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Topological stability and Gromov hyperbolicity

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Abstract. We show that if the geodesic flow of a compact analytic Riemannian manifold M of non-positive curvature is either C^k-topologically stable or satisfies the ϵ-C^k-shadowing property for some $k > 0$ then the universal covering of M is a Gromov hyperbolic space. The same holds for compact surfaces without conjugate points.

0. *Introduction*

The fact that quasi-geodesics are traced or 'shadowed' by true geodesics is one of the most characteristic properties of hyperbolic spaces. This feature noticed by Morse [**11**] in the late 1930s for metrics in the disk, induced by Riemannian metrics of compact surfaces of genus greater than two, plays an outstanding role in the recent theory of Gromov hyperbolic spaces and in geometric group theory. Recall that a complete geodesic metric space (X, d) is said to be *Gromov hyperbolic* if there exists $\delta > 0$ such that for every geodesic triangle $[a_0, a_1]$, $[a_1, a_2]$, $[a_2, a_0]$ made by geodesic segments $[a_i, a_{i+1}]$, $i \in Z/3Z$, in X we have that $\forall p \in [a_i, a_{i+1}]$

$$d(p, [a_{i+1}, a_{i+2}] \cup [a_{i+2}, a_i]) \leq \delta.$$

In other words, every geodesic triangle in the space is δ thin. The name 'shadowing' referred to above is a way of recalling the property of quasi-geodesics of such spaces which are contained in tubular neighborhoods of the geodesics of the space.

This feature is related to some well-known notions of stability of geodesic flows and dynamical systems in general. One of them is the C^1-*structural stability*, which is a necessary and sufficient condition for the hyperbolicity of dynamical systems. However, it is more natural to associate the pseudo-orbit tracing property to a weaker kind of stability. For instance, expansive geodesic flows in compact manifolds without conjugate points may not be Anosov flows, but they still have a local product structure which guarantees, for instance, that these flows are C^1-topologically stable and satisfy the ϵ-C^1-shadowing

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property [**12**]. Let (M, g) be a C^∞ Riemannian manifold, and let $T_1(M, g)$ be its unit tangent bundle. We say that the geodesic flow $\phi_t : T_1(M, g) \longrightarrow T_1(M, g)$ of (M, g) is C^k topologically stable if there exists a C^{k+2} neighborhood V of the metric g such that $\forall \bar{g} \in V$ there is a continuous, surjective map $f : T_1(M, \bar{g}) \longrightarrow T_1(M, g)$ sending the orbits of the geodesic flow ψ_t of $(M, \bar{g})$ into orbits of ϕ_t, i.e. for every $\theta \in T_1(M, \bar{g})$ there exists a continuous surjective map $r : R \longrightarrow R$ depending on θ with $r(0) = 0$ such that

$$\phi_{r(t)}(f(\theta)) = f(\psi_t(\theta))$$

for every $t \in R$. The flow ϕ_t satisfies the ϵ-C^k-shadowing property for some $\epsilon > 0$ if there exists a C^{k+2} neighborhood V of g such that for every $\bar{g} \in V$ there is a continuous map $f : T_1(M, \bar{g}) \longrightarrow T_1(M, g)$ such that for every $\theta \in T_1(M, \bar{g})$ there exists a continuous surjective map $r : R \longrightarrow R$ with $r(0) = 0$ and

$$d(\phi_t(f(\theta)), \psi_{r(t)}(\theta)) \le \epsilon$$

for every $t \in R$. Moreover, compact manifolds without conjugate points and expansive geodesic flows have Gromov hyperbolic universal coverings [**13**], which implies that every quasi-geodesic in the universal covering is shadowed by a geodesic. We would like to point out that it is not known whether a compact manifold without conjugate points whose geodesic flow is expansive admits a metric whose geodesic flow is Anosov. Notice that the map f in the definition of the ϵ-shadowing property is not required to be surjective. Furthermore, the above notions are weaker in some sense than the shadowing of quasi-geodesics in Gromov hyperbolic spaces, since they just involve quasi-geodesics which are, at the same time, geodesics of perturbations of the metric.

Motivated by the previous ideas, we can set up a sort of 'topological stability conjecture': Does the shadowing property (topological stability) of the geodesic flow of a manifold M without conjugate points imply that the universal covering of M is Gromov hyperbolic? The main results of this work are the following.

THEOREM 1. *Let M be a compact, analytic Riemannian manifold of non-positive curvature. If the geodesic flow of M is C^k topologically stable for some $k > 0$ then the universal covering $\tilde{M}$ of M is a Gromov hyperbolic space.*

THEOREM 2. *Let M be a compact, analytic Riemannian manifold of non-positive curvature. Then there exists $\epsilon > 0$ such that if the geodesic flow of M satisfies the ϵ-C^k-shadowing property for some $k > 0$ then the universal covering $\tilde{M}$ of M is Gromov hyperbolic.*

In the case of surfaces we show that, if M is a compact surface without conjugate points whose geodesic flow is either C^k-topologically stable or satisfies the ϵ-C^k-shadowing property for some $k > 0$, then the genus of M has to be greater than two. Theorems 1 and 2 also hold for compact, nonpositively curved manifolds of dimension three regardless of the analyticity of the metric. Although classical examples of topologically stable systems have the shadowing property and *vice versa*, we do not know whether these two properties are equivalent or not. One of the consequences of Theorems 1 and 2 is that they are equivalent for geodesic flows in both compact surfaces with no conjugate points and

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compact analytic manifolds with nonpositive curvature. This is because, on the one hand, Gromov hyperbolicity of $\tilde{M}$ implies the shadowing property for quasi-geodesics of $\tilde{M}$ in the sense of Morse, Eberlein and Gromov, and from this fact it is not hard to deduce that there is a surjective correspondence between geodesics of metrics close to the metric of M and geodesics of M. On the other hand, Theorems 1 and 2 grant that both these properties imply Gromov hyperbolicity of $\tilde{M}$.

