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# TAIL CONDITIONAL EXPECTATIONS FOR GENERALIZED SKEW-ELLIPTICAL DISTRIBUTIONS

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This paper deals with the multivariate tail conditional expectation (MTCE) for generalized skew-elliptical distributions. We present tail conditional expectation for univariate generalized skew-elliptical distributions and MTCE for generalized skew-elliptical distributions. There are many special cases for generalized skew-elliptical distributions, such as generalized skew-normal, generalized skew Student-$t$, generalized skew-logistic and generalized skew-Laplace distributions.

**Keywords:** generalized skew-elliptical distributions, generalized skew-laplace, generalized skew-logistic, generalized skew student-$t$, multivariate risk measures, tail conditional expectations

## 1. INTRODUCTION

Consider a random variable $X$ whose distribution function and tail function are denoted by $F_X(x)$ and $\overline{F}_X(x) = 1 - F_X(x)$, respectively. The tail conditional expectation (TCE) is defined as

$$\mathrm{TCE}_X(x_q) = E(X \mid X > x_q). \tag{1}$$

Given the loss will exceed a particular value $x_q$, generally referred to as the $q$th quantile with

$$\overline{F}_X(x_q) = 1 - q,$$

the TCE defined in formula (1) gives the expected loss that can potentially be experienced (see [9]). There are a number of distributions whose TCE measures have been researched. For instances, the TCE for univariate normal distribution was noticed in Panjer [12]; the TCE for univariate elliptical distributions was provided by Landsman and Valdez [9]; conditional tail expectation for the exponential family and its related distributions were derived by Kim [5]; the TCE for family of symmetric generalized hyperbolic distributions and family of skew generalized hyperbolic distributions was derived by Ignatieva and Landsman [3, 4], respectively.

 

Recently, Landsman *et al.* [7] defined a type of a multivariate tail conditional expectation (MTCE),

$$\begin{aligned}\mathrm{MTCE}_{\boldsymbol{q}}(\mathbf{X}) &= E[\mathbf{X} \mid \mathbf{X} > \mathrm{VaR}_{\boldsymbol{q}}(\mathbf{X})] \\ &= E[\mathbf{X} \mid X_1 > \mathrm{VaR}_{q_1}(X_1), \ldots, X_n > \mathrm{VaR}_{q_n}(X_n)], \quad \boldsymbol{q} = (q_1, \ldots, q_n) \in (0,1)^n,\end{aligned}$$

where $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$ is a $n \times 1$ vector of risks with cumulative distribution function (cdf) $F_{\mathbf{X}}(\boldsymbol{x})$ and tail function $\overline{F}_{\mathbf{X}}(\boldsymbol{x})$,

$$\mathrm{VaR}_{\boldsymbol{q}}(\mathbf{X}) = (\mathrm{VaR}_{q_1}(X_1), \mathrm{VaR}_{q_2}(X_2), \ldots, \mathrm{VaR}_{q_n}(X_n))^T,$$

and $\mathrm{VaR}_{q_k}(X_k),\ k = 1, 2, \ldots, n$ is the value at risk (VaR) measure of the random variable $X_k$, being the $q_k$th quantile of $X_k$ (see [7] or [11]). Landsman *et al.* [6] also define an MTCE, which is the above special case when $\boldsymbol{q} = (q, q, \ldots, q)$. In Mousavi *et al.* [11], MTCE for scale mixtures of skew-normal distribution is discussed. In the present paper, we derive MTCE for generalized skew-elliptical distributions.

The rest of the paper is organized as follows. Section 2 reviews the definitions and properties of the univariate generalized skew-elliptical distributions and provides TCE formula for generalized skew-elliptical random variable. In Section 3, we introduce multivariate generalized skew-elliptical distributions and derive MTCE for generalized skew-elliptical random vector. Some examples are given in Section 4. We present numerical illustration in Section 5. Finally, in Section 6, is the conclusions and directions for further research.

## 2. UNIVARIATE CASES

In this section, we derive TCE for generalized skew-elliptical random variable. Before doing so, let us introduce univariate generalized skew-elliptical distributions.

A random vector $Y \sim \mathrm{GSE}_1(\mu, \sigma^2, g_1, H(\cdot))$ is said to have a univariate generalized skew-elliptical distribution, if its probability density function $f_Y(y)$ exists and satisfies (see [1])

$$f_Y(y) = \frac{2}{\sigma} g_1 \left\{ \frac{1}{2} \left( \frac{y-\mu}{\sigma} \right)^2 \right\} H \left( \frac{y-\mu}{\sigma} \right), \quad y \in \mathbb{R}, \tag{2}$$

where $g_1$ is the density generator of elliptical random variable $X \sim E_1(\mu, \sigma^2, g_1)$ with parameters $\mu$ and $\sigma$ (see [2]). The condition

$$\int_0^\infty t^{-1/2} g_1(t) < \infty$$

guarantees $g_1$ to be the density generator. $H(x), x \in \mathbb{R}$, is called the skewing function satisfying $H(-x) = 1 - H(x)$ and $0 \le H(x) \le 1$. The characteristic function of $X$ takes the form

$$\varphi_X(t) = \exp\{it\mu\} \psi \left( \frac{1}{2} (\sigma t)^2 \right), \quad t \in \mathbb{R},$$

with function $\psi(t) : [0, \infty) \to \mathbb{R}$, called the characteristic generator (see [2]).

To represent TCE for univariate generalized skew-elliptical distributions, a cumulative generator $\overline{G}_1(u)$ is defined as follows:

$$\overline{G}_1(u) = \int_u^{\infty} g_1(v)\,\mathrm{d}v.$$

Tail expectation $\overline{E}_Z^t[h(Z)]$ of a random variable $Z$ is defined as follows:

$$\overline{E}_Z^t[h(Z)] = \int_t^{\infty} h(z) f_Z(z)\,\mathrm{d}z, \quad z, t \in \mathbb{R},$$

where $h(\cdot)$ is an almost differentiable function. So

$$\overline{E}_{X^*}^t[H'(X)] = \int_t^{\infty} H'(x) f_{X^*}(x)\,\mathrm{d}x, \quad x, t \in \mathbb{R},$$

with the probability density function (pdf)

$$f_{X^*}(x) = -\frac{1}{\psi'(0)}\overline{G}_1\left\{\frac{1}{2}x^2\right\},$$

where $X^* \sim E_1(0,1,\overline{G}_1)$ (see [8]), and $\psi'(\cdot)$ is the derivative of characteristic generator $\psi(\cdot)$.

REMARK 1: *In Adcock et al. [1], pdf of a random variable $X^* \sim E_1(\mu, \sigma^2, \overline{G}_1)$ was defined as*

$$f_{X^*}(x) = \frac{1}{E(R^2)\sigma}\overline{G}_1\left\{\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\},$$

*where $R$ is a non-negative random variable with pdf*

$$f_R(r) = 2g_1(r^2/2), \quad r \in [0, \infty).$$

*We further suppose that $E(R^2) < \infty$, in which case the covariance of $X$ exists and $Cov(X) = E[R^2]\sigma^2$. However, in Landsman and Valdez [9], if $|\psi'(0)| < \infty$, the covariance of $X$ exists and is equal to $Cov(X) = -\psi'(0)\sigma^2$. Inspired by this, we define the pdf of $X^* \sim E_1(0,1,\overline{G}_1)$ as above.*

If $h(\cdot) = 1$, we will have $\overline{E}_Z^t[h(Z)] = \overline{F}_Z(t)$, which represents tail function of $Z$.

