ON THE INTEGRAL DEGREE OF INTEGRAL RING EXTENSIONS

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Abstract   Let $A \subset B$ be an integral ring extension of integral domains with fields of fractions K and L, respectively. The integral degree of $A \subset B$, denoted by $d_A(B)$, is defined as the supremum of the degrees of minimal integral equations of elements of B over A. It is an invariant that lies in between $d_K(L)$ and $\mu_A(B)$, the minimal number of generators of the A-module B. Our purpose is to study this invariant. We prove that it is sub-multiplicative and upper-semicontinuous in the following three cases: if $A \subset B$ is simple; if $A \subset B$ is projective and finite and $K \subset L$ is a simple algebraic field extension; or if A is integrally closed. Furthermore, d is upper-semicontinuous if A is noetherian of dimension 1 and with finite integral closure. In general, however, d is neither sub-multiplicative nor upper-semicontinuous.

1. Introduction

Let $A \subset B$ be an integral ring extension, where A and B are two commutative integral domains with fields of fractions $K = Q(A)$ and $L = Q(B)$, respectively. Then, for any element $b \in B$, there exist $n \geq 1$ and $a_i \in A$, such that

$$b^n + a_1 b^{n-1} + a_2 b^{n-2} + \cdots + a_{n-1} b + a_n = 0.$$

The minimum integer $n \geq 1$ satisfying such an equation is called the *integral degree of b over A* and is denoted by $\mathrm{id}_A(b)$. The supremum, possibly infinite, of all the integral

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degrees of elements of B over A, $\sup\{\mathrm{id}_A(b) \mid b \in B\}$, is called the *integral degree of B over A* and is denoted by $\mathrm{d}_A(B)$.

These notions are indeed very natural. They were explicitly considered in [**3**] and, previously, in a different framework, by Kurosch [**11**], Jacobson [**5**], Kaplansky [**9**] and Levitzki [**12**], and more recently by Voight [**20**].

The goal in [**3**] was to study the uniform Artin–Rees property with respect to the set of regular ideals having a principal reduction. It was proved that the integral degree, in fact, provides a uniform Artin–Rees number for such a set of ideals.

The purpose of the present paper is to investigate more deeply the invariant $\mathrm{d}_A(B)$. We first note that $\mathrm{d}_A(B)$ is between $\mathrm{d}_K(L)$, the integral degree of the corresponding algebraic field extension $K \subset L$, and $\mu_A(B)$, the minimal number of generators of the A-module B. That is,

$$\mathrm{d}_K(L) \leq \mathrm{d}_A(B) \leq \mu_A(B).$$

In a sense, $\mathrm{d}_A(B)$ can play the role of, or just substitute for, one of them. For instance, it is a central question in commutative ring theory whether the integral closure $\overline{A}$ of a domain A is a finitely generated A-module. It is well known that, even for one-dimensional noetherian local domains, $\mu_A(\overline{A})$ might be infinite (see, e.g., [**4**, § 4.9], [**15**, § 33]). However, for one-dimensional noetherian local domains $\mathrm{d}_A(\overline{A})$ is finite [**3**, Proposition 6.5]. Hence, in this situation, $\mathrm{d}_A(B)$ would be an appropriate substitute for $\mu_A(B)$. Another positive aspect of $\mathrm{d}_A(B)$, compared with $\mu_A(B)$, is good behaviour with respect to inclusion, that is, if $B_1 \subset B_2$, then $\mathrm{d}_A(B_1) \leq \mathrm{d}_A(B_2)$, while in general we cannot deduce that $\mu_A(B_1)$ is smaller than or equal to $\mu_A(B_2)$.

Similarly, $\mathrm{d}_K(L)$ is a simplification of $\mathrm{d}_A(B)$. Note that $\mathrm{d}_K(L) \leq [L : K]$, the degree of the algebraic field extension $K \subset L$. We will see that $\mathrm{d}_K(L) = [L : K]$ if and only if $K \subset L$ is a simple algebraic field extension.

Of special interest would be to completely characterise when $\mathrm{d}_A(B)$ reaches its maximal or its minimal value. We will say that $A \subset B$ has *maximal integral degree* when $\mathrm{d}_A(B) = \mu_A(B)$. Similarly, we will say that $A \subset B$ has *minimal integral degree* when $\mathrm{d}_K(L) = \mathrm{d}_A(B)$. Examples of maximal integral degree are simple integral extensions $A \subset B = A[b]$, $b \in B$ (Proposition 2.3(b)). Examples of minimal integral degree occur when $A \subset B$ is a projective finite integral ring extension with corresponding simple algebraic field extension $K \subset L$ or when A is integrally closed (cf. Theorem 5.2 and Proposition 6.1). By a projective, respectively free, finite ring extension $A \subset B$ we mean that B is a finitely generated projective, respectively free, A-module. Moreover, integral ring extensions $A \subset B$ of both at the same time minimal and maximal integral degree are precisely free finite integral ring extensions $A \subset B$ with corresponding simple algebraic field extension $K \subset L$ (see Corollary 5.3).

Considering the multiplicativity property of the degree of algebraic field extensions $K \subset L \subset M$, that is, $[M : K] = [L : K][M : L]$, and the sub-multiplicativity property of the minimal number of generators of integral ring extensions $A \subset B \subset C$, namely, $\mu_A(C) \leq \mu_A(B)\mu_B(C)$, it is natural to ask for the same property of $\mathrm{d}_A(B)$. We will say that the integral degree d is *sub-multiplicative with respect to $A \subset B$* if $\mathrm{d}_A(C) \leq \mathrm{d}_A(B)\mathrm{d}_B(C)$, for every integral ring extension $B \subset C$. We prove that d is sub-multiplicative with respect to $A \subset B$ in the following three situations: if $A \subset B$ has maximal integral degree (e.g., if

$A \subset B$ is simple); if $A \subset B$ is projective and finite and $K \subset L$ is simple; or if A is integrally closed (see Corollaries 3.4, 5.5 and 6.7). Note that in the three cases above, $A \subset B$ has either maximal integral degree, or else minimal integral degree. We do not know an instance in which d is sub-multiplicative with respect to $A \subset B$ and $\mathrm{d}_K(L) < \mathrm{d}_A(B) < \mu_A(B)$. We will prove that d is not sub-multiplicative in general. Taking advantage of an example of Dedekind, we find a non-integrally closed noetherian domain A of dimension 1, with finite integral closure B, where B is the ring of integers of a number field, and a degree-two integral extension C of B, such that $\mathrm{d}_A(C) = 6$, whereas $\mathrm{d}_A(B) = 2$ and $\mathrm{d}_B(C) = 2$. In this particular example, $\mathrm{d}_K(L) = 1$, $\mathrm{d}_A(B) = 2$ and $\mu_A(B) = 3$, so $A \subset B$ is neither of maximal nor of minimal integral degree (see Example 6.8).

Another aspect well worth considering is semicontinuity, taking into account that the minimal number of generators is an upper-semicontinuous function (see, e.g., [**10**, Chapter IV, §2, Corollary 2.6]). Note that if $\mathfrak{p}$ is a prime ideal of A, clearly $A_\mathfrak{p} \subset B_\mathfrak{p}$ is integral. Thus one can regard the integral degree as a function $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$, defined by $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p})$. We will say that the integral degree d is *upper-semicontinuous with respect to* $A \subset B$ if $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$ is an upper-semicontinuous function, that is, if $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{d}(\mathfrak{p}) < n\}$ is open, for all $n \geq 1$. We prove (in Proposition 7.1) that d is upper-semicontinuous with respect to $A \subset B$ in the following two situations: if $A \subset B$ is simple; or if $A \subset B$ has minimal integral degree (e.g., if $A \subset B$ is projective and finite and $K \subset L$ is simple; or if A is integrally closed). Note that in the two cases above, $A \subset B$ has either maximal integral degree, or else minimal integral degree. There is a setting in which we can prove that d is upper-semicontinuous with respect to $A \subset B$, yet $\mathrm{d}_A(B)$ might be different from $\mathrm{d}_K(L)$ and $\mu_A(B)$. This happens when A is a non-integrally closed noetherian domain of dimension 1 with finite integral closure (see Theorem 7.2). However, d is not upper-semicontinuous in general, even if A is a noetherian domain of dimension 1 (see Example 7.4).

