Euro. Jnl of Applied Mathematics (2012), *vol.* 23, *pp.* 555–562. © Cambridge University Press 2012
doi:10.1017/S0956792512000113

A note on analytic integrability of planar vector fields

A. ALGABA, C. GARCÍA and M. REYES

Department of Mathematics, Facultad de Ciencias Experimentales, Campus del Carmen, University of Huelva, Spain
e-mail: algaba@uhu.es, cristoba@uhu.es, colume@uhu.es

(*Received 11 June 2009; revised 29 March 2012; accepted 29 March 2012; first published online 23 April 2012*)

We give a new characterisation of integrability of a planar vector field at the origin. This allows us to prove that the analytic systems

$$\dot{x} = \left(\frac{\partial h}{\partial y}(x,y)K(h,y^n) + y^{n-1}\Psi(h,y^n)\right)\xi(x,y), \qquad \dot{y} = -\frac{\partial h}{\partial x}(x,y)K(h,y^n)\xi(x,y),$$

where h, K, Ψ and ξ are analytic functions defined in the neighbourhood of O with $K(O) \neq 0$ or $\Psi(O) \neq 0$ and $n \geqslant 1$, have a local analytic first integral at the origin. We show new families of analytically integrable systems that are held in the above class. In particular, this class includes all the nilpotent and generalised nilpotent integrable centres that we know.

Key words: First integral, Centre, Saddle, Monodromic

1 Introduction

One of the most important problems related to the analytic planar systems differential equations

$$(\dot{x}, \dot{y})^T = \mathbf{X}(x,y) = (P(x,y), Q(x,y))^T \tag{1.1}$$

is determining when the system has an analytic first integral defined in the neighbourhood of a singular point (we can assume that the origin is the singular point).

We recall that a function H is said to be an analytic first integral of (1.1) in an open subset U of $\mathbb{R}^2$ if H is a non-constant analytic function in U which is constant on each solution curve of (1.1) in U. Clearly, in this case such condition is equivalent to $\nabla H \cdot \mathbf{X} \equiv 0$ in U. We say that a non-zero formal power series $H = a_{00} + a_{10}x + a_{01}y + \cdots$ is a formal first integral of (1.1) if it satisfies formally that $\partial H/\partial x P(x,y) + \partial H/\partial y Q(x,y) \equiv 0$. If there exists a formal (analytic) first integral of (1.1) in the neighbourhood of the origin, it says (1.1) is formally (analytically) integrable. The problem of determining the existence of a first integral is known as integrability problem. For isolated singularities, in the analytic case, according to the result of Mattei and Moussu [20], under generic conditions, both formal and analytic integrability are equivalent.

The existence of a first integral can be used to determine the local phase portrait at an isolated singular point, and in particular to characterise when the monodromic

https://doi.org/10.1017/S0956792512000113 Published online by Cambridge University Press

singular point (the orbits of the system close to the isolated singular point turn around it) is a centre or a focus (centre problem). Hence, it will be interesting to know when a monodromic point has a first integral around it, since in such a case it is a centre.

It is known that any analytic differential system with a centre at the origin has an analytic first integral defined in a punctured neighbourhood of it, see Li *et al.* [18]. Also, Mazzi and Sabatini [21] prove that any analytic differential system with a centre in a given singular point has a smooth first integral around it. Therefore, another interesting problem will be to recognize when this first integral is, or is not, analytic.

Let us now do a more detailed review of the known results related to both problems. We know that a system can have a monodromic point at the origin only if it has either linear part of centre type, i.e. with imaginary eigenvalues (non-degenerate point), or nilpotent linear part (nilpotent point) or null linear part (degenerate point).

When the system has linear part of centre type, it follows the classical method of seeking the Liapunov function V gives $V = x^2 + y^2 + O(|x, y|^2)$ defined in the neighbourhood of the origin. It is known (see [23]) that the function V can be constructed such that $\dot{V}$, its rate of change along trajectories, is of the form $\dot{V} = \eta_2(x^2 + y^2) + \eta_4(x^2 + y^2)^2 + \cdots$, where η_j are polynomials in the coefficients of the system. The critical point is a centre if and only if all the focal values η_{2j} are zero; in such a case, there exists an analytic first integral defined at the origin (Non-degenerate centre theorem, due to Poincaré [24] and Liapunov [19]).

