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$\mathbb{Z}^d$ Staircase actions

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Abstract. We define staircase $\mathbb{Z}^d$ actions. We first prove that staircase $\mathbb{Z}^2$ actions satisfying a general condition are mixing. Then we describe how to extend the results to the staircase $\mathbb{Z}^d$ actions. Thus we have constructed explicitly rank one mixing $\mathbb{Z}^d$ actions which include natural analogues to the well-known staircase transformation.

1. *Introduction*

Rank one transformations attracted increasing attention in ergodic theory after Ornstein [**O**] constructed an example of a mixing transformation with no square root. Ornstein's transformation was constructed with a 'random spacer' method. Subsequently, deep results were obtained for all mixing rank one transformations. We refer to [**Fr**] for rank one constructions and [**Fe**] for a recent survey of results and a bibliography on rank one transformations.

In [**AF**] and [**A2**], a family of rank one transformations called the staircase constructions were defined. An algorithm was given in [**AF**] which produced a mixing staircase construction. In [**A2**] a condition was given which implied mixing. Hence, it followed from [**A2**] that the well-known staircase transformation was mixing.

Here our purpose is first, to construct rank one $\mathbb{Z}^2$ actions which are analogous to the staircase constructions discussed in [**AF**] and [**A2**]. Second, to generalize and apply the techniques of [**A2**] to prove that a class of staircase $\mathbb{Z}^2$ actions are mixing. Then we describe how to extend our methods to $\mathbb{Z}^d$ actions.

We denote our $\mathbb{Z}^d$ action as a map $T : \mathbb{Z}^d \times [0, 1) \longrightarrow [0, 1)$ where we agree to write $T^v x$ in place of $T(v, x)$ for $v \in \mathbb{Z}^d$ and $x \in [0, 1)$. The map T will be invertible measure preserving which implies $\mu(T^v A) = \mu(A)$ for all (measurable) sets $A \subset [0, 1)$ and $v \in \mathbb{Z}^d$. Given a sequence $v_n = (v_n^1, \ldots, v_n^d) \in \mathbb{Z}^d$ we say $v_n \to \infty$ if $\sum_{i=1}^d |v_n^i| \to \infty$ as $n \to \infty$. If we view $\mathbb{Z}^d$ as a topological group with the discrete topology then $v \to \infty$ is equivalent to saying v eventually leaves any compact set.

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The action T is (strong) *mixing* if for sets A and B we have

$$\lim_{v\to\infty}\mu(T^vA\cap B)=\mu(A)\mu(B).$$

Before we proceed with a formal proof, let us sketch informally the argument we use to prove mixing for $d=2$. Note that a $\mathbb{Z}^2$ action is given by the commuting transformations $T^{(1,0)}$ and $T^{(0,1)}$. The symmetry in our constructions will imply that $T^{(1,0)}$ and $T^{(0,1)}$ are isomorphic.

With each construction we produce a sequence $h_n\to\infty$ such that

$$\lim_{n\to\infty}\mu(T^{(h_n,0)}A\cap B)=\lim_{n\to\infty}\mu(T^{(0,h_n)}A\cap B)=\mu(A)\mu(B)$$

for all sets A and B. This will be obtained rather easily from the construction and the mean ergodic theorem for the set $\{(i+j,j):i,j\in\mathbb{Z}^+\}$. Hence, both $T^{(1,0)}$ and $T^{(0,1)}$ are weak mixing. Thus, any two-dimensional subgroup of $\mathbb{Z}^2$ will give an ergodic action. This implies the mean ergodic theorem holds over parallelograms of the form $\{iu+jv : u \text{ and } v \text{ are linearly independent and } i,j\in\mathbb{Z}^+\}$. Then we make the connection between $\mu(T^vA\cap B)$ and an average over a parallelogram (possibly with large gaps). Finally, we use the fact that the length of the sides of the parallelogram are comparable to the gap size of the parallelogram in order to reduce the average to an average over a parallelogram with small gaps.

We will make repeated use of the following theorem that is a direct consequence of the mean ergodic theorem for amenable actions as stated in [**OW**].

THEOREM 1.1. (Mean ergodic theorem) *Let T be a finite measure preserving ergodic action of $\mathbb{Z}^d$ on a probability space (X,μ), and $\{F_n\}$ a Følner sequence for $\mathbb{Z}^d$. Then for any sequence of measurable sets $\{A_n\}$ and any measurable set B*

$$\lim_{n\to\infty}\frac{1}{|F_n|}\sum_{v\in F_n}[\mu(T^vA_n\cap B)-\mu(A_n)\mu(B)]=0.$$

2. $\mathbb{Z}^2$ *Staircase actions*

Rank one $\mathbb{Z}^2$ actions were constructed in [**R, PR, A1, MZ**]. The construction in [**R**] generalizes Ornstein's 'random space method' to $\mathbb{Z}^2$. The actions in [**PR**] are $\mathbb{Z}^2$ analogues of the well-known Chacon transformation, and these actions are weakly mixing, but not mixing. They are also shown to have minimal self joinings. The actions in [**A1**] are lightly mixing (and uniformly sweeping out), but not mixing. It is not difficult to construct $\mathbb{Z}^d$ actions with the same properties as those constructed in [**A1**]. The actions in [**MZ**] are infinite measure preserving.

We will follow the setup in [**MZ**] for defining rank one $\mathbb{Z}^2$ actions.

Definition 1. Given a non-negative integer h, a *grid* G of *length* h is a bijection between $\{0,1,\ldots,h-1\}\times\{0,1,\ldots,h-1\}$ and a set of distinct intervals of equal length. An interval I which is in the range of G is called a *level*. When the context is clear we may refer to G as the set which is the union of intervals in the range of G. So if $I\subset G$ is a level then there exists a and b with $0\le a<h$ and $0\le b<h$ such that $G(a,b)=I$. In this case denote $\text{Location}(I)=(a,b)$.

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In visualizing the grids it may be helpful to think of each as a number of intervals, arranged on a square grid in $\mathbb{Z}^2$ so that they project from the plane and are perpendicular to it.

