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THE PROJECTIVE LEAVITT COMPLEX

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Abstract For a finite quiver Q without sources, we consider the corresponding radical square zero algebra A. We construct an explicit compact generator for the homotopy category of acyclic complexes of projective A-modules. We call such a generator the projective Leavitt complex of Q. This terminology is justified by the following result: the opposite differential graded endomorphism algebra of the projective Leavitt complex of Q is quasi-isomorphic to the Leavitt path algebra of Q^{op}. Here, Q^{op} is the opposite quiver of Q, and the Leavitt path algebra of Q^{op} is naturally $\mathbb{Z}$-graded and viewed as a differential graded algebra with trivial differential.

Keywords: Leavitt path algebra; projective Leavitt complex; compact generator; dg quasi-balanced module

2010 *Mathematics subject classification:* Primary 16G20; 16E45; 18E30; 18G35

1. Introduction

In the last decade, Leavitt path algebras of directed graphs (or quivers) [**1**, **5**] were introduced as an algebraization of graph C^*-algebras [**16**, **22**] and, in particular, Cuntz–Krieger algebras [**11**]. This class of algebras has been attracting significant attention, with interest in whether K-theoretic data can be used to classify various classes of Leavitt path algebras, inspired by the Kirchberg–Phillips classification theorem for C^*-algebras [**21**]. One can also find conditions on graphs such that the associated Leavitt path algebras have specific properties, as demonstrated in many papers, for instance [**1**–**3**, **6**].

For a finite quiver Q, Smith [**23**] describes the quotient category

$$\mathrm{QGr}(kQ) := \mathrm{Gr}(kQ)/\mathrm{Fdim}(kQ)$$

of graded kQ-modules modulo those that are the sum of their finite-dimensional submodules in terms of the category of graded modules over the Leavitt path algebra of Q^o over a field k. Here, Q^o is the quiver without sources or sinks that is obtained by repeatedly removing all sinks and sources from Q. The full subcategory qgr(kQ) of finitely presented objects in QGr(kQ) is triangulated equivalent to the singularity category [**8**, **20**] of the corresponding radical square zero algebra; see [**23**, Theorem 7.2].

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Let A be a finite-dimensional algebra over a field k. We denote by $\mathbf{K}_{\mathrm{ac}}(A\text{-Proj})$ the homotopy category of acyclic complexes of projective A-modules. This category is a compactly generated triangulated category whose subcategory of compact objects is triangle equivalent to the opposite category of the singularity category of the opposite algebra A^{op}.

The homotopy category $\mathbf{K}_{\mathrm{ac}}(A\text{-Proj})$ was described as a derived category of the Leavitt path algebra of Q^{op} viewed as a differential graded algebra with trivial differential; see [**10**, Theorem 6.2]. Here, Q^{op} is the opposite quiver of Q. The homotopy category of acyclic complexes of injective modules over A was also described in terms of Leavitt path algebra; see [**10**, Theorem 6.1].

In this paper, we construct an explicit compact generator for the homotopy category $\mathbf{K}_{\mathrm{ac}}(A\text{-Proj})$ in the case where A is an algebra with radical square zero. The compact generator is called the *projective Leavitt complex*. We prove that the opposite differential graded endomorphism algebra of the projective Leavitt complex of a finite quiver without sources is quasi-isomorphic to the Leavitt path algebra of the opposite quiver. Here, the Leavitt path algebra is naturally $\mathbb{Z}$-graded and viewed as a differential graded algebra with trivial differential.

Let Q be a finite quiver without sources, and let $A = kQ/J^2$ be the corresponding algebra with radical square zero. We introduce the projective Leavitt complex $\mathcal{P}^\bullet$ of Q in Definition 2.4. Then we prove that $\mathcal{P}^\bullet$ is acyclic; see Proposition 2.7.

Denote by $L_k(Q)$ the Leavitt path algebra of Q over k, which is naturally $\mathbb{Z}$-graded. We consider the Leavitt path algebra $L_k(Q^{\mathrm{op}})$ of Q^{op} as a differential graded algebra with trivial differential.

The following is the main result, which combines Theorems 3.7 and 5.2.

Theorem. *Let Q be a finite quiver without sources, and $A = kQ/J^2$ be the corresponding finite-dimensional algebra with radical square zero.*

(1) *The projective Leavitt complex $\mathcal{P}^\bullet$ of Q is a compact generator for the homotopy category $\mathbf{K}_{\mathrm{ac}}(A\text{-Proj})$.*

(2) *The opposite differential graded endomorphism algebra of the projective Leavitt complex $\mathcal{P}^\bullet$ of Q is quasi-isomorphic to the Leavitt path algebra $L_k(Q^{\mathrm{op}})$.* □

For the construction of the projective Leavitt complex $\mathcal{P}^\bullet$, we use the basis of the Leavitt path algebra $L_k(Q^{\mathrm{op}})$ given by [**4**, Theorem 1].

For the proof of (1), we construct subcomplexes of $\mathcal{P}^\bullet$. For (2), we actually prove that the projective Leavitt complex has the structure of a differential graded A-$L_k(Q^{\mathrm{op}})$-bimodule, which is right quasi-balanced. Here, we consider A as a differential graded algebra concentrated on degree zero, while $L_k(Q^{\mathrm{op}})$ is naturally $\mathbb{Z}$-graded and viewed as a differential graded algebra with trivial differential.

The paper is structured as follows. In § 2, we introduce the projective Leavitt complex $\mathcal{P}^\bullet$ of Q and prove that it is acyclic. In § 3, we recall some notation and prove that the projective Leavitt complex $\mathcal{P}^\bullet$ is a compact generator of the homotopy category of acyclic complexes of projective A-modules. In § 4, we recall some facts of the Leavitt path algebra and endow the projective Leavitt complex $\mathcal{P}^\bullet$ with a differential graded $L_k(Q^{\mathrm{op}})$-module structure, which makes it become an A-$L_k(Q^{\mathrm{op}})$-bimodule. In § 5, we prove that

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the opposite differential graded endomorphism algebra of $\mathcal{P}^\bullet$ is quasi-isomorphic to the Leavitt path algebra $L_k(Q^{\rm op})$.

2. The projective Leavitt complex of a finite quiver without sources

In this section, we introduce the projective Leavitt complex of a finite quiver without sources, which is an acyclic complex of projective modules over the corresponding finite-dimensional algebra with radical square zero.

2.1. The projective Leavitt complex

Recall that a quiver $Q = (Q_0, Q_1; s, t)$ consists of a set Q_0 of vertices, a set Q_1 of arrows and two maps $s, t : Q_1 \to Q_0$, which associate with each arrow α its starting vertex $s(\alpha)$ and its terminating vertex $t(\alpha)$, respectively. A quiver Q is finite if both the sets Q_0 and Q_1 are finite.

A path in the quiver Q is a sequence $p = \alpha_n \cdots \alpha_2\alpha_1$ of arrows with $t(\alpha_j) = s(\alpha_{j+1})$ for $1 \leq j \leq n-1$. The length of p, denoted by $l(p)$, is n. The starting vertex of p, denoted by $s(p)$, is $s(\alpha_1)$. The terminating vertex of p, denoted by $t(p)$, is $t(\alpha_n)$. We identify an arrow with a path of length one. We associate to each vertex $i \in Q_0$ a trivial path e_i of length zero. Set $s(e_i) = i = t(e_i)$. Denote by Q_n the set of all paths in Q of length n for each $n \geq 0$.

Recall that a vertex of Q is a sink if there is no arrow starting at it and a source if there is no arrow terminating at it. Recall that for a vertex i that is not a sink, we can choose an arrow β with $s(\beta) = i$, which is called the *special arrow* starting at vertex i; see [**4**]. For a vertex i which is not a source, fix an arrow γ with $t(\gamma) = i$. We call the fixed arrow the *associated arrow* terminating at i. For an associated arrow α, we set

$$T(\alpha) = \{\beta \in Q_1 \mid t(\beta) = t(\alpha),\ \beta \neq \alpha\}. \tag{2.1}$$

Definition 2.1. For two paths $p = \alpha_m \cdots \alpha_2\alpha_1$ and $q = \beta_n \cdots \beta_2\beta_1$ with $m, n \geq 1$, we call the pair (p, q) an *associated pair* in Q if $s(p) = s(q)$, and either $\alpha_1 \neq \beta_1$ or $\alpha_1 = \beta_1$ is not associated. In addition, we call $(p, e_{s(p)})$ and $(e_{s(p)}, p)$ *associated pairs* in Q for each path p in Q.

For each vertex $i \in Q_0$ and $l \in \mathbb{Z}$, set

$$\mathbf{\Lambda}_i^l = \{(p, q) \mid (p, q) \text{ is an associated pair with } l(q) - l(p) = l \text{ and } t(p) = i\}. \tag{2.2}$$

Lemma 2.2. *Let Q be a finite quiver without sources. The above set $\mathbf{\Lambda}_i^l$ is not empty for each vertex i and each integer l.*

Proof. Recall that the opposite quiver $Q^{\rm op}$ of the quiver Q has arrows with opposite directions. For each vertex $i \in Q_0$, fix the special arrow of $Q^{\rm op}$ starting at i as the opposite arrow of the associated arrow of Q terminating at i. Observe that for each vertex i and each integer l, $\mathbf{\Lambda}_i^l$ is one-to-one corresponded to $\{(q^{\rm op}, p^{\rm op}) \mid (q^{\rm op}, p^{\rm op})$ is an admissible pair in $Q^{\rm op}$ with $l(p^{\rm op}) - l(q^{\rm op}) = -l$ and $s(p^{\rm op}) = i\}$. Here, refer to [**17**, Definition 2.1] for the definition of an admissible pair. By [**17**, Lemma 2.2], the latter set is not empty. The proof is completed. □

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Let k be a field and Q be a finite quiver. For each $n \geq 0$, denote by kQ_n the k-vector space with basis Q_n. The path algebra kQ of the quiver Q is defined as $kQ = \bigoplus_{n\geq 0} kQ_n$, whose multiplication is given as follows: for two paths p and q, if $s(p) = t(q)$, then the product pq is their concatenation; otherwise, we set the product pq to be zero. Here, we write the concatenation of paths from right to left.

We observe that for any path p and vertex i, $pe_i = \delta_{i,s(p)}p$ and $e_ip = \delta_{i,t(p)}p$. Here, δ denotes the Kronecker symbol. It follows that the unit of kQ equals $\sum_{i\in Q_0} e_i$. Denote by J the two-sided ideal of kQ generated by arrows.

Consider the quotient algebra $A = kQ/J^2$; it is a finite-dimensional algebra with radical square zero. Indeed, $A = kQ_0 \oplus kQ_1$ as a k-vector space, with its Jacobson radical $\mathrm{rad}A = kQ_1$ satisfying $(\mathrm{rad}A)^2 = 0$. For each vertex i and arrow α, we identify e_i and α with their canonical images in A.

Denote by $P_i = Ae_i$ the indecomposable projective left A-module for $i \in Q_0$. We have the following observation.

Lemma 2.3. *Let i, j be two vertices in Q, and let $f : P_i \to P_j$ be a k-linear map. Then f is a left A-module morphism if and only if*

$$\begin{cases} f(e_i) = \delta_{i,j}\lambda e_j + \displaystyle\sum_{\{\beta\in Q_1 \mid s(\beta)=j, t(\beta)=i\}} \mu(\beta)\beta \\ f(\alpha) = \delta_{i,j}\lambda\alpha \end{cases}$$

with λ and $\mu(\beta)$ scalars for all $\alpha \in Q_1$ with $s(\alpha) = i$.

