li_n(z) = \begin{cases} -2(-z)^{\frac{1}{2}}\tan^{-1}(-z)^{\frac{1}{2}}+\log(1-z) & (z \leqq 0),\\ 2z^{\frac{1}{2}}\log(1+z^{\frac{1}{2}})+(1-z^{\frac{1}{2}})\log(1-z) & (0 \leqq z < 1),\\ 2\log 2 & (z = 1), \end{cases} \tag{7}$$

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$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}} Li_n(z) = \begin{cases} 2(-z)^{-\frac{1}{2}} \tan^{-1}(-z)^{\frac{1}{2}} + \log(1-z) - 2 & (z < 0), \\ 2z^{-\frac{1}{2}} \log(1+z^{\frac{1}{2}}) + (1-z^{-\frac{1}{2}}) \log(1-z) - 2 & (0 < z < 1), \\ 2 \log 2 - 2 & (z = 1). \end{cases} \tag{8}$$

Formulae equivalent to (7) and (8) are given in [**4**, pp. 234–235] (see also [**3**]).

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