The proofs of Theorems 1 and 2 are actually extensions of the proofs in the two-dimensional case. The main point in both proofs is the creation of C^k perturbations of the Euclidean metric in the torus having a 'waist' in a certain non-trivial homotopy class, which automatically implies the existence of homoclinic geodesics in the geodesic flow of the perturbation. This well-known result due to Morse [**11**] and Hedlund [**7**] allows us to create dynamical behaviours of geodesics that have no counterpart in the Euclidean geodesic flow. The existence of connecting orbits in Hamiltonian dynamics is an interesting example of the application of variational methods in the study of dynamical systems [**10**]. This paper has three sections: in the first section we state a result about conformal perturbations of immersed tori having only one waist in a certain homotopy class and we prove the main theorems in the two-dimensional case; then in §2 we use a closing lemma of flat submanifolds due to Schroeder [**14**] and Bangert–Schroeder [**2**] to reduce the proof for n-dimensional manifolds to the two-dimensional case, and in §3 we prove the perturbation result used in §1.

1. *Homoclinic orbits of C^k perturbations of the flat metric on T^2*

Let us start by fixing some notation. All geodesics will be parametrized by arc length. The unit tangent bundle of a Riemannian manifold M is denoted by T_1M. There is a canonical coordinate system for T_1M given by $\theta = (p, v) \in T_1M$, where p is a point in M and v is a unit vector tangent to M at p. Let $\pi : T_1M \longrightarrow M$ be the projection $\pi(p, v) = p$ and let $\phi_t : T_1M \longrightarrow T_1M$ be the geodesic flow of M. The purpose of this section is to prove the main theorems in the case of surfaces. The following result will be proved in §3.

PROPOSITION 1.1. *Let (M, g) be a Riemannian manifold and let (T^2, g) be a flat, totally geodesic, immersed torus in M. Let $\gamma(t)$ be a closed geodesic in T^2. For every $k \in N$ and $\delta > 0$ there exists a metric g_δ conformal to g such that:*

(1) *g_δ is δ-C^k close to the metric g.*

(2) *When $l_g(\alpha)$, $l_{g_\delta}(\alpha)$ are the lengths of a closed curve α with respect to g, g_δ, then $l_g(\alpha) \geq l_{g_\delta}(\alpha)$ for every closed curve α, and the lengths are equal if and only if $\alpha = \gamma$.*

(3) *(T^2, g_δ) is totally geodesic in (M, g_δ).*

We start by proving Proposition 1.1 for $M = T^2$.

LEMMA 1.2. *Let (T^2, g) be a Riemannian structure in the torus. Let γ be a closed geodesic and $\sigma : T^2 \longrightarrow R$ be a C^∞ non-negative function such that $\sigma(p) = 0$ if and only if $p \in \gamma$. Let $\bar{g}_p = e^{2\sigma(p)} g_p$. Then $l_{\bar{g}}(\alpha) \geq l_g(\alpha)$ for every closed curve α in T^2 and the lengths coincide if and only if $\alpha = \gamma$. Moreover, if γ minimizes l_g in its (non-trivial) homotopy class, then γ is still a geodesic in $(T^2, \bar{g})$ which minimizes strictly the length $l_{\bar{g}}$ in its homotopy class.*

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Proof. It is easy to see that the length of curves in the new metric $\bar{g}$ are greater than or equal to the length in the metric g, since $e^{\sigma} \geq 1$. Moreover, the only curve having the same length in both metrics is the geodesic $\gamma(t)$, because $e^{\sigma(p)} = 1$ if and only if $p \in \gamma^0$, otherwise it is strictly greater than 1. On the other hand, if γ is a minimizing loop for g in its homotopy class, then it continues to be minimizing for $\bar{g}$: by variational arguments we get that γ is a geodesic in $(T^2, \bar{g})$, but now it is also strictly minimizing in its homotopy class. □

Next, recall the following result by Morse [**11**] and Hedlund [**7**].

LEMMA 1.3. *Let (T^2, g') be a Riemannian structure on the torus T^2. Assume that $a \in \pi_1 T^2$ is a non-trivial homotopy class such that there exists a unique minimizing loop γ in the class, i.e. $l_{g'}(\alpha) \geq l_{g'}(\gamma)$ for every closed curve α with homotopy class a and there is equality if and only if $\alpha = \gamma$. Then, given a parametrization $\gamma : [0, l] \longrightarrow T^2$ by arclength there exists at least two different geodesics β_1, β_2 having the following properties.*

(1) *Any liftings $\tilde{\beta}_1(t)$ of $\beta_1(t)$, $\tilde{\beta}_2(t)$ of $\beta_2(t)$, are contained in a strip bounded by two consecutive liftings of γ.*

(2) *Suppose $\tilde{\gamma}_1(t)$, $\tilde{\gamma}_2(t)$ are two consecutive liftings of $\gamma(t)$ with $d(\tilde{\gamma}_1(t), \tilde{\gamma}_2(t)) \leq D$ $\forall t \in R$ bounding a strip which contains $\tilde{\beta}_1(t)$ and $\tilde{\beta}_2(t)$. Then (up to an interchange of indices),*

$$\lim_{t \to +\infty} d(\tilde{\beta}_1(t), \tilde{\gamma}_1) = 0, \quad \lim_{t \to -\infty} d(\tilde{\beta}_1(t), \tilde{\gamma}_2) = 0$$

and

$$\lim_{t \to +\infty} d(\tilde{\beta}_2(t), \tilde{\gamma}_2) = 0, \quad \lim_{t \to -\infty} d(\tilde{\beta}_2(t), \tilde{\gamma}_1) = 0.$$

Now, we are able to prove Theorem 1 in the case of surfaces.

