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ON THE LATTICE OF CONGRUENCES ON AN EVENTUALLY REGULAR SEMIGROUP

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Abstract

A natural equivalence θ on the lattice of congruences $\Lambda(S)$ of a semigroup S is studied. For any eventually regular semigroup S, it is shown that θ is a congruence, each θ-class is a complete sublattice of $\Lambda(S)$ and the maximum element in each θ-class is determined.

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1. Introduction and summary

The lattice $\Lambda(S)$ of congruences on a semigroup S is investigated using a natural equivalence θ on $\Lambda(S)$, introduced by Reilly and Scheiblich [6]. For S an eventually regular semigroup we show that θ is a congruence on $\Lambda(S)$ and each θ-class is a complete sublattice of $\Lambda(S)$. For a congruence σ on a semigroup S, a congruence $\mu(\sigma)$ is defined on S and if S is an eventually regular semigroup then $\mu(\sigma)$ is shown to be the maximum element of the complete sublattice $\sigma\theta$. The above results include results of Hall [4], Reilly and Scheiblich [6] and Scheiblich [7] for regular semigroups.

Eventually regular semigroups need not be $\mathscr{K}$-regular (for definition see Section 2) for $\mathscr{K}$ any one of Green's relations but eventually regular semigroups are shown to be $\mu(\sigma)$-regular.

2. Preliminaries

Whenever possible the notations and conventions of Clifford and Preston [2] are used. For any semigroup S, the set of idempotents of S will be denoted by $E = E(S)$ and the set of idempotent-separating congruences on S will be denoted by $\Lambda^+(S)$. If ρ is a relation on S, then the restriction of ρ to E, $(\rho \cap (E \times E))$, will be denoted by $\rho|E$. If x is an element of a semigroup S then $V(x)$ will denote the set of inverses of x in S.

An element of S is called group-bound if it has some power that is in a subgroup of S and an element of S is called eventually regular if it has some power that is regular. A semigroup is called group-bound [eventually regular] if all of its elements are group-bound [eventually regular]. Thus the class of eventually regular semigroups includes all regular semigroups and all group-bound semigroups.

Recall from [3] that S is called idempotent-surjective if, for all congruences on S, every idempotent congruence class contains an idempotent. Also recall from [3] the idempotent-separating congruence $\mu = \mu(S)$ on S defined by

$$\mu = \{(a, b) \in S \times S \mid \text{if } x \in S \text{ is regular then each of } x\mathscr{R}xa,\ x\mathscr{R}xb \text{ implies } xa\mathscr{H}xb \text{ and each of } x\mathscr{L}ax,\ x\mathscr{L}bx \textit{ implies } ax\mathscr{H}bx\}.$$

For a congruence σ on a semigroup S and for $\mathscr{K} \in \{\mathscr{L}, \mathscr{R}, \mathscr{H}, \mathscr{D}, \mathscr{J}, \mu\}$, define a relation $\mathscr{K}(\sigma)$ on S by

$$\mathscr{K}(\sigma) = \{(a, b) \in S \times S \mid a\sigma \textit{ and } b\sigma \textit{ are } \mathscr{K}\textit{-related in } S/\sigma\}.$$

It is easy to verify that $\mathscr{K}(\sigma)$ is an equivalence on S, $\sigma \subseteq \mathscr{K}(\sigma)$, and for $\mathscr{K} \neq \mu$, $\mathscr{K} \subseteq \mathscr{K}(\sigma)$. It is also simple to check that $\mu(\sigma)$ is a congruence on S and that $\mu(\sigma)|E = \sigma|E = \mathscr{H}(\sigma)|E$. Remark 2 in Section 3 shows that $\mu \subseteq \mu(\sigma)$ for eventually regular semigroups and that this is not true in general.

The expression $H_a \leqslant H_b$ will denote the conjunction of $L_a \leqslant L_b$ and $R_a \leqslant R_b$.

LEMMA 1 [3, Corollary 2]. *Eventually regular semigroups are idempotent-surjective semigroups. Indeed, if S is an eventually regular semigroup, ρ is a congruence on S and $w\rho$ is an idempotent of S/ρ, then an idempotent e can be found in $w\rho$ such that $H_e \leqslant H_w$.*

From the following lemma and its dual it can be deduced that for any congruence σ on any eventually regular semigroup S, we have $\mu \subseteq \mu(\sigma)$ (equivalently, for all $a, b \in S$, $(a, b) \in \mu$ implies $(a\sigma, b\sigma) \in \mu(S/\sigma)$). This lemma will be used to prove a stronger result (Theorem 3) which will be used to establish the main results of this paper.