The paper is organized as follows. In §2 we recall some definitions and known results given in [**3**]. We also prove that $\mathrm{d}_A(B)$ is a local invariant in the following sense:

$$\mathrm{d}_A(B) = \sup\{\mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(A)\} = \sup\{\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(A)\}.$$

Observe that the analogue for $\mu_A(B)$ is not true in general. Section 3 is mainly devoted to the sub-multiplicativity of the integral degree. Sections 4, 5 and 6 are devoted to the integral degree of, respectively, algebraic field extensions, projective finite ring extensions and integral ring extensions with base ring integrally closed. Finally, §7 is devoted to the upper-semicontinuity of the integral degree.

Notation and conventions. All rings are assumed to be commutative and with unity. Throughout, $A \subset B$ and $B \subset C$ are integral ring extensions. Moreover, we always assume that A, B and C are integral domains, though many definitions and results can easily be extended to the non-integral domain case. The fields of fractions of A, B and C are denoted by $K = Q(A)$, $L = Q(B)$ and $M = Q(C)$, respectively. The integral closure of A in $K = Q(A)$ is denoted by $\overline{A}$ and is simply called the 'integral closure of A'. In the particular case in which A, B and C are fields, we write $A = K$, $B = L$ and $C = M$. Whenever $\{x_1, \ldots, x_r\} \subset N$ is a generating set of an A-module N, we will write $N = \langle x_1, \ldots, x_r \rangle_A$. The minimal number of generators of N as an A-module, understood as the minimum of the cardinalities of generating sets of N, is denoted by $\mu_A(N)$.

2. Preliminaries and first properties

We start by recalling and extending some definitions and results from [**3**, §6]. Recall that $A \subset B$ is an integral ring extension of integral domains, and $K = Q(A)$ and $L = Q(B)$ are their fields of fractions.

Definition 2.1. Let $b \in B$. A *minimal degree polynomial* of b over A (which is not necessarily unique) is a monic polynomial $\mathrm{m}(T) = T^n + a_1 T^{n-1} + \cdots + a_{n-1} T + a_n \in A[T]$, $n \geq 1$, with $\mathrm{m}(b) = 0$, and such that there is no other monic polynomial of lower degree in $A[T]$ and vanishing at b. The *integral degree of b over A*, denoted by $\mathrm{id}_A(b)$, is the degree of a minimal degree polynomial $\mathrm{m}(T)$ of b over A. In other words:

$$\mathrm{id}_A(b) = \deg \mathrm{m}(T) = \min\{n \geq 1 \mid b \text{ satisfies an integral equation over } A \text{ of degree } n\}.$$

The *integral degree of B over A* is defined as the value (possibly infinite)

$$\mathrm{d}_A(B) = \sup\{\mathrm{id}_A(b) \mid b \in B\}.$$

Note that $\mathrm{d}_A(B) = 1$ if and only if $A = B$.

We give a first example, which will be completed subsequently (see Corollary 5.6).

Example 2.2. Let B be an integral domain and let G be a finite group acting as automorphisms on B. Let $A = B^G = \{b \in B \mid \sigma(b) = b, \text{ for all } \sigma \in G\}$. Then $A \subset B$ is an integral ring extension and $\mathrm{d}_A(B) \leq \mathrm{o}(G)$, the order of G.

Proof. Let $G = \{\sigma_1, \ldots, \sigma_n\}$. For every $b \in B$, take $p(T) = (T - \sigma_1(b)) \cdots (T - \sigma_n(b))$. Clearly $p(T) \in A[T]$ and $p(b) = 0$. Thus b is integral over A and $\mathrm{id}_A(b) \leq n = \mathrm{o}(G)$. □

The following is a first list of properties of the integral degree mainly proved in [**3**].

Proposition 2.3. *Let $A \subset B$ be an integral ring extension. The following properties hold.*

(a) $\mathrm{d}_A(B) \leq \mu_A(B)$.

(b) *If $A \subset B = A[b]$ is simple, then $\mathrm{id}_A(b) = \mathrm{d}_A(B) = \mu_A(B)$.*

(c) *If S is a multiplicatively closed subset of A, then $S^{-1}A \subset S^{-1}B$ is an integral ring extension and $\mathrm{d}_{S^{-1}A}(S^{-1}B) \leq \mathrm{d}_A(B)$.*

(d) *If $S = A \setminus \{0\}$, then $S^{-1}B = L$.*

(e) $\mathrm{d}_K(L) \leq \mathrm{d}_{A_{\mathfrak{p}}}(B_{\mathfrak{p}}) \leq \mathrm{d}_A(B)$, *for every $\mathfrak{p} \in \mathrm{Spec}(A)$.*

(f) $\mathrm{d}_K(L) \leq [L : K]$.

Proof. (a), (b) and (c) can be found in [**3**, Corollary 6.3, Corollary 6.2 and Proposition 6.8]. For (d), since $K = S^{-1}A \subset S^{-1}B$ is an integral ring extension, then $S^{-1}B$ is a zero-dimensional domain, hence a field, lying inside $L = Q(B)$. Thus $S^{-1}B = L$. Let

us prove (e). Take $\mathfrak{p} \in \mathrm{Spec}(A)$, so $A \setminus \mathfrak{p} \subseteq S$. Since $A \subset B$ is an integral ring extension, then $A_\mathfrak{p} \subset B_\mathfrak{p}$ and

$$K = S^{-1}A = (A_\mathfrak{p} \setminus \{0\})^{-1} A_\mathfrak{p} \subset (A_\mathfrak{p} \setminus \{0\})^{-1} B_\mathfrak{p} = S^{-1}B = L$$

are integral ring extensions. Applying (c) twice, we get (e). Finally, applying (d) and (a), one has $\mathrm{d}_K(L) = \mathrm{d}_{S^{-1}A}(S^{-1}B)) \leq \mu_{S^{-1}A}(S^{-1}B) = \mu_K(L) = [L : K]$, which proves (f). $\qquad\square$

Notation 2.4. The following picture can help in reading the paper

$$
\begin{array}{ccc}
\mathrm{d}_K(L) & \leq & \mathrm{d}_A(B) \\
\mathrel{|\wedge} & & \mathrel{|\wedge} \\
[L : K] & \leq & \mu_A(B).
\end{array}
$$

We say that $A \subset B$ has *minimal integral degree* when $\mathrm{d}_K(L) = \mathrm{d}_A(B)$. Similarly, we say that $A \subset B$ has *maximal integral degree* when $\mathrm{d}_A(B) = \mu_A(B)$.

Remark 2.5. By Proposition 2.3(b), $A \subset B$ simple implies $A \subset B$ has maximal integral degree. We will see that the converse is true for finite field extensions (see Proposition 4.2). However, in general, $A \subset B$ of maximal integral degree does not imply $A \subset B$ simple. Take for instance $A = \mathbb{Z}$ and B the ring of integers of an algebraic number field L, that is, B is the integral closure of $A = \mathbb{Z}$ in L, a finite field extension of the field of rational numbers $K = \mathbb{Q}$. Then $\mathrm{d}_A(B) = \mu_A(B)$ (see Corollary 5.7). Nevertheless, not every ring of integers B is a simple extension of $A = \mathbb{Z}$. We will take advantage of this fact in Example 6.8.