PROPOSITION 1.4. *The geodesic flow of any flat metric on the torus (T^2, g) is not C^k topologically stable for any k.*

Proof. We argue by contradiction. Suppose that the geodesic flow ϕ_t of (T^2, g) is C^k topologically stable for some k. Let V be a C^{k+2} neighborhood of g where every geodesic flow is semi-equivalent to ϕ_t. Since the arguments are purely topological we can assume without loss of generality that (T^2, g) is the torus $R^2/\langle e_1, e_2 \rangle$ where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Given $\bar{g} \in V$ the semi-equivalence map $f : T_1 M \longrightarrow T_1 M$ induces a linear map f_* in $\pi_1 T^2 = Z \times Z = H_1(T^2, Z)$. Since in every non-trivial homotopy class there exists at least one closed geodesic, and the semi-equivalence is surjective, the induced map f_* is also surjective. Therefore, it is also injective and a linear isomorphism. Now, let γ be the closed geodesic of (T^2, g) tangent to $(0, 1)$ with $\gamma(0) = (0, 0)$. Let g_δ be a conformal C^k perturbation of g obtained from Lemma 1.2 by choosing a factor function σ C^k-close to zero which is zero along γ. Then γ is a strict minimum for the l_{g_δ}-length in (T^2, g_δ) and thus there exist homoclinic geodesics in (T^2, g_δ), β_1, β_2, according to Lemma 1.3.

CLAIM. *There exists a homotopically non-trivial closed orbit of ψ_t whose image under f_* is null-homotopic.*

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Clearly, this will contradict the fact that f_* is an isomorphism. Let $\beta(t)$ be a homoclinic orbit, $\lim_{t\to+\infty} d(\beta(t), \gamma(t)) = 0$ and $\lim_{t\to-\infty} d(\beta(t), \gamma(t)) = 0$. Since the map f is continuous, this implies that $f(\beta) = f(\gamma)$. The liftings of γ in the universal covering R^2 are vertical lines (i, t), for $i \in Z$. Let $\bar{\beta}$ be the lifting of β in R^2 remaining between the vertical lines $\gamma_0(t) = (0, t)$ and $\gamma_1 = (1, t)$. Assume, for instance, that $\bar{\beta}$ is backward asymptotic to γ_0 and forward asymptotic to γ_1. Now, for $k \in N$ big enough, let us consider the curve $\bar{C}_k$ in R^2 formed by the horizontal straight segment $[\gamma_0(-k), p_k]$ lying between $\gamma_0(-k)$ and $\bar{\beta}$, the horizontal segment $[\gamma_1(k), q_k]$ lying between $\gamma_1(k)$ and $\bar{\beta}$, and the subcurve of $\bar{\beta}$ bounded by p_k, q_k. This curve projects by the covering map into a closed curve C_k in T^2 whose homology class is $k'e_1 + e_2$ for some $k' = k'(k)$ that goes to $+\infty$ as $k \to +\infty$. On the other hand, by the continuity of f the homology class $f_*[C_k]$ must be $k' f_*([\gamma]) = k' f_*(e_1)$ and hence

$$f_*(k'e_1 + e_2) = k' f_*(e_1) + f_*(e_2) = k' f_*(e_1),$$

which implies that $f_*(e_2) = 0$, thus proving the claim and the Proposition. □

Now, we proceed to the study of the ϵ-shadowing property in the case of the flat metric of a straight torus. Observe that the definition of ϵ-shadowing gives us a correspondence from the space of orbits of perturbations of the flat geodesic flow into the Euclidean geodesics of T^2. However, this map is not '*a priori*' surjective. Given a compact Riemannian manifold M let us define $\epsilon_0 = \epsilon_0(M)$ to be the supremum of the numbers $\epsilon > 0$ such that every two closed curves α, β in M satisfying $d(\alpha(t), \beta(f(t))) \leq \epsilon$ for some $\epsilon \leq \epsilon_0$ are homotopic, where $f(t)$ is a continuous surjective function of R. This number ϵ_0 depends on the injectivity radius of the manifold. Let $e_1 = (1, 0)$, $e_2 = (0, 1)$.

PROPOSITION 1.5. *Let L be a lattice in R^2 generated by two independent translations T_{v_1}, T_{v_2}, where $v_1 = \lambda_1 e_1$ and $v_2 = \lambda_2 e_2$, $|\lambda_2| \leq |\lambda_1|$. Let $T^2 = R^2/L$ and consider $\epsilon_0 = \epsilon_0(T^2)$. Then, the geodesic flow of T^2 does not satisfy the ϵ-C^k-shadowing property in the set of geodesic flows for any $\epsilon \leq \min\{\epsilon_0, \frac{1}{5}|v_2|\}$ and any $k \in N$.*