THEOREM 2.1: *Assume that a random vector $Y \sim GSE_1(\mu, \sigma^2, g_1, H(\cdot))$ follows a univariate generalized skew-elliptical distribution with pdf (2). We suppose*

$$\lim_{z\to+\infty} H(z)\overline{G}_1\left(\frac{1}{2}z^2\right) = 0. \tag{3}$$

*Then*

$$TCE_Y(y_q) = \mu + 2\sigma H(z_q)\frac{\overline{G}_1\left(\frac{1}{2}z_q^2\right)}{\overline{F}_Z(z_q)} - 2\sigma\psi'(0)\frac{\overline{E}_{X^*}^{z_q}[H'(X^*)]}{\overline{F}_Z(z_q)}, \tag{4}$$

*where $Z \sim GSE_1(0,1,g_1,H(\cdot))$, $X^* \sim E_1(0,1,\overline{G}_1)$, $z_q = (y_q - \mu)/\sigma$, and $H'(\cdot)$ is derivative of function $H(\cdot)$.*

PROOF: Using definition, we obtain

$$\mathrm{TCE}_Y(y_q) = \frac{1}{\overline{F}_Y(y_q)} \int_{y_q}^{+\infty} \frac{2y}{\sigma} g_1 \left\{ \frac{1}{2} \left( \frac{y-\mu}{\sigma} \right)^2 \right\} H\left( \frac{y-\mu}{\sigma} \right) \mathrm{d}y.$$

Applying the transformation $z = (y-\mu)/\sigma$, we have

$$\begin{aligned}
\mathrm{TCE}_Y(y_q) &= \frac{1}{\overline{F}_Z(z_q)} \int_{z_q}^{+\infty} 2(\sigma z + \mu) g_1 \left\{ \frac{1}{2} z^2 \right\} H(z)\, \mathrm{d}z \\
&= \frac{1}{\overline{F}_Z(z_q)} \left[ -2\sigma \int_{z_q}^{+\infty} H(z)\, \mathrm{d}\overline{G}_1 \left\{ \frac{1}{2} z^2 \right\} + \mu \int_{z_q}^{+\infty} 2 g_1 \left\{ \frac{1}{2} z^2 \right\} H(z)\, \mathrm{d}z \right] \\
&= \frac{1}{\overline{F}_Z(z_q)} \left[ 2\sigma \left( H(z_q) \overline{G}_1 \left\{ \frac{1}{2} z_q^2 \right\} + \int_{z_q}^{+\infty} H'(z) \overline{G}_1 \left\{ \frac{1}{2} z^2 \right\} \mathrm{d}z \right) + \mu \overline{F}_Z(z_q) \right] \\
&= \frac{1}{\overline{F}_Z(z_q)} \left[ 2\sigma H(z_q) \overline{G}_1 \left\{ \frac{1}{2} z_q^2 \right\} - 2\sigma \psi'(0) \overline{E}_{Z^*}^{z_q} [H'(Z^*)] + \mu \overline{F}_Z(z_q) \right],
\end{aligned}$$

where the third equality we have used (3).

Therefore, we obtain (4), which completes the proof of Theorem 2.1. ■

REMARK 2: *If $H(\cdot) = \frac{1}{2}$, we will obtain TCE for elliptical distribution:*

$$TCE_Y(y_q) = \mu + \sigma \frac{\overline{G}_1 \left( \frac{1}{2} z_q^2 \right)}{\overline{F}_Z(z_q)}, \tag{5}$$

*where $Z \sim E_1(0, 1, g_1)$. We observe that (5) is a generalization of formula (3) ($c_n = 1$) in Lansman and Valdez [9].*

## 3. MULTIVARIATE CASES

A random vector $\mathbf{Y}$ is called an $n$-dimensional generalized skew-elliptical random vector and denoted by $\mathbf{Y} \sim \mathrm{GSE}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n, H(\cdot))$. If its pdf exists, the form will be (see [1])

$$f_{\mathbf{Y}}(\boldsymbol{y}) = \frac{2}{\sqrt{|\boldsymbol{\Sigma}|}} g_n \left\{ \frac{1}{2} (\boldsymbol{y}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{y}-\boldsymbol{\mu}) \right\} H(\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y}-\boldsymbol{\mu})), \quad \boldsymbol{y} \in \mathbb{R}^n, \tag{6}$$

where

$$f_{\mathbf{X}}(\boldsymbol{x}) := \frac{1}{\sqrt{|\boldsymbol{\Sigma}|}} g_n \left\{ \frac{1}{2} (\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x}-\boldsymbol{\mu}) \right\}, \quad \boldsymbol{x} \in \mathbb{R}^n, \tag{7}$$

is the density of $n$-dimensional elliptical random vector $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$. Here $\boldsymbol{\mu}$ is a $n \times 1$ location vector, $\boldsymbol{\Sigma}$ is a $n \times n$ scale matrix, and $g_n(u)$, $u \geq 0$, is the density generator of $\mathbf{X}$, satisfying the condition

$$\int_0^{\infty} t^{n/2-1} g_n(t) < \infty.$$

$H(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^n$, is called the skewing function satisfying $H(-\boldsymbol{x}) = 1 - H(\boldsymbol{x})$ and $0 \leq H(\boldsymbol{x}) \leq 1$. The characteristic function of $\mathbf{X}$ takes the form $\varphi_{\mathbf{X}}(\boldsymbol{t}) = \exp\{i\boldsymbol{t}^T\boldsymbol{\mu}\}\psi(\frac{1}{2}\boldsymbol{t}^T\boldsymbol{\Sigma}\boldsymbol{t}), \boldsymbol{t} \in \mathbb{R}^n$, with function $\psi(t) : [0,\infty) \to \mathbb{R}$, called the characteristic generator (see [2]). A cumulative generator $\overline{G}_n(u)$ is defined. It takes the form

$$\overline{G}_n(u) = \int_u^\infty g_n(v)\,\mathrm{d}v.$$

Shifted cumulative generator is also defined

$$\overline{G}^*_{n-1}(u) = \int_u^\infty g_n(v+a)\,\mathrm{d}v, \quad a \geq 0,\ n > 1, \tag{8}$$

with $\overline{G}^*_{n-1}(u) < \infty$ (see [6]).

Assume a random vector $\mathbf{Y} \sim \mathrm{GSE}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n, H(\cdot))$ with finite vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^T$, positive defined matrix $\boldsymbol{\Sigma} = (\sigma_{ij})_{i,j=1}^n$ and pdf $f_{\boldsymbol{Y}}(\boldsymbol{y})$.