For a set X and an A-module M, the coproduct $M^{(X)}$ will be understood as $\bigoplus_{x\in X} M\zeta_x$, where each component $M\zeta_x$ is M. For an element $m \in M$, we use $m\zeta_x$ to denote the corresponding element in $M\zeta_x$.

For a path $p = \alpha_n \cdots \alpha_2\alpha_1$ in Q of length $n \geq 2$, we denote by $\widehat{p} = \alpha_{n-1}\cdots\alpha_1$ and $\widetilde{p} = \alpha_n \cdots \alpha_2$ the two *truncations* of p. For an arrow α, denote $\widehat{\alpha} = e_{s(\alpha)}$ and $\widetilde{\alpha} = e_{t(\alpha)}$.

Definition 2.4. Let Q be a finite quiver without sources. The *projective Leavitt complex* $\mathcal{P}^\bullet = (\mathcal{P}^l, \delta^l)_{l\in\mathbb{Z}}$ of Q is defined as follows:

(1) the lth component $\mathcal{P}^l = \bigoplus_{i\in Q_0} P_i^{(\mathbf{\Lambda}_i^l)}$;

(2) the differential $\delta^l : \mathcal{P}^l \longrightarrow \mathcal{P}^{l+1}$ is given by $\delta^l(\alpha\zeta_{(p,q)}) = 0$ and

$$\delta^l(e_i\zeta_{(p,q)}) = \begin{cases} \beta\zeta_{(\widehat{p},q)} & \text{if } p = \beta\widehat{p}, \\ \displaystyle\sum_{\{\beta\in Q_1 \mid t(\beta)=i\}} \beta\zeta_{(e_{s(\beta)},q\beta)} & \text{if } l(p) = 0, \end{cases}$$

for any $i \in Q_0$, $(p, q) \in \mathbf{\Lambda}_i^l$ and $\alpha \in Q_1$ with $s(\alpha) = i$.

Each component $\mathcal{P}^l$ is a projective A-module. The differentials δ^l are A-module morphisms; compare Lemma 2.3. It is straightforward to see that $\delta^{l+1} \circ \delta^l = 0$ for each $l \in \mathbb{Z}$. In summary, $\mathcal{P}^\bullet$ is a complex of projective A-modules.

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2.2. The acyclicity of the projective Leavitt complex

We will show that the projective Leavitt complex is acyclic.

In what follows, $f : V \to V'$ is a k-linear map between two vector spaces V and V'. Suppose that B and B' are k-bases of V and V', respectively. We say that the triple (f, B, B') satisfies condition (X) if $f(B) \subseteq B'$ and the restriction of f on B is injective. In this case, we have $\mathrm{Ker} f = 0$.

We suppose further that there are disjoint unions $B = B_0 \cup B_1 \cup B_2$ and $B' = B'_0 \cup B'_1$. We say that the triple (f, B, B') satisfies condition (W) if the following statements hold.

(W1) $f(b) = 0$ for each $b \in B_0$.

(W2) $f(B_1) \subseteq B'_1$ and (f_1, B_1, B') satisfies condition (X), where f_1 is the restriction of f to the subspace spanned by B_1.

(W3) For $b \in B_2$, $f(b) = b_0 + \sum_{c \in B_1(b)} f(c)$ for some $b_0 \in B'_0$ and some finite subset $B_1(b) \subseteq B_1$. Moreover, if $b, b' \in B_2$ and $b \neq b'$, then $b_0 \neq b'_0$.

We have the following observation. The proof of it is similar to that of [**17**, Lemma 2.7]. We omit it here.

Lemma 2.5. *Assume that (f, B, B') satisfies Condition (W). Then B_0 is a k-basis of $\mathrm{Ker} f$ and $f(B_1) \cup \{b_0 \mid b \in B_2\}$ is a k-basis of $\mathrm{Im} f$.*

From now on, Q is a finite quiver without sources. We consider the differential $\delta^l : \mathcal{P}^l \to \mathcal{P}^{l+1}$ in Definition 2.4. We have the following k-basis of $\mathcal{P}^l$:

$$B^l = \{e_i\zeta_{(p,q)}, \alpha\zeta_{(p,q)} \mid i \in Q_0, (p, q) \in \mathbf{\Lambda}^l_i \text{ and } \alpha \in Q_1 \text{ with } s(\alpha) = i\}.$$

Denote by $B^l_0 = \{\alpha\zeta_{(p,q)} \mid i \in Q_0, (p, q) \in \mathbf{\Lambda}^l_i \text{ and } \alpha \in Q_1 \text{ with } s(\alpha) = i\}$ a subset of B^l. Set

$$B^l_2 = \{e_i\zeta_{(e_i,q)} \mid i \in Q_0, (e_i, q) \in \mathbf{\Lambda}^l_i\}$$

for $l \geq 0$. If $l < 0$, put $B^l_2 = \emptyset$. Take $B^l_1 = B^l \setminus (B^l_0 \cup B^l_2)$. Then we have the disjoint union $B^l = B^l_0 \cup B^l_1 \cup B^l_2$. Set $B'^l_0 = \{\beta\zeta_{(e_{s(q)},q)} \mid i \in Q_0, (e_{s(q)}, q) \in \mathbf{\Lambda}^l_i$ such that $q = \widetilde{q}\beta$ and β is associated$\}$ for $l \in \mathbb{Z}$. We mention that $B'^l_0 = \emptyset$ for $l < 0$. Take $B'^l_1 = B^l \setminus B'^l_0$ for $l \in \mathbb{Z}$. Then we have the disjoint union $B^l = B'^l_0 \cup B'^l_1$ for each $l \in \mathbb{Z}$.

Lemma 2.6. *For each $l \in \mathbb{Z}$, the set B^l_0 is a k-basis of $\mathrm{Ker}\delta^l$ and the set B^{l+1}_0 is a k-basis of $\mathrm{Im}\delta^l$.*

Proof. For $l < 0$, we have $B^l_2 = \emptyset = B'^{l+1}_0$. We observe that the triple (δ^l, B^l, B^{l+1}) satisfies condition (W). Indeed, $\delta^l(b) = 0$ for each $b \in B^l_0$. The differential δ^l induces an injective map $\delta^l : B^l_1 \to B'^{l+1}_1$. Then (W1) and (W2) hold. To see (W3), for $l \geq 0$ and each $i \in Q_0$, $e_i\zeta_{(e_i,q)} \in B^l_2$, we have

$$\delta^l(e_i\zeta_{(e_i,q)}) = \alpha\zeta_{(e_{s(\alpha)},q\alpha)} + \sum_{\beta \in T(\alpha)} \delta^l(e_{s(\beta)}\zeta_{(\beta,q\beta)}),$$

where $\alpha \in Q_1$ such that $t(\alpha) = i$ and α is associated. Here, recall $T(\alpha)$ from (2.1). Thus $(e_i\zeta_{(e_i,q)})_0 = \alpha\zeta_{(e_{s(\alpha)},q\alpha)}$ and the finite subset $B^l_1(e_i\zeta_{(e_i,q)}) = \{e_{s(\beta)}\zeta_{(\beta,q\beta)} \mid \beta \in T(\alpha)\}$.

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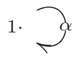
Recall that $B_0^{l+1} = \{\alpha\zeta_{(p,q)} \mid i \in Q_0, (p,q) \in \mathbf{\Lambda}_i^{l+1}$ and $\alpha \in Q_1$ with $s(\alpha) = i\}$. Now we prove that $B_0^{l+1} = \delta^l(B_1^l) \cup \{b_0 \mid b \in B_2^l\}$. We mention that the set $\{b_0 \mid b \in B_2^l\} = \{\alpha\zeta_{(e_{s(\alpha)}, q\alpha)} \mid q \in Q_l$ and α is associated with $t(\alpha) = s(q)\}$. Clearly, $\delta^l(B_1^l) \cup \{b_0 \mid b \in B_2^l\} \subseteq B_0^{l+1}$. Conversely, for each $i \in Q_0$ and $(p,q) \in \mathbf{\Lambda}_i^{l+1}$, we have $\alpha\zeta_{(p,q)} = \delta^l(e_{t(\alpha)}\zeta_{(\alpha p,q)}) \in \delta^l(B_1^l)$ for $\alpha \in Q_1$ with $s(\alpha) = i$, but $\alpha\zeta_{(p,q)} \notin \{b_0 \mid b \in B_2^l\}$. Applying Lemma 2.5 for the triple (δ^l, B^l, B^{l+1}), the proof is complete. □

Proposition 2.7. *Let Q be a finite quiver without sources. Then the projective Leavitt complex $\mathcal{P}^\bullet$ of Q is an acyclic complex.*

Proof. The statement follows directly from Lemma 2.6. □

Example 2.8. Let Q be the following quiver with one vertex and one loop.



The unique arrow α is associated. Set $e = e_1$ and $\mathbf{\Lambda}^l = \mathbf{\Lambda}_1^l$ for each $l \in \mathbb{Z}$. It follows that

$$\mathbf{\Lambda}^l = \begin{cases} \{(\alpha^{-l}, e)\} & \text{if } l < 0, \\ \{(e,e)\} & \text{if } l = 0, \\ \{(e, \alpha^l)\} & \text{if } l > 0. \end{cases}$$

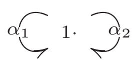
The corresponding algebra A with radical square zero is isomorphic to $k[x]/(x^2)$. Write $A^{(\mathbf{\Lambda}^l)} = A\zeta^l$, where $\zeta^l = \zeta_{(\alpha^{-l}, e)}$ for $l < 0$, $\zeta^0 = \zeta_{(e,e)}$ and $\zeta^l = \zeta_{(e,\alpha^l)}$ for $l > 0$. Then the projective Leavitt complex $\mathcal{P}^\bullet$ of Q is as follows

$$\cdots \longrightarrow A\zeta^{l-1} \xrightarrow{\delta^{l-1}} A\zeta^l \xrightarrow{\delta^l} A\zeta^{l+1} \longrightarrow \cdots,$$

where the differential δ^l is given by $\delta^l(e\zeta^l) = \alpha\zeta^{l+1}$ and $\delta^l(\alpha\zeta^l) = 0$ for each $l \in \mathbb{Z}$.

Observe that A is a self-injective algebra. The projective Leavitt complex $\mathcal{P}^\bullet$ is isomorphic to the injective Leavitt complex $\mathcal{I}^\bullet$ as complexes; compare [**17**, Example 2.11].

Example 2.9. Let Q be the following quiver with one vertex and two loops.



We choose α_1 to be the associated arrow terminating at the unique vertex. Set $e = e_1$ and $\mathbf{\Lambda}^l = \mathbf{\Lambda}_1^l$ for each $l \in \mathbb{Z}$. A pair (p,q) of paths lies in $\mathbf{\Lambda}^l$ if and only if $l(q) - l(p) = l$ and p, q do not start with α_1 simultaneously.

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We denote by A the corresponding radical square zero algebra. The projective Leavitt complex $\mathcal{P}^\bullet$ of Q is as follows:

$$\cdots \xrightarrow{\delta^{-1}} A^{(\mathbf{\Lambda}^0)} \xrightarrow{\delta^0} A^{(\mathbf{\Lambda}^1)} \xrightarrow{\delta^1} \cdots$$

We write the differential δ^0 explicitly: $\delta^0(\alpha_k\zeta_{(p,q)}) = 0$ and

$$\delta^0(e\zeta_{(p,q)}) = \begin{cases} \alpha_k\zeta_{(\widehat{p},q)} & \text{if } p = \alpha_k\widehat{p}, \\ \alpha_1\zeta_{(e,q\alpha_1)} + \alpha_2\zeta_{(e,q\alpha_2)} & \text{if } p = e, \end{cases}$$

for $k = 1, 2$ and $(p, q) \in \mathbf{\Lambda}^0$.

