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Symmetries, ansätze and exact solutions of nonlinear second-order evolution equations with convection terms, II

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New results concerning Lie symmetries of nonlinear reaction-diffusion-convection equations, which supplement in a natural way the results published in the *European Journal of Applied Mathematics* (**9**(1998), 527–542) are presented.

1 Introduction

Recently we established new results, which supplement those published earlier [1]. We remind the reader that § 2 of Cherniha & Serov [1] is devoted to a complete description of Lie symmetries of the reaction-diffusion-convection (RDC) equation

$$u_0 = \partial_1[A(u)u_1] + B(u)u_1 + C(u), \tag{1.1}$$

where $u = u(x_0, x_1)$ is an unknown function and $A(u), B(u), C(u)$ are arbitrary smooth functions. Hereinafter the 0 and 1 subscripts to functions denote differentiation with respect to the variables x_0 and x_1, and $\partial_1 = \frac{\partial}{\partial x_1}$.

The main result was presented in Theorem 2.1 of Cherniha & Serov [1], and proved under the following restriction: formula (2.29) in Cherniha & Serov [1] presents the most general form of transformations reducing any given nonlinear RDC equation with a non-trivial Lie symmetry (we remind that the Lie algebra spanned by the operators of time and space translations was called the trivial Lie algebra) to one of those given in Table 1 of Cherniha & Serov [1] or to the well known Burgers equation.

We also assumed that the function $B(u) \neq 0$ since the case $B(u) = 0$ has been completely described in Dorodnitsyn [2]. It turns out that some RDC equations listed in Table 1 of Cherniha & Serov [1] can be reduced to reaction-diffusion equations, i.e., equations of the form (1.1) with $B(u) = 0$, if one applies some local substitutions not belonging to (2.29) in Cherniha & Serov [1]. In other words, the list of nonlinear RDC equations with a non-trivial Lie symmetry can be shortened. Moreover, we established that there are two nonlinear RDC equations with non-trivial Lie symmetries which are not listed in that table.

The paper is organized as follows. In the next section, we show how to reduce a number of RDC equations arising in Table 1 of Cherniha & Serov [1] using new local substitutions. The final result is formulated as a new table. In § 3 we discuss how to construct a standard procedure for reducing the given list of differential equations with the known Lie algebra invariance to a list of "canonical" equations and the corresponding Lie symmetries.

2 Main results

The list of RDC equations admitting non-trivial Lie symmetries obtained in section 2 of Cherniha & Serov [1] can be given in the form of Table 1 (here some misprints arising in Cherniha & Serov [1] are corrected).

In Table 1, the following designations are introduced: $\lambda_1 \neq 0, \lambda_2, \lambda_3, \alpha_1 \neq 0, k \neq 0, m$ are arbitrary constants, $\lambda_4 = 2\alpha_1^2(k+2)$, and:

$$\begin{aligned}
D_1 &= 2mx_0\partial_0 + mx_1\partial_1 - u\partial_u,\\
D_2 &= (2m-k)x_0\partial_0 + (m-k)x_1\partial_1 - u\partial_u,\\
D_3^* &= kx_0\partial_0 - u\partial_u,\\
D_4 &= (2m-1)x_0\partial_0 + (m-1)x_1\partial_1 - \partial_u,\\
D_5 &= x_0\partial_0 - \partial_u,\\
D_6 &= 2x_0\partial_0 + x_1\partial_1 - \partial_u,\\
G &= x_0\partial_1 - \frac{1}{\lambda_1}u\partial_u,\\
Y &= \exp\left[\left(\frac{\lambda_1^2}{4} + \lambda_3\right)x_0 - \frac{\lambda_1}{2}x_1\right]u\partial_u,\\
\mathcal{G}_1 &= \exp(\lambda_2 x_0)\left(\partial_1 - \frac{\lambda_2}{\lambda_1}\partial_u\right),\\
\mathcal{G}_2 &= \exp(\lambda_3 x_0)\left(\partial_1 - \frac{\lambda_3}{\lambda_1}u\partial_u\right),\\
T_1 &= \exp(-\lambda_2 k x_0)(\partial_0 + \lambda_2 u\partial_u),\\
T_2 &= \exp(-\lambda_2 x_0)(\partial_0 + \lambda_2\partial_u),\\
X_1 &= \exp(-k\alpha_1 x_1)(\partial_1 - 2\alpha_1 u\partial_u),\\
X_2 &= \exp(-\alpha_1 x_1)(\partial_1 - 2\alpha_1\partial_u)
\end{aligned} \tag{2.1}$$