Given any two intervals of the same length, if we think of them in the real line, there is a unique translation that sends the interval on the left to the one on the right. We think of a grid G as partially defining transformations $T^{(1,0)}$ and $T^{(0,1)}$ in the following way. $T^{(1,0)}$ is the translation map taking level $G(i, j)$ onto level $G(i+1, j)$, for $0 \le i < h-1$ and $0 \le j \le h$, and $T^{(0,1)}$ maps $G(i, j)$ onto $G(i, j+1)$, for $0 \le i < h$ and $0 \le j < h-1$. We visualize $T^{(1,0)}$ moving a level to the one on its right and $T^{(0,1)}$ moving a level to the one above; if no interval exists to the right or above a given interval the transformations remain undefined at this stage.

Let G and H be grids of length g and h respectively. Given non-negative integers a and b we say the subgrid G' defined by $G'(i, j) = H(a+i, b+j)$, for $0 \le i < g$ and $0 \le j < g$, is a *copy* of G *in* H if $G'(i, j) \subset G(i, j)$ for $0 \le i < g$ and $0 \le j < g$, and

$$T^{(1,0)}(G'(i, j)) = G'(i+1, j), \quad T^{(0,1)}(G'(i, j)) = G'(i, j+1).$$

This last condition guarantees that the definitions of $T^{(1,0)}$ and $T^{(0,1)}$ on a copy of a grid agree with their definitions on a grid. Let Location$(G') = (a, b)$.

Given a positive integer c, a grid H is a *staircase c-cut* of grid G of length g if $G \subset H$ and H contains $(c+1)^2$ copies of G located at

$$\left(ig + \frac{i(i-1)}{2} + ij,\, jg + \frac{j(j-1)}{2} + ij\right)$$

for $0 \le i \le c$ and $0 \le j \le c$, where the length of H is

$$h = (c+1)g + \frac{c(c-1)}{2} + c^2.$$

Locations in H that do not correspond to copies of G are assigned intervals not previously used, called *spacers*; all intervals in H have the same length. Figure 1 below shows the staircase cut for $c = 3$. This corresponds to cutting each level of G (or slicing the grid G) into $(3+1)^2 = 16$ subintervals of equal length. While keeping all subintervals in their proper position according to $T^{(1,0)}$ and $T^{(0,1)}$, we place copies of G in the locations $(ig + i(i-1)/2 + ij,\, jg + j(j-1)/2 + ij)$ and place spacers in the remaining locations.

A *staircase action* is defined by giving a sequence of positive numbers $\{c_n\}$ and a sequence of grids $\{G_n\}$ such that G_{n+1} is a staircase c_n-cut of G_n. Each grid G_n has length h_n; G_0 is a fixed grid of length one. We assume that $\lim_{n\to\infty} c_n = \infty$, and that the total length of the spacers that are added is finite and normalize the measure of the space to be one. If $\limsup_{n\to\infty} c_n < \infty$ then one can show that the action is partially rigid, hence not mixing.

The following notation will be used throughout. Let $G_n^{[i,j]}$ denote the copy of G_n in G_{n+1} at location $(ih_n + i(i-1)/2 + ij,\, jh_n + j(j-1)/2 + ij)$; we will say that this copy is *indexed* by (i, j), $0 \le i, j \le c_n$, (recall that $G_n(i, j)$ stands for the level of G_n at location (i, j)). For any set A let

$$A[i, j; n] = A \cap G_n^{[i,j]}.$$

With each $n \in \mathbb{N}$ and each (a, b) as specified below, we associate two vectors:

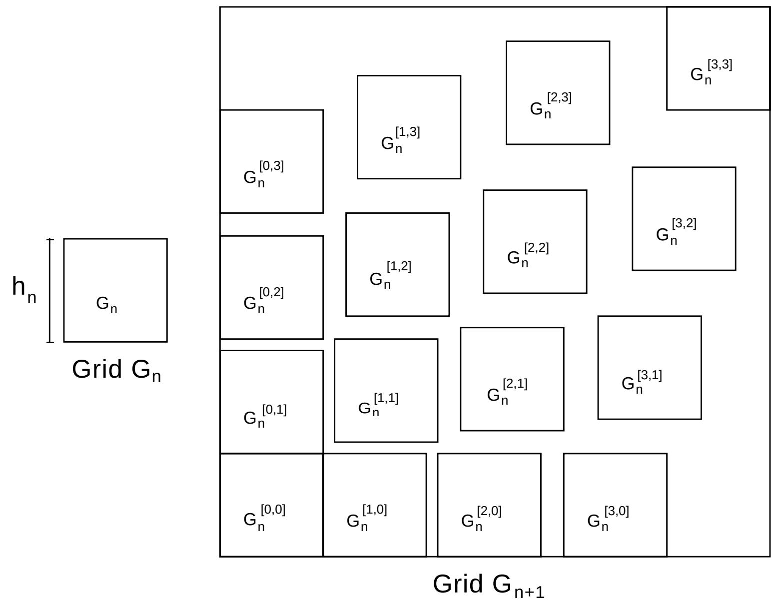

FIGURE 1.

1. for $(a, b) \in \{0, \ldots, c_n - 1\} \times \{0, \ldots, c_n\}$ let $\sigma_n(a, b) = (\sigma_1, \sigma_2)$ be such that

$$T^{(h_n,0)} G_n^{[a,b]} = T^{(-\sigma_1,-\sigma_2)} G_n^{[a+1,b]}$$

and

2. for $(a, b) \in \{0, \ldots, c_n\} \times \{0, \ldots, c_n - 1\}$ let $\tau_n(a, b) = (\tau_1, \tau_2)$ be such that

$$T^{(0,h_n)} G_n^{[a,b]} = T^{(-\tau_1,-\tau_2)} G_n^{[a,b+1]}.$$

If we use the definition of the staircase cut we get

$$\begin{aligned}
&\sigma_n(a, b) + (h_n, 0) \\
&\quad = \left((a+1)h_n + \frac{(a+1)a}{2} + (a+1)b,\, bh_n + \frac{b(b-1)}{2} + (a+1)b \right) \\
&\qquad - \left(ah_n + \frac{a(a-1)}{2} + ab,\, bh_n + \frac{b(b-1)}{2} + ab \right) \\
&\quad = (h_n + a + b, b).
\end{aligned}$$

Hence, $\sigma_n(a, b) = (a + b, b)$ and similarly $\tau_n(a, b) = (a, a + b)$. (These are special cases of Lemma 2.3.) Below we use this property of the spacers, along with the mean ergodic theorem for $\mathbb{Z}^2$ actions to prove that both $T^{(1,0)}$ and $T^{(0,1)}$ are weak mixing.