$\mathbf{X}^* \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \overline{G}_n)$ (see [8]) is called an elliptical random vector with generator $\overline{G}_n(u)$, if its density function (if it exists) defined by

$$f_{\mathbf{X}^*}(\boldsymbol{x}) = \frac{-1}{\psi'(0)\sqrt{|\boldsymbol{\Sigma}|}}\overline{G}_n\left\{\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}, \quad \boldsymbol{x} \in \mathbb{R}^n. \tag{9}$$

Remark 3: *In Adcock et al. [1], pdf of a random vector* $\mathbf{X}^* \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \overline{G}_n)$ *was defined as*

$$f_{\mathbf{X}^*}(\boldsymbol{x}) = \frac{n}{E(R^2)\sqrt{|\boldsymbol{\Sigma}|}}\overline{G}_n\left\{\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}, \quad \boldsymbol{x} \in \mathbb{R}^n,$$

*where $R$ is a non-negative random variable with pdf*

$$f_R(r) = \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1} g_n(r^2/2), \quad r \in [0,\infty).$$

*We further suppose that $E(R^2) < \infty$, in which case the covariance of $\mathbf{X}$ exists and $Cov(\mathbf{X}) = (E[R^2]/n)\boldsymbol{\Sigma}$. However, in Landsman and Valdez [9], if $|\psi'(0)| < \infty$, the covariance of $\mathbf{X}$ exists and is equal to $Cov(\mathbf{X}) = -\psi'(0)\boldsymbol{\Sigma}$. Inspired by this, we define the pdf of $\mathbf{X}^* \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \overline{G}_n)$ as above.*

*$\mathbf{Y}^* \sim GSE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \overline{G}_n, H(\cdot))$ is a generalized skew-elliptical random vector. Let $\mathbf{Z} = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{Y}-\boldsymbol{\mu}) \sim GSE_n(\mathbf{0}, \boldsymbol{I_n}, g_n, H(\cdot))$. Writing*

$$\boldsymbol{z_q} = (z_{1,\boldsymbol{q}}, z_{2,\boldsymbol{q}}, \ldots, z_{n,\boldsymbol{q}})^T = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y_q}-\boldsymbol{\mu}),$$

*where $\boldsymbol{y_q} = VaR_{\boldsymbol{q}}(\boldsymbol{Y})$, and $\boldsymbol{z}_{-k,\boldsymbol{q}} = (z_{1,\boldsymbol{q}}, z_{2,\boldsymbol{q}}, \ldots, z_{k-1,\boldsymbol{q}}, z_{k+1,\boldsymbol{q}}, \ldots, z_{n,\boldsymbol{q}})^T$.*

To derive the formula for MTCE, we define tail expectation $\overline{E}^{\boldsymbol{t}}_{\mathbf{Z}}[h(\boldsymbol{Z})]$ of $n$-dimensional random vector $\mathbf{Z}$:

$$\overline{E}^{\boldsymbol{t}}_{\mathbf{Z}}[h(\boldsymbol{Z})] = \int_{\boldsymbol{t}}^\infty h(\boldsymbol{z}) f_{\mathbf{Z}}(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}, \quad \boldsymbol{z}, \boldsymbol{t} \in \mathbb{R}^n,$$

where $h$ is an almost differentiable function and $f_{\mathbf{Z}}(\boldsymbol{z})$ is pdf of $\mathbf{Z}$.

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So tail expectation $\overline{E}^{\boldsymbol{t}}_{\mathbf{W}_{-k}}[H^*(\boldsymbol{W_{-k}})]$ of $(n-1)$-dimensional random vector

$$\mathbf{W}_{-k} = (W_1, W_2, \ldots, W_{k-1}, W_{k+1}, \ldots, W_n)^T \sim E_{n-1}(\mathbf{0}, \mathbf{I_{n-1}}, \overline{G}^*_{n-1})$$

enable to be expressed as

$$\overline{E}^{\boldsymbol{t}}_{\mathbf{W}_{-k}}[H^*(\boldsymbol{W_{-k}})] = \int_{\boldsymbol{t}}^{\infty} H^*(\boldsymbol{w}_{-k}) f_{\mathbf{W}_{-k}}(\boldsymbol{w}_{-k})\, \mathrm{d}\boldsymbol{w}_{-k}, \quad \boldsymbol{w}_{-k}, \boldsymbol{t} \in \mathbb{R}^{n-1},$$

$\mathrm{d}\boldsymbol{w}_{-k} = \mathrm{d}w_1\, \mathrm{d}w_2 \cdots \mathrm{d}w_{k-1}\, \mathrm{d}w_{k+1} \cdots \mathrm{d}w_n$, with the pdf

$$\begin{aligned} f_{\mathbf{W}_{-k}}(\boldsymbol{w}_{-k}) &= -\frac{1}{\psi^{*\prime}(0)} \overline{G}^*_{n-1,k} \left\{ \frac{1}{2} \boldsymbol{w}^T_{-k} \boldsymbol{w}_{-k} \right\} \\ &= -\frac{1}{\psi^{*\prime}(0)} \overline{G}_n \left\{ \frac{1}{2} \boldsymbol{w}^T_{-k} \boldsymbol{w}_{-k} + \frac{1}{2} z^2_{k,\boldsymbol{q}} \right\}, \quad k = 1, 2, \ldots, n, \end{aligned}$$

where $H^*(\boldsymbol{w}_{-k}) = H(\boldsymbol{\xi}_{k,\boldsymbol{q}})$ with $\boldsymbol{\xi}_{k,\boldsymbol{q}} = (w_1, w_2, \ldots, w_{k-1}, z_{k,\boldsymbol{q}}, w_{k+1}, \ldots, w_n)^T$, and $\overline{G}^*_{n-1,k}$ is defined by (8). In addition, $\psi^{*'}(\cdot)$ is derivative of characteristic generator corresponding to $\overline{G}^*_{n-1,k}$.

Remark 4: *From Landsman et al. [6], we know that* $-1/\psi^{*'}(0) = c^*_{n-1,k}$*, and* $c^*_{n-1,k}$ *is the normalizing constant, that is to say,*

$$c^*_{n-1,k} = \int_0^{\infty} t^{(n-1)/2-1} \overline{G}^*_{n-1,k}(t)\, dt.$$

*If* $h(\cdot) = 1$*, we will have* $\overline{E}^{\boldsymbol{t}}_{\mathbf{Z}}[h(\cdot)] = \overline{F}_{\boldsymbol{Z}}(\boldsymbol{t})$*, which represents tail function of* $\boldsymbol{Z}$*. Its tail function as follows:*

$$\overline{F}_{\mathbf{Z}}(\boldsymbol{t}) = \int_{\boldsymbol{t}}^{\infty} f_{\mathbf{Z}}(\boldsymbol{z})\, d\boldsymbol{z}, \quad \boldsymbol{z}, \boldsymbol{t} \in \mathbb{R}^n,$$

$d\boldsymbol{z} = dz_1\, dz_2 \cdots dz_n$*, and* $f_{\mathbf{Z}}(\boldsymbol{z})$ *is pdf of* $\boldsymbol{Z}$ *as above.*

In the following, we formulate the theorem that gives MTCE for generalized skew-elliptical distributions.

Theorem 3.1: *Assume that a random vector* $\mathbf{Y} \sim GSE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n, H(\cdot))$ *follows a n-variate generalized skew-elliptical distribution with pdf (6). We suppose*

$$\lim_{z_k \to +\infty} H(\boldsymbol{z}) \overline{G}_n \left( \frac{1}{2} \boldsymbol{z}^T \boldsymbol{z} \right) = 0, \quad k = 1, 2, \ldots, n. \tag{10}$$

*Then*

$$MTCE_{\mathbf{Y}}(\boldsymbol{y_q}) = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2} \frac{\boldsymbol{\delta_q}}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})}, \tag{11}$$

*where*

$$\boldsymbol{\delta_q} = (\delta_{1,\boldsymbol{q}}, \delta_{2,\boldsymbol{q}}, \ldots, \delta_{n,\boldsymbol{q}})^T$$

*with*

$$\delta_{k,\boldsymbol{q}} = -2\psi^{*'}(0) \overline{E}^{\boldsymbol{z}_{-k,\boldsymbol{q}}}_{\boldsymbol{W}_{-k}}[H^*(\boldsymbol{W}_{-k})] - 2\psi'(0) \overline{E}^{\boldsymbol{z_q}}_{\mathbf{X}^*}[\partial_k H(\boldsymbol{X}^*)], \quad k = 1, 2, \ldots, n,$$