Proof. We will show that the only way for a flat geodesic flow on the torus to satisfy the ϵ-shadowing property is the trivial way, i.e. the torus must be small. By Lemmas 1.2 and 1.3 applied to T^2, there is an arbitrarily small C^k perturbation $\bar{g}$ of the Euclidean metric g such that the geodesic $\gamma(t) = (0, t)$, $0 \leq t \leq \lambda_2$, is strictly $l_{\bar{g}}$-minimizing in its homotopy class, and thus there exists a homoclinic geodesic β whose α and ω limits are precisely the points of γ. Let $\Pi : R^2 \longrightarrow T^2$ be the covering map. We claim that each two consecutive liftings of $\gamma(t)$ by Π are parallel straight lines at distance at most 2ϵ. Indeed, since by hypotheses the flat geodesic flow on T^2 satisfies the ϵ-C^k-shadowing property we have that the following statement holds: for the geodesic $\beta(t)$ there exists a geodesic $\beta_0(t)$ in the flat metric on T^2 and a reparametrization $f : R \longrightarrow R$ of $\beta_0(t)$ such that

$$d(\beta(t), \beta_0(f(t))) \leq \epsilon$$

for every $t \in R$. Note that $\beta_0(t)$ is the projection of a straight line of the plane by Π, so the set of liftings of $\beta_0(t)$ by Π is a countable collection of parallel straight lines in R^2. By the choice of ϵ the liftings of $\beta(t)$ are curves in the plane which are at a distance at most

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ϵ from some lifting of $\beta_0(t)$. So let $\bar{\beta}$ be a lifting of β, and let $\bar{\beta_0}$ be the lifting of β_0 such that

$$d(\bar{\beta}(t), \bar{\beta_0}(f(t))) \leq \epsilon$$

for every t. Recall that by Lemma 1.3 we have that there exist two consecutive liftings $\bar{\gamma}_1$ and $\bar{\gamma}_2$ of $\gamma(t)$ such that

$$\lim_{t\to+\infty} d(\bar{\beta}(t), \bar{\gamma}_1) = 0, \qquad \lim_{t\to-\infty} d(\bar{\beta}(t), \bar{\gamma}_2) = 0.$$

Moreover, we have that the distance between $\bar{\gamma}_1$ and $\bar{\gamma}_2$ is exactly λ_1, because two consecutive liftings of γ differ precisely by an iterate of the translation T_{v_1}. This implies that $\bar{\beta_0}$ has to be parallel to the liftings of γ. From the above equations we deduce that $d(\bar{\gamma}_1, \bar{\beta_0}) \leq \epsilon$ and $d(\bar{\gamma}_2, \bar{\beta_0}) \leq \epsilon$. Thus,

$$\lambda_1 = d(\bar{\gamma}_1, \bar{\gamma}_2) \leq d(\bar{\gamma}_1, \bar{\beta_0}) + d(\bar{\gamma}_2, \bar{\beta_0}) \leq 2\epsilon,$$

which proves the claim.

But now, the above claim leads to a contradiction, since by hypotheses

$$\lambda_2 \leq \lambda_1 \leq 2\epsilon < \tfrac{1}{2}\lambda_2. \qquad \square$$

The proof of Proposition 1.5 can be generalized to any Euclidean metric in the torus. Recall the following canonical property of lattices in the plane.

LEMMA 1.6. *Let L be a lattice in the plane generated by two linearly independent vectors* v_1 *and* v_2*, where* $|v_1| = min\{|v|, v \in L\}$*. Then there exists* $\bar{v}_2$ *in L such that:*
(1) v_1 *and* $\bar{v}_2$ *generate L;*
(2) $\frac{1}{3}\pi \leq \angle(v_1, \bar{v}_2) \leq \frac{2}{3}\pi$.

COROLLARY 1.7. *The geodesic flow of any flat metric on* T^2 *does not satisfy the* ϵ-C^k-*shadowing property for* ϵ *defined in Proposition 1.5 for any* $k > 0$.

Proof. Let $T^2 = R^2/L$ be a Euclidean structure on the torus, where L is a lattice in the plane, and assume that its geodesic flow satisfies the ϵ-C^k-shadowing property for some $k > 0$ and ϵ is as in Proposition 1.5. Let v_1, $v_2 = \bar{v}_2$ be a pair of generating vectors of the lattice L satisfying the statement of Lemma 1.6. Let $\gamma(t)$ be a closed geodesic tangent to v_1. The distance between two consecutive liftings $\bar{\gamma}_1(t)$, $\bar{\gamma}_2(t)$ of $\gamma(t)$ is at least $\frac{1}{2}|v_2|$, since the angle between v_1 and v_2 is at least $\frac{1}{3}\pi$. On the other hand, by Proposition 1.1 and Lemmas 1.2 and 1.3, there are small C^k perturbations of this Riemannian structure on T^2 having homoclinic geodesic $\beta(t)$ whose α and ω limits are precisely $\gamma(t)$. So, by the same reasoning as in the proof of Proposition 1.5 we deduce that the distance between $\bar{\gamma}_1(t)$ and $\bar{\gamma}_2(t)$ is at most 2ϵ. Thus $\frac{1}{2}|v_2| \leq d(\bar{\gamma}_1, \bar{\gamma}_2) \leq 2\epsilon \leq \frac{2}{5}|v_2|$, leading to a contradiction. $\square$

Recall that a systole of (M, g) is the shortest closed geodesic of (M, g). Theorem 1.8 follows from the fact that the only metrics without conjugate points in the two-dimensional torus are the flat metrics [**8**].