3. The projective Leavitt complex as a compact generator

In this section, we prove that the projective Leavitt complex is a compact generator of the homotopy category of acyclic complexes of projective A-modules.

3.1. The cokernel complex

Let Q be a finite quiver without sources, and let A be the corresponding algebra with radical square zero. For each $i \in Q_0$, $l \in \mathbb{Z}$ and $n \geq 0$, denote

$$\mathbf{\Lambda}_i^{l,n} = \{(p,q) \mid (p,q) \in \mathbf{\Lambda}_i^l \text{ with } p \in Q_n\}.$$

Refer to (2.2) for the definition of the set $\mathbf{\Lambda}_i^l$.

Recall the projective Leavitt complex $\mathcal{P}^\bullet = (\mathcal{P}^l, \delta^l)_{l\in\mathbb{Z}}$ of Q. For each $l \geq 0$, we denote $\mathcal{K}^l = \bigoplus_{i\in Q_0} P_i^{(\mathbf{\Lambda}_i^{l,0})} \subseteq \mathcal{P}^l$, where $P_i = Ae_i$. Observe that the differential $\delta^l : \mathcal{P}^l \to \mathcal{P}^{l+1}$ satisfies $\delta^l(\mathcal{K}^l) \subseteq \mathcal{K}^{l+1}$. Then we have a subcomplex $\mathcal{K}^\bullet$ of $\mathcal{P}^\bullet$, whose components $\mathcal{K}^l = 0$ for $l < 0$. Let $\phi^\bullet = (\phi^l)_{l\in\mathbb{Z}} : \mathcal{K}^\bullet \longrightarrow \mathcal{P}^\bullet$ be the inclusion chain map by setting $\phi^l = 0$ for $l < 0$. We set $\mathcal{C}^\bullet$ to be the cokernel of $\phi^\bullet$.

We now describe the cokernel $\mathcal{C}^\bullet = (\mathcal{C}^l, \widetilde{\delta}^l)$ of $\phi^\bullet$. For each vertex $i \in Q_0$ and $l \in \mathbb{Z}$, set

$$\mathbf{\Lambda}_i^{l,+} = \bigcup_{n>0} \mathbf{\Lambda}_i^{l,n}.$$

Observe that we have the disjoint union $\mathbf{\Lambda}_i^l = \mathbf{\Lambda}_i^{l,0} \cup \mathbf{\Lambda}_i^{l,+}$ for $l \geq 0$ and $\mathbf{\Lambda}_i^l = \mathbf{\Lambda}_i^{l,+}$ for $l < 0$. The component of $\mathcal{C}^\bullet$ is $\mathcal{C}^l = \bigoplus_{i\in Q_0} P_i^{(\mathbf{\Lambda}_i^{l,+})}$ for each $l \in \mathbb{Z}$. We have $\mathcal{C}^l = \mathcal{P}^l$ for $l < 0$ and the differential $\widetilde{\delta}^l = \delta^l$ for $l \leq -2$. The differential $\widetilde{\delta}^l : \mathcal{C}^l \to \mathcal{C}^{l+1}$ for $l \geq -1$ is given as follows: $\widetilde{\delta}^l(\alpha\zeta_{(p,q)}) = 0$ and

$$\widetilde{\delta}^l(e_i\zeta_{(p,q)}) = \begin{cases} 0 & \text{if } l(p) = 1, \\ \delta^l(e_i\zeta_{(p,q)}) & \text{otherwise}, \end{cases}$$

for any $i \in Q_0$, $(p,q) \in \mathbf{\Lambda}_i^{l,+}$ and $\alpha \in Q_1$ with $s(\alpha) = i$. The restriction of $\widetilde{\delta}^l$ to $\bigoplus_{i\in Q_0} P_i^{(\mathbf{\Lambda}_i^{l,1})}$ is zero for $l \geq -1$. We emphasize that the differentials $\widetilde{\delta}^l$ for $l \geq -1$ are induced by the differentials δ^l in Definition 2.4.

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We observe the inclusions

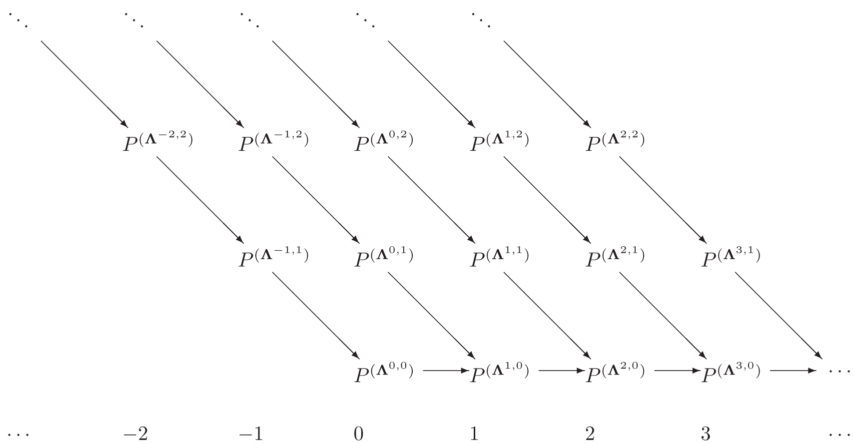
$$\widetilde{\delta}^l\Big(\bigoplus_{i\in Q_0} P_i^{(\mathbf{\Lambda}_i^{l,n})}\Big) \subseteq \bigoplus_{i\in Q_0} P_i^{(\mathbf{\Lambda}_i^{l+1,n-1})}$$

inside the complex $\mathcal{C}^\bullet$ for each $l\in\mathbb{Z}$ and $n\geq 2$. Then, for each $n\geq 0$, the following complex, denoted by $\mathcal{C}_n^\bullet$,

$$\cdots \xrightarrow{\delta^{n-4}} \bigoplus_{i\in Q_0} P_i^{(\mathbf{\Lambda}_i^{n-3,3})} \xrightarrow{\delta^{n-3}} \bigoplus_{i\in Q_0} P_i^{(\mathbf{\Lambda}_i^{n-2,2})} \xrightarrow{\delta^{n-2}} \bigoplus_{i\in Q_0} P_i^{(\mathbf{\Lambda}_i^{n-1,1})} \to 0$$

is a subcomplex of $\mathcal{C}^\bullet$ satisfying $\mathcal{C}_n^l = 0$ for $l\geq n$. The differential δ^l for $l\leq n-2$ is the differential of $\mathcal{P}^\bullet$.

We visually represent the projective Leavitt complex $\mathcal{P}^\bullet$ and the cokernel complex $\mathcal{C}^\bullet$ of $\phi^\bullet$. For each $l\in\mathbb{Z}$ and $n\geq 0$, we denote $\bigoplus_{i\in Q_0} P_i^{(\mathbf{\Lambda}_i^{l,n})}$ by $P^{(\mathbf{\Lambda}^{l,n})}$ for simplicity.



Remark 3.1.

(1) For each $l\in\mathbb{Z}$, the lth component of the projective Leavitt complex $\mathcal{P}^\bullet$ is the coproduct of the objects in the lth column of the above figure. The differentials of $\mathcal{P}^\bullet$ are coproducts of the maps in the figure.

(2) The horizontal line of the above figure is the subcomplex $\mathcal{K}^\bullet$, while the other part gives the cokernel $\mathcal{C}^\bullet$ of $\phi^\bullet:\mathcal{K}^\bullet\to\mathcal{P}^\bullet$. The diagonal lines (not including the intersection with the horizontal line) of the figure are the subcomplexes $\mathcal{C}_n^\bullet$ of $\mathcal{C}^\bullet$. For example, the first diagonal line on the left (not including $P^{(\mathbf{\Lambda}^{0,0})}$) is the subcomplex $\mathcal{C}_0^\bullet$.

We have the following observation immediately.

Proposition 3.2. *The complex* $\mathcal{C}^\bullet = \bigoplus_{n\geq 0}\mathcal{C}_n^\bullet$.

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Proof. Observe that for each $n \geq 0$, the lth component of $\mathcal{C}_n^\bullet$ is

$$\mathcal{C}_n^l = \begin{cases} \bigoplus_{i \in Q_0} P_i^{(\mathbf{\Lambda}_i^{l,n-l})} & \text{if } l < n \\ 0 & \text{otherwise.} \end{cases}$$

Then the lth component of $\bigoplus_{n\geq 0} \mathcal{C}_n^\bullet$ is $\bigoplus_{n \geq 0} \mathcal{C}_n^l = \bigoplus_{i \in Q_0} P_i^{(\mathbf{\Lambda}_i^{l,+})} = \mathcal{C}^l$. Recall the differential $\widetilde{\delta}^l : \mathcal{C}^l \to \mathcal{C}^{l+1}$ of $\mathcal{C}^\bullet$. The restriction of $\widetilde{\delta}^l$ to $\bigoplus_{i \in Q_0} P_i^{(\mathbf{\Lambda}_i^{l,1})}$ is zero, and the restriction of $\widetilde{\delta}^l$ to $\bigoplus_{i \in Q_0} P_i^{(\mathbf{\Lambda}_i^{l,n})}$ for $n > 1$ is δ^l. Thus, $\widetilde{\delta}^l : \mathcal{C}^l \to \mathcal{C}^{l+1}$ is the coproduct of the differentials $\delta^l : \mathcal{C}_n^l \to \mathcal{C}_n^{l+1}$ for $n \geq 0$. □

3.2. An explicit compact generator of the homotopy category

We consider the category A-Mod of left A-modules. Denote by $\mathbf{K}(A\text{-Mod})$ its homotopy category. We will always view a module as a stalk complex concentrated on degree zero.

For $X^\bullet = (X^i, d_X^i)_{i \in \mathbb{Z}}$, a complex of A-modules, we denote by $X^\bullet[1]$ the complex given by $(X^\bullet[1])^i = X^{i+1}$ and $d_{X[1]}^i = -d_X^{i+1}$ for $i \in \mathbb{Z}$. For a chain map $f^\bullet : X^\bullet \to Y^\bullet$, its *mapping cone* $\mathrm{Con}(f^\bullet)$ is a complex such that $\mathrm{Con}(f^\bullet) = X^\bullet[1] \oplus Y^\bullet$ with the differential

$$d_{\mathrm{Con}(f^\bullet)}^i = \begin{pmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{pmatrix}.$$

Each triangle in $\mathbf{K}(A\text{-Mod})$ is isomorphic to

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \mathrm{Con}(f^\bullet) \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} X^\bullet[1]$$

for some chain map $f^\bullet$.

Denote by $I_i = D(e_i A_A)$ the injective left A-module for each $i \in Q_0$, where $(e_i A)_A$ is the indecomposable projective right A-module and $D = \mathrm{Hom}_k(-, k)$ denotes the standard k-duality. Denote by $\{e_i^\sharp\} \cup \{\alpha^\sharp \mid \alpha \in Q_1, t(\alpha) = i\}$ the basis of I_i, which is dual to the basis $\{e_i\} \cup \{\alpha \mid \alpha \in Q_1, t(\alpha) = i\}$ of $e_i A$.