If a nilpotent or degenerate monodromic singular point has an analytic first integral, then it is a centre. However, there are some nilpotent and degenerate centres which are non-integrable. If the linear part of the system is nilpotent, theoretically, Moussu [22] characterises the centres and Strozyna and Zoladek [25] obtain the orbital normal form of the centres with analytic first integral. Chavarriga *et al.* [6] prove that if a nilpotent centre of an analytic system has a formal (analytic) first integral, then it has a formal (analytic) first integral of the form $H = y^2 + F(x, y)$, where F starts with terms of order higher than two. This result is used as a tool in order to detect if an analytic nilpotent centre has an analytic first integral or not. Thus, for instance, the system $\dot{x} = y + x^2$, $\dot{y} = -x^3$ has a centre at the origin, but there exists neither an analytic first integral defined at the origin nor a formal first integral. In that paper, the authors also prove that the monodromic nilpotent systems time reversible under the change of variables $(x, y, t) \to (x, -y, -t)$ have a local analytic first integral at the origin, see [17] for a short proof of the same result.

Giacomini *et al.* [12, 13] prove that the analytic nilpotent systems with a centre can be expressed as limit of non-degenerate systems with a centre, and consequently the Poincaré-Liapunov method can be used to find the nilpotent centres.

Others papers related to the analytic integrability problem of nilpotent centre are [7–10 and 15] and related to the nilpotent centres are [3] and [11].

Analogously, in general, if the linear part of the system is null, we cannot expect to have integrability at a centre. An example of a degenerate centre (null linear part) which does not have an analytic first integral at the centre is given in [23] (p. 122). The proof of this fact can be seen in [14]. The above example is a reversible system. Recently, Algaba *et al.* ([1] p. 426, Proposition 12) gave a family of systems with a degenerate singular point. This family contains, in particular, non-integrable, non-Hamiltonian and non-reversible centres.

https://doi.org/10.1017/S0956792512000113 Published online by Cambridge University Press

On the other hand, there are also integrable systems with a non-monodromic critical point. In this sense, Algaba *et al.* [2], using the normal form theory, solve the integrability problem of a family of planar systems that can be expressed as a perturbation of a Hamiltonian quasi-homogeneous system with high-degree quasi-homogeneous terms and whose Hamiltonian satisfies some generic conditions.

Others papers related to integrability of degenerate systems are [14] and [16].

The next section contains our contribution. We give a new characterisation of analytically integrable systems (Theorem 2.1). This fact allows us to find a wide family of systems (Theorem 2.2), which includes, among others, the above-mentioned vector fields. Furthermore, we give some new families of analytically integrable non-degenerate, nilpotent and degenerate systems (Theorem 2.3). We conclude that all the integrable nilpotent or generalised nilpotent centres that we know belong to the family (2.2).

2 Some families of analytically integrable planar systems

Now we prove the main results obtained. The first is a new characterisation of analytically integrable systems.

Theorem 2.1 *System (1.1) has a local analytic first integral defined in a neighbourhood of O if and only if there exist two analytic functions u and v with $\frac{\partial(u,v)}{\partial(x,y)}(x,y) \neq 0$ in a neighbourhood of O, except possibly in a null Lebesgue measure set, such that $\nabla u \cdot \mathbf{X} = F(u,v)\nabla v \cdot \mathbf{X}$ where F is an analytic function in a neighbourhood of $(u(O), v(O))$.*

Proof We assume that (1.1) has an analytic first integral H. Choosing $u = H(x,y)$, v any analytic function defined in a neighbourhood of the origin satisfying $\frac{\partial(u,v)}{\partial(x,y)}(x,y) \neq 0$ except perhaps in the null Lebesgue measure set, and $F(u,v) \equiv 0$, the result follows.