LEMMA 2.1. *The transformations $T^{(1,0)}$ and $T^{(0,1)}$ are weak mixing.*

Proof. Since $T^{(1,0)}$ and $T^{(0,1)}$ are isomorphic, it is enough to prove $T^{(1,0)}$ is weak mixing. Let A and B each be a union of levels in grid G_k. For $n > k$, let $F_n = \{(-i - j, -j) : 0 \leq$

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$i < c_n, 0 \le j \le c_n\}$. We note that

$$A \cap G_{n+1} = \bigcup_{i=0}^{c_n} \bigcup_{j=0}^{c_n} [A \cap G_n^{[i,j]}]$$

and similarly for B.

Since $\sigma_n(i, j) = (i + j, j)$, then for each $(i, j) \in \{0, \dots, c_n - 1\} \times \{0, \dots, c_n\}$,

$$T^{(-i-j,-j)} A[i+1, j; n] \subset T^{(h_n,0)} A$$

with $\mu(A[i, j; n]) = 1/(c_n + 1)^2 \mu(A)$.

We have

$$\begin{aligned}
\mu(T^{(h_n,0)} A \cap B) &= \mu\bigg(T^{(h_n,0)}\bigg(\bigcup_{i=0}^{c_n} \bigcup_{j=0}^{c_n} A[i, j; n]\bigg) \cap \bigg(\bigcup_{i=0}^{c_n} \bigcup_{j=0}^{c_n} B[i, j; n]\bigg)\bigg) \\
&\ge \mu\bigg(\bigcup_{i=0}^{c_n-1} \bigcup_{j=0}^{c_n} T^{(h_n,0)} A[i, j; n] \cap B[i+1, j; n]\bigg) \\
&= \sum_{i=0}^{c_n-1} \sum_{j=0}^{c_n} \mu(T^{(h_n,0)} A[i, j; n] \cap B[i+1, j; n]) \\
&= \sum_{i=0}^{c_n-1} \sum_{j=0}^{c_n} \mu(T^{(-i-j,-j)} A[i+1, j; n] \cap B[i+1, j; n]) \\
&\ge \sum_{i=0}^{c_n-1} \sum_{j=0}^{c_n} \frac{1}{(c_n+1)^2} \bigg[\mu(T^{(-i-j,-j)} A \cap B) - \frac{i+2j}{h_n}\bigg] \\
&\ge \bigg[\frac{1}{(c_n+1)^2} \sum_{i=0}^{c_n-1} \sum_{j=0}^{c_n} \mu(T^{(-i-j,-j)} A \cap B)\bigg] - \frac{3c_n}{h_n}.
\end{aligned}$$

The fact that T is finite measure preserving is sufficient to imply $\lim_{n\to\infty}(c_n/h_n) = 0$. Therefore using the mean ergodic theorem

$$\begin{aligned}
\liminf_{n\to\infty} \mu(T^{(h_n,0)} A \cap B) &\ge \liminf_{n\to\infty} \frac{1}{(c_n+1)^2} \sum_{i=0}^{c_n-1} \sum_{j=0}^{c_n} \mu(T^{(-i-j,-j)} A \cap B) \\
&= \lim_{n\to\infty} \frac{1}{(c_n+1)^2} \sum_{(i,j)\in F_n} \mu(T^{(i,j)} A \cap B) = \mu(A)\mu(B).
\end{aligned}$$

Hence, for all sets A and B,

$$\liminf_{n\to\infty} \mu(T^{(h_n,0)} A \cap B) \ge \mu(A)\mu(B).$$

Thus, $\limsup_{n\to\infty} \mu(T^{(h_n,0)} A \cap B) \le \mu(A)\mu(B)$, for all sets A and B, which completes the proof. □

Thus, we have that each of $T^{(1,0)}$ and $T^{(0,1)}$ is totally ergodic. Therefore, by the following proposition the mean ergodic theorem will hold along any two-dimensional subgroup of $\mathbb{Z}^2$.

PROPOSITION 2.2. *Any two-dimensional subgroup of a $\mathbb{Z}^2$ action is ergodic if at least one of the generating directions of T is totally ergodic.*

Proof. Suppose the direction generated by $(1, 0)$ is totally ergodic. Let F be the subgroup generated by linearly independent vectors (a, b) and (c, d). Then

$$d(a, b) - b(c, d) = (ad - bc, 0) \in F$$

and since $T^{(ad-bc,0)}$ is ergodic, so is the action of F. □

We refer to σ and τ as the spacer functions for $T^{(1,0)}$ and $T^{(0,1)}$ respectively. We introduce the *cumulative spacer functions* $\phi_{k,\ell} : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}^2$, defined by

$$\phi_{1,0}(a, b) = (a + b, b) \quad \text{and} \quad \phi_{0,1}(a, b) = (a, a + b)$$

and for positive integers k and ℓ

$$\phi_{k,\ell}(a, b) = \sum_{i=0}^{k-1} \phi_{1,0}(a + i, b) + \sum_{j=0}^{\ell-1} \phi_{0,1}(a + k, b + j).$$

We extend the definition to negative integers by setting, for all positive integers k and ℓ

$$\begin{aligned} \phi_{-k,\ell}(a, b) &= -\phi_{k,\ell}(a - k, b), \\ \phi_{k,-\ell}(a, b) &= -\phi_{k,\ell}(a, b - \ell), \\ \phi_{-k,-\ell}(a, b) &= -\phi_{k,\ell}(a - k, b - \ell). \end{aligned}$$

The main consequence of our spacer arrangement is that $\phi_{k,\ell}(a, b) - \phi_{k,\ell}(0, 0)$ is the two-dimensional subgroup of $\mathbb{Z}^2$ generated by $(k + \ell, \ell)$ and $(k, k + \ell)$.