are operators of Lie symmetry. The substitutions

$$\begin{aligned}
x_0 &\to c_0 x_0,\\
x_1 &\to c_1 x_1 + c_2 x_0 + c_3 x_0^2 + c_7\exp(c_8 x_1),\\
A_0(u) &\to c_4 + c_5 x_0 + c_6 A_0(u)\exp(c_9 x_0) + c_{10} x_1,
\end{aligned} \tag{2.2}$$

with correctly-specified constants $c_i, i = 0, \ldots, 10$ and $A_0(u) = \int A(u)du$ reduce any given nonlinear RDC equation with a non-trivial Lie symmetry either to one of those given in this table or to the well-known Burgers equation $u_0 = u_{11} + \lambda_1 u u_1$.

The main idea leading to the new substitutions is based on the following observation: the operators T_1, T_2, X_1 and X_2 arising in the cases 8–11, 13–16 of Table 1 can be mapped to the operators of time and space translation.

Table 1. *Lie symmetries of equation (1.1)[1]*

N	$A(u)$	$B(u)$	$C(u)$	Maximal algebra of invariance
1.	1	$\lambda_1 u^m$	$\lambda_2 u^{2m+1}, (m \neq 0)$	$\partial_0, \partial_1, D_1$
2.	1	$\lambda_1 u$	$\lambda_2 u, (\lambda_2 \neq 0)$	$\partial_0, \partial_1, \mathcal{G}_1$
3.	1	$\lambda_1 \ln u$	$\lambda_2 u$	$\partial_0, \partial_1, G$
4.	1	$\lambda_1 \ln u$	$u(\lambda_2 + \lambda_3 \ln u), (\lambda_3 \neq 0)$	$\partial_0, \partial_1, \mathcal{G}_2$
5.	1	$\lambda_1 \ln u$	$u(\lambda_2 + \lambda_3 \ln u + \frac{\lambda_1^2}{4} \ln^2 u)$	$\partial_0, \partial_1, Y$
6.	1	$\lambda_1 \exp u$	$\lambda_2 \exp(2u)$	$\partial_0, \partial_1, D_6$
7.	u^k	$\lambda_1 u^m, (m \neq 0)$	$\lambda_2 u^{2m-k+1}$	$\partial_0, \partial_1, D_2$
8.	u^k	$\alpha_1 u^k$	$u(\lambda_2 + \lambda_3 u^k), (\lambda_2 \neq 0)$	$\partial_0, \partial_1, T_1$
9.	$u^k, (k \neq -\frac{4}{3})$	$\lambda_1 u^{k/2} + (3k+4)\alpha_1 u^k$	$u(\lambda_2 + \lambda_4 u^k + 2\alpha_1 \lambda_1 u^{k/2})$	$\partial_0, \partial_1, X_1$
10.	$u^k, (k \neq -\frac{4}{3})$	$\alpha_1(3k+4)u^k$	$\lambda_4 u^{k+1}$	$\partial_0, \partial_1, D_3^*, X_1$
11.	$u^k, (k \neq -\frac{4}{3})$	$\alpha_1(3k+4)u^k$	$u(\lambda_3 + \lambda_4 u^k), (\lambda_3 \neq 0)$	$\partial_0, \partial_1, T_1, X_1$
12.	$\exp u$	$\lambda_1 \exp(mu)$	$\lambda_2 \exp[(2m-1)u]$	$\partial_0, \partial_1, D_4$
13.	$\exp u$	$\alpha_1 \exp u$	$\lambda_2 + \lambda_3 \exp u, (\lambda_2 \neq 0)$	$\partial_0, \partial_1, T_2$
14.	$\exp u$	$\lambda_1 \exp \frac{u}{2} + 3\alpha_1 \exp u$	$\lambda_2 + 2\alpha_1^2 \exp u + 2\alpha_1\lambda_1 \exp \frac{u}{2}$	$\partial_0, \partial_1, X_2$
15.	$\exp u$	$3\alpha_1 \exp u$	$2\alpha_1^2 \exp u$	$\partial_0, \partial_1, D_5, X_2$
16.	$\exp u$	$3\alpha_1 \exp u,$	$\lambda_2 + 2\alpha_1^2 \exp u$	$\partial_0, \partial_1, T_2, X_2$