Clearly, there are integral ring extensions of non-maximal integral degree. This can already happen with affine domains, as shown in the next example.

Example 2.6. Let k be a field and t a variable over k. Let $A = k[t^3, t^8, t^{10}]$ and $B = k[t^3, t^4, t^5]$. Then $A \subset B$ is a finite ring extension with $\mathrm{d}_A(B) = 2$ and $\mu_A(B) = 3$.

Proof. Since $k[t^3] \subset A$, then $B = \langle 1, t^4, t^5 \rangle_A$ and $A \subset B$ is a finite ring extension with $\mu_A(B) \leq 3$. If $x = a + bt^4 + ct^5 \in B$, with $a, b, c \in A$, then $x^2 - 2ax \in A$. Therefore $\mathrm{d}_A(B) = 2$. Let us see that $\mu_A(B) = 3$. Suppose that there exist $f, g \in B$ such that $B = \langle f, g \rangle_A$, that is, $1, t^4, t^5 \in \langle f, g \rangle_A$. Write $f = a_0 + t^3 f_1$ and $g = b_0 + t^3 g_1$, with $a_0, b_0 \in k$ and $f_1, g_1 \in k[t]$. Since $1 \in \langle f, g \rangle_A$, one can suppose that $a_0 = 1$ and $b_0 = 0$. Thus $f = 1 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \cdots$ and $g = b_3 t^3 + b_4 t^4 + b_5 t^5 + \cdots$. In particular, every element of $\langle f, g \rangle_A$ is of the form:

$$
\begin{aligned}
&(\lambda_0 + \lambda_3 t^3 + \lambda_6 t^6 + \lambda_8 t^8 + \cdots)(1 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \cdots) \\
&+ (\mu_0 + \mu_3 t^3 + \mu_6 t^6 + \mu_8 t^8 + \cdots)(b_3 t^3 + b_4 t^4 + b_5 t^5 + \cdots) \\
&= (\lambda_0) + (\lambda_3 + \lambda_0 a_3 + \mu_0 b_3) t^3 + (\lambda_0 a_4 + \mu_0 b_4) t^4 + (\lambda_0 a_5 + \mu_0 b_5) t^5 + \cdots.
\end{aligned}
$$

From $t^4 \in \langle f, g \rangle_A$, one deduces that $(\lambda_0 = 0$ and$)$ $b_4 \neq 0$. Hence, one can suppose that $a_4 = 0$. From $1 \in \langle f, g \rangle_A$, it follows that $(\lambda_0 = 1, \mu_0 = 0$ and$)$ $a_5 = 0$. From $t^4 \in \langle f, g \rangle_A$

again, now it follows $b_5 = 0$. But from $t^5 \in \langle f, g \rangle_A$, one must have $b_5 \neq 0$, a contradiction. Hence $\mu_A(B) = 3$. $\qquad\square$

Remark 2.7. In the example above $K = L$ and so $A \subset B$ does not have minimal integral degree. We will prove that if $A \subset B$ is projective and finite with $K \subset L$ simple, or if A is integrally closed, then $A \subset B$ has minimal integral degree (see Theorem 5.2 and Proposition 6.1).

As for the finiteness of the integral degree, we recall the following.

Remark 2.8. There exist one-dimensional noetherian local domains A with integral closure $\overline{A}$ such that $\mathrm{d}_A(\overline{A})$ is finite while $\mu_A(\overline{A})$ is infinite (see [**3**, Proposition 6.5]). There exist one-dimensional noetherian domains A such that $\mathrm{d}_A(\overline{A})$ is infinite (see [**3**, Example 6.6]).

Next we prove that the integral degree coincides with the supremum of the integral degrees of the localizations. (The analogue for $\mu_A(B)$ is not true in general.)

Proposition 2.9. *Let $A \subset B$ be an integral ring extension. For any $b \in B$, there exists a maximal ideal $\mathfrak{m} \in \mathrm{Max}(A)$ such that $\mathrm{id}_A(b) = \mathrm{id}_{A_\mathfrak{m}}(b/1)$. In particular,*

$$\mathrm{d}_A(B) = \sup\{\mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(A)\} = \sup\{\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(A)\}.$$

Furthermore, if $\mathrm{d}_A(B)$ is finite, then there exists $\mathfrak{m} \in \mathrm{Max}(A)$ such that $\mathrm{d}_A(B) = \mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m})$.

Proof. If $\mathrm{id}_A(b) = 1$, then $b \in A$. Thus, for any $\mathfrak{m} \in \mathrm{Max}(A)$, $b/1 \in A_\mathfrak{m}$, so $\mathrm{id}_{A_\mathfrak{m}}(b/1) = 1$ and $\mathrm{id}_A(b) = \mathrm{id}_{A_\mathfrak{m}}(b/1)$. Suppose that $\mathrm{id}_A(b) = n \geq 2$. Then

$$A[b]/\langle 1, b, \ldots, b^{n-2} \rangle \neq 0 \quad \text{and} \quad A[b]/\langle 1, b, \ldots, b^{n-1} \rangle = 0.$$

Clearly, for every $\mathfrak{p} \in \mathrm{Spec}(A)$, and for every $m \geq 1$,

$$(A[b]/\langle 1, b, \ldots, b^m \rangle)_\mathfrak{p} = A_\mathfrak{p}[b/1]/\langle 1, b/1, \ldots, b^m/1 \rangle.$$

In particular, $A_\mathfrak{p}[b/1]/\langle 1, b/1, \ldots, b^{n-1}/1 \rangle = 0$, for every $\mathfrak{p} \in \mathrm{Spec}(A)$. Since $A[b]/$ $\langle 1, b, \ldots, b^{n-2} \rangle \neq 0$, then there exists a maximal ideal $\mathfrak{m} \in \mathrm{Max}(A)$ with

$$A_\mathfrak{m}[b/1]/\langle 1, b/1, \ldots, b^{n-2}/1 \rangle \neq 0 \quad \text{and} \quad A_\mathfrak{m}[b/1]/\langle 1, b/1, \ldots, b^{n-1}/1 \rangle = 0.$$

Therefore, $\mathrm{id}_{A_\mathfrak{m}}(b/1) = n$ and $\mathrm{id}_A(b) = \mathrm{id}_{A_\mathfrak{m}}(b/1)$. In particular,

$$\mathrm{d}_A(B) \leq \sup\{\mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(A)\} \leq \sup\{\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(A)\}.$$

On the other hand, by Proposition 2.3, $\sup\{\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(A)\} \leq \mathrm{d}_A(B)$. Finally, if $\mathrm{d}_A(B)$ is finite, then there exists $b \in B$ such that $\mathrm{id}_A(b) = \mathrm{d}_A(B)$. We have just shown above that there exists a maximal ideal $\mathfrak{m} \in \mathrm{Max}(A)$ with $\mathrm{id}_A(b) = \mathrm{id}_{A_\mathfrak{m}}(b/1)$. Therefore

$$\mathrm{d}_A(B) = \mathrm{id}_A(b) = \mathrm{id}_{A_\mathfrak{m}}(b/1) \leq \mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m}) \leq \mathrm{d}_A(B)$$

and the equality holds. $\qquad\square$

Remark 2.10. Suppose that $A \subset B$ is finite. Since A is a domain, by generic flatness, there exists $f \in A \setminus \{0\}$ such that $A_f \subset B_f$ is a finite free extension (see, e.g., [**14**, Theorem 22.A]). In particular, for every $\mathfrak{p} \in D(f) = \mathrm{Spec}(A) \setminus V(f)$, $A_\mathfrak{p} \subset B_\mathfrak{p}$ is a finite free ring extension. So $\mathrm{d}_A(B) = \max\{\mathrm{d}_1, \mathrm{d}_2\}$, where $\mathrm{d}_1 = \sup\{\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \mid \mathfrak{p} \in V(f)\}$ and $\mathrm{d}_2 = \sup\{\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \mid A_\mathfrak{p} \subset B_\mathfrak{p} \text{ is free}\}$. Therefore, if one is able to control the integral degree for finite free ring extensions, the calculation of $\mathrm{d}_A(B)$ is reduced to find $\mathrm{d}_1 = \sup\{\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \mid \mathfrak{p} \in V(f)\}$, where $V(f)$ is a proper closed set of $\mathrm{Spec}(A)$. We will come back to this question in Theorem 5.3.