$\mathbf{Z} \sim GSE_n(\mathbf{0}, \boldsymbol{I_n}, g_n, H(\cdot))$, $\mathbf{X}^* \sim E_n(\mathbf{0}, \mathbf{I_n}, \overline{G}_n)$ *and* $\mathbf{W}_{-k} \sim E_{n-1}(\mathbf{0}, \mathbf{I_{n-1}}, \overline{G}^*_{n-1,k})$*. Furthermore,* $\partial_k H(\boldsymbol{X}^*) = dH(\boldsymbol{x}^*)/\, dx^*_k$*.*

Proof: Using definition, we obtain

$$\begin{aligned}&\mathrm{MTCE}_{\mathbf{Y}}(\boldsymbol{y_q})\\&=\frac{1}{\overline{F}_{\mathbf{Y}}(\boldsymbol{y_q})}\int_{\boldsymbol{y_q}}^{+\infty}\frac{2\boldsymbol{y}}{\sqrt{|\boldsymbol{\Sigma}|}}g_n\left\{\frac{1}{2}(\boldsymbol{y}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\right\}H(\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y}-\boldsymbol{\mu}))\,\mathrm{d}\boldsymbol{y}.\end{aligned}$$

Applying the transformation $\boldsymbol{z}=\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y}-\boldsymbol{\mu})$, we have

$$\begin{aligned}\mathrm{MTCE}_{\mathbf{Y}}(\boldsymbol{y_q})&=\frac{1}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})}\int_{\boldsymbol{z_q}}^{+\infty}2(\boldsymbol{\Sigma}^{1/2}\boldsymbol{z}+\boldsymbol{\mu})g_n\left\{\frac{1}{2}\boldsymbol{z}^T\boldsymbol{z}\right\}H(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\\&=\frac{1}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})}\left[\boldsymbol{\Sigma}^{1/2}\int_{\boldsymbol{z_q}}^{+\infty}2H(\boldsymbol{z})\boldsymbol{z}g_n\left\{\frac{1}{2}\boldsymbol{z}^T\boldsymbol{z}\right\}\mathrm{d}\boldsymbol{z}\right.\\&\qquad\left.+\boldsymbol{\mu}\int_{\boldsymbol{z_q}}^{+\infty}2g_n\left\{\frac{1}{2}\boldsymbol{z}^T\boldsymbol{z}\right\}H(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\right]\\&=\frac{1}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})}[\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta_q}+\boldsymbol{\mu}\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})].\end{aligned}$$

Since

$$\begin{aligned}\delta_{k,\boldsymbol{q}}&=\int_{\boldsymbol{z_q}}^{+\infty}2H(\boldsymbol{z})z_kg_n\left\{\frac{1}{2}\boldsymbol{z}^T\boldsymbol{z}\right\}\mathrm{d}\boldsymbol{z}\\&=-2\int_{\boldsymbol{z}_{-k,\boldsymbol{q}}}^{+\infty}\mathrm{d}\boldsymbol{z}_{-k}\int_{z_{k,\boldsymbol{q}}}^{+\infty}H(\boldsymbol{z})\partial_k\overline{G}_n\left\{\frac{1}{2}\boldsymbol{z}_{-k}^T\boldsymbol{z}_{-k}+\frac{1}{2}z_k^2\right\}\\&=2\int_{\boldsymbol{z}_{-k,\boldsymbol{q}}}^{+\infty}H(\boldsymbol{\xi}_{k,\boldsymbol{q}})\overline{G}_n\left\{\frac{1}{2}\boldsymbol{z}_{-k}^T\boldsymbol{z}_{-k}+\frac{1}{2}z_{k,\boldsymbol{q}}^2\right\}\mathrm{d}\boldsymbol{z}_{-k}\\&\quad+2\int_{\boldsymbol{z_q}}^{+\infty}\partial_kH(\boldsymbol{z})\overline{G}_n\left\{\frac{1}{2}\boldsymbol{z}^T\boldsymbol{z}\right\}\mathrm{d}\boldsymbol{z}\\&=-2\psi^{*'}(0)\overline{E}_{\boldsymbol{W}_{-k}}^{\boldsymbol{z}_{-k,\boldsymbol{q}}}[H^*(\boldsymbol{W}_{-k})]-2\psi'(0)\overline{E}_{\mathbf{X}^*}^{\boldsymbol{z_q}}[\partial_kH(\boldsymbol{X}^*)],\end{aligned}$$

where $\boldsymbol{\xi}_{k,\boldsymbol{q}}=(z_1,z_2,\ldots,z_{k-1},z_{k,\boldsymbol{q}},z_{k+1},\ldots,z_n)^T$, and in the third equality, we have used (10). So that

$$\begin{aligned}\boldsymbol{\delta_q}&=\int_{\boldsymbol{z_q}}^{+\infty}2H(\boldsymbol{z})\boldsymbol{z}g_n\left\{\frac{1}{2}\boldsymbol{z}^T\boldsymbol{z}\right\}\mathrm{d}\boldsymbol{z}\\&=(\delta_{1,\boldsymbol{q}},\delta_{2,\boldsymbol{q}},\ldots,\delta_{n,\boldsymbol{q}})^T.\end{aligned}$$

Therefore, we obtain (11), which completes the proof of Theorem 3.1. ■

Remark 5: *Letting $H(\cdot)=\frac{1}{2}$ in Theorem 3.1, we will obtain generalized formula of Theorem 2.1 ($c_n=1$) in Landsman et al. [6]:*

$$MTCE_{\mathbf{Y}}(\boldsymbol{y_q})=\boldsymbol{\mu}+\boldsymbol{\Sigma}^{1/2}\frac{\boldsymbol{\delta_q}}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})},\tag{12}$$

*where $\boldsymbol{\delta_q}=(\delta_{1,\boldsymbol{q}},\delta_{2,\boldsymbol{q}},\ldots,\delta_{n,\boldsymbol{q}})^T$, $\delta_{k,\boldsymbol{q}}=-\psi^{*'}(0)\overline{F}_{\boldsymbol{W}_{-k}}(\boldsymbol{z}_{-k,\boldsymbol{q}})$, $k=1,2,\ldots,n$.*

## 4. EXAMPLES

We now provide MTCE for special cases of the generalized skew-elliptical distributions, such as generalized skew-normal, generalized skew Student-$t$, generalized skew-logistic and generalized skew-Laplace distributions.

EXAMPLE 4.1 (Generalized skew-normal distribution): *A $n$-dimensional generalized skew-normal random vector $\mathbf{Y}$, with location parameter $\boldsymbol{\mu}$, scale matrix $\boldsymbol{\Sigma}$ and skewing function $H(\cdot):\mathbb{R}\to\mathbb{R}$, has its density function as*

$$f_{\boldsymbol{Y}}(\boldsymbol{y})=\frac{2}{\sqrt{|\boldsymbol{\Sigma}|}(2\pi)^{n/2}}\exp\left\{-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\right\}H(\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y}-\boldsymbol{\mu})),$$

*$\boldsymbol{y}\in\mathbb{R}^n$, where $\boldsymbol{\gamma}=(\gamma_1,\gamma_2,\ldots,\gamma_n)^T\in\mathbb{R}^n$. We denote it by $\mathbf{Y}\sim GSN_n(\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\gamma},H(\cdot))$. In this case, $\overline{G}_n(u)=g_n(u)=(2\pi)^{-n/2}\exp\{-u\}$ and*

$$H(\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y}-\boldsymbol{\mu}))=H(\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y}-\boldsymbol{\mu})).$$

*MTCE is given by*

$$MTCE_{\mathbf{Y}}(\boldsymbol{y_q})=\boldsymbol{\mu}+\boldsymbol{\Sigma}^{1/2}\frac{\boldsymbol{\delta_q}}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})},\tag{13}$$