Let us start from the operator $T_1 = \exp(-\lambda_2 k x_0)(\partial_0 + \lambda_2 u \partial_u)$ arising in cases 8 and 11 of Table 1. The Lie algebra with the basic operators ∂_0, ∂_1 and T_1 by the simple renaming

$$-\frac{T_1}{\lambda_2} \to \partial_0^*, \quad \partial_1 \to \partial_1^*, \quad \partial_0 \to D_2^*$$

is taken to the algebra with the basic operators $\partial_0^*, \partial_1^*$ and D_2^*, which satisfies the same commutation relations as the algebra $\langle \partial_0, \partial_1, D_2 \rangle$ with $m = k$ arising in case 7 of Table 1. In other words, we deal with two representations of the same Lie algebra. So one can expect that the relevant equations are related by the same substitution which transforms the algebra $\langle \partial_0, \partial_1, T_1 \rangle$ into the algebra $\langle \partial_0, \partial_1, D_2 \rangle$ with $m = k$. The operator T_1 is transformed to the operator of time translations $\partial_0^* = \frac{\partial}{\partial t}$ by the substitution

$$t = \frac{1}{\lambda_2 k} \exp(\lambda_2 k x_0), \quad w = u \exp(-\lambda_2 x_0). \tag{2.3}$$

Simultaneously, this substitution maps the operator ∂_0 to $\lambda_2 D_2^* = \lambda_2(kt\partial_t - w\partial_w)$. Now one easily checks that the relevant RDC equations, arising in cases 8 and 11 of Table 1 are taken to

$$w_t = \partial_1[w^k w_1] + \alpha_1 w^k w_1 + \lambda_3 w^{k+1}, \tag{2.4}$$

and

$$w_t = \partial_1[w^k w_1] + \alpha_1(3k+4)w^k w_1 + 2\alpha_1^2(k+2)w^{k+1}, \tag{2.5}$$

respectively, by substitution (2.3).

Now one observes that (2.4) is a particular case of the equation arising in case 7 of Table 1 from Cherniha & Serov [1]. So, case 8 can be combined with case 7. Equation (2.5) coincides with the equation arising in case 10. On the other hand, it will be shown that (2.5) admits the further simplification.

In fact, since the algebras arising in case 9 and 7 with $2m = k$ are two representations of the same Lie algebra, we found the substitution

$$x = -\frac{1}{\alpha_1 k}\exp(\alpha_1 k x_1), \quad U = w\exp(2\alpha_1 x_1), \tag{2.6}$$

which reduces the operator $X_1 = \exp(-k\alpha_1 x_1)(\partial_1 - 2\alpha_1 w\partial_w)$ to the operator of space translations $\partial_1^* = \frac{\partial}{\partial x}$. Thus, (2.6) can be also applicable for simplifying the equation (see case 9 of Table 1)

$$w_t = \partial_1[w^k w_1] + (\lambda_1 w^{k/2} + \alpha_1(3k+4)w^k)w_1 + w(\lambda_2 + 2\alpha_1\lambda_1 w^{k/2} + 2\alpha_1^2(k+2)w^k). \tag{2.7}$$

Substituting (2.6) into (2.7) and making cumbersome calculations, one obtains the equation

$$U_t = \partial_x[U^k U_x] - \lambda_1 U^{k/2} U_x + \lambda_2 U. \tag{2.8}$$

We again observe that (2.8) is a particular case of the equation arising in case 7 of Table 1. So, case 9 can be combined with case 7.

It turns out, that substitution (2.6) also reduces (2.5) to the known nonlinear heat equation

$$U_t = \partial_x[U^k U_x], \tag{2.9}$$

which is invariant under a 4-dimensional Lie algebra (see, for example, Dorodnitsyn [2]).