COROLLARY 1.8. *Let M be a compact surface without conjugate points. Let* $\epsilon_0(M)$ *be as before and let* ϵ_1 = *length of a systole in M. If the geodesic flow* ϕ_t *of M satisfies either the* ϵ-C^k-*shadowing property for* $\epsilon = min\{\epsilon_0, \frac{1}{5}\epsilon_1\}$ *and some* $k \in N$*, or is topologically stable in the set of geodesic flows, then the genus of M is greater than two.*

2. *The proof in the general case*

We start by stating some theorems concerning the geometry of non-positively curved manifolds.

THEOREM 2.1. (Eberlein [**5**]) *Let M be a compact Riemannian manifold of non-positive curvature. If the universal covering $\bar{M}$ of M has no isometric immersion of an Euclidean plane then $\bar{M}$ is a visibility manifold.*

THEOREM 2.2. (Bangert and Schroeder [**2**]) *Let M be a compact analytic manifold with non-positive curvature. If $\bar{M}$ contains an isometric immersion of R^k for some $k \geq 2$ then M contains a q-dimensional immersed flat torus for some $q \geq 2$.*

The idea now is to extend the arguments in §1 to totally geodesic immersed tori in compact Riemannian manifolds (M, g). Proposition 1.1 is already a generalization of Lemma 1.2, so given (T^2, g) totally geodesic in (M, g) we are able to produce perturbations g_δ of g such that (T^2, g_δ) is still totally geodesic and has a waist, i.e. a closed loop of minimal length in (T^2, g_δ). Now we would like to apply Hedlund's result to deduce the existence of homoclinic geodesics in (T^2, g_δ). This is possible because there exists an embedding $\bar{T}^2$ of the torus covering (T^2, g_δ) that is locally isometric to T^2. Indeed, since T^2 is totally geodesic in (M, g_δ), there is a collection of totally geodesic submanifolds diffeomorphic to R^2 in $\tilde{M}$ covering T^2. Let $P \subset \tilde{M}$ be one of these planes and let $\alpha, \beta \in \pi_1(M)$ be two generators of the representation of $\pi_1(T^2)$ in $\pi_1(M)$. The maps α and β act as isometries in $(\tilde{M}, g_\delta)$, the subgroup $\langle \alpha, \beta \rangle$ acts properly discontinuously on P and the quotient manifold $\bar{T}^2 = P/\langle \alpha, \beta \rangle$ is a finite covering of (T^2, g_δ) having a waist in a certain homotopy class. Thus, Hedlund's result applies to $\bar{T}^2$ and, since it is locally isometric to (T^2, g_δ), the homoclinic geodesics of $\bar{T}^2$ project onto homoclinic geodesics of (T^2, g_δ) which are also geodesics in (M, g_δ). Therefore, we can apply Lemma 1.3 without loss of generality to immersed totally geodesic tori.

Now, we are ready to extend Proposition 1.4 to n-dimensional manifolds.

Proof of Theorem 1. Suppose that the geodesic flow of M is C^k topologically stable for some $k \in N$. By Theorems 2.1 and 2.2 it is enough to show that M does not contain an immersed flat torus. Assume by contradiction that M contains a torus T^m. Then it contains a flat, immersed, two-dimensional torus T^2 covered by flat planes in $\tilde{M}$. Let γ be a systole in T^2. From Proposition 1.1 and Hedlund's theorem (Lemma 1.3 applied to immersions) we get C^k perturbations g_δ of g for any $k \in N$ such that:

- γ is still a geodesic of (M, g_δ);
- there are at least two homoclinic geodesics connecting any two consecutive parallel liftings of γ in the universal covering of T^2.

Since T^2 is totally geodesic, it is incompressible in M—i.e. its fundamental group injects in $\pi_1(M)$—so $H_1(T^2, Z)$ is a two-dimensional submodule of $H_1(M, Z)$. If $f : T_1M \longrightarrow T_1M$ is any semi-conjugation between the geodesic flow of (M, g_δ) and the geodesic flow of (M, g), then the induced map f_* in homology must be an isomorphism. On the other hand, the argument in Proposition 1.4 shows the existence of a nonzero homology class in $H_1(T^2, Z)$ in the kernel of f_*. This concludes the proof of Theorem 1.

Proof of Theorem 2. Again, the argument reduces essentially to the two-dimensional case, although the ϵ-shadowing hypotheses does not imply in general the existence of a surjective correspondence between orbits. Suppose by contradiction that (M, g) contains a flat immersed torus T^2. Let $P \subset \tilde{M}$ be a flat plane covering T^2, so $\pi_1(T^2) \subset \pi_1(M)$ leaves P invariant and generates a lattice L such that $T^2 = P/L$. Let v_1, v_2 be two generators of L satisfying the conditions of Lemma 1.6, and let γ be a closed geodesic in the homotopy class v_1. From Proposition 1.1 and Lemma 1.3 we have C^k perturbations g_δ of g such that (T^2, g_δ) is totally geodesic, γ is a strictly minimizing geodesic in the class v_1, and there exists a homoclinic geodesic β in (T^2, g_δ) whose α and ω limits are the points of γ. Let $\phi_t, \bar{\phi}_t$ be the geodesic flows of (M, g) and (M, g_δ), respectively. Let $\Pi : T_1M \longrightarrow M$ be the canonical projection, and let $\Pi(\bar{\phi}_t(\psi)) = \beta(t)$.