3. Sub-multiplicativity

In this section we study the sub-multiplicativity of the integral degree with respect to $A \subset B$, that is, whether $\mathrm{d}_A(C) \leq \mathrm{d}_A(B)\mathrm{d}_B(C)$ holds for every integral ring extension $B \subset C$. Observe that, in this situation, $A \subset C$ is an integral ring extension too and, by definition, $\mathrm{d}_B(C) \leq \mathrm{d}_A(C)$. We start with a useful criterion to determine possible bounds $\nu \in \mathbb{N}$ in the inequality $\mathrm{d}_A(C) \leq \nu \, \mathrm{d}_B(C)$.

Lemma 3.1. *Let $A \subset B$ and $B \subset C$ be two integral ring extensions. Set $\nu \in \mathbb{N}$. The following conditions are equivalent:*

(i) $\mathrm{d}_A(D) \leq \nu \, \mathrm{d}_B(D)$, *for every ring D such that $B \subseteq D \subseteq C$;*

(ii) $\mathrm{d}_A(D) \leq \nu \, \mathrm{d}_B(D)$, *for every ring D such that $D = B[\alpha]$ for some $\alpha \in C$;*

(iii) $\mathrm{id}_A(\alpha) \leq \nu \, \mathrm{id}_B(\alpha)$, *for every element $\alpha \in C$.*

In particular, if (iii) holds, then $\mathrm{d}_A(C) \leq \nu \, \mathrm{d}_B(C)$.

Proof. Clearly, (i) $\Rightarrow$ (ii). Let $\alpha \in C$; in particular, α is integral over A. Since $A[\alpha] \subset B[\alpha]$, then $\mathrm{id}_A(\alpha) = \mathrm{d}_A(A[\alpha]) \leq \mathrm{d}_A(B[\alpha])$. By hypothesis (ii), $\mathrm{d}_A(B[\alpha]) \leq \nu \, \mathrm{d}_B(B[\alpha]) = \nu \, \mathrm{id}_B(\alpha)$. Therefore, $\mathrm{id}_A(\alpha) \leq \nu \, \mathrm{id}_B(\alpha)$, which proves (ii) $\Rightarrow$ (iii). To see (iii) $\Rightarrow$ (i), take D with $B \subseteq D \subseteq C$ and $\alpha \in D$, which will be integral over B and, hence, integral over A. By hypothesis (iii), $\mathrm{id}_A(\alpha) \leq \nu \, \mathrm{id}_B(\alpha) \leq \nu \, \mathrm{d}_B(D)$. Taking supremum over all $\alpha \in D$, $\mathrm{d}_A(D) \leq \nu \, \mathrm{d}_B(D)$.

Finally, if (iii) holds, then (i) holds for $D = C$, so $\mathrm{d}_A(C) \leq \nu \, \mathrm{d}_B(C)$. $\qquad\square$

The next result shows that we can take $\nu = \mu_A(B)$ as a particular $\nu \in \mathbb{N}$, understanding that if $A \subset B$ is not finite, then $\mu_A(B) = \infty$ and the inequality is trivial.

Theorem 3.2. *Let $A \subset B$ and $B \subset C$ be two integral ring extensions. Then, for every $\alpha \in C$,*

$$\mathrm{id}_A(\alpha) \leq \mu_A(B)\mathrm{id}_B(\alpha).$$

In particular,

$$\mathrm{d}_A(C) \leq \mu_A(B)\mathrm{d}_B(C).$$

Proof. Let $\alpha \in C$, which is integral over B and A. Then

$$\mathrm{id}_A(\alpha) = \mathrm{d}_A(A[\alpha]) \le \mathrm{d}_A(B[\alpha]) \le \mu_A(B[\alpha]) \le \mu_A(B)\mu_B(B[\alpha]) = \mu_A(B)\mathrm{id}_B(\alpha).$$

To finish, apply Lemma 3.1. $\qquad\qquad\square$

Remark 3.3. A proof of Theorem 3.2 using the standard 'determinantal trick' would be as follows. Suppose $B = \langle b_1, \ldots, b_n\rangle_A$, with $\mu_A(B) = n$, and consider $\alpha \in C$ with $\mathrm{id}_B(\alpha) = m$. Let X be the following $nm \times 1$ vector, whose entries form an A-module generating set of $B[\alpha]$,

$$X^\top = (1, \alpha, \ldots, \alpha^{m-1}, b_1, b_1\alpha, \ldots, b_1\alpha^{m-1}, \ldots, b_n, b_n\alpha, \ldots, b_n\alpha^{m-1}).$$

Then there exists an nm square matrix P with coefficients in A, such that $\alpha X = PX$. Therefore, $(\alpha\mathrm{I} - P)X = 0$. Multiplying by the adjugate matrix (that is, the transpose of the cofactor matrix) leads to $Q_P(\alpha) = \det(\alpha\mathrm{I} - P) = 0$, where $Q_P(T)$ is the characteristic polynomial of P (recall that C is a domain). Hence $\mathrm{id}_A(\alpha) \le \deg Q_P(T) = nm = \mu_A(B)\mathrm{id}_B(\alpha)$.

As an immediate consequence of Theorem 3.2, we obtain the sub-multiplicativity of the integral degree with respect to integral ring extensions of maximal integral degree.

Corollary 3.4. *Let $A \subset B$ and $B \subset C$ be two integral ring extensions. If $\mathrm{d}_A(B) = \mu_A(B)$, then*

$$\mathrm{d}_A(C) \le \mathrm{d}_A(B)\mathrm{d}_B(C).$$

To finish this section we recover part of a result shown in [3], but now with a slightly different proof.

Corollary 3.5 (see [3, Proposition 6.7]). *Let $A \subset B$ and $B \subset C$ be two integral ring extensions. Then, for every $\alpha \in C$,*

$$\mathrm{id}_A(\alpha) \le \mathrm{d}_A(B)^{\mathrm{d}_B(C)}\mathrm{id}_B(\alpha).$$

In particular,

$$\mathrm{d}_A(C) \le \mathrm{d}_A(B)^{\mathrm{d}_B(C)}\mathrm{d}_B(C).$$

Furthermore, if $A \subset B$ and $B \subset C$ have finite integral degrees, then $A \subset C$ has finite integral degree.