Thus, cases 8–11 of Table 1 from Cherniha & Serov [1] can be omitted, if one applies a combination of the local substitutions (2.3) and (2.6) not belonging to (2.29) in Cherniha & Serov [1].

Remark 1. Substitution (2.6), mapping (2.5) to (2.9), can be also obtained from King [3] if one puts $N = 1$ in the relevant formulas on P.52.

In a similar way, one can deal with the RDC equations arising in cases 13–16 of Table 1. It was again established that the operator $T_2 = \exp(-\lambda_2 x_0)(\partial_0 + \lambda_2\partial_u)$ arising in cases 13 and 16 is transformed to the operator of time translation P_t by the substitution

$$t = \frac{1}{\lambda_2}\exp(\lambda_2 x_0), \quad w = u - \lambda_2 x_0. \tag{2.10}$$

Simultaneously, the equations arising in cases 13 and 16 of Table 1 take the forms

$$w_t = \partial_1[\exp w\ w_1] + \alpha_1 \exp w\ w_1 + \lambda_3 \exp w, \tag{2.11}$$

Table 2. *Lie symmetries of equation (1.1)*

N	$A(u)$	$B(u)$	$C(u)$	Maximal algebra of invariance
1.	1	$\lambda_1 u$	0	$\partial_0, \partial_1, D_1, G_1, \Pi$
2.	1	$\lambda_1 u$	$\lambda_2 u, (\lambda_2 \neq 0)$	$\partial_0, \partial_1, \mathscr{G}_1$
3.	1	$\lambda_1 \log u$	$\lambda_2 u$	$\partial_0, \partial_1, G_2$
4.	1	$\lambda_1 \log u$	$\lambda_3 u \log u, (\lambda_3 \neq 0)$	$\partial_0, \partial_1, \mathscr{G}_2$
5.	1	$\lambda_1 \log u$	$u(\lambda_2 + \frac{\lambda_1^2}{4} \log^2 u)$	$\partial_0, \partial_1, Y$
6.	u^k	$\lambda_1 u^m, (m \neq 0)$	$\lambda_2 u^{2m-k+1}$	$\partial_0, \partial_1, D_2$
7.	$\exp ku$	$\lambda_1 \exp(mu), (m \neq 0)$	$\lambda_2 \exp((2m-k)u)$	$\partial_0, \partial_1, D_3$
8.	$\exp(ku), (k \neq 0)$	$\lambda_1 u$	$\lambda_2 \exp(-ku)$	$\partial_0, \partial_1, Z_1$
9.	$u^k, (k \neq 0),$	$\lambda_1 \log u$	$\lambda_2 u^{1-k}$	$\partial_0, \partial_1, Z_2$

and

$$w_t = \partial_1[\exp w \; w_1] + 3\alpha_1 \exp w \; w_1 + 2\alpha_1^2 \exp w, \tag{2.12}$$

respectively. While (2.11) is a particular case of the equation arising in case 12 of Table 1, equation (2.12) (one coincides with the equation arising in case 15) admits the further simplification by applying the substitution

$$x = -\frac{1}{\alpha_1} \exp(\alpha_1 x_1), \quad U = w + 2\alpha_1 x_1. \tag{2.13}$$

It can be checked by direct calculations that this substitution transforms (2.12) into the known nonlinear heat equation

$$U_t = \partial_x[\exp U \; U_x], \tag{2.14}$$

which is invariant under 4-dimensional Lie algebra (see, for example, Dorodnitsyn [2]). Moreover, (2.13) reduces the equation (see case 14 of Table 1)

$$w_t = \partial_1[\exp w \; w_1] + (\lambda_1 \exp(w/2) + 3\alpha_1 \exp w)w_1 + \lambda_2 + 2\alpha_1\lambda_1 \exp(w/2) + 2\alpha_1^2 \exp w \tag{2.15}$$

to the form

$$U_t = \partial_x[\exp U \; U_x] - \lambda_1 \exp(U/2) U_x + \lambda_2, \tag{2.16}$$

which is nothing else but another particular case of the equation arising in case 12 of Table 1.