*where $\boldsymbol{\delta_q}=(\delta_{1,\boldsymbol{q}},\delta_{2,\boldsymbol{q}},\ldots,\delta_{n,\boldsymbol{q}})^T$,*

$$\delta_{k,\boldsymbol{q}}=\sqrt{\frac{2}{\pi}}\exp\left\{-\frac{1}{2}z_{k,\boldsymbol{q}}^2\right\}\overline{E}_{\boldsymbol{W}_{-k}}^{\boldsymbol{z}_{-k,\boldsymbol{q}}}[H(\boldsymbol{\gamma}^T\boldsymbol{\xi}_{k,\boldsymbol{q}})]+2\gamma_k\overline{E}_{\boldsymbol{X}^*}^{\boldsymbol{z_q}}[H'(\boldsymbol{\gamma}^T\boldsymbol{X}^*)],$$

*$k=1,2,\ldots,n$, $\mathbf{Z}\sim GSN_n(\mathbf{0},\mathbf{I_n},\boldsymbol{\gamma},H(\cdot))$, $\mathbf{X}^*\sim N_n(\mathbf{0},\mathbf{I_n})$ and $\mathbf{W}_{-k}\sim N_{n-1}(\mathbf{0},\mathbf{I_{n-1}})$.*

REMARK 6: *If $H(\cdot)=\Phi(\cdot)$ (the cdf of 1-dimensional standard normal distribution) in Example 4.1, MTCE for n-dimensional skew-normal distribution will be obtained:*

$$MTCE_{\mathbf{Y}}(\boldsymbol{y_q})=\boldsymbol{\mu}+\boldsymbol{\Sigma}^{1/2}\frac{\boldsymbol{\delta_q}}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})},\tag{14}$$

*where $\boldsymbol{\delta_q}=(\delta_{1,\boldsymbol{q}},\delta_{2,\boldsymbol{q}},\ldots,\delta_{n,\boldsymbol{q}})^T$,*

$$\delta_{k,\boldsymbol{q}}=\sqrt{\frac{2}{\pi}}\exp\left\{-\frac{1}{2}z_{k,\boldsymbol{q}}^2\right\}\overline{E}_{\boldsymbol{W}_{-k}}^{\boldsymbol{z}_{-k,\boldsymbol{q}}}[\Phi(\boldsymbol{\gamma}^T\boldsymbol{\xi}_{k,\boldsymbol{q}})]+2\gamma_k\overline{E}_{\boldsymbol{X}^*}^{\boldsymbol{z_q}}[\phi(\boldsymbol{\gamma}^T\mathbf{X}^*)],\quad k=1,2,\ldots,n,$$

*$\mathbf{Z}\sim GSN_n(\mathbf{0},\mathbf{I_n},\boldsymbol{\gamma},\Phi(\cdot))$, $\mathbf{X}^*\sim N_n(\mathbf{0},\mathbf{I_n})$ and $\mathbf{W}_{-k}\sim N_{n-1}(\mathbf{0},\mathbf{I_{n-1}})$. In addition, $\phi(\cdot)$ is the pdf of 1-dimensional standard normal distribution.*

REMARK 7: *If $H(\cdot)=\frac{1}{2}$ in Example 4.1, we will obtain MTCE for n-dimensional normal distribution as follows:*

$$MTCE_{\mathbf{Y}}(\boldsymbol{y_q})=\boldsymbol{\mu}+\boldsymbol{\Sigma}^{1/2}\frac{\boldsymbol{\delta_q}}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})},\tag{15}$$

*where $\boldsymbol{\delta_q}=(\delta_{1,\boldsymbol{q}},\delta_{2,\boldsymbol{q}},\ldots,\delta_{n,\boldsymbol{q}})^T$,*

$$\delta_{k,\boldsymbol{q}}=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}z_{k,\boldsymbol{q}}^2\right\}\overline{F}_{\boldsymbol{W}_{-k}}(\boldsymbol{z}_{-k,\boldsymbol{q}}),\quad k=1,2,\ldots,n,$$

*$\mathbf{Z}\sim N_n(\mathbf{0},\mathbf{I_n})$, and $\mathbf{W}_{-k}$ is the same as in Example 4.1.*

We observe that (15) is generalization of MTCE for normal distribution in Landsman *et al.* [6].

EXAMPLE 4.2 (Generalized skew Student-$t$ distribution): *The density function of a $n$-dimension generalized skew Student-t random vector $\mathbf{Y}$, with location parameter $\boldsymbol{\mu}$, scale matrix $\boldsymbol{\Sigma}$, $m > 0$ degrees of freedom and skewing function $H(\cdot) : \mathbb{R} \to \mathbb{R}$, is given by*

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = \frac{2c_n}{\sqrt{|\boldsymbol{\Sigma}|}} \left[1 + \frac{(\boldsymbol{y}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{y}-\boldsymbol{\mu})}{m}\right]^{-(m+n)/2} H(\gamma^T \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y}-\boldsymbol{\mu})), \quad \boldsymbol{y} \in \mathbb{R}^n,$$

*where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T \in \mathbb{R}^n$ and $c_n = (\Gamma((m+n)/2))/(\Gamma(m/2)(m\pi)^{n/2})$. We denote it by $\mathbf{Y} \sim GSSt_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma}, m, H(\cdot))$. The density generator in this case is*

$$g_n(u) = c_n \left(1 + \frac{2u}{m}\right)^{-(m+n)/2}, \quad \textit{and} \quad H(\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y}-\boldsymbol{\mu})) = H(\gamma^T \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y}-\boldsymbol{\mu}))$$

*and $\overline{G}_n(u) = (c_n m/(m+n-2))(1+2u/m)^{-(m+n-2)/2}$. MTCE is given by*

$$MTCE_{\mathbf{Y}}(\boldsymbol{y_q}) = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2} \frac{\boldsymbol{\delta_q}}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})}, \tag{16}$$

*where $\boldsymbol{\delta_q} = (\delta_{1,\boldsymbol{q}}, \delta_{2,\boldsymbol{q}}, \ldots, \delta_{n,\boldsymbol{q}})^T$,*

$$\delta_{k,\boldsymbol{q}} = b^*_{n,k} \overline{E}^{\boldsymbol{z}_{-k,\boldsymbol{q}}}_{\boldsymbol{W}_{-k}} [H(\gamma^T \boldsymbol{\xi}_{k,\boldsymbol{q}})] + \frac{2m\gamma_k}{m+n-2} \overline{E}^{\boldsymbol{z_q}}_{\boldsymbol{X}^*} \left[\left(1 + \frac{\boldsymbol{X}^{*T}\boldsymbol{X}^*}{m}\right) H'(\boldsymbol{\gamma^T X^*})\right],$$

*$k = 1, 2, \ldots, n$, $\mathbf{Z} \sim GSSt_n(\mathbf{0}, \mathbf{I_n}, \boldsymbol{\gamma}, m, H(\cdot))$, $\mathbf{X}^* \sim St_n(\mathbf{0}, \mathbf{I_n}, m)$ (Student-t distribution), $\mathbf{W}_{-k} \sim St_{n-1}(\mathbf{0}, \triangle_k, m-1)$, $\triangle_k = [m(1 + z^2_{k,q}/m)/(m-1)]\mathbf{I}_{n-1}$ and*

$$b^*_{n,k} = \frac{\Gamma(\frac{m-1}{2})(\frac{m-1}{m})^{(n-1)/2}}{\Gamma(\frac{m}{2})\sqrt{\pi/m}} \left(1 + \frac{z^2_{k,q}}{m}\right)^{-(m+n-2)/2}.$$