Proof. Let $\alpha \in C$; in particular, α is integral over B and over A. Let $\mathrm{m}(T)$ be a minimal degree polynomial of α over B, $\mathrm{m}(T) = T^n + b_1 T^{n-1} + \cdots + b_n \in B[T]$, so that $n = \mathrm{id}_B(\alpha) \le \mathrm{d}_B(C)$. Set $E = A[b_1, \ldots, b_n]$, where $A \subseteq E \subseteq B$. Therefore,

$$\mathrm{id}_A(\alpha) = \mathrm{d}_A(A[\alpha]) \le \mathrm{d}_A(E[\alpha]) \le \mu_A(E[\alpha]) \le \mu_A(E)\mu_E(E[\alpha]),$$

where clearly $\mu_A(E) \le \prod_{i=1}^n \mathrm{id}_A(b_i) \le \mathrm{d}_A(B)^{\mathrm{d}_B(C)}$, and $\mu_E(E[\alpha]) = \mathrm{id}_E(\alpha) = \mathrm{id}_B(\alpha)$. To finish, apply Lemma 3.1. $\qquad\square$

4. Integral degree of algebraic field extensions

In this section, we suppose that $A = K$, $B = L$ and $C = M$ are fields. For ease of reading, we begin by recalling some definitions and basic facts (see, e.g., [**2**, Chapter V] and [**6**]).

Reminder 4.1. Let $K \subset L$ be a finite field extension.

- A polynomial is separable if it has no multiple roots (in any field extension). The extension $K \subset L$ is separable if every element of L is the root of a separable polynomial of $K[T]$. A field K is perfect if either has characteristic zero or else, when it has characteristic $p > 0$, every element is a pth power in K. If K is perfect, then $K \subset L$ is separable.

- The primitive element theorem states that a finite separable field extension is simple. Even more, there exists an 'extended' version which affirms that a simple algebraic field extension of a finite separable field extension is again simple (cf. [**6**, III, Chapter I, §11, Theorem 14]).

- Let K_s be the separable closure of K in L, that is, the set of all elements of L which are separable over K. Then K_s is a field and $K \subset K_s$ is a separable extension. Its degree $[K_s : K]$ is called the separable degree and is denoted by $[L : K]_s := [K_s : K]$.

For the rest of the reminder, suppose that K has characteristic $p > 0$ and let K_s be the separable closure of K in L.

- Then $K_s \subset L$ is a purely inseparable field extension, that is, for every element $\alpha \in L$, there exists an integer $m \geq 1$ such that $\alpha^{p^m} \in K_s$. The least such integer m is called the height of α over K_s. Let $\mathrm{ht}_{K_s}(\alpha)$ stand for the height of α over K_s. Set $h = \sup\{\mathrm{ht}_{K_s}(\alpha) \mid \alpha \in L\}$ and call h the *height of the purely inseparable extension* $K_s \subset L$.

- Given $\alpha \in L$ with $\mathrm{ht}_{K_s}(\alpha) = m$, setting $a = \alpha^{p^m}$, one proves that $T^{p^m} - a$ is irreducible in $K_s[T]$ (see, e.g., [**2**, Chapter V, §5]). Thus $[K_s(\alpha) : K_s] = p^m$. Since $K_s \subset L$ is a finite extension, then $L = K_s(\alpha_1, \ldots, \alpha_r)$, where each α_i is purely inseparable over $K_s(\alpha_1, \ldots, \alpha_{i-1})$, $i = 2, \ldots, r$. Hence $[L : K_s] = p^e$, for some $e \geq 1$. Call e the *exponent of the purely inseparable extension* $K_s \subset L$. Note that, since $[K_s(\alpha) : K_s]$ (which is p^m) divides $[L : K_s] = [L : K_s(\alpha)][K_s(\alpha) : K_s]$ (which is p^e), then $m \leq e$ and $h \leq e$.

Our first result characterizes simple finite field extensions as finite field extensions of maximal integral degree.

Proposition 4.2. *Let $K \subset L$ be a finite field extension. Then $K \subset L$ is simple if and only if $\mathrm{d}_K(L) = [L : K]$.*

Proof. Since $K \subset L$ is an algebraic extension, $K(\alpha) = K[\alpha]$, for any $\alpha \in L$. By Proposition 2.3(b), $\mathrm{id}_K(\alpha) = \mathrm{d}_K(K(\alpha)) = [K(\alpha) : K]$. Therefore,

$$[L : K] = [L : K(\alpha)][K(\alpha) : K] = [L : K(\alpha)]\mathrm{d}_K(K(\alpha)) = [L : K(\alpha)]\mathrm{id}_K(\alpha). \quad (1)$$

If $K \subset L$ is simple, then $L = K(\alpha)$, for some $\alpha \in L$. Using (1), it follows that

$$[L : K] = [L : K(\alpha)]\mathrm{d}_K(K(\alpha)) = \mathrm{d}_K(L).$$

Conversely, if $\mathrm{d}_K(L) = [L : K] < \infty$, by definition, there exists $\alpha \in L$ with $\mathrm{id}_K(\alpha) = [L : K]$. By (1) again, it follows that $[L : K(\alpha)] = 1$ and $K \subset L$ is simple. $\qquad\square$

Using the primitive element theorem we obtain the following result.

Corollary 4.3. *Let $K \subset L$ be a finite separable field extension. Then $\mathrm{d}_K(L) = [L : K]$.*

The 'extended' version of the primitive element theorem will be very useful in proving the next result.

Proposition 4.4. *Let $K \subset L$ be a finite field extension. Suppose that K has characteristic $p > 0$ and let K_s be the separable closure of K in L. Set h and e to be the height and exponent, respectively, of the finite purely inseparable field extension $K_s \subset L$. Then the following hold.*

(a) $\mathrm{d}_K(L) = \mathrm{d}_K(K_s)\mathrm{d}_{K_s}(L)$.

(b) *For every $\alpha \in L$, $\mathrm{d}_{K_s}(\alpha) = p^m$, where m is the height of α over K_s.*

(c) $\mathrm{d}_{K_s}(L) = p^h$ *and* $[L : K_s] = p^e$.

(d) $[L : K]_s$ *divides* $\mathrm{d}_K(L)$ *and* $\mathrm{d}_K(L)$ *divides* $[L : K]$. *Concretely,*

$$\mathrm{d}_K(L) = [L : K]_s p^h \text{ and } [L : K] = p^{e-h}\mathrm{d}_K(L).$$

Proof. By Corollary 4.3, $\mathrm{d}_K(K_s) = [K_s : K]$. By Corollary 3.4, $\mathrm{d}_K(L) \le \mathrm{d}_K(K_s)$ $\mathrm{d}_{K_s}(L)$. To see the other inequality, take $\alpha \in L$ with $\mathrm{id}_{K_s}(\alpha) = \mathrm{d}_{K_s}(L)$. By Proposition 2.3(b),

$$\mathrm{id}_{K_s}(\alpha) = \mathrm{d}_{K_s}(K_s[\alpha]) = \mu_{K_s}(K_s[\alpha]).$$

By the extended primitive element theorem, $K \subset K_s[\alpha]$ is a simple algebraic field extension (cf. [6, III, Chapter I, §11, Theorem 14]). Hence, by Proposition 2.3(b), $\mathrm{d}_K(K_s[\alpha]) = [K_s[\alpha] : K]$. Since $K \subset K_s$ is a finite separable extension, by Corollary 4.3, $\mathrm{d}_K(K_s) = [K_s : K]$. Writing all together:

$$\mathrm{d}_K(K_s)\mathrm{d}_{K_s}(L) = [K_s : K]\mathrm{id}_{K_s}(\alpha) = [K_s : K]\mu_{K_s}(K_s[\alpha]) = [K_s[\alpha] : K]$$
$$= \mathrm{d}_K(K_s[\alpha]) \le \mathrm{d}_K(L).$$

This proves (a). Let $\alpha \in L$ with $\mathrm{ht}_{K_s}(\alpha) = m$. Set $a = \alpha^{p^m}$. Then $T^{p^m} - a \in K_s[T]$ is irreducible in $K_s[T]$ and hence it is the minimal polynomial of α over K_s. It follows that

$\mathrm{id}_{K_s}(\alpha) = p^m$. This proves (b). Therefore,

$$\mathrm{d}_{K_s}(L) = \sup\{\mathrm{id}_{K_s}(\alpha) \mid \alpha \in L\} = \sup\{p^{\mathrm{ht}_{K_s}(\alpha)} \mid \alpha \in L\} = p^{\sup\{\mathrm{ht}_{K_s}(\alpha) \mid \alpha \in L\}} = p^h,$$

which proves (c). Finally, (d) follows from (a), (c) and Corollary 4.3 applied repeatedly. Indeed,

$$\mathrm{d}_K(L) = \mathrm{d}_K(K_s)\mathrm{d}_{K_s}(L) = [K_s : K]p^h = [L : K]_s p^h$$

and

$$[L : K] = [L : K_s][K_s : K] = p^e \mathrm{d}_K(K_s) = p^{e-h}\mathrm{d}_{K_s}(L)\mathrm{d}_K(K_s) = p^{e-h}\mathrm{d}_K(L). \qquad \square$$

Here there is an example of a finite field extension of non-maximal integral degree.