Finally, the proof of Theorem 2.1 [1] has been once more checked and two new cases found, which were lost in Cherniha & Serov [1]. They are added below as cases 8 and 9. Thus, we can formulate the following theorem.

Theorem 1 *All possible maximal algebras of invariance of the nonlinear RDC equation* (1.1) *for any fixed triplet of functions A, B, C, where $AB \neq 0$, are presented in Table 2. Any other equation of the form* (1.1) *with non-trivial Lie symmetry is reduced by a local substitution* *of the form*

$$\begin{aligned} x_0 &\to c_0x_0 + c_1\exp(c_2x_0) \\ x_1 &\to c_3x_1 + c_4\exp(c_5x_1) + c_6x_0 + c_7x_0^2 \\ U &\to c_8U + c_9x_0 + c_{10}x_1 + c_{11}U\exp(c_{12}x_0 + c_{13}x_1) + c_{14} \end{aligned} \tag{2.17}$$

either to one of those given in Table 1 or to an equation of the form (1.1) *with* $B = 0$ *(the constants* $c_i, i = 1,\dots,14$ *are determined by the form of the equation in question, many of them necessarily being zero in any given case).*

In Table 2, the following designations are introduced: $\lambda_1 \neq 0, \lambda_2, \lambda_3, k$ *and* m *are arbitrary constants, and*

$$\begin{aligned} \Pi &= x_0^2P_0 + x_0x_1P_1 - (\tfrac{x_1}{\lambda_1} + x_0u)\partial_u, \\ G_1 &= x_0\partial_1 - \tfrac{1}{\lambda_1}\partial_u, \\ G_2 &= x_0\partial_1 - \tfrac{1}{\lambda_1}u\partial_u, \\ Y &= \exp[\tfrac{\lambda_1^2}{4}x_0 - \tfrac{\lambda_1}{2}x_1]u\partial_u \\ D_3 &= (2m-k)x_0\partial_0 + (m-k)x_1\partial_1 - \partial_u, \\ Z_1 &= kx_0\partial_0 + (kx_1 - \lambda_1x_0)\partial_1 + \partial_u, \\ Z_2 &= kx_0\partial_0 + (kx_1 - \lambda_1x_0)\partial_1 + u\partial_u \end{aligned} \tag{2.18}$$

are operators of Lie symmetry, the operators $D_1, D_2, \mathscr{G}_1$ *and* $\mathscr{G}_2$ *are defined in* (2.1).

Remark 2. In Table 2, the Burgers equation and its Lie symmetry (case 1) are included (see formulas (1.7)–(1.8) in Cherniha & Serov [1]). Cases 1 and 7 of Table 1 are combined here in case 6, while cases 6 and 12 of Table 1 are combined in case 7. Similarly, the equations arising in cases 4 and 5 of Table 1 are reduced to the same equations but with $\lambda_2 = 0$ and $\lambda_3 = 0$, respectively, by applying the substitution $x_1 \to x_1 + c_6x_0$, $u \to c_{11}u$ with correctly-specified c_6 and c_{11} (see formulas (2.17)).

It should be noted that the set of substitutions (2.17) (in contrary to (2.2)) does not contain the function $A_0(u) = \int A(u)du$. In fact, we have proved in Cherniha & Serov [1] that any equation of the form (1.1) with a non-constant $A(u)$ can admit a non-trivial Lie symmetry only in the cases $A(u) = c_0(u+c_1)^k$ and $A(u) = c_0\exp(ku)$. We have also checked that each of those equations can be reduced to the relevant equation from Table 2 (see cases 6–9) by a linear substitution with respect to the variable u. Simultaneously, this substitution can be also presented in a form containing $A_0(u)$. For example, the second formula from (2.3) can be presented in the form

$$A_0(w) \equiv \frac{w^{k+1}}{k+1} = \frac{u^{k+1}}{k+1}\exp(-\lambda_2x_0) \equiv A_0(u)\exp(-\lambda_2x_0), \quad k \neq -1$$

or

$$A_0(w) \equiv \exp w = \exp u\exp(-\lambda_2x_0) \equiv A_0(u)\exp(-\lambda_2x_0), \quad k = -1$$