LEMMA 2.3. *For the cumulative spacer function the following formula holds for integers k, ℓ, a and b:*

$$\phi_{k,\ell}(a, b) - \phi_{k,\ell}(0, 0) = a(k + \ell, \ell) + b(k, k + \ell).$$

Proof. We verify the formula for the case k and ℓ are positive integers. The other cases may be verified in the same manner. We get

$$\begin{aligned} &\phi_{k,\ell}(a, b) - \phi_{k,\ell}(0, 0) \\ &\quad = \sum_{i=0}^{\ell-1} \tau_n(a + k, b + i) + \sum_{i=0}^{k-1} \sigma_n(a + i, b) - \sum_{i=0}^{\ell-1} \tau_n(k, i) - \sum_{i=0}^{k-1} \sigma_n(i, 0) \\ &\quad = \sum_{i=0}^{\ell-1} [\tau_n(a + k, b + i) - \tau(k, i)] + \sum_{i=0}^{k-1} [\sigma_n(a + i, b) - \sigma_n(i, 0)] \\ &\quad = \sum_{i=0}^{\ell-1} (a, a + b) + \sum_{i=0}^{k-1} (a + b, b) \\ &\quad = a(k + \ell, \ell) + b(k, k + \ell). \end{aligned}$$

□

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LEMMA 2.4. *If $\{p_n\}$ a sequence of positive integers tending to infinity, $u_n = (y_n, z_n)$ is a sequence of vectors such that $y_n \geq z_n \geq 0$ and $h_{p_n} \leq y_n \leq 2h_{p_n}$, then for any sets A and B and positive integer k we have*

$$\lim_{n\to\infty} \mu(T^{ku_n}A \cap B) = \mu(A)\mu(B).$$

Proof. It is sufficient to prove the lemma when A and B are each a union of levels in some grid. Fix k and choose k_n and ℓ_n such that $ky_n = k_n h_{p_n} + q_n$ and $kz_n = \ell_n h_{p_n} + r_n$ where $0 \leq q_n, r_n < h_{p_n}$. For each n, we will partition the set A into four subsets $A_1^{(p_n)}, \ldots, A_4^{(p_n)}$. First let

$$\Lambda_1^{(p_n)} = \{(i, j) : h_{p_n} - q_n \leq i < h_{p_n}, 0 \leq j < h_{p_n} - r_n\}$$
$$\Lambda_2^{(p_n)} = \{(i, j) : 0 \leq i < h_{p_n} - q_n, h_{p_n} - r_n \leq j < h_{p_n}\}$$
$$\Lambda_3^{(p_n)} = \{(i, j) : h_{p_n} - q_n \leq i \leq h_{p_n}, h_{p_n} - r_n \leq j < h_{p_n}\}$$
$$\Lambda_4^{(p_n)} = \{(i, j) : 0 \leq i < h_{p_n} - q_n, 0 \leq j < h_{p_n} - r_n\}$$

and for $d = 1, \ldots, 4$ let

$$D_d^{(p_n)} = \bigcup_{(i,j)\in\Lambda_d^{(p_n)}} G_{p_n}(i, j).$$

Let $A_d^{(p_n)} = A \cap D_d^{(p_n)}$. Below we prove that for all $\epsilon > 0$ there exists N such that for all $n \geq N$,

$$\mu(T^{ku_n}A_4^{(p_n)} \cap B) \geq \mu(A_4^{(p_n)})\mu(B) - \epsilon.$$

A similar argument will hold for sets $A_1^{(p_n)}$, $A_2^{(p_n)}$ and $A_3^{(p_n)}$. It follows that

$$\liminf_{n\to\infty} \mu(T^{ku_n}A \cap B) \geq \mu(A)\mu(B).$$

Since this is true for all sets, by considering B and B^c, we get that $\lim_{n\to\infty} \mu(T^{ku_n}A \cap B) = \mu(A)\mu(B)$, which will complete the proof of the lemma.

Now define $A_4^{\prime(p_n)}[i, j; p_n] = T^{(q_n,r_n)}A_4^{(p_n)}[i, j; p_n]$. We use the definition of the cumulative spacer function to get that for all n sufficiently large,

$$\begin{aligned}T^{ku_n}A_4^{(p_n)}[i, j; p_n] &= T^{(q_n,r_n)}T^{(k_n h_{p_n}, \ell_n h_{p_n})}A_4^{(p_n)}[i, j; p_n] \\ &= T^{(q_n,r_n)}T^{-\phi_{k_n,\ell_n}(i,j)}A_4^{(p_n)}[i + k_n, j + \ell_n; p_n] \\ &= T^{-\phi_{k_n,\ell_n}(i,j)}A_4^{\prime(p_n)}[i + k_n, j + \ell_n; p_n].\end{aligned}$$

We now apply the general formula of the cumulative spacer functions. Let $R_n = c_{p_n} - (k_n + 1)$ and $S_n = c_{p_n} - (\ell_n + 1)$. Using that A and B are each a union of levels of the (P_n)th grid,

$$\begin{aligned}&\mu(T^{ku_n}A_4^{(p_n)} \cap B) \\ &\quad = \mu\left(T^{ku_n}\left(\bigcup_{i=0}^{c_{p_n}}\bigcup_{j=0}^{c_{p_n}} A_4^{(p_n)}[i, j; p_n]\right) \cap \left(\bigcup_{i=0}^{c_{p_n}}\bigcup_{j=0}^{c_{p_n}} B[i, j; p_n]\right)\right)\end{aligned}$$

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$$\geq \mu\left(\bigcup_{i=0}^{R_n}\bigcup_{j=0}^{S_n} T^{ku_n} A_4^{(p_n)}[i, j; p_n] \cap B[i+k_n, j+\ell_n; p_n]\right)$$
$$= \sum_{i=0}^{R_n}\sum_{j=0}^{S_n} \mu(T^{ku_n} A_4^{(p_n)}[i, j; p_n] \cap B[i+k_n, j+\ell_n; p_n])$$
$$= \sum_{i=0}^{R_n}\sum_{j=0}^{S_n} \mu(T^{-\phi_{k_n,\ell_n}(i,j)} A_4^{\prime(p_n)}[i+k_n, j+\ell_n; p_n] \cap B[i+k_n, j+\ell_n; p_n])$$
$$\geq \sum_{i=0}^{R_n}\sum_{j=0}^{S_n} \frac{1}{(c_n+1)^2}\left[\mu(T^{-\phi_{k_n,\ell_n}(i,j)} A_4^{\prime(p_n)} \cap B) - \frac{2k_n}{h_{p_n}}\right]$$
$$= \frac{1}{(c_n+1)^2}\sum_{(i,j)\in F_n}\left[\mu(T^{-\phi_{k_n,\ell_n}(i,j)} A_4^{\prime(p_n)} \cap B) - \frac{2k_n}{h_{p_n}}\right],$$

where $F_n = \{-\phi_{k_n,\ell_n}(i, j) : 0 \leq i < R_n, 0 \leq j < S_n\}$.