REMARK 8: *If $H(\cdot) = T(\cdot)$ (the cdf of 1-dimensional standard Student-t distribution) in Example 4.2, MTCE for n-dimensional skew Student-t distribution will be obtained:*

$$MTCE_{\mathbf{Y}}(\boldsymbol{y_q}) = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2} \frac{\boldsymbol{\delta_q}}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})}, \tag{17}$$

*where $\boldsymbol{\delta_q} = (\delta_{1,\boldsymbol{q}}, \delta_{2,\boldsymbol{q}}, \ldots, \delta_{n,\boldsymbol{q}})^T$,*

$$\delta_{k,\boldsymbol{q}} = b^*_{n,k} \overline{E}^{\boldsymbol{z}_{-k,\boldsymbol{q}}}_{\boldsymbol{W}_{-k}} [T(\gamma^T \boldsymbol{\xi}_{k,\boldsymbol{q}})] + \frac{2m\gamma_k}{m+n-2} \overline{E}^{\boldsymbol{z_q}}_{\boldsymbol{X}^*} \left[\left(1 + \frac{\boldsymbol{X}^{*T}\boldsymbol{X}^*}{m}\right) \boldsymbol{t}(\boldsymbol{\gamma^T X^*})\right],$$

*$k = 1, 2, \ldots, n$, $\mathbf{Z} \sim GSSt_n(\mathbf{0}, \mathbf{I_n}, \boldsymbol{\gamma}, m, T(\cdot))$, and $\boldsymbol{t}(\cdot)$ is the pdf of 1-dimensional standard Student-t distribution. Furthermore, $b^*_{n,k}$, $\mathbf{X}^*$ and $\mathbf{W}_{-k}$ are the same as in Example 4.2.*

REMARK 9: *If* $H(\cdot)=\frac{1}{2}$ *in Example 4.2, we will obtain a generalized formula of MTCE for Student-t distribution in Landsman et al. [6]:*

$$MTCE_{\mathbf{Y}}(\boldsymbol{y_q})=\boldsymbol{\mu}+\boldsymbol{\Sigma}^{1/2}\frac{\boldsymbol{\delta_q}}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})},$$

*where* $\boldsymbol{\delta_q}=(\delta_{1,\boldsymbol{q}},\delta_{2,\boldsymbol{q}},\ldots,\delta_{n,\boldsymbol{q}})^T$,

$$\delta_{k,\boldsymbol{q}}=\frac{b^*_{n,k}}{2}\overline{F}_{\boldsymbol{W}_{-k}}(\boldsymbol{z}_{-k,\boldsymbol{q}}),\quad k=1,2,\ldots,n,$$

*and* $\mathbf{Z}\sim St_n(\mathbf{0},\mathbf{I_n},m)$. *In addition,* $b^*_{n,k}$ *and* $\mathbf{W}_{-k}$ *are the same as in Example 4.2.*

EXAMPLE 4.3 (Generalized skew-logistic distribution): *The density function of a generalized skew-logistic random vector* $\mathbf{Y}$*, with location parameter* $\boldsymbol{\mu}$*, scale matrix* $\boldsymbol{\Sigma}$ *and skewing function* $H(\cdot):\mathbb{R}\to\mathbb{R}$*, is given by*

$$f_{\mathbf{Y}}(\boldsymbol{y})=\frac{2c_n}{\sqrt{|\boldsymbol{\Sigma}|}}\frac{\exp\{-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\}}{[1+\exp\{-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\}]^2}H(\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y}-\boldsymbol{\mu})),$$

$\boldsymbol{y}\in\mathbb{R}^n$, *where* $\boldsymbol{\gamma}=(\gamma_1,\gamma_2,\ldots,\gamma_n)^T$,

$$\begin{aligned}c_n&=\frac{\Gamma(n/2)}{(2\pi)^{n/2}}\left[\int_0^\infty t^{n/2-1}\frac{\exp(-t)}{[1+\exp(-t)]^2}\,dt\right]^{-1}\\&=\frac{1}{(2\pi)^{n/2}\Psi_2^*(-1,\frac{n}{2},1)}.\end{aligned}$$

*We denote it by* $\mathbf{Y}\sim GSLo_n(\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\gamma},H(\cdot))$. *The density generator in this case is*

$$g_n(u)=c_n\frac{\exp\{-u\}}{[1+\exp\{-u\}]^2},\quad \text{and so } \overline{G}_n(u)=c_n\frac{\exp\{-u\}}{1+\exp\{-u\}},$$

*and* $H(\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y}-\boldsymbol{\mu}))=H(\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y}-\boldsymbol{\mu}))$. *MTCE is given by*

$$MTCE_{\mathbf{Y}}(\boldsymbol{y_q})=\boldsymbol{\mu}+\boldsymbol{\Sigma}^{1/2}\frac{\boldsymbol{\delta_q}}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})},\tag{18}$$

*where* $\boldsymbol{\delta_q}=(\delta_{1,\boldsymbol{q}},\delta_{2,\boldsymbol{q}},\ldots,\delta_{n,\boldsymbol{q}})^T$,

$$\delta_{k,\boldsymbol{q}}=\frac{2c_n}{c^*_{n-1,k}}\overline{E}^{\boldsymbol{z}_{-k,\boldsymbol{q}}}_{\boldsymbol{W}_{-k}}[H(\boldsymbol{\gamma}^T\boldsymbol{\xi}_{k,\boldsymbol{q}})]+2\gamma_k\overline{E}^{\boldsymbol{z_q}}_{\boldsymbol{X}^*}\left[\left(1+\exp\left\{-\frac{\boldsymbol{X}^{*T}\boldsymbol{X}^*}{2}\right\}\right)H'(\boldsymbol{\gamma^T}\boldsymbol{X}^*)\right],$$

$k=1,2,\ldots,n$, $\mathbf{Z}\sim GSLo_n(\mathbf{0},\mathbf{I_n},\pi(\cdot))$, $\mathbf{X}^*\sim Lo_n(\mathbf{0},\mathbf{I_n})$ *(logistic distribution),*

$$\begin{aligned}c^*_{n-1,k}&=\frac{\Gamma((n-1)/2)\exp\{\frac{z^2_{k,\boldsymbol{q}}}{2}\}}{(2\pi)^{(n-1)/2}}\left[\int_0^\infty\frac{t^{(n-3)/2}\exp\{-t\}}{1+\exp\{-\frac{z^2_{k,\boldsymbol{q}}}{2}\}\exp\{-t\}}\,dt\right]^{-1}\\&=\frac{\exp\{\frac{z^2_{k,\boldsymbol{q}}}{2}\}}{(2\pi)^{(n-1)/2}\Psi_1^*(-\exp\{-\frac{z^2_{k,\boldsymbol{q}}}{2}\},\frac{n-1}{2},1)},\end{aligned}$$

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*and pdf of* $\mathbf{W}_{-k}$*:*

$$f_{\mathbf{W}_{-k}}(\boldsymbol{t}) = c^*_{n-1,k}\frac{\exp\{-\frac{\boldsymbol{t}^T\boldsymbol{t}}{2} - \frac{z^2_{k,\boldsymbol{q}}}{2}\}}{1+\exp\{-\frac{\boldsymbol{t}^T\boldsymbol{t}}{2} - \frac{z^2_{k,\boldsymbol{q}}}{2}\}}, \quad k = 1, 2, \ldots, n, \boldsymbol{t} \in \mathbb{R}^{n-1}.$$