Conversely, making the change of variables (possibly singular), $u = u(x,y)$, $v = v(x,y)$, the system (1.1) becomes

$$\dot{u} = \nabla u \cdot \mathbf{X} = F(u,v)\nabla v \cdot \mathbf{X},$$
$$\dot{v} = \nabla v \cdot \mathbf{X}.$$

If $\nabla v \cdot \mathbf{X} \equiv 0$, then $v(x,y)$ is a first integral of $\mathbf{X}$, otherwise, by redefining the time variable by $d\tau = \nabla v \cdot \mathbf{X} dt$ and by denoting $\frac{d}{d\tau} =' $ it turns out that

$$u' = F(u,v), \quad v' = 1. \tag{2.1}$$

From the Cauchy–Arnold's theorem (see [5], p. 98), the system (2.1) has an analytic first integral $H(u,v)$ defined in the neighbourhood of $(u(O), v(O))$, that is, $\nabla H \cdot (F,1)(u,v) = \frac{\partial H}{\partial u}F + \frac{\partial H}{\partial v} = 0$. We define $\tilde{H}(x,y) := H(u(x,y), v(x,y))$, which is an analytic function in a neighbourhood N of the origin, since it is a composition of analytic functions. Also, $\nabla\tilde{H} \cdot \mathbf{X}(x,y) = \frac{\partial H}{\partial u}(\nabla u \cdot \mathbf{X}) + \frac{\partial H}{\partial v}(\nabla v \cdot \mathbf{X}) = \nabla v \cdot \mathbf{X}(x,y)[\nabla H \cdot (F,1)(u,v)] = 0$, for all $(x,y) \in N$ (by reducing N if necessary). Moreover, $\tilde{H}$ is non-constant since H is non-constant and $\frac{\partial(u,v)}{\partial(x,y)}(x,y) = 0$ except possibly in the null Lebesgue measure set. Therefore, $\tilde{H}$ is an analytic first integral of (1.1) in N. □

The next result provides a new class of systems with a local analytic first integral.

https://doi.org/10.1017/S0956792512000113 Published online by Cambridge University Press

Theorem 2.2 *The family of systems of differential equations*

$$\begin{aligned}\dot{x} &= \left(\frac{\partial h}{\partial y}(x,y)K(h,y^n) + y^{n-1}\Psi(h,y^n)\right)\xi(x,y),\\ \dot{y} &= -\frac{\partial h}{\partial x}(x,y)K(h,y^n)\xi(x,y),\end{aligned} \tag{2.2}$$

where h, K, Ψ *and* ξ *are analytic functions defined in a neighbourhood of* O *with* $n \geqslant 1$, $\frac{\partial h}{\partial x}(x,y) \not\equiv 0$ *and* $K(O) \neq 0$ *or* $\Psi(O) \neq 0$ *is analytically integrable at the origin.*

Proof If $\Psi(O) \neq 0$, by taking

$$u = y^n,\ v = h,\ F(u,v) = \frac{-nK(u,v)}{\Psi(u,v)}$$

and by applying Theorem 2.1 the result follows. In the case $\Psi(O) = 0$, $K(O) \neq 0$, and we can choose

$$u = h,\ v = y^n,\ F(u,v) = \frac{-\Psi(u,v)}{nK(u,v)}.$$

□

This class is wide enough and includes some systems and families of interest. We now give several subfamilies of (2.2).

Theorem 2.3 *The following systems have a local analytic first integral at the origin:*

(a)

$$\begin{aligned}\dot{x} &= x + a_1y + a_2xy + a_3y^2 + a_4x^2y + a_5xy^2 + a_6y^3,\\ \dot{y} &= -y + (b_1 + b_2x + b_3y)y^2,\end{aligned} \tag{2.3}$$

with a_k *and* b_k *being arbitrary constants such that* $a_4 + b_2 = 0$.

(b)

$$\begin{aligned}\dot{x} &= y^3 + a_1x^2 + a_1(4a_4 - b_2)x^2y^2 + a_3y^5 + a_4x^4y,\\ \dot{y} &= -x^3 - 2a_1xy + b_2x^3y^2 + 2a_1b_2xy^3,\end{aligned} \tag{2.4}$$

with a_1, a_3, a_4 *and* b_2 *being arbitrary constants.*

(c)

$$\dot{x} = y^{n-1}\bar{\Psi}(x^m, y^n), \quad \dot{y} = x^{m-1}\bar{K}(x^m, y^n), \tag{2.5}$$

with $\bar{\Psi}$ *and* $\bar{K}$ *being analytic functions such that* $\bar{\Psi}(O) \neq 0$ *or* $\bar{K}(O) \neq 0$ *and* $n, m \geqslant 1$.