By Lemma 2.3 $\{\phi_{k_n,\ell_n}(i, j) : i, j \in \mathbb{Z}\}$ is a translate of the subgroup generated by $(k_n + \ell_n, \ell_n)$ and $(k_n, k_n + \ell_n)$. By Proposition 2.2, each subgroup acts ergodically. Since $h_{p_n} \leq y_n \leq 2h_{p_n}$, there are at most $(2k + 1)(k + 1)$ subgroups generated by the pairs $(k_n + \ell_n, \ell_n)$ and $(k_n, k_n + \ell_n)$ which vary with n. We apply the mean ergodic theorem to the set of Følner sequences F_n corresponding to the six subgroups to obtain

$$\frac{1}{|F_n|}\sum_{(i,j)\in F_n}[\mu(T^{(i,j)} A_4^{\prime(p_n)} \cap B) - \mu(A_4^{\prime(p_n)})\mu(B)] \to 0$$

as $n \to \infty$.

Thus, for any ϵ, for all n sufficiently large,

$$\mu(T^{ku_n} A_4^{(p_n)} \cap B) \geq \mu(A_4^{(p_n)})\mu(B) - \epsilon,$$

which by an earlier remark completes the proof of the lemma. □

LEMMA 2.5. *If T is a finite measure preserving staircase action then*

$$\lim_{p\to\infty} \frac{h_{p-1}^2}{h_p} = \infty.$$

Proof. We have

$$\lim_{p\to\infty} \frac{h_{p-1}^2}{h_p} = \lim_{p\to\infty}\left(\frac{(c_{p-1}+1)h_{p-1}}{h_p}\right)\left(\frac{h_{p-1}}{c_{p-1}+1}\right) = 1 \cdot \infty.$$ □

3. *Consequences of the mean ergodic theorem*

We start with an extension to the case of $\mathbb{Z}^2$ actions of a lemma in [**A2**].

LEMMA 3.1. *Let u and v be linearly independent vectors in $\mathbb{Z}^2$. For positive integers R, S, ρ and L, and any measurable set B we have*

$$\int_{[0,1)} \left| \frac{1}{RS} \sum_{i=0}^{R-1} \sum_{j=0}^{S-1} \mathcal{X}_B(T^{iu+jv}x) - \mu(A) \right| d\mu \leq \int_{[0,1)} \left| \frac{1}{L} \sum_{i=0}^{L-1} \mathcal{X}_B(T^{i\rho u}x) - \mu(B) \right| d\mu + \frac{\rho L}{R}.$$

Proof. Separate the averaging set $\Gamma = \{iu + jv : 0 \leq i \leq R-1, 0 \leq j \leq S-1\}$ into two disjoint sets: $\Gamma_1 = \{iu + jv : 0 \leq i \leq \lfloor R/\rho L \rfloor \rho L - 1, 0 \leq j \leq S-1\}$ and $\Gamma_2 = \{iu + jv : \rho L \lfloor R/\rho L \rfloor \leq i \leq R-1, 0 \leq j \leq S-1\}$. Let $\Gamma_2 = \emptyset$ if $R/\rho L$ is a natural number. Thus,

$$\begin{aligned} \int_{[0,1)} \frac{1}{|\Gamma|} \left| \sum_{w \in \Gamma} (\mathcal{X}_B(T^w x) - \mu(B)) \right| d\mu &\leq \int_{[0,1)} \frac{1}{|\Gamma|} \left| \sum_{w \in \Gamma_1} (\mathcal{X}_B(T^w x) - \mu(B)) \right| d\mu \\ &\quad + \int_{[0,1)} \frac{1}{|\Gamma|} \left| \sum_{w \in \Gamma_2} (\mathcal{X}_B(T^w x) - \mu(B)) \right| d\mu. \end{aligned}$$

First we see that Γ_1 may be covered by disjoint translates of the set $\Gamma_3 = \{i\rho u + jv : 0 \leq i \leq L-1, 0 \leq j \leq S-1\}$. Using the fact that T is measure preserving we obtain that the first term on the right-hand side of the previous inequality will be less than or equal to the mean ergodic average over Γ_3.

Now the second term has less than ρLS terms in the sum. Hence, this term is less than $\rho LS/|\Gamma| = \rho L/R$. □

LEMMA 3.2. *Let u_n and v_n be sequences of linearly independent vectors in $\mathbb{Z}^2$. Suppose R_n and ρ_n are sequences of positive integers such that $\lim_{n\to\infty}(R_n/\rho_n) = \infty$. If for some set B and positive integer i we have*

$$\lim_{n\to\infty} \mu(T^{i\rho_n u_n} B \cap B) = \mu(B)^2$$

then for any sequence S_n of positive integers,

$$\lim_{n\to\infty} \int_{[0,1)} \left| \frac{1}{R_n S_n} \sum_{i=0}^{R_n} \sum_{j=0}^{S_n} \mathcal{X}_B(T^{iu_n+jv_n}x) - \mu(B) \right| d\mu = 0.$$

Proof. A technique of Blum–Hanson [**BH**] implies that given $\epsilon > 0$ there exists $\delta > 0$ and a positive integer L so that if $|\mu(T^{i\rho_n u_n} B \cap B) - \mu(B)^2| < \delta$ for $0 < i \leq L-1$ then

$$\int_{[0,1)} \left| \frac{1}{L} \sum_{i=0}^{L-1} \mathcal{X}_B(T^{i\rho_n u_n}x) - \mu(B) \right| d\mu < \epsilon.$$

Hence, if R_n is chosen so that also $\rho_n L/R_n < \epsilon$, then Lemma 3.1 implies that for all S_n,

$$\int_{[0,1)} \left| \frac{1}{R_n S_n} \sum_{i=0}^{R_n-1} \sum_{j=0}^{S_n-1} \mathcal{X}_B(T^{iu_n+jv_n}x) - \mu(B) \right| d\mu < 2\epsilon. \qquad \square$$

4. *Mixing staircase actions*

THEOREM 4.1. *Let T be a finite measure preserving staircase $\mathbb{Z}^2$ action with cuts $\{c_n\}$ and lengths $\{h_n\}$ such that $\lim_{n\to\infty} c_n = \infty$. If*

$$\lim_{n\to\infty} \frac{c_n^2}{h_n} = 0$$

then T is mixing.