*In addition,* $\Psi^*_\mu(z, s, a)$*is the generalized Hurwitz–Lerch zeta function defined by (cf. [10])*

$$\Psi^*_\mu(z, s, a) = \frac{1}{\Gamma(\mu)}\sum_{n=0}^{\infty}\frac{\Gamma(\mu+n)}{n!}\frac{z^n}{(n+a)^s},$$

*which has an integral representation*

$$\Psi^*_\mu(z, s, a) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-at}}{(1-ze^{-t})^\mu}\, dt,$$

*where* $\mathcal{R}(a) > 0$*;* $\mathcal{R}(s) > 0$ *when* $|z| \le 1$ $(z \ne 1)$*;* $\mathcal{R}(s) > 1$ *when* $z = 1$*. Therefore,*

$$\frac{c_n}{c^*_{n-1,k}} = \frac{\Psi^*_1\left(-\exp\{-\frac{z^2_{k,\boldsymbol{q}}}{2}\}, \frac{n-1}{2}, 1\right)\phi(z_{k,\boldsymbol{q}})}{\Psi^*_2(-1, \frac{n}{2}, 1)},$$

*where* $\phi(\cdot)$ *is pdf of* $1$*-dimensional standard normal distribution.*

Remark 10: *If* $H(\cdot) = Lo(\cdot)$*(the cdf of* $1$*-dimensional standard logistic) in Example 4.3 , MTCE for* $n$*-dimensional skew-logistic distribution will be obtained:*

$$MTCE_{\mathbf{Y}}(\boldsymbol{y}_{\boldsymbol{q}}) = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\frac{\boldsymbol{\delta}_{\boldsymbol{q}}}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z}_{\boldsymbol{q}})}, \tag{19}$$

*where* $\boldsymbol{\delta}_{\boldsymbol{q}} = (\delta_{1,\boldsymbol{q}}, \delta_{2,\boldsymbol{q}}, \ldots, \delta_{n,\boldsymbol{q}})^T$*,*

$$\delta_{k,\boldsymbol{q}} = \frac{2c_n}{c^*_{n-1,k}}\overline{E}^{\boldsymbol{z}_{-k,\boldsymbol{q}}}_{\boldsymbol{W}_{-k}}[Lo(\boldsymbol{\gamma}^T\boldsymbol{\xi}_{k,\boldsymbol{q}})] + 2\gamma_k\overline{E}^{\boldsymbol{z}_{\boldsymbol{q}}}_{\boldsymbol{X}^*}\left[\left(1+\exp\left\{-\frac{\boldsymbol{X}^{*T}\boldsymbol{X}^*}{2}\right\}\right)\boldsymbol{lo}(\boldsymbol{\gamma}^{\boldsymbol{T}}\boldsymbol{X}^*)\right],$$

$k = 1, 2, \ldots, n,$ $\mathbf{Z} \sim GSLo_n(\mathbf{0}, \mathbf{I_n}, \boldsymbol{\gamma}, Lo(\cdot))$*, and* $\boldsymbol{lo}(\cdot)$ *is pdf of* $1$*-dimensional standard logistic distribution. Furthermore,* $c_n$*,* $c^*_{n-1,k}$*,* $\mathbf{X}^*$ *and* $\mathbf{W}_{-k}$ *are the same as in Example 4.3.*

Remark 11: *If* $H(\cdot) = \frac{1}{2}$ *in Example 4.3, we will obtain a generalization of MTCE for logistic distribution in Landsman et al. [6]:*

$$MTCE_{\mathbf{Y}}(\boldsymbol{y}_{\boldsymbol{q}}) = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\frac{\boldsymbol{\delta}_{\boldsymbol{q}}}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z}_{\boldsymbol{q}})},$$

*where* $\boldsymbol{\delta}_{\boldsymbol{q}} = (\delta_{1,\boldsymbol{q}}, \delta_{2,\boldsymbol{q}}, \ldots, \delta_{n,\boldsymbol{q}})^T$*,*

$$\delta_{k,\boldsymbol{q}} = \frac{c_n}{c^*_{n-1,k}}\overline{F}_{\boldsymbol{W}_{-k}}(\boldsymbol{z}_{-k,\boldsymbol{q}}), \quad k = 1, 2, \ldots, n,$$

*and* $\mathbf{Z} \sim Lo_n(\mathbf{0}, \mathbf{I_n})$*. In addition,* $c_n$*,* $c^*_{n-1,k}$ *and* $\mathbf{W}_{-k}$ *are the same as in Example 4.3.*

EXAMPLE 4.4 (Generalized skew-Laplace distribution): *A n-variate generalized skew-Laplace random vector* $\mathbf{Y}$*, with location parameter* $\boldsymbol{\mu}$*, scale matrix* $\boldsymbol{\Sigma}$ *and skewing function* $H(\cdot): \mathbb{R} \to \mathbb{R}$*, has its density function as*

$$f_{\mathbf{Y}}(\boldsymbol{y}) = \frac{2c_n}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\{-[(\boldsymbol{y}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})]^{1/2}\} H(\boldsymbol{\gamma}^T \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y}-\boldsymbol{\mu})), \quad \boldsymbol{y} \in \mathbb{R}^n,$$

*where* $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T$ *and* $c_n = \Gamma(n/2)/2\pi^{n/2}\Gamma(n)$*. We denote it by* $\mathbf{Y} \sim GSLa_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma}, H(\cdot))$*. In this case,* $g_n(u) = c_n \exp\{-\sqrt{2u}\}$*,* $H(\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y}-\boldsymbol{\mu})) = H(\boldsymbol{\gamma}^T \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y}-\boldsymbol{\mu}))$ *and* $\overline{G}_n(u) = c_n(1+\sqrt{2u})\exp\{-\sqrt{2u}\}$*. MTCE is given by*

$$MTCE_{\mathbf{Y}}(\boldsymbol{y_q}) = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2} \frac{\boldsymbol{\delta_q}}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})}, \tag{20}$$

*where* $\boldsymbol{\delta_q} = (\delta_{1,\boldsymbol{q}}, \delta_{2,\boldsymbol{q}}, \ldots, \delta_{n,\boldsymbol{q}})^T$,

$$\delta_{k,\boldsymbol{q}} = \frac{2c_n}{c^*_{n-1,k}} \overline{E}^{\boldsymbol{z}_{-k,\boldsymbol{q}}}_{\boldsymbol{W}_{-k}}[H(\boldsymbol{\gamma}^T \boldsymbol{\xi}_{k,\boldsymbol{q}})] + 2\gamma_k \overline{E}^{\boldsymbol{z_q}}_{\boldsymbol{X}^*}[(1+\sqrt{\boldsymbol{X}^{*T}\boldsymbol{X}^*}) H'(\boldsymbol{\gamma}^T \boldsymbol{X}^*)],$$

$k = 1, 2, \ldots, n$, $\mathbf{Z} \sim GSLa_n(\mathbf{0}, \mathbf{I_n}, \boldsymbol{\gamma}, H(\cdot))$, $\mathbf{X}^* \sim La_n(\mathbf{0}, \mathbf{I_n})$ *(Laplace distribution),*

$$c^*_{n-1,k} = \frac{\Gamma\left(\frac{n-1}{2}\right)}{(2\pi)^{(n-1)/2}} \left[\int_0^\infty t^{\frac{n-3}{2}} (1+\sqrt{t+z^2_{k,\boldsymbol{q}}}) \exp\{-\sqrt{t+z^2_{k,\boldsymbol{q}}}\}\, dt\right]^{-1},$$

*and pdf of* $\mathbf{W}_{-k}$*:*

$$f_{\mathbf{W}_{-k}}(\boldsymbol{t}) = c^*_{n-1,k}(1+\sqrt{\boldsymbol{t}^T\boldsymbol{t}+z^2_{k,\boldsymbol{q}}}) \exp\{-\sqrt{\boldsymbol{t}^T\boldsymbol{t}+z^2_{k,\boldsymbol{q}}}\}, \quad k = 1, 2, \ldots, n,$$

$\boldsymbol{t} \in \mathbb{R}^{n-1}$.