We stress that the local substitutions presented above do not coincide with the set of equivalence transformations of the RDC equation (1.1)

$$x_0 \to c_0x_0 + c_1, \quad x_1 \to c_2x_1 + c_3x_0 + c_4, \quad U \to c_5U + c_6, \tag{2.19}$$

which was calculated using the well-known procedure (see, for example, Akhatov *et al.* [4]). A similar situation takes place in the case of systems of reaction-diffusion equations. Using a set of new local substitutions, it was established in a recent paper [5] that there are only 10 non-equivalent reaction-diffusion systems with variable diffusivities admitting non-trivial Lie symmetries. Those substitutions cannot be obtained from the relevant equivalence transformations, too. A natural question is: Can we claim that nine equations listed in Table 2 are non-equivalent up to any local substitutions? The answer is positive.

Theorem 2 *The nine reaction-diffusion-convection equations listed in Table 2 are non-equivalent (up to local substitutions).*

Sketch of the proof The proof is based on the known fact that a given equation from the class (1.1) can be transformed into another equation from this class only by a substitution of the form

$$t = \alpha(x_0), \quad x = \beta(x_0, x_1), \quad w = \gamma(x_0, x_1)u + \kappa(x_0, x_1), \tag{2.20}$$

where $\alpha(x_0), \beta(x_0, x_1), \gamma(x_0, x_1)$ and $\kappa(x_0, x_1)$ are correctly-specified functions. Formulas (2.20) can be found by direct calculations of u_0, u_1, u_{11} using the most general form of local transformations and by substitution of the expressions obtained into (1.1). The equation obtained can have the form (1.1) (with respect to w) only under condition (2.20).

Of course, the Burgers equation (see case 1 of Table 2) cannot be mapped to any other equation from this table because it is invariant a under five-dimension Lie algebra.

Consider the equations and three-dimension Lie algebras arising in cases 2–9 of Table. The algebras $\langle \partial_0, \partial_1, \mathcal{G}_1 \rangle$ and $\langle \partial_0, \partial_1, \mathcal{G}_2 \rangle$ are two representations of the same algebra. However, the first representation can be mapped to the second one only by the nonlinear transformation $u \to \exp u$. Obviously, this transformation is not belonging to the set of substitutions (2.20). Similarly, the algebras $\langle \partial_0, \partial_1, D_3 \rangle$ and $\langle \partial_0, \partial_1, Z_1 \rangle$ can be mapped to those $\langle \partial_0, \partial_1, D_2 \rangle$ and $\langle \partial_0, \partial_1, Z_2 \rangle$, respectively, by the same nonlinear transformation.

Thus, RDC equations from Table 2 that are invariant under three-dimension Lie algebras and contain the term ∂_u are not reducible to the corresponding equations and Lie algebras with term $u\partial_u$.

Consider three RDC equations and corresponding Lie algebras listed in cases 2, 7 and 8 of Table 2. These algebras contain only the term ∂_u, however, there is one non-zero commutation relation for basic operators of the algebra $\langle \partial_0, \partial_1, \mathcal{G}_1 \rangle$, while two other algebras possess two non-zero commutation relations. Moreover, Lie algebras $\langle \partial_0, \partial_1, D_3 \rangle$ and $\langle \partial_0, \partial_1, Z_1 \rangle$ are not reducible one to another by the substitution of the form (2.20). Thus, the equations listed in cases 2,7 and 8 are non-equivalent up to local substitutions.

Finally we should consider five RDC equation and corresponding Lie algebras listed in cases 3–6 and 9 of Table 2. In a quite similar way, we can again establish that equations arising in cases 4,6 and 9 of Table 2 are not reducible one to another. Moreover, algebra $\langle \partial_0, \partial_1, G_2 \rangle$ from case 3 is nothing else but Galilei algebra without unit operator [1], which cannot be reduced to the algebra $\langle \partial_0, \partial_1, \mathcal{G}_2 \rangle$ from case 4 by any local substitution. Algebra $\langle \partial_0, \partial_1, Y \rangle$ arising in case 5 can be rewritten as $\langle \partial_0, \partial_0 + \frac{\lambda_1}{2}\partial_1, Y \rangle$ and one easily checks that algebras $\langle \partial_0, \partial_1, \mathcal{G}_2 \rangle$ and $\langle \partial_0, \partial_0 + \frac{\lambda_1}{2}\partial_1, Y \rangle$ possess the same commutation relations. However, they are not reducible one to another by any substitution of the form (2.20). It can be checked by direct calculations. Thus, the RDC equations listed in cases 3–6 and 9 of Table 2 are non-equivalent up to local substitutions.