Proof. It is sufficient to prove that every sequence of vectors converging to infinity has a subsequence (s_n, t_n) which is a mixing sequence, i.e. that for all measurable sets A and B we have $\lim_{n\to\infty} \mu(T^{(s_n,t_n)}A \cap B) = \mu(A)\mu(B)$. Since $T^{(1,0)}$ and $T^{(0,1)}$ are isomorphic, by taking a subsequence if necessary, it suffices to prove that (s_n, t_n) is a mixing sequence where $s_n \geq |t_n|$. Moreover, by passing to a subsequence if necessary and for convenience of notation renaming it, we may assume $h_n \leq s_n < h_{n+1}$ for all positive integers n, and the ratios k_n/c_n, ℓ_n/c_n, q_n/h_n and r_n/h_n all converge as $n \to \infty$. Choose integers k_n, ℓ_n, q_n and r_n so that

$$s_n = k_n h_n + q_n \quad \text{and} \quad t_n = \ell_n h_n + r_n$$

where $1 \leq k_n \leq c_n$, $0 \leq q_n < h_n$, $0 \leq |\ell_n| \leq c_n$ and $0 \leq |r_n| < h_n$. Let A and B be measurable sets which appear as a union of levels in G_n for sufficiently large n. We fix such an n and consider the grid G_n. First assume ℓ_n and r_n are non-negative. Partition G_{n+1} into four subsets: $D^{(n)}$, $E_1^{(n)}$, $E_2^{(n)}$, and $E_3^{(n)}$. When $t_n \geq 0$ define

$$\begin{aligned}
\Gamma_1^{(n)} &= \{(a,b) : h_{n+1} - s_n \leq a \leq h_{n+1}, 0 \leq b < h_{n+1} - t_n\} \\
\Gamma_2^{(n)} &= \{(a,b) : 0 \leq a \leq h_{n+1} - s_n, h_{n+1} - t_n \leq b \leq h_{n+1}\} \\
\Gamma_3^{(n)} &= \{(a,b) : h_{n+1} - s_n \leq a \leq h_{n+1}, h_{n+1} - t_n \leq b \leq h_{n+1}\} \\
\Lambda^{(n)} &= \{(a,b) : 0 \leq a < h_{n+1} - s_n, 0 \leq b < h_{n+1} - t_n\}.
\end{aligned}$$

Let

$$E_i^{(n)} = \bigcup_{(a,b)\in\Gamma_i^{(n)}} G_{n+1}(a,b)$$

and

$$D^{(n)} = \bigcup_{(a,b)\in\Lambda^{(n)}} G_{n+1}(a,b).$$

Mixing on the four subsets can be shown separately. The argument for sets $E_1^{(n)}$, $E_2^{(n)}$, and $E_3^{(n)}$ are similar and follows from the method used in Lemma 2.1. Thus, we only give the complete argument for the set $E_3^{(n)}$. The proof of mixing on $D^{(n)}$ is more intricate and requires techniques similar to those utilized in [**A2**].

Case 1: Mixing on $E_3^{(n)}$. Let $A_3^{(n)} = A \cap E_3^{(n)}$. Note that $A_3^{(n)}[i,j;n+1] = A_3^{(n)} \cap G_{n+1}^{[i,j]}$, and $B[i,j;n+1] = B \cap G_{n+1}^{[i,j]}$, but define $A_3'^{(n)}[i,j;n+1] = T^{(s_n-h_{n+1},t_n-h_{n+1})}A_3^{(n)}[i,j;n+1]$. Our cumulative spacer function was defined so that

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for i and j satisfying $0 \le i, j < c_{n+1}$ we have,

$$\begin{aligned} T^{(s_n,t_n)} A_3^{(n)}[i, j; n+1] &= T^{(s_n - h_{n+1}, t_n - h_{n+1})} T^{(h_{n+1}, h_{n+1})} A_3^{(n)}[i, j; n+1] \\ &= T^{(s_n - h_{n+1}, t_n - h_{n+1})} T^{-\phi_{1,1}(i,j)} A_3^{(n)}[i+1, j+1; n+1] \\ &= T^{-\phi_{1,1}(i,j)} A_3'^{(n)}[i+1, j+1; n+1]. \end{aligned}$$

Now we give a computation similar to that shown in Lemma 2.1:

$$\begin{aligned} &\mu(T^{(s_n,t_n)} A_3^{(n)} \cap B) \\ &\quad = \mu\bigg(T^{(s_n,t_n)} \bigg(\bigcup_{i=0}^{c_{n+1}} \bigcup_{j=0}^{c_{n+1}} A_3^{(n)}[i, j; n+1]\bigg) \cap \bigg(\bigcup_{i=0}^{c_{n+1}} \bigcup_{j=0}^{c_{n+1}} B[i, j; n+1]\bigg)\bigg) \\ &\quad \ge \mu\bigg(\bigcup_{i=0}^{c_{n+1}-1} \bigcup_{j=0}^{c_{n+1}-1} T^{(s_n,t_n)} A_3^{(n)}[i, j; n+1] \cap B[i+1, j+1; n+1]\bigg) \\ &\quad = \sum_{i=0}^{c_{n+1}-1} \sum_{j=0}^{c_{n+1}-1} \mu(T^{(s_n,t_n)} A_3^{(n)}[i, j; n+1] \cap B[i+1, j+1; n+1]) \\ &\quad = \sum_{i=0}^{c_{n+1}-1} \sum_{j=0}^{c_{n+1}-1} \mu(T^{-\phi_{1,1}(i,j)} A_3'^{(n)}[i+1, j+1; n+1] \cap B[i+1, j+1; n+1]) \\ &\quad \ge \sum_{i=0}^{c_{n+1}-1} \sum_{j=0}^{c_{n+1}-1} \frac{1}{(c_{n+1}+1)^2} \bigg[\mu(T^{-\phi_{1,1}(i,j)} A_3'^{(n)} \cap B) - \frac{3i+3j+1}{h_{n+1}}\bigg] \\ &\quad \ge \bigg[\frac{1}{(c_{n+1}+1)^2} \sum_{i=0}^{c_{n+1}-1} \sum_{j=0}^{c_{n+1}-1} \mu(T^{-\phi_{1,1}(i,j)} A_3'^{(n)} \cap B)\bigg] - \frac{6c_{n+1}+1}{h_{n+1}}. \end{aligned}$$