(d)

$$\dot{x} = y + \frac{\partial h_{2m}}{\partial y}h_{2m}^p + \sum_{k=0}^{p} a_k h_{2m}^{p-k} y^{2m(k+1)-1}, \quad \dot{y} = -\frac{\partial h_{2m}}{\partial x}h_{2m}^p, \tag{2.6}$$

where $p \geqslant 0$, h_{2m} *is a homogeneous polynomial of degree* $2m$ *and* a_k *are an arbitrary constants,* $k = 0, \ldots, p$.

(e)

$$\dot{x} = y^{l-1} + \Omega(x^m), \quad \dot{y} = -x^{m-1}, \tag{2.7}$$

where Ω *is an analytic function and* $m, l \geqslant 1$.

https://doi.org/10.1017/S0956792512000113 Published online by Cambridge University Press

Proof

(a) For $n = 1, \xi \equiv 1,\ h(x,y) = xy$ and

$$K(h,y) = 1 + a_4 h - b_1 y - b_3 y^2,$$
$$\Psi(h,y) = a_1 + (a_2 + b_1)h + a_3 y^2 + (a_5 + b_3)hy + a_6 y^3,$$

in (2.2) becomes (2.3). Therefore, the origin of these systems is an integrable saddle point.

(b) Taking $n = 2, \xi \equiv 1$ and

$$h(x,y) = \frac{1}{4}x^4 + \frac{1}{4}y^4 + a_1 x^2 y,$$
$$K(h,y^2) = 1 - b_2 y^2,$$
$$\Psi(h,y^2) = 4a_4 h + (a_3 + b_2 - a_4)y^4$$

in (2.2) becomes (2.4).

We note that if $a_1 = 0$, the origin is a monodromic point and therefore it is a centre. Otherwise, O is a saddle point.

(c) For $h(x,y) = x^m,\ \xi \equiv 1,\ \Psi(h,y^n) = \bar{\Psi}(mh,y^n)$ and $-mK(h,y^n) = \bar{K}(h,y^n)$ in (2.2) becomes (2.5).

(d) Taking $n = 2,\ \xi \equiv 1,\ h(x,y) = h_{2m}(x,y)$ and

$$K(y^2,h) = h_{2m}^p,\ \ \Psi(y^2,h) = 1 + \sum_{k=0}^{p} a_k h_{2m}^{p-k} y^{2m(k+1)-2}$$

in (2.2) becomes (2.6).

(e) Taking $K \equiv 1,\ n = 1,\ \xi \equiv 1$ and

$$h(x,y) = \frac{1}{l}y^l + \frac{1}{m}x^m,\ \ \Psi(h,y) = \Omega(mh - \frac{m}{l}y^l)$$

in (2.2) becomes (2.7). □

3 Conclusions

In the following remarks, we emphasise the wideness of the family (2.2).

• In [16] Giné studies the systems called generalised nilpotent systems, $(\dot{x},\dot{y})^T = (y^r + P_s, Q_s)^T$ with a formal first integral of the form $y^k + F(x,y)$, where P_s and Q_s are homogeneous polynomials of degree $s = 2,3,4,5,\ r = 1,2,3,\ r < s$ and, F starts with terms of order higher than $k = 1,2,3,4,5,6,7$. All are included in the family (2.5).

In particular, if $m = 1,\ n = 2$ and $\bar{\Psi}(x,y^2) = 1 + f(x,y^2)$, the system (2.5) turns out to be

$$\dot{x} = y + yf(x,y^2), \quad \dot{y} = \bar{K}(x,y^2). \tag{3.1}$$

So the nilpotent systems time reversible under the change of variables $(x,y,t) \to (x,-y,-t)$ are analytically integrable. In the case of a centre, this fact is proved in [6]. If the singular point is non-monodromic, Chavarriga's [6] result is also true. For instance, system

$$\dot{x} = y + yf(x,y^2), \quad \dot{y} = x^2 + g(x,y^2),$$

https://doi.org/10.1017/S0956792512000113 Published online by Cambridge University Press

where f and g are analytic functions with $f(O) = g(O) = 0$, has an integrable cuspidal point, see [17].