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LEMMA 2. *Let σ be a congruence on an eventually regular semigroup S and let $x\sigma\mathscr{L}(ax)\sigma$ with $x\sigma$ regular in S/σ. Then there exists u in S such that*

(i) $u \in x\sigma$,

(ii) $u\mathscr{L}au$ *in* S,

(iii) u *is regular.*

PROOF. Let $x\sigma = X$ and $(ax)\sigma = Y$. Take $X' \in V(X)$, $Y' \in V(Y)$, $x' \in X'$, $y' \in Y'$ and put $y = ax$. Since S is eventually regular there exists $n > 1$ such that $(x'xy'y)^n$ is regular. Take $z \in V((x'xy'y)^n)$ and put $c = z(x'xy'y)^{n-1}x'xy'$, $d = xc$, $w = ca$ and $u = dy$. It is a matter of routine to verify that $w \in V(u)$ and $dau = u$, whence u is regular and $u\mathscr{L}au$.

Finally, since $X\mathscr{L}Y$ there exists $B \in S/\sigma$ such that $X = BY$, whence

$$(*) \qquad X = BY = BYY'Y = XY'Y.$$

Therefore since $z \in V((x'xy'y)^n)$, we have $z\sigma \in V((X'XY'Y)^n) = V((X'X)^n) = V(X'X)$, using $(*)$, and so $z\sigma$ may be denoted by $[X'X]'$.

Thus

$$\begin{aligned} u\sigma &= X[X'X]'(X'XY'Y)^{n-1}X'XY'Y \\ &= X[X'X]'(X'X)^{n-1}X'X \qquad \text{by } (*) \\ &= XX'X[X'X]'X'X \\ &= X. \end{aligned}$$

THEOREM 3. *Let ρ and σ be any congruences on an eventually regular semigroup S such that $\rho|E \subseteq \mu(\sigma)|E$. Then $\rho \subseteq \mu(\sigma)$.*

PROOF. Take $(a, b) \in \rho$. It will be shown that $(a\sigma, b\sigma) \in \mu(S/\sigma)$, whence $(a, b) \in \mu(\sigma)$ and so $\rho \subseteq \mu(\sigma)$.

Assume that $x\sigma\mathscr{L}(ax)\sigma$ with $x\sigma$ regular in S/σ. By Lemma 2 there exists $u \in x\sigma$ such that $u\mathscr{L}au$ and u is regular. Take $y \in V(au)$. Then since $(a, b) \in \rho$, $(yau, ybu) \in \rho$ and $(ybu)\rho = (yau)\rho$ is idempotent in S/ρ. Therefore by Lemma 1 there exists an idempotent $e \in (ybu)\rho$ such that $H_e \leqslant H_{ybu}$, whence $H_{e\sigma} \leqslant H_{(ybu)\sigma}$. Since $(e, yau) \in \rho|E$ it follows that $(e, yau) \in \mu(\sigma)|E$, whence $e\sigma = (yau)\sigma$ because $\mu(\sigma)|E = \sigma|E$. Thus $H_{(yau)\sigma} = H_{e\sigma} \leqslant H_{(ybu)\sigma}$ and so $L_{u\sigma} = L_{(au)\sigma} = L_{(yau)\sigma} \leqslant L_{(ybu)\sigma} \leqslant L_{(bu)\sigma} \leqslant L_{u\sigma}$. Since $u\sigma = x\sigma$ it now follows that $(ax)\sigma\mathscr{L}(bx)\sigma$ and that $x\sigma\mathscr{L}(bx)\sigma$ with $x\sigma$ regular.

Since $(a, b) \in \rho$, we have $(auy, buy) \in \rho$ and $(buy)\rho = (auy)\rho$ is idempotent in S/ρ. Therefore by Lemma 1 there exists an idempotent $f \in (buy)\rho$ such that $H_f \leqslant H_{buy}$, whence $H_{f\sigma} \leqslant H_{(buy)\sigma}$. Since $(f, auy) \in \rho|E$ it follows that $(f, auy) \in \mu(\sigma)|E = \sigma|E$, whence $f\sigma = (auy)\sigma$. Thus $H_{(auy)\sigma} = H_{f\sigma} \leqslant H_{(buy)\sigma}$, whence $R_{(au)\sigma} = R_{(auy)\sigma} \leqslant R_{(buy)\sigma} \leqslant R_{(bu)\sigma}$. Therefore since $u\sigma = x\sigma$, $R_{(ax)\sigma} \leqslant R_{(bx)\sigma}$.