The sketch of the proof is now completed.

3 Discussion

In this paper, the complete list of canonical reaction-diffusion equations with convection terms and corresponding Lie symmetry algebras is constructed and it's shown that the equations listed in Table 2 are locally non-equivalent, i.e., this list cannot be reduced.

It should be stressed that the classical method of Lie symmetry classification (the group classification problem) of differential equations suggested by Ovsiannikov [6] is based on the classical Lie scheme and a set of equivalence transformations of a given equation. The formal application of this method to equations containing several arbitrary functions leads to a lot equations with non-trivial Lie algebras of invariance. For example, there are about 27 non-linear equations of the form (1.1) admitting three-, four- and five-dimensional Lie algebras. Our approach of Lie symmetry classification of differential equations is based on the classical Lie scheme and on finding and then making essential use of the sets of local substitutions that reduce any differential equation with a non-trivial Lie algebra to one given in the relevant list. This approach was also applied for reaction-diffusion systems [5, 7, 8]. In the particular case, it was found that there are only 10 non-equivalent reaction-diffusion systems with variable diffusivities admitting non-trivial Lie symmetries [5].

Finally, we note an open problem: How to find those local substitutions using a standard procedure? The steps of the procedure could be as follows:

(1) construction of the set of equivalence transformations, i.e., an analog of (2.19);

(2) construction of the most general form of substitutions transforming a given equation into another equation from the same class, i.e., an analog of (2.20);

(3) analysis of the Lie algebras (with the same dimensionality) obtained by the direct application of the Lie scheme;

(4) checking whether substitutions obtained in items (1)–(2) can transform different representations of the same algebra one into another;

(5) construction of the complete list of canonical equations with non-trivial Lie algebras of invariance.

Of course, each step may contain new difficulties. For example, the third step is difficult to realize if there are several six or more dimensional Lie algebras because only Lie algebras of low dimensionality are completely described at the present time (see Popovytch *et al.* [9] and references therein). We are going to return to this problem in a forthcoming paper.

References

[1] Cherniha, R. & Serov, M. (1998) Symmetries, ansätze and exact solutions of nonlinear second-order evolution equations with convection terms. *Euro. J. Appl. Math.* **9**, 527–542.

[2] Dorodnitsyn, V. A. (1982) On invariant solutions of non-linear heat conduction with a source. *USSR Comput. Math. and Math. Phys.* **22**, 115–122.

[3] King, J .R. (1993) Exact polynomial solutions to some nonlinear diffusion equations. *Physica D*, **64** 35–65.

[4] Akhatov, I. S., Gazizov, R. K. & Ibragimov, N. H. (1991) Nonlocal symmetries. Heuristic approach. *J. Sov. Math.* **55**, 1401–1450.

[5] Cherniha, R. & King, J. R. (2005) Nonlinear Reaction-Diffusion Systems with Variable Diffusivities: Lie Symmetries, Ansätze and Exact Solutions. *J. Math. Anal. Appl.* **308**, 11–35.

[6] Ovsiannikov, L. V. (1978) *The Group Analysis of Differential Equations.* Nauka, Moscow (in Russian).

[7] Cherniha, R. & King, J. R. (2000) Lie Symmetries of Nonlinear Multidimensional Reaction-Diffusion Systems: I. *J. Phys. A: Math. Gen.* **33**, 267–282, 7839–41.

[8] Cherniha, R. & King, J. R. (2003) Lie Symmetries of Nonlinear Multidimensional Reaction-Diffusion Systems: II. *J. Phys. A: Math. Gen.* **36**, 405–425.

[9] Popovych, R., Boyko, V., Nesterenko, M. & Lutfullin, M. (2003) Realization of Real Low-dimensional Lie Algebras *J. Phys. A: Math. Gen.* **36**, 7337–7360.