Example 4.5. Let $p > 1$ be a prime and $K = \mathbb{F}_p(u_1^p, u_2^p)$, where u_1, u_2 are algebraically independent over $\mathbb{F}_p$. Set $L = K[u_1, u_2]$. Then $K \subset L$ is a finite purely inseparable field extension with $\mathrm{d}_K(L) = p$. However $[L : K] = p^2$.

Proof. Any $\beta \in L$ is of the form $\beta = \sum_{0 \leq i,j \leq p-1} a_{i,j} u_1^i u_2^j$, with $a_{i,j} \in K$. So

$$\beta^p = \sum_{0 \leq i,j \leq p-1} a_{i,j}^p u_1^{ip} u_2^{jp} = \sum_{0 \leq i,j \leq p-1} a_{i,j}^p (u_1^p)^i (u_2^p)^j,$$

which is an element of K. Therefore $\beta^p \in K$ and $\mathrm{id}_K(\beta) \leq p$. Since $\mathrm{id}_K(u_1) = p$, it follows that $\mathrm{d}_K(L) = p$. Since $K \subsetneq K(u_1) \subsetneq L$ are finite field extensions, each one of degree p, by the multiplicative formula for algebraic field extensions, $[L : K] = [L : K(u_1)][K(u_1) : K] = p^2$. $\qquad \square$

Similarly, we obtain an example of an infinite field extension with finite integral degree (see also Remark 2.8).

Example 4.6. Let $p > 1$ be a prime and $K = \mathbb{F}_p(u_1^p, u_2^p, \ldots)$, where $u_1, u_2, \ldots$ are algebraically independent over $\mathbb{F}_p$. Set $L = K[u_1, u_2, \ldots]$. Then $\mathrm{d}_K(L) = p$ but $[L : K] = \infty$.

Now we prove the sub-multiplicativity of the integral degree with respect to an algebraic field extension $K \subset L$.

Theorem 4.7. *Let $K \subset L$ and $L \subset M$ be two algebraic field extensions. Then, for every $\alpha \in M$,*

$$\mathrm{id}_K(\alpha) \leq \mathrm{d}_K(L)\mathrm{id}_L(\alpha).$$

In particular,

$$\mathrm{d}_K(M) \leq \mathrm{d}_K(L)\mathrm{d}_L(M).$$

Proof. We can assume that $\mathrm{d}_K(L)$ and $\mathrm{d}_L(M)$ are finite.

Let $\alpha \in M$ and let $\mathrm{m}(T) = T^n + b_1 T^{n-1} + \cdots + b_n \in L[T]$ be a minimal degree polynomial of α over L. Let h be the height of the purely inseparable field extension $K_s \subset L$,

where K_s is the separable closure of K in L. Let $p = \operatorname{char}(K)$. If K has characteristic 0, we understand that $p^h = 1$. Then $0 = \mathrm{m}(\alpha)^{p^h} = \alpha^{np^h} + b_1^{p^h}\alpha^{(n-1)p^h} + \cdots + b_n^{p^h}$. It follows that α is a root of a monic polynomial in $K_s[T]$ of degree $p^h\mathrm{id}_L(\alpha)$. So we have

$$\mathrm{id}_K(\alpha) = \mathrm{d}_K(K[\alpha]) \le \mathrm{d}_K(K_s[\alpha]) \le \mu_K(K_s[\alpha]) \le \mu_K(K_s) \cdot \mu_{K_s}(K_s[\alpha])$$
$$= \mathrm{d}_K(K_s) \cdot \mathrm{id}_{K_s}(\alpha) \le \mathrm{d}_K(K_s)p^h\mathrm{id}_L(\alpha) = \mathrm{d}_K(L)\mathrm{id}_L(\alpha).$$

To finish, apply Lemma 3.1. $\qquad\square$

Though sub-multiplicative, the integral degree might not be multiplicative, even for two simple algebraic field extensions.

Example 4.8. Let $p > 1$ be a prime and let $K = \mathbb{F}_p(u_1^p, u_2^p)$, where u_1, u_2 are algebraically independent over $\mathbb{F}_p$. Set $L = K[u_1]$ and $M = L[u_2]$. Then $K \subset L$ and $L \subset M$ are two finite field extensions with $\mathrm{d}_K(M) = p$ and $\mathrm{d}_K(L)\mathrm{d}_L(M) = \mathrm{id}_K(u_1)\mathrm{id}_L(u_2) = p^2$ (see Example 4.5 and Proposition 2.3).

However, for finite separable field extensions, multiplicativity holds.

Remark 4.9. Let $K \subset L$ be a finite separable field extension and $L \subset M$ be a simple algebraic field extension. Then

$$\mathrm{d}_K(M) = \mathrm{d}_K(L)\mathrm{d}_L(M).$$

Proof. By the extended primitive element theorem, $K \subset M$ is simple. Hence, by Proposition 4.2, $\mathrm{d}_K(M) = [M : K]$, $\mathrm{d}_K(L) = [L : K]$ and $\mathrm{d}_L(M) = [M : L]$. $\qquad\square$

5. Integral degree of projective finite ring extensions

We return to the general hypotheses: $A \subset B$ and $B \subset C$ are integral ring extensions of integral domains, and K, L and M are their fields of fractions, respectively. In this section we are interested in the integral degree of projective finite ring extensions (by a projective, respectively free, ring extension $A \subset B$ we understand that B is a projective, respectively free, A-module). We begin by recalling some well-known definitions and facts (see, e.g., [10, Chapter IV, §2, 3]).

Reminder 5.1. Let A be a domain and let N be a finitely generated A-module.

- N is a *free* A-module if it has a basis, that is, a linearly independent system of generators. The *rank of a free module* N, $\operatorname{rank}_A(N)$, is defined as the cardinality of (indeed, any) a basis. Clearly, N is free of rank n if and only if $N \cong A^n$. If N is a free A-module, the minimal generating sets are just the bases of N. In particular, $\mu_A(N) = \operatorname{rank}_A(N)$.

- N is a *projective* A-module if there exists an A-module N' such that $N \oplus N'$ is free. One has that N is projective if and only if N is finitely presentable and locally free. The *rank of a projective module* N *at a prime* $\mathfrak{p}$, $\operatorname{rank}_{\mathfrak{p}}(N)$, is defined as

the rank of the free $A_{\mathfrak{p}}$-module $N_{\mathfrak{p}}$, that is, $\operatorname{rank}_{\mathfrak{p}}(N) = \operatorname{rank}_{A_{\mathfrak{p}}}(N_{\mathfrak{p}}) = \mu_{A_{\mathfrak{p}}}(N_{\mathfrak{p}}) = \dim_{k(\mathfrak{p})}(N \otimes k(\mathfrak{p}))$, where $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ stands for the residue field of A at $\mathfrak{p}$.