REMARK 12: *If* $H(\cdot) = La(\cdot)$ *(the cdf of 1-dimensional standard Laplace) in Example 4.4, MTCE for n-dimensional skew-Laplace distribution will be obtained:*

$$MTCE_{\mathbf{Y}}(\boldsymbol{y_q}) = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2} \frac{\boldsymbol{\delta_q}}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})}, \tag{21}$$

*where* $\boldsymbol{\delta_q} = (\delta_{1,\boldsymbol{q}}, \delta_{2,\boldsymbol{q}}, \ldots, \delta_{n,\boldsymbol{q}})^T$,

$$\delta_{k,\boldsymbol{q}} = \frac{2c_n}{c^*_{n-1,k}} \overline{E}^{\boldsymbol{z}_{-k,\boldsymbol{q}}}_{\boldsymbol{W}_{-k}}[La(\boldsymbol{\gamma}^T \boldsymbol{\xi}_{k,\boldsymbol{q}})] + 2\gamma_k \overline{E}^{\boldsymbol{z_q}}_{\boldsymbol{X}^*}[(1+\sqrt{\boldsymbol{X}^{*T}\boldsymbol{X}^*}) \boldsymbol{la}(\boldsymbol{\gamma}^T \boldsymbol{X}^*)],$$

$k = 1, 2, \ldots, n$, $\mathbf{Z} \sim GSLa_n(\mathbf{0}, \mathbf{I_n}, \boldsymbol{\gamma}, La(\cdot))$, *and* $\boldsymbol{la}(\cdot)$ *is pdf of 1-dimensional standard Laplace distribution. Furthermore,* $c_n$*,* $c^*_{n-1,k}$*,* $\mathbf{X}^*$ *and* $\mathbf{W}_{-k}$ *are the same as in Example 4.4.*

REMARK 13: *If* $H(\cdot) = \frac{1}{2}$ *in Example 4.4, we will obtain a generalized formula of MTCE for Laplace distribution in Landsman et al. [6]:*

$$MTCE_{\mathbf{Y}}(\boldsymbol{y_q}) = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2} \frac{\boldsymbol{\delta_q}}{\overline{F}_{\mathbf{Z}}(\boldsymbol{z_q})},$$

*where* $\boldsymbol{\delta_q} = (\delta_{1,\boldsymbol{q}}, \delta_{2,\boldsymbol{q}}, \ldots, \delta_{n,\boldsymbol{q}})^T$,

$$\delta_{k,\boldsymbol{q}} = \frac{c_n}{c^*_{n-1,k}} \overline{F}_{\boldsymbol{W}_{-k}}(\boldsymbol{z}_{-k,\boldsymbol{q}}), \quad k = 1, 2, \ldots, n,$$

*and* $\mathbf{Z} \sim La_n(\mathbf{0}, \mathbf{I_n})$*. In addition,* $c_n$*,* $c^*_{n-1,k}$ *and* $\mathbf{W}_{-k}$ *are the same as in Example 4.4.*

## 5. NUMERICAL ILLUSTRATION

We provide a numerical illustration of the TCE risk measure for the normal ($N$), skew-normal (SN), Student-$t$ ($St$) and skew Student-$t$ (SS$t$) distributions. In addition, we also consider MTCE risk measure for the normal ($N$) and skew-normal (SN) distributions.

We consider $N_1(\mu, \sigma^2)$, $\mathrm{SN}_1(\mu, \sigma^2, \gamma)$, $St_1(\mu, \sigma^2, m)$ and $\mathrm{SS}t_1(\mu, \sigma^2, m, \gamma)$, with $\mu = 1.4$, $\sigma = 1.33$, $m = 4$ and $\gamma = -1.0, 2.0$. The results are presented in Tables 1 and 2:

TABLE 1. The TCE of $N$ and SN ($\gamma = -1.0, 2.0$) distributions for $q = 0.90, 0.95, 0.98$

| TCE | | Distribution | |
|---|---|---|---|
| $q$ | SN($-1.0$) | $N$ | SN (2.0) |
| 0.90 | 2.521028 | 3.734046 | 4.132160 |
| 0.95 | 2.830467 | 4.143596 | 4.450346 |
| 0.98 | 3.209941 | 4.619739 | 4.871836 |

TABLE 2. The TCE of $St$ and SS$t$ ($\gamma = -1.0, 2.0$) distributions for $q = 0.90$, 0.95, 0.98

| TCE | | Distribution | |
|---|---|---|---|
| $q$ | SS$t$($-1.0$) | $St$ | SS$t$(2.0) |
| 0.90 | 2.691679 | 4.724058 | 5.664056 |
| 0.95 | 3.097983 | 5.659712 | 6.705451 |
| 0.98 | 3.601481 | 7.080627 | 8.273033 |

In addition, we consider $\mathbf{U} = (U_1, U_2, U_3)^T \sim N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\mathbf{V} = (V_1, V_2, V_3)^T \sim \mathrm{SN}_3(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma})$, with

$$\boldsymbol{\mu} = \begin{pmatrix} 1.4 \\ 1.1 \\ 3.4 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} 1.33 & -0.067 & 2.63 \\ -0.067 & 0.25 & -0.50 \\ 2.63 & -0.50 & 5.76 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\gamma} = \begin{pmatrix} 2.2 \\ -0.046 \\ -1.38 \end{pmatrix}.$$

We let $\boldsymbol{q} = (0.90, 0.95, 0.98)^T$, then the results are presented in Table 3:

TABLE 3. The MTCE of $N$ and SN distributions for $\boldsymbol{q} = (0.90, 0.95, 0.98)^T$

| MTCE | | Vector | |
|---|---|---|---|
| Distribution | $U_1(V_1)$ | $U_2(V_2)$ | $U_3(V_3)$ |
| $N$ | 3.4240 | 2.0915 | 9.1187 |
| SN | $1.3825 \times 10^3$ | $0.5788 \times 10^3$ | $1.7689 \times 10^3$ |

REMARK 14: *From Tables 1 and 2, we can observe that TCE increases with the increase in $\gamma$. From Table 3, we can find that the component-wise MTCE for skewed Normal distributions are greater than that for corresponding non skewed distribution. As one of the reviewers pointed out that it maybe a deep theoretical fact. However, we do not prove this statement at this moment.*

## 6. CONCLUSIONS AND DIRECTIONS FOR FURTHER RESEARCH

This paper has introduced MTCE for generalized skew-elliptical distributions, which is a generalization of MTCE for elliptical distributions [6]. As special cases, generalized skew-normal, generalized skew Student-$t$, generalized skew-logistic and generalized skew-Laplace distributions are considered. To illustrate our results can be computed in the theorems, the numerical illustrations of the obtained results are given. Furthermore, in [7], the authors introduced and provided expressions for multivariate tail covariance (MTCov) and multivariate tail correlation (MTCorr) matrices for the case of elliptical distribution. These matrices are very important for the tail analysis of the data, it is worthwhile extending these presentations to the generalized skew-elliptical distributions. We hope that these important problems can be addressed in future research.

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*Conflicts of Interest*

The authors declare that they have no conflicts of interest.

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