Since T is finite measure preserving $\lim_{n\to\infty}(c_n/h_n) = 0$. By Lemma 2.3, $\{\phi_{1,1}(i, j) : i, j \in \mathbb{Z}\}$ is a translation of the two-dimensional subgroup generated by $(2, 1)$ and $(1, 2)$ and by Proposition 2.2 acts ergodically. We apply the mean ergodic theorem to the Følner sequence $F_{n+1} = \{-\phi_{1,1}(i, j) : 0 \le i < c_{n+1}, 0 \le j < c_{n+1}\}$ for the subgroup action to obtain

$$\bigg|\frac{1}{(c_{n+1}+1)^2} \sum_{(i,j)\in F_{n+1}} \mu(T^{(i,j)} A_3'^{(n)} \cap B) - \mu(A_3'^{(n)})\mu(B)\bigg| \to 0.$$

Case 2: Mixing on $D^{(n)}$. Define the following sets:

$$\begin{aligned} \Lambda_1^{(n)} &= \{(i, j) : h_n - q_n \le i < h_n, 0 \le j < h_n - r_n\} \\ \Lambda_2^{(n)} &= \{(i, j) : 0 \le i < h_n - q_n, h_n - r_n \le j < h_n\} \\ \Lambda_3^{(n)} &= \{(i, j) : h_n - q_n \le i \le h_n, h_n - r_n \le j < h_n\} \\ \Lambda_4^{(n)} &= \{(i, j) : 0 \le i < h_n - q_n, 0 \le j < h_n - r_n\} \end{aligned}$$

and for $p = 1, \ldots, 4$ let

$$D_p^{(n)} = \bigg(\bigcup_{(i,j)\in\Lambda_p^{(n)}} G_n(i, j)\bigg) \cap D^{(n)}.$$

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Let $A_4^{(n)} = A \cap D^{(n)}$ and $A_{4,p}^{(n)} = A \cap D_p^{(n)}$ for $p = 1, \ldots, 4$. This produces a partition of $A_4^{(n)}$. Below we show mixing on $A_{4,4}^{(n)}$ and a similar argument works for sets $A_1^{(n)}$, $A_2^{(n)}$ and $A_3^{(n)}$. Note that $A_{4,4}^{(n)}[i, j; n] = A_{4,4}^{(n)} \cap G_n^{[i,j]}$, and $B[i, j; n] = B \cap G_n^{[i,j]}$ but define $A_{4,4}^{\prime(n)}[i, j; n] = T^{(q_n, r_n)} A_{4,4}^{(n)}[i, j; n]$. Use the general formula of the cumulative spacer function to obtain the following.

$$\begin{aligned} T^{(s_n,t_n)} A_{4,4}^{(n)}[i, j; n] &= T^{(q_n,r_n)} T^{(k_n h_n, \ell_n h_n)} A_{4,4}^{(n)}[i, j; n] \\ &= T^{(q_n,r_n)} T^{-\phi_{k_n,\ell_n}(i,j)} A_{4,4}^{(n)}[i + k_n, j + \ell_n; n] \\ &= T^{-\phi_{k_n,\ell_n}(i,j)} A_{4,4}^{\prime(n)}[i + k_n, j + \ell_n; n]. \end{aligned}$$

Once again we use the idea from Lemma 2.1 which was also used to prove mixing on $E_3^{(n)}$. However, in this case we will need the condition that $\lim_{n\to\infty}(c_n^2/h_n) = 0$. For the following expression let $R_n = c_n - (k_n + 1)$ and $S_n = c_n - (\ell_n + 1)$.

$$\begin{aligned} &\mu(T^{(s_n,t_n)} A_{4,4}^{(n)} \cap B) \\ &\quad = \mu\bigg(T^{(s_n,t_n)}\bigg(\bigcup_{i=0}^{c_n}\bigcup_{j=0}^{c_n} A_{4,4}^{(n)}[i, j; n]\bigg) \cap \bigg(\bigcup_{i=0}^{c_n}\bigcup_{j=0}^{c_n} B[i, j; n]\bigg)\bigg) \\ &\quad \geq \mu\bigg(\bigcup_{i=0}^{R_n}\bigcup_{j=0}^{S_n} T^{(s_n,t_n)} A_{4,4}^{(n)}[i, j; n] \cap B[i + k_n, j + \ell_n; n]\bigg) \\ &\quad = \sum_{i=0}^{R_n}\sum_{j=0}^{S_n} \mu(T^{(s_n,t_n)} A_{4,4}^{(n)}[i, j; n] \cap B[i + k_n, j + \ell_n; n]) \\ &\quad = \sum_{i=0}^{R_n}\sum_{j=0}^{S_n} \mu(T^{-\phi_{k_n,\ell_n}(i,j)} A_{4,4}^{\prime(n)}[i + k_n, j + \ell_n; n] \cap B[i + k_n, j + \ell_n; n]) \\ &\quad \geq \sum_{i=0}^{R_n}\sum_{j=0}^{S_n} \frac{1}{R_n S_n}\bigg[\mu(T^{-\phi_{k_n,\ell_n}(i,j)} A_{4,4}^{\prime(n)} \cap B) - \frac{10c_n^2}{h_n}\bigg] \\ &\quad \geq \bigg[\frac{1}{R_n S_n}\sum_{i=0}^{R_n}\sum_{j=0}^{S_n} \mu(T^{-\phi_{k_n,\ell_n}(i,j)} A_{4,4}^{\prime(n)} \cap B)\bigg] - \frac{10c_n^2}{h_n}. \end{aligned}$$