We denote by $\mathcal{M}^\bullet$ the following complex

$$0 \to \bigoplus_{i \in Q_0} I_i^{(\mathbf{\Lambda}_i^{0,0})} \xrightarrow{d^0} \bigoplus_{i \in Q_0} I_i^{(\mathbf{\Lambda}_i^{1,0})} \longrightarrow \cdots \longrightarrow \bigoplus_{i \in Q_0} I_i^{(\mathbf{\Lambda}_i^{l,0})} \xrightarrow{d^l} \bigoplus_{i \in Q_0} I_i^{(\mathbf{\Lambda}_i^{l+1,0})} \longrightarrow \cdots$$

of A-modules satisfying $\mathcal{M}^l = 0$ for $l < 0$, where the differential d^l for $l \geq 0$ is given by $d^l(e_i^\sharp \zeta_{(e_i,p)}) = 0$ and $d^l(\alpha^\sharp \zeta_{(e_i,p)}) = e_{s(\alpha)}^\sharp \zeta_{(e_{s(\alpha)}, p\alpha)}$ for $i \in Q_0$, $(e_i, p) \in \mathbf{\Lambda}_i^{l,0}$ and $\alpha \in Q_1$ with $t(\alpha) = i$. Consider the semisimple left A-module $kQ_0 = A/\mathrm{rad}A$.

Lemma 3.3. *The left A-module $kQ_0 = A/\mathrm{rad}A$ is quasi-isomorphic to the complex $\mathcal{M}^\bullet$. In other words, $\mathcal{M}^\bullet$ is an injective resolution of the A-module kQ_0.*

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Proof. Define a left A-module map $f^0 : kQ_0 \longrightarrow \mathcal{M}^0$ such that $f^0(e_i) = e_i^\sharp \zeta_{(e_i,e_i)}$ for each $i \in Q_0$. Then we obtain a chain map $f^\bullet = (f^l)_{l\in\mathbb{Z}} : kQ_0 \longrightarrow \mathcal{M}^\bullet$ such that $f^l = 0$ for $l \neq 0$. We observe the following k-basis of $\mathcal{M}^l$ for $l \geq 0$:

$$\Gamma^l = \{e_i^\sharp \zeta_{(e_i,q)}, \alpha^\sharp \zeta_{(e_i,q)} \mid i \in Q_0, (e_i,q) \in \mathbf{\Lambda}_i^{l,0} \text{ and } \alpha \in Q_1 \text{ with } t(\alpha) = i\}.$$

Set $\Gamma_0^l = \{e_i^\sharp \zeta_{(e_i,q)} \mid i \in Q_0, (e_i,q) \in \mathbf{\Lambda}_i^{l,0}\}$, $\Gamma_1^l = \Gamma^l \setminus \Gamma_0^l$, and $\Gamma_1'^l = \Gamma^l$. We have the disjoint union $\Gamma^l = \Gamma_0^l \cup \Gamma_1^l$. The triple $(d^l, \Gamma^l, \Gamma^{l+1})$ satisfies condition (W). By Lemma 2.5, the set Γ_0^l is a k-basis of $\mathrm{Ker} d^l$ and the set $d^l(\Gamma_1^l)$ is a k-basis of $\mathrm{Im} d^l$. For each $l \geq 0$, $i \in Q_0$ and $(e_i,q) \in \mathbf{\Lambda}_i^{l+1,0}$, write $q = \widetilde{q}\alpha$ with $\alpha \in Q_1$. Then we have $e_i^\sharp \zeta_{(e_i,q)} = d^l(\alpha^\sharp \zeta_{(e_{t(\alpha)},\widetilde{q})})$. Thus $d^l(\Gamma_1^l) = \Gamma_0^{l+1}$. Hence $\mathrm{Im} d^l = \mathrm{Ker} d^{l+1}$ for each $l \geq 0$ and $\mathrm{Ker} d^0 \cong kQ_0$. The statement follows directly. $\square$

We now recall some terminology and facts on triangulated categories. For a triangulated category $\mathcal{T}$, a *thick* subcategory of $\mathcal{T}$ is a triangulated subcategory of $\mathcal{T}$ which is closed under direct summands. Let $\mathcal{S}$ be a class of objects in $\mathcal{T}$. We denote by $\mathrm{thick}\langle\mathcal{S}\rangle$ the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{S}$. If $\mathcal{T}$ has arbitrary coproducts, we denote by $\mathrm{Loc}\langle\mathcal{S}\rangle$ the smallest triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{S}$ and is closed under arbitrary coproducts. By [7, Proposition 3.2], we have that $\mathrm{thick}\langle\mathcal{S}\rangle \subseteq \mathrm{Loc}\langle\mathcal{S}\rangle$.

For a triangulated category $\mathcal{T}$ with arbitrary coproducts, an object M in $\mathcal{T}$ is *compact* if the functor $\mathrm{Hom}_{\mathcal{T}}(M,-)$ commutes with arbitrary coproducts. Denote by $\mathcal{T}^c$ the full subcategory consisting of compact objects; it is a thick subcategory.

A triangulated category $\mathcal{T}$ with arbitrary coproducts is *compactly generated* [13, 18] if there exists a set $\mathcal{S}$ of compact objects such that any non-zero object T satisfies $\mathrm{Hom}_{\mathcal{T}}(S, T[n]) \neq 0$ for some $S \in \mathcal{S}$ and $n \in \mathbb{Z}$. This is equivalent to the condition that $\mathcal{T} = \mathrm{Loc}\langle\mathcal{S}\rangle$, in which case we have $\mathcal{T}^c = \mathrm{thick}\langle\mathcal{S}\rangle$; see [18, Lemma 3.2]. If the above set $\mathcal{S}$ consists of a single object S, we call S a *compact generator* of $\mathcal{T}$.

The following is [17, Lemma 3.9].

Lemma 3.4. *Suppose that $\mathcal{T}$ is a compactly generated triangulated category with a compact generator X. Let $\mathcal{T}' \subseteq \mathcal{T}$ be a triangulated subcategory closed under arbitrary coproducts. Assume that there exists a triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

such that $Y \in \mathcal{T}'$ and Z satisfies $\mathrm{Hom}_{\mathcal{T}}(Z, T') = 0$ for each $T' \in \mathcal{T}'$. Then Y is a compact generator of $\mathcal{T}'$.

Let A-Inj and A-Proj be the categories of injective and projective A-modules, respectively. Denote by $\mathbf{K}(A\text{-Inj})$ and $\mathbf{K}(A\text{-Proj})$ the homotopy categories of complexes of injective and projective A-modules, respectively. These homotopy categories are triangulated subcategories of $\mathbf{K}(A\text{-Mod})$ which are closed under coproducts. By [15, Proposition 2.3(1)], $\mathbf{K}(A\text{-Inj})$ is a compactly generated triangulated category.

Recall that the Nakayama functor $\nu = DA \otimes_A - : A\text{-Proj} \longrightarrow A\text{-Inj}$ is an equivalence, whose quasi-inverse $\nu^{-1} = \mathrm{Hom}_A(D(A_A), -)$. Thus we have a triangle equivalence $\mathbf{K}(A\text{-Inj}) \xrightarrow{\sim} \mathbf{K}(A\text{-Proj})$. The category $\mathbf{K}(A\text{-Proj})$ is a compactly generated triangulated category; see [12, Theorem 2.4] and [19, Proposition 7.14].

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Lemma 3.5. *The complex $\mathcal{K}^\bullet$ is a compact generator of* $\mathbf{K}(A\text{-Proj})$.

Proof. Recall that $\mathrm{Hom}_A(D(A_A), I_i) \cong Ae_i$ for each $i \in Q_0$. Then we have $\mathcal{K}^\bullet = (\nu^{-1}(\mathcal{M}^i), \nu^{-1}(d^i))_{i\in\mathbb{Z}}$. By Lemma 3.3, $\mathcal{M}^\bullet = ikQ_0$ in $\mathbf{K}(A\text{-Mod})$. It follows from [**15**, Proposition 2.3] that $\mathcal{M}^\bullet$ is a compact object in $\mathbf{K}(A\text{-Inj})$ and $\mathrm{Loc}\langle\mathcal{M}^\bullet\rangle = \mathbf{K}(A\text{-Inj})$. Since $\mathbf{K}(A\text{-Inj}) \xrightarrow{\sim} \mathbf{K}(A\text{-Proj})$ is a triangle equivalence which sends $\mathcal{M}^\bullet$ to $\mathcal{K}^\bullet$, we have $\mathrm{Loc}\langle\mathcal{K}^\bullet\rangle = \mathbf{K}(A\text{-Proj})$. □

Lemma 3.6. *Suppose that* $\mathrm{P}^\bullet \in \mathbf{K}(A\text{-Proj})$ *is a bounded-above complex. Then we have*

$$\mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(\mathrm{P}^\bullet, X^\bullet) = 0$$

for any acyclic complex $X^\bullet$ of A-modules.

Proof. Directly check that any chain map $f^\bullet : \mathrm{P}^\bullet \longrightarrow X^\bullet$ is null-homotopic. □

Denote by $\mathbf{K}_{\mathrm{ac}}(A\text{-Proj})$ the full subcategory of $\mathbf{K}(A\text{-Mod})$ which is formed by acyclic complexes of projective A-modules. Applying [**19**, Propositions 7.14 and 7.12] and the localization theorem in [**14**, 1.5], we have that the category is a compactly generated triangulated category with the triangle equivalence

$$\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathbf{K}_{\mathrm{ac}}(A\text{-Proj})^c.$$

Here, for a category $\mathcal{C}$, we denote by $\mathcal{C}^{\mathrm{op}}$ its opposite category; the category $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}})$ is the singularity category of algebra A^{op} in the sense of [**8**, **20**].

Theorem 3.7. *Let Q be a finite quiver without sources. Then the projective Leavitt complex $\mathcal{P}^\bullet$ of Q is a compact generator of the homotopy category* $\mathbf{K}_{\mathrm{ac}}(A\text{-Proj})$.

Proof. Recall from Proposition 2.7 that $\mathcal{P}^\bullet$ is an object of $\mathbf{K}_{\mathrm{ac}}(A\text{-Proj})$. The complex $\mathcal{C}^\bullet = \mathrm{Coker}(\phi^\bullet)$, where $\phi^\bullet : \mathcal{K}^\bullet \longrightarrow \mathcal{P}^\bullet$ is the inclusion chain map. Then we have the following exact sequence

$$0 \longrightarrow \mathcal{K}^\bullet \xrightarrow{\phi^\bullet} \mathcal{P}^\bullet \longrightarrow \mathcal{C}^\bullet \longrightarrow 0,$$

which splits in each component. This gives rise to a triangle

$$\mathcal{K}^\bullet \xrightarrow{\phi^\bullet} \mathcal{P}^\bullet \longrightarrow \mathcal{C}^\bullet \longrightarrow X[1] \qquad (3.1)$$

in the category $\mathbf{K}(A\text{-Proj})$.

By Proposition 3.2 and Lemma 3.6, the following equality holds

$$\mathrm{Hom}_{\mathbf{K}(A\text{-Proj})}(\mathcal{C}^\bullet, X^\bullet) = \prod_{n\geq 0} \mathrm{Hom}_{\mathbf{K}(A\text{-Proj})}(\mathcal{C}^\bullet_n, X^\bullet) = 0$$

for any $X^\bullet \in \mathbf{K}_{\mathrm{ac}}(A\text{-Proj})$. Recall from Lemma 3.5 that $\mathcal{K}^\bullet$ is a compact generator of $\mathbf{K}(A\text{-Proj})$. By the triangle (3.1) and Lemma 3.4, the proof is completed. □

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4. The projective Leavitt complex as a differential graded bimodule

In this section, we endow the projective Leavitt complex with a differential graded bimodule structure over the corresponding Leavitt path algebra.