Furthermore, the integrable systems $(\dot{x}, \dot{y})^T = (y + P_n, Q_n)^T$, where P_n and Q_n are homogeneous polynomials of degree $n = 2, 3, 4, 5$, studied in [15], also belong to the family (3.1).

• Algaba *et al.* [3] study the centre problem of the analytic system of differential equations on the plane whose origin is a nilpotent singular point

$$\dot{x} = y + \sum_{i=1}^{\infty} P_{q-p+2is}(x, y), \qquad \dot{y} = \sum_{i=1}^{\infty} Q_{q-p+2is}(x, y), \tag{3.2}$$

where $p, q, n \in \mathbb{N}$, $p \leqslant q$, $s = (n+1)p - q > 0$, and $\mathbf{F}_i = (P_i, Q_i)^T$ is a quasi-homogeneous vector field of type (p, q) and degree i with $Q_{(2n+1)p-q}(1, 0) < 0$ (necessary condition of monodromy). In particular, the authors solve the centre problem for several subfamilies of (3.2) and show that the centres obtained have an analytic first integral at the origin. We note that these systems with a centre are Hamiltonian, time-reversible (of the form (3.1)) or can be expressed as systems (2.2) with K, ξ, Ψ and h polynomials.

• Systems (2.6) include the analytically integrable nilpotent family given in Lemma 2 of [4].

• Systems (2.7) are a family of generalised nilpotent systems. If $l = 2$ and $m = 2M$, we have the systems

$$\dot{x} = y + \Omega(x^{2M}), \quad \dot{y} = -x^{2M-1}. \tag{3.3}$$

Hence, nilpotent systems (3.3) are analytically integrable.

We conjecture that, in general, the integrable nilpotent systems and the integrable generalised nilpotent systems are included in family (2.2).

• If a vector field has a local analytic inverse integrating factor V, i.e. $\nabla V \cdot \mathbf{X} = div(\mathbf{X}).V$, with $V(O) \neq 0$, its associated system can be written as

$$\dot{x} = \frac{\partial H}{\partial y}(x, y)V(x, y), \;\; \dot{y} = -\frac{\partial H}{\partial x}(x, y)V(x, y),$$

where $H = \int P/V dy + g(x)$ is an analytic first integral with $\frac{\partial g}{\partial x}(x, y) = -Q/V$. These systems are included in family (2.2), taking $K \equiv 1, \;\Psi \equiv 0, \;h = H$ and $V = \xi$.

Also, a normal form of a non-degenerate centre is

$$(\dot{x}, \dot{y})^T = (-y, x)^T \left(1 + \sum_{i=1}^{\infty} \beta_j (x^2 + y^2)^j\right), \tag{3.4}$$

where β_j are real numbers. This normal form is analytic, since it satisfies Brunos's conditions 'A' and 'ω', see [5]. Furthermore, it is included in family (2.2); concretely, for

https://doi.org/10.1017/S0956792512000113 Published online by Cambridge University Press

the case $K \equiv 1$, $\Psi \equiv 0$ and

$$h(x, y) = -\frac{(x^2 + y^2)}{2}, \ \xi(x, y) = 1 + \sum_{i=1}^{\infty} \beta_j (x^2 + y^2)^j.$$

Therefore, the non-degenerate centres have a local analytic first integral (Non-degenerate centre theorem, see [19, 24]).

Acknowledgements

This work has been partially supported by *Ministerio de Ciencia e Innovación,* co-financed with FEDER funds, in the frame of the project MTM2010-20907-C02-02 and by *Consejería de Educación y Ciencia de la Junta de Andalucía* (FQM-276 and EXC/2008/FQM-872).

References

[1] Algaba, A., Fuentes, N. & García, C. (2012) Centres of quasi-homogeneous polynomial planar systems. *Nonlinear Anal.* **13**, 419–431.

[2] Algaba, A., Gamero, E. & García, C. (2009) The integrability problem for a class of planar systems. *Nonlinearity* **22**, 395–420.

[3] Algaba, A., García, C. & Reyes, M. (2008) The centre problem for a family of systems of differential equations having a nilpotent singular point. *J. Math. Anal. Appl.* **340**, 32–43.