- If N is projective, then $\mathfrak{p} \mapsto \operatorname{rank}_{\mathfrak{p}}(N)$ is constant (since A is a domain, $\operatorname{Spec}(A)$ is connected) and is simply denoted by $\operatorname{rank}_A(N)$. In particular, on taking the prime ideal (0), then $\operatorname{rank}_A(N) = \mu_K(N \otimes K) = \operatorname{rank}_{\mathfrak{p}}(N)$, for every prime ideal $\mathfrak{p}$. Clearly, when N is free both definitions of rank coincide.

Theorem 5.2. *Let $A \subset B$ be a projective finite ring extension. Then*

$$\mathrm{d}_K(L) \leq \mathrm{d}_A(B) \leq \operatorname{rank}_A(B) = [L : K].$$

If moreover $K \subset L$ is simple, then

$$\mathrm{d}_K(L) = \mathrm{d}_A(B) = \operatorname{rank}_A(B) = [L : K].$$

Proof. By Proposition 2.9, there exists a maximal ideal $\mathfrak{m}$ of A such that $\mathrm{d}_A(B) = \mathrm{d}_{A_{\mathfrak{m}}}(B_{\mathfrak{m}})$. By Proposition 2.3 and using that $B_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$-free and B is A-projective, then

$$\mathrm{d}_K(L) \leq \mathrm{d}_A(B) = \mathrm{d}_{A_{\mathfrak{m}}}(B_{\mathfrak{m}}) \leq \mu_{A_{\mathfrak{m}}}(B_{\mathfrak{m}}) = \operatorname{rank}_{A_{\mathfrak{m}}}(B_{\mathfrak{m}})$$
$$= \operatorname{rank}_A(B) = \mu_K(B \otimes K) = [L : K].$$

To finish, recall that if $K \subset L$ is simple, then $\mathrm{d}_K(L) = [L : K]$ (see Propositions 2.3 or 4.2). □

The next result characterizes finite ring extensions of maximal and minimal integral degree at the same time.

Corollary 5.3. *Let $A \subset B$ be a finite ring extension.*

(a) *$A \subset B$ is free if and only if $[L : K] = \mu_A(B)$.*

(b) *$A \subset B$ is free and $K \subset L$ is simple if and only if $\mathrm{d}_K(L) = \mathrm{d}_A(B) = \mu_A(B)$.*

Proof. If $A \subset B$ is free, then $\mu_A(B) = \operatorname{rank}_A(B) = [L : K]$ (see Reminder 5.1 and Theorem 5.2). Reciprocally, suppose that $[L : K] = \mu_A(B)$. Set $\mu_A(B) = n$ and let $u_1, \ldots, u_n$ be a system of generators of the A-module B. Thus, $u_1, \ldots, u_n$ is a system of generators of the K-module L, where $n = [L : K]$ (recall that, if $S = A \setminus \{0\}$, then $K = S^{-1}A$ and $L = S^{-1}B$, cf. Proposition 2.3). Hence, they are a K-basis of L, so K-linearly independent. In particular, $u_1, \ldots, u_n$ are A-linearly independent. Since they also generate B, one concludes that $u_1, \ldots, u_n$ is an A-basis of B and that B is a free A-module. This shows (a). Since $\mathrm{d}_K(L) \leq \mathrm{d}_A(B) \leq \mu_A(B)$ and $\mathrm{d}_K(L) \leq [L : K] \leq \mu_A(B)$ (see Notation 2.4), part (b) follows from part (a) and Proposition 4.2. □

Corollary 5.4. *Let $A \subset B = A[b]$ be a projective simple integral ring extension. Then $A \subset B$ is free and $1, b, \ldots, b^{n-1}$ is a basis, where*

$$n = \operatorname{id}_A(b) = \mathrm{d}_A(B) = \mu_A(B) \quad \text{and} \quad n = \operatorname{id}_K(b/1) = \mathrm{d}_K(L) = [L : K].$$

Proof. If $S = A \setminus \{0\}$, then $K = S^{-1}A$ and $L = S^{-1}B = S^{-1}A[b] = K[b/1]$. Thus $K \subset L = K[b/1]$ is a simple algebraic field extension. By Proposition 2.3, $\mathrm{id}_A(b) = \mathrm{d}_A(B) = \mu_A(B) = n$, say, and $\mathrm{id}_K(b/1) = \mathrm{d}_K(L) = [L : K] = m$, say. By Theorem 5.2 and Corollary 5.3, $n = m$ and $A \subset B$ is free. Since $\{1, b, \ldots, b^{n-1}\}$ is a minimal system of generators of $B = A[b]$, then it is a basis of the free A-module B (see Reminder 5.1). $\square$

Sub-multiplicativity holds in the case of projective finite ring extensions $A \subset B$ with $K \subset L$ being simple.

Corollary 5.5. *Let $A \subset B$ and $B \subset C$ be two finite ring extensions. If $A \subset B$ is projective and $K \subset L$ is simple, then*

$$\mathrm{d}_A(C) \leq \mathrm{d}_A(B)\mathrm{d}_B(C).$$

If moreover, $K \subset L$ is separable, $B \subset C$ is projective and $L \subset M$ is simple, then

$$\mathrm{d}_A(C) = \mathrm{d}_A(B)\mathrm{d}_B(C).$$

Proof. By Proposition 2.9, there exists a maximal ideal $\mathfrak{m}$ of A such that $\mathrm{d}_A(C) = \mathrm{d}_{A_\mathfrak{m}}(C_\mathfrak{m})$. Since $A \subset B$ is projective, then $A_\mathfrak{m} \subset B_\mathfrak{m}$ is free with fields of fractions $Q(A_\mathfrak{m}) = Q(A) = K$ and $Q(B_\mathfrak{m}) = Q(B) = L$, respectively, where $K \subset L$ is simple by hypothesis. By Corollary 5.3, $\mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m}) = \mu_{A_\mathfrak{m}}(B_\mathfrak{m})$. Therefore, by Corollary 3.4 and Proposition 2.3,

$$\mathrm{d}_A(C) = \mathrm{d}_{A_\mathfrak{m}}(C_\mathfrak{m}) \leq \mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m})\mathrm{d}_{B_\mathfrak{m}}(C_\mathfrak{m}) \leq \mathrm{d}_A(B)\mathrm{d}_B(C).$$

As for the second part of the statement, by hypothesis, $A \subset C$ is projective and $K \subset M$ is simple (again, we use the extended primitive element theorem). By Theorem 5.2, $\mathrm{d}_K(L) = \mathrm{d}_A(B)$, $\mathrm{d}_L(M) = \mathrm{d}_B(C)$ and $\mathrm{d}_K(M) = \mathrm{d}_A(C)$. By Remark 4.9, $\mathrm{d}_K(M) = \mathrm{d}_K(L)\mathrm{d}_L(M)$, so $\mathrm{d}_A(C) = \mathrm{d}_A(B)\mathrm{d}_B(C)$. $\square$

Now we can complement Example 2.2. Let $A \subset B$ a ring extension. Let G be a finite group acting as A-algebra automorphisms on B. Define B^G as the subring $B^G = \{b \in B \mid \sigma(b) = b, \text{ for all } \sigma \in G\}$. It is said that $A \subset B$ is a *Galois extension with group G* if $B^G = A$, and for any maximal ideal $\mathfrak{n}$ in B and any $\sigma \in G \setminus \{1\}$, there is a $b \in B$ such that $\sigma(b) - b \notin \mathfrak{n}$ (see, e.g., [8, Definition 4.2.1]).