4.1. The Leavitt path algebra and module structure

Let k be a field and Q be a finite quiver. We will endow the projective Leavitt complex of Q with a Leavitt path algebra module structure. Recall from [**1**, **5**] the notion of the Leavitt path algebra.

Definition 4.1. The *Leavitt path algebra* $L_k(Q)$ of Q is the k-algebra generated by the set $\{e_i \mid i \in Q_0\} \cup \{\alpha \mid \alpha \in Q_1\} \cup \{\alpha^* \mid \alpha \in Q_1\}$ subject to the following relations:

(0) $e_i e_j = \delta_{i,j} e_i$ for every $i, j \in Q_0$;

(1) $e_{t(\alpha)}\alpha = \alpha e_{s(\alpha)} = \alpha$ for all $\alpha \in Q_1$;

(2) $e_{s(\alpha)}\alpha^* = \alpha^* e_{t(\alpha)} = \alpha^*$ for all $\alpha \in Q_1$;

(3) $\alpha\beta^* = \delta_{\alpha,\beta} e_{t(\alpha)}$ for all $\alpha, \beta \in Q_1$;

(4) $\sum_{\{\alpha \in Q_1 \mid s(\alpha)=i\}} \alpha^*\alpha = e_i$ for $i \in Q_0$ which is not a sink.

Here, δ is the Kronecker symbol. The relations (3) and (4) are called *Cuntz–Krieger relations*. The elements α^* for $\alpha \in Q_1$ are called *ghost arrows*.

There is an alternative description of $L_k(Q)$. Let $\overline{Q}$ be the *double quiver* obtained from Q by adding an arrow α^* in the opposite direction for each arrow α in Q. Then the Leavitt path algebra $L_k(Q)$ is isomorphic to the quotient algebra of the path algebra $k\overline{Q}$ of $\overline{Q}$ modulo the ideal generated by $\{\alpha\beta^* - \delta_{\alpha,\beta}e_{t(\alpha)}, \sum_{\{\gamma \in Q_1 \mid s(\gamma)=i\}} \gamma^*\gamma - e_i \mid \alpha, \beta \in Q_1, i \in Q_0\}$.

If $p = \alpha_n \cdots \alpha_2\alpha_1$ is a path in Q of length $n \geq 1$, we define $p^* = \alpha_1^*\alpha_2^* \cdots \alpha_n^*$. By convention, we set $e_i^* = e_i$ for $i \in Q_0$. We observe by (2) that for paths p, q in Q, $p^*q = 0$ for $t(p) \neq t(q)$. Consider the relation (3). We have the following fact; see [**24**, Lemma 3.1].

Lemma 4.2. *Let p, q, γ and η be paths in Q with $t(p) = t(q)$ and $t(\gamma) = t(\eta)$. Then in $L_k(Q)$ we have*

$$(p^*q)(\gamma^*\eta) = \begin{cases} (\gamma'p)^*\eta & \text{if } \gamma = \gamma'q, \\ p^*\eta & \text{if } q = \gamma, \\ p^*(q'\eta) & \text{if } q = q'\gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Here, γ' and q' are some non-trivial paths in Q.

By the above lemma, we deduce that the Leavitt path algebra $L_k(Q)$ is spanned by the following set: $\{p^*q \mid p, q \text{ are paths in } Q \text{ with } t(p) = t(q)\}$; see [**1**, Lemma 1.5], [**24**, Corollary 3.2] or [**9**, Corollary 2.2]. By (4), this set is not k-linearly independent in general.

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For each vertex which is not a sink, we fix a special arrow starting at it. The following result is [4, Theorem 1].

Lemma 4.3. *The following elements form a k-basis of the Leavitt path algebra $L_k(Q)$:*

(1) e_i, $i \in Q_0$;

(2) p, p^*, *where p is a non-trivial path in Q;*

(3) p^*q *with $t(p) = t(q)$, where $p = \alpha_m \cdots \alpha_1$ and $q = \beta_n \cdots \beta_1$ are non-trivial paths of Q such that $\alpha_m \neq \beta_n$, or $\alpha_m = \beta_n$ that is not special.*

From now on, Q is a finite quiver without sources. For notation, Q^{op} is the opposite quiver of Q. For a path p in Q, denote by p^{op} the corresponding path in Q^{op}. The starting and terminating vertices of p^{op} are $t(p)$ and $s(p)$, respectively. By convention, $e_j^{\text{op}} = e_j$ for each vertex $j \in Q_0$. The opposite quiver Q^{op} has no sinks.

For the opposite quiver Q^{op} of Q, choose α^{op} to be the special arrow of Q^{op} starting at vertex i, where α is the associated arrow in Q terminating at i. By Lemma 4.3, there exists a k-basis of the Leavitt path algebra $L_k(Q^{\text{op}})$, denoted by Γ. Define a map $\chi : \bigcup_{l\in\mathbb{Z}, i\in Q_0} \mathbf{\Lambda}_i^l \to \Gamma$ such that $\chi(p,q) = (p^{\text{op}})^* q^{\text{op}}$. Here, $(p^{\text{op}})^* q^{\text{op}}$ is the multiplication of $(p^{\text{op}})^*$ and q^{op} in $L_k(Q^{\text{op}})$. The map χ is a bijection. We identify Γ with the set of associated pairs in Q. A non-zero element x in $L_k(Q^{\text{op}})$ can be written in the unique form

$$x = \sum_{i=1}^{m} \lambda_i (p_i^{\text{op}})^* q_i^{\text{op}}$$

with $\lambda_i \in k$ non-zero scalars and (p_i, q_i) pairwise distinct associated pairs in Q.

In what follows, $B = L_k(Q^{\text{op}})$. We write ab for the multiplication of a and b in B for $a, b \in B$. Recall that the projective Leavitt complex $\mathcal{P}^\bullet = (\mathcal{P}^l, \delta^l)_{l\in\mathbb{Z}}$ and $\mathcal{P}^l = \bigoplus_{i\in Q_0} P_i^{(\mathbf{\Lambda}_i^l)}$.

We define a right B-module action on $\mathcal{P}^\bullet$. For each vertex $j \in Q_0$ and each arrow $\alpha \in Q_1$, define right actions '$\cdot$' on $\mathcal{P}^l$ for any $l \in \mathbb{Z}$ as follows. For any element $x\zeta_{(p,q)} \in P_i\zeta_{(p,q)}$ with $i \in Q_0$ and $(p,q) \in \mathbf{\Lambda}_i^l$, we set

$$x\zeta_{(p,q)}\cdot e_j = \delta_{j,t(q)} x\zeta_{(p,q)}; \tag{4.1}$$

$$x\zeta_{(p,q)}\cdot\alpha^{\text{op}} = \begin{cases} x\zeta_{(\widetilde{p}, e_{t(\alpha)})} - \displaystyle\sum_{\beta\in T(\alpha)} x\zeta_{(\widetilde{p}\beta,\beta)} & \text{if } l(q)=0, p=\widetilde{p}\alpha \text{ and } \alpha \text{ is associated,} \\ \delta_{s(\alpha),t(q)} x\zeta_{(p,\alpha q)} & \text{otherwise;} \end{cases} \tag{4.2}$$

$$x\zeta_{(p,q)}\cdot(\alpha^{\text{op}})^* = \begin{cases} \delta_{\alpha,\alpha_1} x\zeta_{(p,\widehat{q})} & \text{if } q = \alpha_1\widehat{q} \\ \delta_{s(p),t(\alpha)} x\zeta_{(p\alpha, e_{s(\alpha)})} & \text{if } l(q) = 0. \end{cases} \tag{4.3}$$

Here, regarding the notation, a path $p = \alpha_n \cdots \alpha_2\alpha_1$ in Q of length $n \geq 2$ has two truncations, $\widehat{p} = \alpha_{n-1} \cdots \alpha_1$ and $\widetilde{p} = \alpha_n \cdots \alpha_2$. For an arrow α, $\widehat{\alpha} = e_{s(\alpha)}$ and $\widetilde{\alpha} = e_{t(\alpha)}$. The set $T(\alpha) = \{\beta \in Q_1 \mid t(\beta) = t(\alpha), \beta \neq \alpha\}$ for an associated arrow α.

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We observe the following fact:

$$\begin{cases} x\zeta_{(p,q)}\cdot\alpha^{\mathrm{op}} = 0 & \text{if } s(\alpha) \neq t(q), \\ x\zeta_{(p,q)}\cdot(\alpha^{\mathrm{op}})^* = 0 & \text{if } t(\alpha) \neq t(q). \end{cases} \tag{4.4}$$

Lemma 4.4. *The above actions make the projective Leavitt complex $\mathcal{P}^\bullet$ of Q a right B-module.*

Proof. We prove that the above right actions satisfy the defining relations of the Leavitt path algebra $L_k(Q^{\mathrm{op}})$. We fix $x\zeta_{(p,q)} \in P_i\zeta_{(p,q)} \subseteq \mathcal{P}^l$.

For (0), we observe that $x\zeta_{(p,q)}\cdot(e_j \circ e_{j'}) = \delta_{j,j'}x\zeta_{(p,q)}\cdot e_j$.

For (1), for each $\alpha \in Q_1$ we have

$$\begin{aligned} x\zeta_{(p,q)}\cdot(\alpha^{\mathrm{op}}e_{t(\alpha)}) &= (x\zeta_{(p,q)}\cdot\alpha^{\mathrm{op}})\cdot e_{t(\alpha)} \\ &= x\zeta_{(p,q)}\cdot\alpha^{\mathrm{op}}. \end{aligned}$$

We have

$$\begin{aligned} x\zeta_{(p,q)}\cdot(e_{s(\alpha)}\alpha^{\mathrm{op}}) &= (x\zeta_{(p,q)}\cdot e_{s(\alpha)})\cdot\alpha^{\mathrm{op}} \\ &= \delta_{s(\alpha),t(q)}x\zeta_{(p,q)}\cdot\alpha^{\mathrm{op}} \\ &= x\zeta_{(p,q)}.\alpha^{\mathrm{op}}, \end{aligned}$$

where the last equality uses (4.4). Similar arguments prove the relation (2).

For (3), we have that for $\alpha, \beta \in Q_1$

$$\begin{aligned} x\zeta_{(p,q)}\cdot(\alpha^{\mathrm{op}}(\beta^{\mathrm{op}})^*) &= (x\zeta_{(p,q)}\cdot\alpha^{\mathrm{op}})\cdot(\beta^{\mathrm{op}})^* \\ &= \begin{cases} \delta_{t(\alpha),t(\beta)}x\zeta_{(\widetilde{p}\beta,e_{s(\beta)})} - \sum\limits_{\gamma\in T(\alpha)}\delta_{\gamma,\beta}x\zeta_{(\widetilde{p}\gamma,e_{s(\gamma)})} & \begin{array}{l}\text{if } l(q)=0, p=\widetilde{p}\alpha \\ \text{and } \alpha \text{ is associated,}\end{array} \\ \delta_{s(\alpha),t(q)}\delta_{\alpha,\beta}x\zeta_{(p,q)} & \text{otherwise,} \end{cases} \\ &= \delta_{s(\alpha),t(q)}\delta_{\alpha,\beta}x\zeta_{(p,q)} \\ &= x\zeta_{(p,q)}\cdot(\delta_{\alpha,\beta}e_{s(\alpha)}). \end{aligned}$$

Here, we use the fact that in the case where $l(q) = 0$, $p = \widetilde{p}\alpha$ and α is associated, if $\alpha = \beta$, then $s(\alpha) = t(q)$ and $\gamma \neq \beta$ for each $\gamma \in T(\alpha)$; and if $\alpha \neq \beta$ with $t(\alpha) = t(\beta)$, then there exists an arrow $\gamma \in T(\alpha)$ such that $\gamma = \beta$.