[4] Andreev, A., Sadovskii, A. P. & Tskialyuk, V. A. (2003) The centre-focus problem for a system with homogeneous nonlinearities in the case zero eigenvalues of the linear part. *Diff. Equ.* **39**(2), 155–164.

[5] Bruno, A. D. (1989) *Local Methods in Nonlinear Differential Equations,* Springer-Verlag, New York.

[6] Chavarriga, J., Giacomini, H., Giné, J. & Llibre, J. (2003) Local analytic integrability for nilpotent centres. *Ergod. Th. Dynam. Sys.* **23**, 417–428.

[7] Chavarriga, J., García, I. & Giné, J. (1997) Integrability of centres perturbed by quasihomogeneous polynomials. *J. Math. Anal. Appl.* **210**(1), 268–278.

[8] Chavarriga, J. & Giné, J. (1999) Local integrability for nilpotent critical point. *Dyn. Syst. Plasmas Gravit. (Lect. Notes Phys.)* **518**, 278–287 (Springer, Berlin, Germany).

[9] Chavarriga, J., Giné, J. & Sorolla, J. (2002) Analytic integrability of a class of nilpotent cubic systems. *Math. Comput. Simul.* **59**, 489–495.

[10] García, I. & Giné, J. (2010) Analytic nilpotent centres with analytic first integral. *Nonlinear Anal.* **72**(9–10), 3732–3738.

[11] Gasull, A. & Torregrosa, J. (2001) A new algorithm for the computation of the Lyapunov constants for some degenerate critical points. *Nonlinear Anal.* **47**, 4479–4490.

[12] Giacomini, H., Giné, J. & Llibre, J. (2006) The problem of distinguishing between a centre and a focus for nilpotent and dgenerate analytic systems. *J. Differ. Equ.* **227**(2), 406–426.

[13] Giacomini, H., Giné, J. & Llibre, J. (2007) Corrigendum to: "The problem of distinguishing between a centre and a focus for nilpotent and dgenerate analytic systems" (J. Differential Equations **227**(2) (2006), 406–426). *J. Differ. Equ.* **232**(2), 702.

[14] Giné, J. (2002) Sufficient conditions for a centre at a completely degenerate critical point. *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **12**(7), 1659–1666.

[15] Giné, J. (2004) Analytic integrability and characterization of centre for nilpotent singular point. *Z. Angew. Math. Phys.* **55**, 725–740.

https://doi.org/10.1017/S0956792512000113 Published online by Cambridge University Press

[16] Giné, J. (2004) Analytic integrability and characterization of centre for generalized nilpotent singular point. *Appl. Math. Comput.* **148**, 849–868.

[17] Giné, J. & Maza, S. (2011) The reversibility and the centre problem. *Nonlinear Anal.* **74**(2), 695–704.

[18] Li, W., Llibre, J., Nicolau, M. & Zhang, X. (2002) On the differentiability of first integrals of two-dimensional flows. *Proc. Am. Math. Soc.* **130**, 2079–2088.

[19] Liapunov, M. A. (1949) Problème général de la stabilité du mouvement. *Annals of Mathematics Studies*, vol. 17, Princeton University Press, Princeton, NY.

[20] Mattei, J. F. & Moussu, R. (1980) *Holonomie et intégrales premières. Ann. Sci. Ecole Normale Superieure* **13**, 469–523.

[21] Mazzi, L. & Sabatini, M. (1988) A characterization of centres via first integrals. *J. Differ. Equ.* **76**, 222–237.

[22] Moussu, R. (1982) Symétrie et forme normale des centres et foyers dégénérés. *Ergodic Theory Dynam. Sys.* **2**, 241–251.

[23] Nemytskii, V. V. & Stepanov, V. V. (1960) *Qualitative Theory of Differential Equations,* Princeton University Press, Princeton, NY.

[24] Poincaré, H. (1881) Mémoire sur les courbes définies par les équations différentielles. *J. Math.* **37**, 375–422.

[25] Strozyna, E. & Zoladek, H. (2002) The analytic and formal normal form for the nilpotent singularity. *J. Differ. Equ.* **179**, 479–537.

https://doi.org/10.1017/S0956792512000113 Published online by Cambridge University Press