CLAIM 1. *Given two consecutive liftings $\gamma_1(t)$, $\gamma_2(t)$ of $\gamma(t)$ in P, there exists a lifting $\bar{\beta}(t)$ of β in P such that*

$$\lim_{t\to+\infty} d(\bar{\beta}(t), \gamma_1) = 0, \qquad \lim_{t\to-\infty} d(\bar{\beta}(t), \gamma_2) = 0.$$

According to the hypotheses of Theorem 2 the orbit $\bar{\phi}_t(\psi)$ has a 'shadow' $\phi_t(f(\psi))$, that satisfies $d(\bar{\phi}_{h(t)}(\psi), \phi_t(f(\psi))) \leq \epsilon$ for every real t and for some reparametrization $h(t)$ of $\bar{\phi}_t(\psi)$.

CLAIM 2. *Let P, γ_1, γ_2 be as in Claim 1. Then there exists a lifting η in $\tilde{M}$ of $\Pi(\phi_t(f(\psi)))$ such that $d(\gamma_1, \eta) \leq \epsilon$ and $d(\gamma_2, \eta) \leq \epsilon$ where the distance above means Hausdorff distance.*

By the choice of ϵ we have that any lifting $\eta(t)$ in $\tilde{M}$ of $\Pi(\phi_t(f(\psi)))$ must be at a distance at most ϵ from a lifting $\bar{\beta}$ in $\tilde{M}$ of the homoclinic orbit $\beta(t)$. Since β is contained in a strip of a certain plane P covering T^2 and distances from geodesics to totally geodesic submanifolds are convex functions in spaces of nonpositive curvature, the distance from the geodesic $\eta(t)$ to the plane P is constant. Moreover, the distances $d(\eta(t), \gamma_1(t))$ and $d(\eta(t), \gamma_2(t))$ must be constant by the same reason, for suitable parametrizations of these geodesics. On the other hand, we have

$$\lim_{t\to+\infty} d(\gamma_1(t), \eta(t)) \leq \lim_{t\to+\infty} d(\gamma_1(t), \bar{\beta}(t)) + d(\bar{\beta}(t), \eta(t)) \leq \epsilon$$

and hence $d(\eta(t), \gamma_1(t)) \leq \epsilon \ \forall t \in R$. Analogously, we get $d(\eta(t), \gamma_2(t)) \leq \epsilon \ \forall t \in R$.

Claims 1 and 2 imply that $d(\gamma_1, \gamma_2) \leq 2\epsilon$ so any two consecutive liftings in $\tilde{M}$ of the generating geodesic $\gamma(t)$ in the homotopy class v_1 have Hausdorff distance less than 2ϵ. Now, by Lemma 1.6 and the proof of Corollary 1.7 we get again that $\frac{1}{2}|v_2| \leq \frac{2}{5}\epsilon$, leading to a contradiction. This ends the proof of Theorem 2. □

3. *Conformal metric changes of the flat torus*

The goal of this section is to show Proposition 1.1. We start with some basic definitions. A metric $\bar{g}$ is conformal to a metric g defined in a manifold M of dimension n if there exists a positive C^∞ function $f : M \longrightarrow M$ such that $\bar{g}(p) = f(p)g(p) \ \forall p \in M$. Writing $\bar{g}(p) = e^{2\sigma(p)}g(p)$ we get the usual formula of the Levi–Civita connection $\bar{\nabla}$ of $\bar{g}$ in terms

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of the Levi–Civita connection ∇ of g:

$$(\bar{\nabla}_X Y)_p = (\nabla_X Y)_p + g(p)(\mathrm{grad}_p(\sigma), X)Y(p) + g(p)(\mathrm{grad}_p(\sigma), Y)X(p) - g(p)(X, Y)\,\mathrm{grad}_p(\sigma)$$

where $\mathrm{grad}_p(\sigma)$ is the gradient vector field of σ at p and X, Y are two differentiable vector fields.

LEMMA 3.1. *Let N be a totally geodesic, embedded submanifold of (M, g). Let $f : M \longrightarrow R$ be a positive C^∞ function such that* $\mathrm{grad}_p(f)$ *is tangent to N at every $p \in N$. Then N is totally geodesic in $(M, \bar{g})$, where $\bar{g}_p = f(p)g_p$.*

Proof. Let $X(M)$, $X(N)$ be the set of differentiable vector fields in M, N, respectively. To show that N is totally geodesic in $(M, \bar{g})$ it is enough to show that the connection $\bar{\nabla} : X(M) \times X(M) \longrightarrow X(M)$ sends $X(N) \times X(N)$ into $X(N)$. So let $X, Y \in X(M)$ be two differentiable vector fields such that $X|_N \in X(N)$, $Y|_N \in X(N)$. We will show that the vector field $\bar{\nabla}_X Y$ is always tangent to N. This vector field has four components, according to the conformal connection formula. The first one is $\nabla_X Y$ which is tangent to N since N is totally geodesic in (M, g). The second and third components are scalar multiples of the vector fields X and Y which are assumed to be tangent to N. Finally, the last term in the formula is a multiple of $\mathrm{grad}_p(\sigma)$, where $\sigma(p) = \frac{1}{2}\log f(p)$. Since $\mathrm{grad}_p(f) \in T_pN$ for every $p \in N$ the same happens to $\mathrm{grad}_p(\sigma)$. □

Lemma 3.1 gives us a very simple method to construct conformal perturbations that preserve totally geodesic immersed submanifolds: it is enough to consider factor functions which are critical at the points of the submanifold. We will first consider two simple cases of immersed submanifolds. The general construction will be derived from these cases.