For (4), for each $j \in Q_0$, we have that if $\alpha \in Q_1$ with $t(\alpha) = j$ is associated, then

$$\begin{aligned} x\zeta_{(p,q)}\cdot((\alpha^{\mathrm{op}})^*\alpha^{\mathrm{op}}) &= (x\zeta_{(p,q)}\cdot(\alpha^{\mathrm{op}})^*)\cdot\alpha^{\mathrm{op}} \\ &= \begin{cases} \delta_{\alpha,\alpha_1}x\zeta_{(p,q)} & \text{if } q = \alpha_1\widehat{q}, \\ \delta_{j',s(p)}\left(x\zeta_{(p,e_{s(p)})} - \sum\limits_{\beta\in T(\alpha)} x\zeta_{(p\beta,\beta)}\right) & \text{if } l(q) = 0. \end{cases} \end{aligned}$$

If $\alpha \in Q_1$ with $t(\alpha) = j$ is not associated, then

$$x\zeta_{(p,q)}\cdot((\alpha^{\mathrm{op}})^*\alpha^{\mathrm{op}}) = \begin{cases} \delta_{\alpha,\alpha_1}x\zeta_{(p,q)} & \text{if } q = \alpha_1\widehat{q}, \\ \delta_{j,s(p)}x\zeta_{(p\alpha,\alpha)} & \text{if } l(q) = 0. \end{cases}$$

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Thus, we have the following equality

$$
\begin{aligned}
x\zeta_{(p,q)}\cdot\Bigg(\sum_{\{\alpha\in Q_1\ \mid\ t(\alpha)=j\}}(\alpha^{\text{op}})^*\alpha^{\text{op}}\Bigg)
&= \begin{cases} \delta_{j,t(q)}x\zeta_{(p,q)} & \text{if } q=\alpha_1\widehat{q},\\ \delta_{j,s(p)}x\zeta_{(p,e_{s(p)})} & \text{if } l(q)=0,\end{cases}\\
&= \delta_{j,t(q)}x\zeta_{(p,q)}\\
&= x\zeta_{(p,q)}\cdot e_j.
\end{aligned}
$$

□

The following observation gives an intuitive description of the B-module action on $\mathcal{P}^\bullet$.

Lemma 4.5. *Let (p,q) be an associated pair in Q.*

(1) *We have $\sum_{i\in Q_0} e_i\zeta_{(e_i,e_i)}\cdot(p^{\text{op}})^*q^{\text{op}} = e_{t(p)}\zeta_{(p,q)}$.*

(2) *For each arrow $\beta\in Q_1$, the following equality holds:*

$$\beta\zeta_{(e_{s(\beta)},e_{s(\beta)})}\cdot(p^{\text{op}})^*q^{\text{op}} = \delta_{s(\beta),t(p)}\beta\zeta_{(p,q)}.$$

Proof. Since (p,q) is an associated pair in Q, we are in the second subcases of (4.3) and (4.2) for the right action of $(p^{\text{op}})^*q^{\text{op}}$. Then the statements follow from direct calculation. □

4.2. The differential graded bimodule

We first recall from [**13**] some notation on differential graded modules. Let $A=\bigoplus_{n\in\mathbb{Z}}A^n$ be a $\mathbb{Z}$-graded algebra. For a (left) graded A-module $M=\bigoplus_{n\in\mathbb{Z}}M^n$, elements m in M^n are said to be homogeneous of degree n, denoted by $|m|=n$.

A *differential graded algebra* (dg algebra) is a $\mathbb{Z}$-graded algebra A with a differential $d:A\to A$ of degree one such that $d(ab)=d(a)b+(-1)^{|a|}ad(b)$ for homogeneous elements $a,b\in A$.

A *(left) differential graded* A-module (dg A-module) M is a graded A-module $M=\bigoplus_{n\in\mathbb{Z}}M^n$ with a differential $d_M:M\to M$ of degree one such that $d_M(a{\cdot}m)=d(a){\cdot}m+(-1)^{|a|}a{\cdot}d_M(m)$ for homogeneous elements $a\in A$ and $m\in M$. A morphism of dg A-modules is a morphism of A-modules preserving degrees and commuting with differentials. A *right differential graded* A-module (right dg A-module) N is a right graded A-module $N=\bigoplus_{n\in\mathbb{Z}}N^n$ with a differential $d_N:N\to N$ of degree one such that $d_N(m\cdot a)=d_N(m)\cdot a+(-1)^{|m|}m\cdot d(a)$ for homogeneous elements $a\in A$ and $m\in N$. Here, we use central dots to denote the A-module action.

Let B be another dg algebra. Recall that a *dg A-B-bimodule* M is a left dg A-module as well as a right dg B-module such that $(a\cdot m)\cdot b=a\cdot(m\cdot b)$ for $a\in A$, $m\in M$ and $b\in B$.

Recall that Q is a finite quiver without sources. In what follows, we write $B=L_k(Q^{\text{op}})$, which is naturally $\mathbb{Z}$-graded by the length of paths. We view B as a dg algebra with trivial differential.

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Consider $A = kQ/J^2$ as a dg algebra concentrated on degree zero. Recall the projective Leavitt complex $\mathcal{P}^\bullet = \bigoplus_{l\in\mathbb{Z}} \mathcal{P}^l$, which is a left dg A-module. By Lemma 4.4, $\mathcal{P}^\bullet$ is a right B-module. We observe from (4.1)–(4.3) that $\mathcal{P}^\bullet$ is a right graded B-module.

The following result states that $\mathcal{P}^\bullet$ is a dg A-B-bimodule. It is evident that $\mathcal{P}^\bullet$ is a graded A-B-bimodule. Recall that the differentials on $\mathcal{P}^\bullet$ are denoted by δ^l.

Proposition 4.6. *For each $l \in \mathbb{Z}$, let $x\zeta_{(p,q)} \in P_i\zeta_{(p,q)}$ with $i \in Q_0$ and $(p,q) \in \mathbf{\Lambda}_i^l$. Then for each vertex $j \in Q_0$ and each arrow $\beta \in Q_1$, we have:*

(1) $\delta^l(x\zeta_{(p,q)}\cdot e_i) = \delta^l(x\zeta_{(p,q)})\cdot e_i$;

(2) $\delta^{l+1}(x\zeta_{(p,q)}\cdot \beta^{\mathrm{op}}) = \delta^l(x\zeta_{(p,q)})\cdot \beta^{\mathrm{op}}$;

(3) $\delta^{l-1}(x\zeta_{(p,q)}\cdot (\beta^{\mathrm{op}})^*) = \delta^l(x\zeta_{(p,q)})\cdot (\beta^{\mathrm{op}})^*$.

In other words, the right B-action makes $\mathcal{P}^\bullet$ a right dg B-module and thus a dg A-B-bimodule.

We make some preparation for the proof of the above proposition. There is a unique right B-module morphism $\phi : B \to \mathcal{P}^\bullet$ with $\phi(1) = \sum_{i\in Q_0} e_i\zeta_{(e_i,e_i)}$. Here, 1 is the unit of B. For each arrow $\beta \in Q_1$, there is a unique right B-module morphism $\phi_\beta : B \to \mathcal{P}^\bullet$ with $\phi_\beta(1) = \beta\zeta_{(e_{s(\beta)},e_{s(\beta)})}$. By Lemma 4.5, we have

$$\phi((p^{\mathrm{op}})^* q^{\mathrm{op}}) = e_{t(p)}\zeta_{(p,q)} \text{ and } \phi_\beta((p^{\mathrm{op}})^* q^{\mathrm{op}}) = \delta_{s(\beta),t(p)}\beta\zeta_{(p,q)} \tag{4.5}$$

for $(p^{\mathrm{op}})^* q^{\mathrm{op}} \in \Gamma$. Here, we emphasize that Γ is the k-basis of $B = L_k(Q^{\mathrm{op}})$. Then ϕ is injective and the restriction of ϕ_β to $e_{s(\beta)}B$ is injective. Observe that both ϕ and ϕ_β are graded B-module morphisms.

Lemma 4.7. *For each $i \in Q_0$, $l \in \mathbb{Z}$ and $(p,q) \in \mathbf{\Lambda}_i^l$, we have*

$$(\delta^l \circ \phi)((p^{\mathrm{op}})^* q^{\mathrm{op}}) = \sum_{\{\alpha\in Q_1 \mid t(\alpha)=i\}} \phi_\alpha(\alpha^{\mathrm{op}}(p^{\mathrm{op}})^* q^{\mathrm{op}}).$$

From this, we conclude that $(\delta^l \circ \phi)(b) = \sum_{\alpha\in Q_1} \phi_\alpha(\alpha^{\mathrm{op}} b)$ for $b \in B^l$.

Proof. For each arrow $\alpha \in Q_1$ and $(p^{\mathrm{op}})^* q^{\mathrm{op}} \in \Gamma$, we observe that

$$\alpha^{\mathrm{op}}(p^{\mathrm{op}})^* q^{\mathrm{op}} = \begin{cases} \delta_{\alpha,\alpha_1}(\widehat{p}^{\mathrm{op}})^* q^{\mathrm{op}} & \text{if } p = \alpha_1\widehat{p}; \\ (q\alpha)^{\mathrm{op}} & \text{if } l(p) = 0, \end{cases} \tag{4.6}$$

which are combinations of basis elements of $L_k(Q^{\mathrm{op}})$. Then we have that

$$\begin{aligned}(\delta^l \circ \phi)((p^{\mathrm{op}})^* q^{\mathrm{op}}) &= \delta^l(e_i\zeta_{(p,q)}) \\ &= \begin{cases} \alpha_1\zeta_{(\widehat{p},q)} & \text{if } p = \alpha_1\widehat{p}; \\ \sum_{\{\alpha\in Q_1 \mid t(\alpha)=i\}} \alpha\zeta_{(e_{s(\alpha)},q\alpha)} & \text{if } l(p) = 0, \end{cases} \\ &= \sum_{\{\alpha\in Q_1 \mid t(\alpha)=i\}} \phi_\alpha(\alpha^{\mathrm{op}}(p^{\mathrm{op}})^* q^{\mathrm{op}}).\end{aligned}$$

The last equality uses (4.6). □

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Proof of Proposition 4.6. Recall that $\delta^l(\alpha\zeta_{(p,q)}) = 0$ for $\alpha \in Q_1$ with $s(\alpha) = i$. It follows that (1–3) hold for $x = \alpha$. It suffices to prove that (1–3) hold for $x = e_i$. We recall that $(p, q) \in \mathbf{\Lambda}_i^l$, and thus $t(p) = i$.