Now we will work out the combinatorics on the set $\{\phi_{k_n,\ell_n}(i, j) : i, j \in \mathbb{Z}\}$. Note that $\{\phi_{k_n,\ell_n}(i, j) : i, j \in \mathbb{Z}\}$ is a translation of the subgroup generated by $(k_n + \ell_n, \ell_n)$ and $(k_n, k_n + \ell_n)$. In particular, Lemma 2.3 implies $\phi_{k_n,\ell_n}(i, j) = i(k_n + \ell_n, \ell_n) + j(k_n, k_n + \ell_n) + \phi_{k_n,\ell_n}(0, 0)$. Choose positive integers p_n so that $h_{p_n-1} < k_n + \ell_n \leq h_{p_n}$. If $k_n/c_n \to 1$ as $n \to \infty$ then $\mu(A_{4,4}^{\prime(n)}) \to 0$ as $n \to \infty$. Otherwise Lemma 2.5 implies

$$\begin{aligned} \lim_{n\to\infty} \frac{(c_n - k_n - 1)(k_n + \ell_n)}{h_{p_n}} &= \lim_{n\to\infty}\bigg(\frac{c_n - k_n - 1}{2k_n}\bigg)\frac{(2k_n)(k_n + \ell_n)}{h_{p_n}} \\ &\geq \lim_{n\to\infty}\bigg(\frac{c_n - k_n - 1}{2k_n}\bigg)\bigg(\frac{h_{p_n-1}^2}{h_{p_n}}\bigg) = \infty. \end{aligned}$$

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Let $\rho_n = \inf\{\rho \in \mathbb{N} : \rho(k_n + \ell_n) \geq h_{p_n}\}$. Hence $\lim_{n\to\infty}(R_n/\rho_n) = \infty$. If we let $u_n = (k_n+\ell_n, \ell_n)$, then by Lemma 2.4, for all i and B, $\lim_{n\to\infty} \mu(T^{i\rho_n u_n} B\cap B) = \mu(B)^2$. Therefore by Lemma 3.2

$$|\mu(T^{(s_n,t_n)} A_{4,4}^{(n)} \cap B) - \mu(A_{4,4}^{(n)})\mu(B)| = |\mu(T^{(s_n,t_n)} A_{4,4}^{(n)} \cap B) - \mu(A_{4,4}^{\prime(n)})\mu(B)| \to 0.$$

This completes the proof where $t_n \geq 0$. When $t_n < 0$ the argument may be handled in the same manner. The positions of the sets $E_i^{(n)}$ and $D_i^{(n)}$ change but the basic ideas are the same. In fact, in this case we define

$$\begin{aligned}
\Gamma_1^{(n)} &= \{(a,b) : h_{n+1} - s_n \leq a \leq h_{n+1}, 0 \leq b < -t_n\}\\
\Gamma_2^{(n)} &= \{(a,b) : 0 \leq a \leq h_{n+1} - s_n, 0 \leq b \leq -t_n\}\\
\Gamma_3^{(n)} &= \{(a,b) : h_{n+1} - s_n \leq a \leq h_{n+1}, -t_n \leq b \leq h_{n+1}\}\\
\Lambda^{(n)} &= \{(a,b) : 0 \leq a < h_{n+1} - s_n, -t_n \leq b < h_{n+1}\}.
\end{aligned}$$

Then let

$$E_i^{(n)} = \bigcup_{(a,b)\in\Gamma_i^{(n)}} G_{n+1}(a,b)$$

and

$$D^{(n)} = \bigcup_{(a,b)\in\Lambda^{(n)}} G_{n+1}(a,b)$$

and finally define the sets $\Lambda_p^{(n)}$ in the following way to obtain the corresponding sets $D_p^{(n)}$.

$$\begin{aligned}
\Lambda_1^{(n)} &= \{(i,j) : h_n - q_n \leq i < h_n, 0 \leq j < -r_n\}\\
\Lambda_2^{(n)} &= \{(i,j) : 0 \leq i < h_n - q_n, -r_n \leq j < h_n\}\\
\Lambda_3^{(n)} &= \{(i,j) : h_n - q_n \leq i \leq h_n, -r_n \leq j < h_n\}\\
\Lambda_4^{(n)} &= \{(i,j) : 0 \leq i < h_n - q_n, 0 \leq j < -r_n\}.
\end{aligned}$$

□

5. $\mathbb{Z}^d$ *Staircase actions*

In this section we extend our construction to the case of $\mathbb{Z}^d$ actions. Given a positive integer c, a grid H is a *staircase c-cut* of grid G of length g if $G \subset H$ and for each $(a_1,\ldots,a_d) \in \{0,\ldots,c\}^d$, H contains a copy of G located at $(b_1,\ldots,b_d)$ where

$$b_i = \left(a_i g + [a_i(a_i-1)/2] + \sum_{j\neq i}^{d} a_j\right)$$

and the length of H is

$$h = (c+1)g + \frac{c(c-1)}{2} + (d-1)c^2.$$

For the cumulative spacer function, the jth coordinate of $\phi_{e_i}(a_1,\ldots,a_d)$ is a_j if $j \neq i$ and $\sum_{t=1}^{d} a_t$ if $j = i$. We give the general formula below in Lemma 5.1.

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LEMMA 5.1. *For the cumulative spacer functions, the set of vectors*

$$V = \{\phi_{(k_1,\ldots,k_d)}(a_1,\ldots,a_d) - \phi_{(k_1,\ldots,k_d)}(0,\ldots,0)\colon (a_1,\ldots,a_d) \in \mathbb{Z}^d\}$$

is generated by vectors $v_1,\ldots,v_d$ *where the* j*th component of* v_i *is* k_j *for* $j \neq i$ *and* $\sum_{t=1}^{d} k_t$ *for* $j = i$.

Since for each j, $F_n = \{\phi_{e_j}(a) : a = (a_1,\ldots,a_d) \in \{0,\ldots,c_n\}^d\}$ is a Følner sequence, applying the mean ergodic theorem as in Lemma 2.1 gives that each direction is weak mixing. Hence, the mean ergodic theorem holds on all d-dimensional subgroups of $\mathbb{Z}^d$. This allows one to use an averaging argument similar to that in Theorem 4.1 to show that the action is mixing.

THEOREM 5.2. *Let* T *be a finite measure preserving staircase* $\mathbb{Z}^d$ *action with cuts* $\{c_n\}$ *and lengths* $\{h_n\}$ *such that* $\lim_{n\to\infty} c_n = \infty$. *If*

$$\lim_{n\to\infty} \frac{c_n^2}{h_n} = 0,$$

then T *is mixing.*

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