Case 1. Consider an embedded submanifold N. Let $S(N) = \cup_{p\in N} S_pN$ be the normal subbundle of N, S_pN the normal subspace at p, and let $\exp_p^\perp : S(N) \longrightarrow M$ be the exponential map restricted to $S(N)$ at $p \in N$. We assume that N is totally geodesic, has compact closure, is diffeomorphic to an open set in R^k for some $k \in N$, and there exists a number $\delta > 0$ such that the set

$$V_\delta(N) = \{q = \exp_p^\perp(v),\ \ p \in N, v \in S(N), d(q, N) < \delta\}$$

is an embedded submanifold of M. Decreasing δ if necessary (for instance, to make $V_\delta(N)$ a subset of a union of normal neighborhoods of the points of N), we can define a coordinate system

$$\Phi : V_\delta(N) \longrightarrow U^k \times U^{n-k}, \quad \Phi(q) = (x_1(q), x_2(q), \dots, x_n(q))$$

where U^i is the open subset $\{(x_1, x_2, \dots, x_i), |x_i| < 1\}$ such that $\Phi(N) = U^k \times \{0\}$ and $D\Phi|_{S_pN} = \{0\} \times R^{n-k}$ for every $p \in N$.

Now, take a C^∞ bump function $f : R^k \longrightarrow R$ with support in U^k, such that $f(x_1, \dots, x_k) = 1$ for every point in U^k with $|x_i| \le \frac{1}{2}$, and $0 < f(z) \le 1$ elsewhere in U^k. Take another bump function $h : R^{n-k} \longrightarrow R$ with support in U^{n-k}, with

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$h(x_1, \ldots, x_{n-k}) = 1$ for every point in U^k with $|x_i| \leq \frac{1}{2}$, and $0 < h(z) \leq 1$ elsewhere in U^{n-k}. For $r > 0$ define $F_r : V_\delta(N) \longrightarrow R$:

$$F_r(q) = rf(x_1(q), x_2(q), \ldots, x_k(q))h(x_{k+1}(q), \ldots, x_n(q)).$$

It is clear that the points of N are critical for F_r restricted to the normal directions of N, since F_r restricted to a normal fibre $(x_1(q), \ldots, x_k(q)) = (c_1, \ldots, c_k)$ is just a multiple of the function $h(z)$ which is critical at $z = 0 \in R^{n-k}$. Notice that given $\epsilon > 0$ and $k \in N$ there exists $r = r(\epsilon, k, \delta)$ such that F_r is a C^k perturbation of the zero function.

Case 2. Ramified submanifolds. Here we consider an immersed totally geodesic flat submanifold N of M satisfying the following conditions.

(1) N is the union of a finite collection $N_1, \ldots, N_l$ of totally geodesic submanifolds each of which is diffeomorphic to a two-dimensional disk.

(2) The submanifolds N_i intersect along an embedded geodesic segment $\gamma : (-L, L) \longrightarrow M$, so N is diffeomorphic to a union of two-dimensional planes with a line in common.

(3) N has compact closure.

Assume that $V_\delta(\gamma)$ is embedded, $V_\delta(\gamma)$ is diffeomorphic to a cylinder for δ small enough. We will construct a factor function with support in $V_\delta(\gamma)$ with the required properties by taking cylindrical coordinates around γ and constructing a real-valued function whose level sets are cylinders having γ as an axis. Let $\{e_i(t)\}$, $t \in (-L, L)$, $i = 1, 2, \ldots, n$, be an orthonormal parallel frame defined along γ with $e_1(t) = \gamma'(t)$. Consider the Fermi coordinate system $\Phi : V_\delta(\gamma) \longrightarrow (-L, L) \times B_\delta$ associated to this frame, where B_δ is the open ball of radius δ in R^{n-1}. Then for every point $q \in V_\delta(\gamma)$ we have that

$$\Phi(q) = (t(q), (\exp^{\perp}_{\gamma(t(q))})^{-1}(q))$$

where $t(q)$ is given by $q \in \exp^{\perp}_{\gamma(t(q))}$. Note that the submanifolds N_i have coordinates

$$\Phi(N_i) = (L, L) \times (-\delta, \delta)v_i$$

where $v_i = v_i(t) \in R^{n-1}$ is a parallel vector field perpendicular to $\gamma'(t) = e_1(t)$. Let $f : R \longrightarrow R$ be a C^∞ bump function with support in $[-1, 1]$ such that $f(x) = 1\ \forall |x| \leq \frac{1}{2}$ and $0 < f(x) \leq 1$ for every $x \in (-1, 1)$. For $r > 0$ define $G_r : V_\delta(\gamma) \longrightarrow M$ by

$$G_r(q) = rf\left(\frac{t(q)}{L}\right) f\left(\frac{d(q, \gamma)}{\delta}\right).$$

Notice that G_r is constant along the $n - 2$ spheres $\{d(q, \gamma) = c\} \cap \{t(q) = t_0\}$ which are perpendicular to the planes N_i. So the gradient of G_r has no nonzero components in the normal directions of the N_i and therefore the N_i's are totally geodesic for the metric $\bar{g} = G_r g$. Again, it is possible to chose $r = r(\delta, k, \epsilon)$ arbitrarily small in order to make $\bar{g}$ an ϵ-C^k perturbation of g.