For (1), we have that

$$\begin{aligned}\delta^l(e_i\zeta_{(p,q)}\cdot e_j) &= \delta^l(\phi((p^{\mathrm{op}})^* q^{\mathrm{op}})e_j)\\ &= (\delta^l \circ \phi)((p^{\mathrm{op}})^* q^{\mathrm{op}} e_j)\\ &= \sum_{\{\alpha\in Q_1 \mid t(\alpha)=i\}} \phi_\alpha(\alpha^{\mathrm{op}}(p^{\mathrm{op}})^* q^{\mathrm{op}} e_j)\\ &= \sum_{\{\alpha\in Q_1 \mid t(\alpha)=i\}} \phi_\alpha(\alpha^{\mathrm{op}}(p^{\mathrm{op}})^* q^{\mathrm{op}})\cdot e_j\\ &= \delta^l(e_i\zeta_{(p,q)})\cdot e_j.\end{aligned}$$

Here, the second and fourth equalities hold because ϕ and ϕ_α are right B-module morphisms; the third and last equalities use Lemma 4.7. Similar arguments prove (2) and (3). □

5. The differential graded endomorphism algebra of the projective Leavitt complex

In this section, we prove that the opposite differential graded endomorphism algebra of the projective Leavitt complex of a finite quiver without sources is quasi-isomorphic to the Leavitt path algebra of the opposite quiver. Here, the Leavitt path algebra is naturally $\mathbb{Z}$-graded and viewed as a differential graded algebra with trivial differential.

5.1. The quasi-balanced dg bimodule

We first recall some notation on quasi-balanced dg bimodules. Let A be a dg algebra and M, N be (left) dg A-modules. We have a $\mathbb{Z}$-graded vector space $\mathrm{Hom}_A(M, N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_A(M, N)^n$ such that each component $\mathrm{Hom}_A(M, N)^n$ consists of k-linear maps $f : M \to N$ satisfying $f(M^i) \subseteq N^{i+n}$ for all $i \in \mathbb{Z}$ and $f(a \cdot m) = (-1)^{n|a|}a \cdot f(m)$ for all homogenous elements $a \in A$. The differential on $\mathrm{Hom}_A(M, N)$ sends $f \in \mathrm{Hom}_A(M, N)^n$ to $d_N \circ f - (-1)^n f \circ d_M \in \mathrm{Hom}_A(M, N)^{n+1}$. Furthermore, $\mathrm{End}_A(M) := \mathrm{Hom}_A(M, M)$ becomes a dg algebra with this differential and the usual composition as multiplication. The dg algebra $\mathrm{End}_A(M)$ is usually called the *dg endomorphism algebra* of M.

We denote by A^{opp} the *opposite dg algebra* of a dg algebra A. More precisely, $A^{\mathrm{opp}} = A$ as graded spaces with the same differential, and the multiplication '$\circ$' on A^{opp} is given by $a \circ b = (-1)^{|a||b|}ba$.

Let B be another dg algebra. Recall that a right dg B-module is a left dg B^{opp}-module. For a dg A-B-bimodule M, the canonical map $A \to \mathrm{End}_{B^{\mathrm{opp}}}(M)$ is a homomorphism of dg algebras, sending a to l_a with $l_a(m) = a \cdot m$ for $a \in A$ and $m \in M$. Similarly, the canonical map $B \to \mathrm{End}_A(M)^{\mathrm{opp}}$ is a homomorphism of dg algebras, sending b to r_b with $r_b(m) = (-1)^{|b||m|}m \cdot b$ for homogeneous elements $b \in B$ and $m \in M$.

A dg A-B-bimodule M is called *right quasi-balanced* provided that the canonical homomorphism $B \to \mathrm{End}_A(M)^{\mathrm{opp}}$ of dg algebras is a quasi-isomorphism; see [**10**, 2.2].

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Denote by $\mathbf{K}(A)$ the homotopy category and by $\mathbf{D}(A)$ the derived category of left dg A-modules; they are triangulated categories with arbitrary coproducts. For a dg A-B-bimodule M and a left dg A-module N, $\mathrm{Hom}_A(M, N)$ has a natural structure of a left dg B-module.

The following lemma is [10, Proposition 2.2]; compare [13, 4.3] and [15, Appendix A].

Lemma 5.1. *Let M be a dg A-B-bimodule that is right quasi-balanced. Recall that $\mathrm{Loc}\langle M\rangle \subseteq \mathbf{K}(A)$ is the smallest triangulated subcategory of $\mathbf{K}(A)$ which contains M and is closed under arbitrary coproducts. Assume that M is a compact object in $\mathrm{Loc}\langle M\rangle$. Then we have a triangle equivalence*

$$\mathrm{Hom}_A(M, -) : \mathrm{Loc}\langle M\rangle \xrightarrow{\sim} \mathbf{D}(B).$$

In what follows, Q is a finite quiver without sources and $A = kQ/J^2$ is the corresponding algebra with radical square zero. Consider A as a dg algebra concentrated on degree zero. Recall that the Leavitt path algebra $B = L_k(Q^{\mathrm{op}})$ is naturally $\mathbb{Z}$-graded, and that it is viewed as a dg algebra with trivial differential.

Recall from Proposition 4.6 that the projective Leavitt complex $\mathcal{P}^\bullet$ is a dg A-B-bimodule. The following statement establishes a connection between the projective Leavitt complex and the Leavitt path algebra.

Theorem 5.2. *Let Q be a finite quiver without sources. Then the dg A-B-bimodule $\mathcal{P}^\bullet$ is right quasi-balanced.*

In particular, the opposite dg endomorphism algebra of the projective Leavitt complex of Q is quasi-isomorphic to the Leavitt path algebra $L_k(Q^{\mathrm{op}})$. Here, Q^{op} is the opposite quiver of Q; $L_k(Q^{\mathrm{op}})$ is naturally $\mathbb{Z}$-graded and viewed as a dg algebra with trivial differential.

We will prove Theorem 5.2 in § 5.2. The following equivalence has been given by [10, Theorem 6.2].

Corollary 5.3. *Let Q be a finite quiver without sources. Then there is a triangle equivalence*

$$\mathrm{Hom}_A(\mathcal{P}^\bullet, -) : \mathbf{K}_{\mathrm{ac}}(A\text{-Proj}) \xrightarrow{\sim} \mathbf{D}(B)$$

such that $\mathrm{Hom}_A(\mathcal{P}^\bullet, \mathcal{P}^\bullet) \cong B$ in $\mathbf{D}(B)$.

Proof. Recall from Theorem 3.7 that $\mathbf{K}_{\mathrm{ac}}(A\text{-Proj}) = \mathrm{Loc}\langle \mathcal{P}^\bullet\rangle$. Then the triangle equivalence follows from Theorem 5.2 and Lemma 5.1. The canonical map $B \to \mathrm{End}_A(\mathcal{P}^\bullet)^{\mathrm{opp}}$, which is a quasi-isomorphism, identifies $\mathrm{Hom}_A(\mathcal{P}^\bullet, \mathcal{P}^\bullet)$ with B in $\mathbf{D}(B)$. □

5.2. The proof of Theroem 5.2

We follow the notation in § 5.1.

Lemma 5.4 (see [24, Theorem 4.8]). *Let A be a $\mathbb{Z}$-graded algebra and $\varphi : L_k(Q) \to A$ be a graded algebra homomorphism with $\varphi(e_i) \neq 0$ for all $i \in Q_0$. Then φ is injective.*

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Let Z^n and C^n denote the nth cocycle and coboundary of the dg algebra $\mathrm{End}_A(\mathcal{P}^\bullet)^{\mathrm{opp}}$. We have the following observation.

Lemma 5.5. *Any element $f : \mathcal{P}^\bullet \longrightarrow \mathcal{P}^\bullet$ in C^n satisfies $f(\mathcal{P}^l) \subseteq \mathrm{Ker}\delta^{n+l}$ for each integer l and n.*

Proof. For any $f \in C^n$ there exists $h = (h^l)_{l\in\mathbb{Z}} \in \mathrm{End}_A(\mathcal{P}^\bullet)^{\mathrm{opp}}$ such that $f^l = \delta^{n+l-1} \circ h^l - (-1)^{n-1}h^{l+1} \circ \delta^l$ for each $l \in \mathbb{Z}$. By Proposition 2.7, we have $\mathrm{Im}\delta^{n+l-1} \subseteq \mathrm{Ker}\delta^{n+l}$ and $\mathrm{Im}\delta^l \subseteq \mathrm{Ker}\delta^{l+1}$. Then it suffices to prove that $h(\mathrm{Ker}\delta^{l+1}) \subseteq \mathrm{Ker}\delta^{n+l}$. Recall from Lemma 2.6 that $\{\alpha\zeta_{(p,q)} \mid i \in Q_0, (p,q) \in \mathbf{\Lambda}_i^{l+1}$ and $\alpha \in Q_1$ with $s(\alpha) = i\}$ is a k-basis of $\mathrm{Ker}\delta^{l+1}$. By Lemma 2.3, the proof is complete. □

Recall that the projective Leavitt complex $\mathcal{P}^\bullet$ is a dg A-B-bimodule; see Proposition 4.6. Let $\rho : B \to \mathrm{End}_A(\mathcal{P}^\bullet)^{\mathrm{opp}}$ be the canonical map which is induced by the right B-action. Since B is a dg algebra with trivial differential, we have that $\rho(L_k(Q^{\mathrm{op}})^n) \subseteq Z^n$ for $n \in \mathbb{Z}$. Taking cohomologies, we have the graded algebra homomorphism

$$H(\rho) : B \longrightarrow H(\mathrm{End}_A(\mathcal{P}^\bullet)^{\mathrm{opp}}). \tag{5.1}$$

Lemma 5.6. *The graded algebra homomorphism $H(\rho)$ is an embedding.*

Proof. By Lemma 5.4, it suffices to prove that $H(\rho)(e_i) \neq 0$ for all $i \in Q_0$. For each vertex $i \in Q_0$, $H(\rho)(e_i)(e_i\zeta_{(e_i,e_i)}) = e_i\zeta_{(e_i,e_i)}$. By Lemma 2.6, we have $e_i\zeta_{(e_i,e_i)} \notin \mathrm{Ker}\delta^0$. By Lemma 5.5, $H(\rho)(e_i) \notin C^0$. This implies $H(\rho)(e_i) \neq 0$. □

We will prove that the graded algebra homomorphism $H(\rho)$ is surjective. For each $y \in Z^n$, we will find an element $x \in B^n$ with $y - \rho(x) \in C^n$.

In what follows, we fix $y \in Z^n$ for $n \in \mathbb{Z}$. Then $y \in Z^n$ implies $\delta^\bullet \circ y - (-1)^n y \circ \delta^\bullet = 0$. Recall that $\mathcal{P}^l = \bigoplus_{i\in Q_0} P_i^{(\mathbf{\Lambda}_i^l)}$ for each $l \in \mathbb{Z}$. The set $\{e_i\zeta_{(p,q)}, \alpha\zeta_{(p,q)} \mid i \in Q_0, (p,q) \in \mathbf{\Lambda}_i^l$ and $\alpha \in Q_1$ with $s(\alpha) = i\}$ is a k-basis of $\mathcal{P}^l$. For each $i \in Q_0$, $l \in \mathbb{Z}$ and $(p,q) \in \mathbf{\Lambda}_i^l$, we have

$$\begin{cases} (\delta^{n+l} \circ y)(e_i\zeta_{(p,q)}) = (-1)^n(y \circ \delta^l)(e_i\zeta_{(p,q)}) \\ (\delta^{n+l} \circ y)(\alpha\zeta_{(p,q)}) = 0, \end{cases} \tag{5.2}$$

where $\alpha \in Q_1$ with $s(\alpha) = i$.

Observe that y is an A-module morphism. By Lemma 2.3, we may assume that

$$\begin{cases} y(e_i\zeta_{(p,q)}) = \phi(y_{(p,q)}) + \displaystyle\sum_{\{\gamma\in Q_1 \mid t(\gamma)=i\}} \phi_\gamma(\mu^\gamma_{(p,q)}) \\ y(\alpha\zeta_{(p,q)}) = \phi_\alpha(y_{(p,q)}), \end{cases} \tag{5.3}$$

where $y_{(p,q)} \in e_iL_k(Q^{\mathrm{op}})^{n+l}$ and $\mu^\gamma_{(p,q)} \in e_{s(\gamma)}L_k(Q^{\mathrm{op}})^{n+l}$.