It has been established that $x\sigma\mathscr{L}(bx)\sigma$ with $x\sigma$ regular. Using this fact and a similar argument to the above yields that $R_{(bx)\sigma} \leqslant R_{(ax)\sigma}$. Thus $(ax)\sigma\mathscr{R}(bx)\sigma$ and it has already been shown that $(ax)\sigma\mathscr{L}(bx)\sigma$. Therefore $(ax)\sigma\mathscr{H}(bx)\sigma$. Dually the assumption that $x\sigma\mathscr{R}(xa)\sigma$ with $x\sigma$ regular yields $(xa)\sigma\mathscr{H}(xb)\sigma$. It follows from the above and symmetry that $(a\sigma, b\sigma) \in \mu(S/\sigma)$ as required.

REMARK 1. Let γ be an equivalence on a semigroup S. Then S is called γ-regular [5, Definition 1] if, for any congruence ρ on S, $\rho|E \subseteq \gamma|E$ implies $\rho \subseteq \gamma$. Theorem 3 above shows that eventually regular semigroups are $\mu(\sigma)$-regular. It follows from considering the case of $\sigma = 1_S$ that eventually regular semigroups are μ-regular and thus μ is the maximum idempotent-separating congruence on an eventually regular semigroup. The last assertion is Theorem 11 of [3]. Let $\mathscr{K}$ be one of Green's relations. Then regular semigroups are $\mathscr{K}(\sigma)$-regular [4, Theorem 1] for any congruence σ and thus are also $\mathscr{K}$-regular [5, Theorem 2.3]. Finally, let S be the two element null semigroup, $\sigma = 1_S$ and let $\mathscr{K}$ be one of Green's relations. Then $\mathscr{K}(\sigma) = \mathscr{K}$ and the eventually regular semigroup S is not $\mathscr{K}(\sigma)$-regular.

3. The lattice of congruences

Let S be a semigroup and define a relation $\theta = \theta(S)$ on $\Lambda(S)$ as in [6], by

$$\theta = \{(\rho_1, \rho_2) \in \Lambda(S) \times \Lambda(S) \,|\, \rho_1|E = \rho_2|E\}.$$

It is easy to verify that θ is a meet compatible equivalence on $\Lambda(S)$. Clearly, if $\sigma \in \Lambda(S)$ then $\sigma\theta$ has a least element. This element will be denoted $1(\sigma)$.

It has been shown [6, Theorem 5.1] that θ is a congruence if S is an inverse semigroup. Subsequently, it was shown ([4], [7]) that θ is a congruence if S is a regular semigroup. It is also known [6] that when S is regular each θ-class is a complete modular sublattice of $\Lambda(S)$.

In this section it will be shown that θ is a congruence if S is an eventually regular semigroup. Moreover, if S is an eventually regular semigroup and $\sigma \in \Lambda(S)$ then $\sigma\theta$ is a complete sublattice of $\Lambda(S)$ and has maximum element $\mu(\sigma)$. The sublattice $\sigma\theta = [1(\sigma), \mu(\sigma)]$ is isomorphic to $[1_{S/1(\sigma)}, \mu(S/1(\sigma))] = \Lambda^+(S/1(\sigma))$. The θ-class $1_S\theta = \Lambda^+(S)$ need not be modular when S is a finite semigroup.

We begin with a corollary to Theorem 3.

COROLLARY 4. *Let S be an eventually regular semigroup and $\sigma \in \Lambda(S)$. Then $\sigma\theta$ is a complete sublattice of $\Lambda(S)$ with maximum element $\mu(\sigma)$.*

PROOF. Since $\mu(\sigma)|E = \sigma|E$, it follows from Theorem 3 that $\mu(\sigma)$ is the maximum element of $\sigma\theta$. It is now clear that $\sigma\theta$ is a complete sublattice of $\Lambda(S)$ with maximum element $\mu(\sigma)$ and minimum element $1(\sigma)$.

REMARK 2. Let S be an eventually regular semigroup and $\sigma, \gamma \in \Lambda(S)$. It is clear that $\mu(\sigma) \subseteq \mu(\gamma)$ if and only if $\sigma|E \subseteq \gamma|E$. Also $\mu(\mu) = \mu$. It follows from the above or Theorem 3 that $\mu \subseteq \mu(\sigma)$. However, this result is not true for arbitrary semigroups as the following example shows. Let F be a free semigroup and let $T = F^0$. Consider the congruence σ on T given by the partition F, $\{0\}$ and take $x \in F$. Then as $\mu = T \times T$, $(0, x) \in \mu$ but $(0, x) \notin \mu(\sigma)$ since $(0\sigma, x\sigma) \notin \mu(S/\sigma)$. This last assertion is true since $x\sigma = F$ is an idempotent in S/σ and of course $\mu(S/\sigma)$ is idempotent-separating. This example also shows that $1_S \subseteq \sigma$ but $\mu(1_S) = \mu \not\subseteq \mu(\sigma)$.