Proof of Proposition 1.1. Let (M, g) be a compact Riemannian manifold and let (T^2, g) be a flat, totally geodesic immersed torus. Let us chose a closed geodesic γ_0 in T^2. The self-intersections of T^2 take place along closed geodesics $\gamma_1, \gamma_2, \ldots, \gamma_m$ and at points $p_1, \ldots, p_a$. The geodesic γ_0 may belong to the collection of γ_i's for $i \geq 1$. If this is the case, we just reorder the geodesics γ_i to include γ_0 so $0 \leq i \leq m$. Notice that

$$T^2 - \bigcup_{i,j} \gamma_i \cup p_j$$

is a disjoint union of embedded, open submanifolds S_k, $k = 1, 2, \ldots, s$, all with compact closures. Add to the collection $\{p_j\}$ the finite set of points of crossings of the geodesics γ_i. Then the set $\cup_i \gamma_i - \cup_j p_j$ is a collection of embedded geodesic arcs. If there are some closed geodesics in this subset, add some extra points to the collection $\{p_j\}$ in order to break $\cup_i \gamma_i - \cup_j p_j$ into disjoint, embedded geodesic segments. Let B_i be a (disjoint) collection of open balls in M of radius $\frac{1}{4} \inf_{i \neq j}\{d(p_i, p_j)\}$ centered at the points p_i such that $T^2 \cap B_i$ is diffeomorphic to a finite set of planes with a common point. Such a collection B_i exists by the compactness of T^2. By the choice of the B_i's there is no closed geodesic of M contained in $\cup_{i=1}^a B_i$. The set $\cup_{i=0}^m \gamma_i - \cup_{i=1}^a B_i$ is a disjoint union of embedded geodesic arcs β_k, $k = 1, 2, \ldots, b$. If γ_0 belongs to the collection $\{\gamma_i\}$ we can assume, without loss of generality, that β_k, $k = 1, 2, \ldots, b_0$ corresponds to the geodesic subsegments of γ_0 in the collection of the β_i's. Let $\delta > 0$ be such that the collection of tubular neighborhoods V_k of radius δ of the segments β_k, $k = 1, 2, \ldots, b$, is also a disjoint collection of embedded subsets with the property that every connected component S_j meeting V_k contains β_k. Thus

$$V = \cup_{i=1}^a B_i \bigcup \cup_{k=1}^b V_k$$

is a thin tubular neighborhood of the collection of γ_i's satisfying:

(1) each subset V_k is a tubular neighborhood of a geodesic in T^2 such that $T^2 \cap V_k$ is an immersed surface of the type considered in Case 2 above;

(2) the complement $T^2 - V$ is a disjoint collection of embedded closed surfaces $\bar{S}_k \subset S_k$.

Therefore, we can apply the conformal changes constructed in Cases 1 and 2 above. First, take a triangulation in each $\bar{S}_k$ in order to decompose it into a finite union of cells $\bar{P}_{ki}$. Consider a normal neighborhood W_{ki} of radius $\delta/2$ of $\bar{P}_{ki}$ in M and let $F_{r,ki} : M \longrightarrow R$ be a C^∞ function of the type considered in Case 1, supported in W_{ki}, with the property that $F_{r,ki}(p) = r \; \forall p \in \bar{P}_{ki}$ and $F_{r,ki}(p) \geq 0 \; \forall p \in W_{ki} \cap S_k$. Adding up over the i's we obtain a factor function

$$\sum_i F_{r,ki} = F_{r,k} : M \longrightarrow R$$

supported in a normal neighborhood W_k of $\bar{S}_k$ such that $r \leq F_{r,k}(p) \leq 2r$ for every $p \in \bar{S}_k$.

Next, for each connected geodesic segment β_i not in γ_0 consider a function $G_{r,i} : M \longrightarrow R$ of the type observed in Case 2 with support in V_i and such that $G_{r,i}(p) = r$ for every p in a tubular neighborhood of $\beta_i \cap V_i$ of radius $\delta/2$, and $G_{r,i}(p) \geq 0$ for every $p \in V_i$. If β_i is a subset of γ_0, then consider the function $H_{r,i} : M \longrightarrow R$ supported in V_i and given by

$$H_{r,i}(p) = d(p, \beta_i)^l G_{r,i}(p)$$

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where $d(p, \beta_i)^l$ is the distance from p to β_i (which is a well defined function in the normal neighborhood V_i) raised to an even power l to eliminate the singularities of $d(p, \beta_i)$ at the points $p \in \beta_i$. Finally, define $\sigma_r : M \longrightarrow R$ by

$$\sigma_r(p) = \sum_{i,j,k} (F_{r,i}(p) + G_{r,j}(p) + H_{r,k}(p))$$

and observe that:

(1) the support of σ_r is a thin tubular neighborhood of (T^2, g) and $\sigma_r \geq 0$;

(2) we have that $\sigma_r(\gamma_0(t)) = 0$ for every t;

(3) the zeros of σ_r not in γ_0 are contained in the balls B_j.

Item (3) implies that given any geodesic $\alpha : (-\infty, +\infty) \longrightarrow T^2$ different from γ_0 there exists t such that $\sigma_r(\alpha(t)) \neq 0$. In fact, a geodesic $\alpha \neq \gamma_0$ is either one of the γ_i, $i > 0$—and then $G_{r,i}$ is nonzero at some points in α—or it must hit one of the regions S_k since there is no geodesic completely contained in $\cup_i B_i$. In the latter case, the function $\sum_{i,j,k}(F_{r,i} + G_{r,j} + H_{r,k})$ is positive in $\alpha \cap S_k$. Consider the metric

$$g_r(p) = e^{2\sigma_r(p)} g(p).$$

Then (M, g_r) is conformal to (M, g), the torus (T^2, g_r) is totally geodesic by the properties of σ_r, and given $\epsilon > 0$, $k \in N$ there exists $r > 0$ such that g_r is an ϵ-C^k perturbation of g. This concludes the proof of Proposition 1.1.

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