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By (5.3) and Lemma 4.7, we have that

$$(\delta^{n+l}\circ y)(e_i\zeta_{(p,q)}) = \delta^{n+l}(\phi(y_{(p,q)})) = \sum_{\{\gamma\in Q_1\ \mid\ t(\gamma)=i\}} \phi_\gamma(\gamma^{\mathrm{op}} y_{(p,q)})$$

and that

$$(y\circ\delta^l)(e_i\zeta_{(p,q)}) = y\Bigg(\sum_{\{\gamma\in Q_1\ \mid\ t(\gamma)=i\}} \phi_\gamma(\gamma^{\mathrm{op}}(p^{\mathrm{op}})^* q^{\mathrm{op}})\Bigg).$$

By (5.2), we have

$$\sum_{\{\gamma\in Q_1\ \mid\ t(\gamma)=i\}} \phi_\gamma(\gamma^{\mathrm{op}} y_{(p,q)}) = (-1)^n y\Bigg(\sum_{\{\gamma\in Q_1\ \mid\ t(\gamma)=i\}} \phi_\gamma(\gamma^{\mathrm{op}}(p^{\mathrm{op}})^* q^{\mathrm{op}})\Bigg). \tag{5.4}$$

We recall that for each arrow $\beta\in Q_1$, the restriction of ϕ_β to $e_{s(\beta)}B$ is injective. If $l(p)=0$, then by (5.4) and (5.3), for any $\gamma\in Q_1$ with $t(\gamma)=i$ we have

$$y_{(e_{s(\gamma)},q\gamma)} = (-1)^n \gamma^{\mathrm{op}} y_{(p,q)}. \tag{5.5}$$

If $l(p)>0$, write $p=a\widehat{p}$ with $a\in Q_1$ and $t(a)=i$. By (5.4) and (5.3), we have $a^{\mathrm{op}}y_{(p,q)} = (-1)^n y_{(\widehat{p},q)}$ and $\gamma^{\mathrm{op}}y_{(p,q)}=0$ for $\gamma\in Q_1$ with $t(\gamma)=i$ and $\gamma\neq a$. The following equality holds:

$$y_{(p,q)} = \sum_{\{\gamma\in Q_1\ \mid\ t(\gamma)=i\}} (\gamma^{\mathrm{op}})^*\gamma^{\mathrm{op}} y_{(p,q)} = (-1)^n (a^{\mathrm{op}})^* y_{(\widehat{p},q)}. \tag{5.6}$$

Lemma 5.7. *Keep the notation as above. Take $x=\sum_{j\in Q_0} y_{(e_j,e_j)}\in L(Q^{\mathrm{op}})^n$. Then $y_{(p,q)} = (-1)^{nl}(p^{\mathrm{op}})^* q^{\mathrm{op}} x$ in $L_k(Q^{\mathrm{op}})$ for each $(p,q)\in\mathbf{\Lambda}_i^l$.*

Proof. Clearly, we have $y_{(e_i,e_i)}=e_i x$ in $L_k(Q^{\mathrm{op}})$. For $l(p)=0$ and $l(q)>0$, write $q=\widetilde{q}\gamma$ with $\gamma\in Q_1$. We use (5.5) to obtain $y_{(e_{s(q)},q)}=(-1)^{nl}q^{\mathrm{op}}x$ in $L_k(Q^{\mathrm{op}})$ by induction on $l(q)$. For $l(p)>0$, write $p=\beta_m\cdots\beta_1$ with all β_k arrows in Q. By (5.6), we have $y_{(p,q)} = (-1)^n(\beta_m^{\mathrm{op}})^* y_{(\widehat{p},q)}$. We obtain $y_{(p,q)}=(-1)^{nm}(\beta_m^{\mathrm{op}})^*\cdots(\beta_1^{\mathrm{op}})^* y_{(e_{s(q)},q)}$ by iterating (5.6). Then, by $y_{(e_{s(q)},q)} = (-1)^{n(m+l)}q^{\mathrm{op}}x$ in $L_k(Q^{\mathrm{op}})$, the proof is complete. □

We will construct a map $h:\mathcal{P}^\bullet\to\mathcal{P}^\bullet$ of degree $n-1$, which will be used to prove $y-\rho(x)\in C^n$. To define h, we first assign to each pair $(p,q)\in\mathbf{\Lambda}_i^l$ an element $\theta_{(p,q)}$ in $L_k(Q^{\mathrm{op}})$.

For each $i\in Q_0$, define $\theta_{(e_i,e_i)} = \sum_{\{\gamma\in Q_1\ \mid\ s(\gamma)=i\}} \mu^\gamma_{(\gamma,e_{s(\gamma)})}\in e_iL_k(Q^{\mathrm{op}})^{n-1}$. Here, refer to (5.3) for the element $\mu^\gamma_{(\gamma,e_{s(\gamma)})}$. We define $\theta_{(e_{s(q)},q)}$ inductively by

$$\theta_{(e_{s(q)},q)} = (-1)^{n-1}(\gamma^{\mathrm{op}}\theta_{(e_{s(\widetilde{q})},\widetilde{q})} - \mu^\gamma_{(e_{s(\widetilde{q})},\widetilde{q})}) \in e_{s(q)}L_k(Q^{\mathrm{op}})^{n+l-1}, \tag{5.7}$$

where $q=\widetilde{q}\gamma$ with $l(q)=l$ and $\gamma\in Q_1$. Let $(p,q)\in\mathbf{\Lambda}_i^l$ with $l(p)>0$. We define $\theta_{(p,q)}$ by induction on the length of p as follows:

$$\theta_{(p,q)} = (-1)^{n-1}(\beta^{\mathrm{op}})^*\theta_{(\widehat{p},q)} + \sum_{\{\gamma\in Q_1\ \mid\ t(\gamma)=i\}} (\gamma^{\mathrm{op}})^*\mu^\gamma_{(p,q)} \in e_iL_k(Q^{\mathrm{op}})^{n+l-1}, \tag{5.8}$$

where $p=\beta\widehat{p}$ with $\beta\in Q_1$ is of length $l(q)-l$.

We define a k-linear map $h : \mathcal{P}^\bullet \longrightarrow \mathcal{P}^\bullet$ such that

$$h(e_i\zeta_{(p,q)}) = \phi(\theta_{(p,q)}) \quad \text{and} \quad h(\alpha\zeta_{(p,q)}) = \phi_\alpha(\theta_{(p,q)})$$

for each $i \in Q_0$, $l \in \mathbb{Z}$, $(p,q) \in \mathbf{\Lambda}_i^l$ and $\alpha \in Q_1$ with $s(\alpha) = i$.

Lemma 5.8. *Let x be the element in Lemma 5.7, and let h be the above map. For each $i \in Q_0$, $l \in \mathbb{Z}$, $(p,q) \in \mathbf{\Lambda}_i^l$, we have*

$$\begin{cases} (y - \rho(x))(e_i\zeta_{(p,q)}) = (\delta^{n+l-1} \circ h - (-1)^{n-1} h \circ \delta^l)(e_i\zeta_{(p,q)}) \\ (y - \rho(x))(\alpha\zeta_{(p,q)}) = (\delta^{n+l-1} \circ h - (-1)^{n-1} h \circ \delta^l)(\alpha\zeta_{(p,q)}) = 0, \end{cases}$$

where $\alpha \in Q_1$ with $s(\alpha) = i$.

Proof. Recall from (4.5) the right B-module morphisms ϕ and ϕ_β for $\beta \in Q_1$. By (5.3) and Lemma 5.7, we have

$$\begin{cases} \rho(x)(e_i\zeta_{(p,q)}) = (-1)^{nl}\phi(p^{\mathrm{op}*}q^{\mathrm{op}})\cdot x = \phi(y_{(p,q)}) \\ \rho(x)(\alpha\zeta_{(p,q)}) = (-1)^{nl}\phi_\alpha(p^{\mathrm{op}*}q^{\mathrm{op}})\cdot x = \phi_\alpha(y_{(p,q)}) \\ (y - \rho(x))(e_i\zeta_{(p,q)}) = \displaystyle\sum_{\{\gamma \in Q_1 \mid t(\gamma) = i\}} \phi_\gamma(\mu^\gamma_{(p,q)}). \end{cases}$$

Recall that $\delta^l \circ \phi_\beta = 0$ for each arrow $\beta \in Q_1$. It remains to prove $(\delta^{n+l-1} \circ h - (-1)^{n-1} h \circ \delta^l)(e_i\zeta_{(p,q)}) = \sum_{\{\gamma \in Q_1 \mid t(\gamma) = i\}} \phi_\gamma(\mu^\gamma_{(p,q)})$.

By the definition of δ^l, we have

$$(h \circ \delta^l)(e_i\zeta_{(p,q)}) = \begin{cases} \phi_\beta(\theta_{(\widehat{p},q)}) & \text{if } p = \beta\widehat{p}, \\ \displaystyle\sum_{\{\gamma \in Q_1 \mid t(\gamma) = i\}} \phi_\gamma(\theta_{(e_{s(\gamma)}, q\gamma)}) & \text{if } l(p) = 0. \end{cases}$$

By Lemma 4.7, we have $(\delta^{n+l-1} \circ h)(e_i\zeta_{(p,q)}) = \sum_{\{\gamma \in Q_1 \mid t(\gamma) = i\}} \phi_\gamma(\gamma^{\mathrm{op}}\theta_{(p,q)})$. Then the following equalities hold:

$$\begin{aligned} &(\delta^{n+l-1} \circ h)(e_i\zeta_{(p,q)}) - (-1)^{n-1}(h \circ \delta^l)(e_i\zeta_{(p,q)}) \\ &= \begin{cases} \displaystyle\sum_{\{\gamma \in Q_1 \mid t(\gamma) = i\}} \phi_\gamma(\gamma^{\mathrm{op}}\theta_{(p,q)}) - (-1)^{n-1}\phi_\beta(\theta_{(\widehat{p},q)}) & \text{if } p = \beta\widehat{p} \\ \displaystyle\sum_{\{\gamma \in Q_1 \mid t(\gamma) = i\}} \phi_\gamma(\gamma^{\mathrm{op}}\theta_{(p,q)} - (-1)^{n-1}\theta_{(e_{s(\gamma)}, q\gamma)}) & \text{if } l(p) = 0 \end{cases} \\ &= \sum_{\{\gamma \in Q_1 \mid t(\gamma) = i\}} \phi_\gamma(\mu^\gamma_{(p,q)}). \end{aligned}$$

The last equality uses (5.7) and (5.8). □

Proof of Theorem 5.2. It suffices to prove that $H(\rho)$ in (5.1) is an isomorphism. By Lemma 5.6, it remains to prove that $H^n(\rho)$ is surjective for any $n \in \mathbb{Z}$. For any element

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$\overline{y} = y + C^n$ with $y \in Z^n$, take $x = \sum_{j \in Q_0} y_{(e_j, e_j)} \in B^n = L_k(Q^{\mathrm{op}})^n$. By Lemma 5.8, we have $y - \rho(x) \in C^n$. Then it follows that $\overline{y} = \overline{\rho(x)}$ in $H^n(\mathrm{End}_A(\mathcal{P}^\bullet)^{\mathrm{opp}})$. □

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