THEOREM 5. *Let S be an eventually regular semigroup. Then θ is a complete congruence on $\Lambda(S)$.*

PROOF. Suppose that for each element i of some index set I, $(\rho_i, \rho_i') \in \theta$. Put $\sigma = \bigvee\{\rho_i | i \in I\}$ and $\sigma' = \bigvee\{\rho_i' | i \in I\}$. To show that θ is a complete congruence on S it will suffice to show that $(\sigma, \sigma') \in \theta$ since the meet case is trivial. For each $i \in I$, $\rho_i \subset \sigma$ and so $\mu(\rho_i) \subseteq \mu(\sigma)$ by Corollary 4, whence $\bigvee\{\mu(\rho_i) | i \in I\} \subseteq \mu(\sigma)$. Thus $\sigma|E \subseteq (\bigvee\{\mu(\rho_i) | i \in I\})|E \subseteq \mu(\sigma)|E = \sigma|E$, whence $(\bigvee\{\mu(\rho_i) | i \in I\})|E = \sigma|E$. Similarly, $(\bigvee\{\mu(\rho_i') | i \in I\})|E = \sigma'|E$. Now, by Corollary 4, $\mu(\rho_i) = \mu(\rho_i')$ for each $i \in I$, and so it follows that $\sigma|E = (\bigvee\{\mu(\rho_i) | i \in I\})|E = (\bigvee\{\mu(\rho_i') | i \in I\})|E = \sigma'|E$, whence $(\sigma, \sigma') \in \theta$ as required.

The following theorem is now clear and includes the main results of this paper.

THEOREM 6. *Let S be an eventually regular semigroup. Then*

(i) *θ is a complete congruence on $\Lambda(S)$;*

(ii) *the θ-class $\sigma\theta$ is a complete sublattice of $\Lambda(S)$ with maximum element $\mu(\sigma)$;*

(iii) *the quotient lattice $\Lambda(S)/\theta$ is complete and the natural morphism $\theta^\natural$ of $\Lambda(S)$ onto $\Lambda(S)/\theta$ is a complete lattice morphism.*

REMARK 3. Let S be an eventually regular semigroup and T an arbitrary semigroup and let $\sigma \in \Lambda(S)$. Define (in the notation of [6], page 352) for $\rho \in \sigma\theta$ a relation,

$$\rho/1(\sigma) = \{(a1(\sigma), b1(\sigma)) \in S/1(\sigma) \times S/1(\sigma) | (a, b) \in \rho\}.$$

Then the relation $\rho/1(\sigma) \in \Lambda^+(S/1(\sigma))$ and the mapping $\sigma^+ \colon \sigma\theta \to \Lambda^+(S/1(\sigma))$ defined by $(\rho)\sigma^+ = \rho/1(\sigma)$ for each $\rho \in \sigma\theta$, is an isomorphism of the complete lattices $\sigma\theta$ and $\Lambda^+(S/1(\sigma))$. It is well known that a sublattice L, of $\Lambda(T)$, is modular if all of the elements of L commute [1, page 86, Theorem 3]. Thus if the elements of $\Lambda^+(S/1(\sigma))$ commute then $\sigma\theta$ is a complete modular sublattice of $\Lambda(S)$. Let the set of all congruences contained in $\mathscr{H}$ on T be denoted $\Sigma(\mathscr{H})$. Lallement [5] has shown that the elements of $\Sigma(\mathscr{H})$ commute and that if T is regular then $\Sigma(\mathscr{H}) = \Lambda^+(T)$. It is therefore clear that if S is regular then $\Lambda^+(S/1(\sigma))$ is modular and σ^+ above is an isomorphism of complete modular lattices. That $\sigma\theta$ is a complete modular sublattice of $\Lambda(S)$ when S is regular is Theorem 3.4(ii) of [6].

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References

[1] Garrett Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. Publ., Vol. 25, Amer. Math. Soc., Providence, R. I., 1964.

[2] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., Vol. I, 1961; Vol. II, 1967.

[3] P. M. Edwards, 'Eventually regular semigroups', *Bull. Austral. Math. Soc.* **28** (1983), 23–38.

[4] T. E. Hall, 'On the lattice of congruences on a regular semigroup', *Bull. Austral. Math. Soc.* **1** (1969), 231–235.

[5] Gérard Lallement, 'Congruences et équivalences de Green sur un demi-groupe régulier', *C. R. Acad. Sci. Paris Sér. A* **262** (1966), 613–616.

[6] N. R. Reilly and H. E. Scheiblich, 'Congruences on regular semigroups', *Pacific J. Math.* **23** (1967), 349–360.

[7] H. E. Scheiblich, 'Certain congruence and quotient lattices related to completely 0-simple and primitive regular semigroups', *Glasgow Math. J.* **10** (1969), 21–24.

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