Corollary 5.6. *Let G be a finite group and let $A \subset B$ be a Galois ring extension with group G. Then $A \subset B$ is a projective finite ring extension and $\mathrm{d}_K(L) = \mathrm{d}_A(B) = [L : K] = \mathrm{o}(G)$.*

Proof. Since $A \subset B$ is a Galois ring extension of domains with group G, then $A \subset B$ is a projective finite ring extension, $K \subset L$ is a Galois field extension with group G and $[L : K] = \mathrm{o}(G)$ (see, e.g., [8, subsequent Remark to Definition 4.2.1 and Lemma 4.2.5]). In particular, $K \subset L$ is separable and hence simple. By Theorem 5.2, $\mathrm{d}_K(L) = \mathrm{d}_A(B) = [L : K] = \mathrm{o}(G)$. $\square$

Next we calculate the integral degree when A is a Dedekind domain and $K \subset L$ is simple, for example, when B is the ring of integers of an algebraic number field (see Remark 2.5).

Corollary 5.7. *Let $A \subset B$ be a finite ring extension. Suppose that A is Dedekind and that $K \subset L$ is simple. Then $A \subset B$ is projective and $\mathrm{d}_K(L) = \mathrm{d}_A(B) = \mathrm{rank}_A(B) = [L : K]$. If moreover A is a principal ideal domain, then $A \subset B$ is free and $\mathrm{d}_K(L) = \mathrm{d}_A(B) = \mu_A(B)$.*

Proof. From the structure theorem of finitely generated modules over a Dedekind domain, and since B is a torsion-free A-module, it follows that $A \subset B$ is a projective finite ring extension (see, e.g., [**16**, Corollary to Theorem 1.32, p. 30]). Since $A \subset B$ is projective finite and $K \subset L$ is simple, then $\mathrm{d}_K(L) = \mathrm{d}_A(B) = \mathrm{rank}_A(B) = [L : K]$ (see Theorem 5.2). Finally, if A is a principal ideal domain, then $A \subset B$ must be free and we apply Corollary 5.3. □

6. Integrally closed base ring

As always, $A \subset B$ and $B \subset C$ are integral ring extensions of domains, and K, L and M are their fields of fractions, respectively. Recall that $\overline{A}$ denotes the integral closure of A in K. In this section we focus our attention on the case where A is integrally closed. We begin by noting that, in such a situation, $A \subset B$ has minimal integral degree.

Proposition 6.1. *Let $A \subset B$ be an integral ring extension. Then, for every $b \in B$, $\mathrm{id}_K(b) = \mathrm{id}_{\overline{A}}(b)$. In particular, if A is integrally closed, then $\mathrm{d}_K(L) = \mathrm{d}_A(B)$.*

Proof. Since $K \supset \overline{A}$, $\mathrm{id}_K(b) \leq \mathrm{id}_{\overline{A}}(b)$. On the other hand, it is well known that the minimal polynomial of b over K has coefficients in $\overline{A}$ (see, e.g., [**1**, Chapter V, §1.3, Corollary to Proposition 11]), which forces $\mathrm{id}_{\overline{A}}(b) \leq \mathrm{id}_K(b)$. So $\mathrm{id}_K(b) = \mathrm{id}_{\overline{A}}(b)$.

Suppose now that A is integrally closed. Then, for every $b \in B$, $\mathrm{id}_A(b) = \mathrm{id}_{\overline{A}}(b) = \mathrm{id}_K(b) \leq \mathrm{d}_K(L)$. Thus $\mathrm{d}_A(B) \leq \mathrm{d}_K(L)$. The equality follows from Proposition 2.3. □

Certainly, $\mathrm{id}_{\overline{A}}(b)$ may not be equal to $\mathrm{id}_A(b)$, as the next example shows.

Example 6.2. Let $A = \mathbb{Z}[\sqrt{-3}]$. Then $K = Q(A) = \mathbb{Q}(\sqrt{-3})$. Let $b = (1 + \sqrt{-3})/2 \in K$. Clearly, b is integral over A, and the minimal polynomial of b over A is $T^2 - T + 1$. Thus $\mathrm{id}_A(b) = 2$, whereas $\mathrm{id}_K(b) = 1$.

Recall that a simple integral ring extension $B = A[b]$ over an integrally closed domain A is free. Indeed, as said above, the minimal polynomial $p(T)$ of b over K has coefficients in A. Therefore $1, b, \ldots, b^{n-1}$ is a set of generators of the A-module $A[b]$ (where $n = \deg p(T)$). Moreover, since they are linearly independent over K, they are also linearly independent over A. The next result, which is a rephrasing of this fact, is obtained as a direct consequence of Proposition 6.1.

Corollary 6.3. *Let $A \subset B$ be a finite ring extension. Suppose that A is an integrally closed domain. Then, $\mathrm{d}_A(B) = \mu_A(B)$ is equivalent to $A \subset B$ free and $K \subset L$ simple. In particular, if $A \subset B$ is simple and A is integrally closed, then $A \subset B$ is free.*

Proof. By Proposition 6.1, one has $\mathrm{d}_K(L) = \mathrm{d}_A(B)$. Thus, $\mathrm{d}_A(B) = \mu_A(B)$ is equivalent to $\mathrm{d}_K(L) = [L : K] = \mu_A(B)$ (see Notation 2.4). The latter is equivalent to $A \subset B$ free and $K \subset L$ simple (see Corollary 5.3). To finish apply Proposition 2.3. $\square$

This corollary suggests how to find a finite integral extension $A \subset B$ with $\mathrm{d}_K(L) = \mathrm{d}_A(B)$ and $[L : K] < \mu_A(B)$. It suffices to take, as in the next example, an extension of number fields $K \subset L$ which does not admit a relative integral basis (see also Final comments §8).

Example 6.4. Let $K = \mathbb{Q}(\sqrt{-14})$ and $L = K(\sqrt{-7})$. Let A be the integral closure of $\mathbb{Z}$ in K and let B be the integral closure of $\mathbb{Z}$ in L. Then $A \subset B$ is a finite integral extension, A is integrally closed, $K \subset L$ is simple, but $A \subset B$ is not free (see [**13**]). Hence, by Corollary 6.3, $\mathrm{d}_A(B) < \mu_A(B)$. Note that $\mathrm{d}_K(L) = \mathrm{d}_A(B) = [L : K] = 2$. Moreover, it is well known that $A = \mathbb{Z}[\sqrt{-14}]$ and $B = \mathbb{Z}[(1 + \sqrt{-7})/2, \sqrt{2}]$ (see, e.g., [**7**, Theorem 9.5]). An easy calculation shows that $B = \langle 1, (1 + \sqrt{-7})/2, \sqrt{2}\rangle_A$. Thus $\mu_A(B) = 3$.

Now, we return to the sub-multiplicativity question.

Theorem 6.5. *Let $A \subset B$ and $B \subset C$ be two integral ring extensions. Then, for every $\alpha \in C$,*

$$\mathrm{id}_A(\alpha) \le \mu_A(\overline{A})\mathrm{d}_A(B)\mathrm{id}_B(\alpha).$$

In particular,

$$\mathrm{d}_A(C) \le \mu_A(\overline{A})\mathrm{d}_A(B)\mathrm{d}_B(C).$$

Proof. Let $\alpha \in C$. Consider the integral extensions $A \subset \overline{A}$ and $\overline{A} \subset \overline{A}[C]$, where $\overline{A}[C]$ stands for the $\overline{A}$-algebra generated by the elements of C. By Theorem 3.2, $\mathrm{id}_A(\alpha) \le \mu_A(\overline{A})\mathrm{id}_{\overline{A}}(\alpha)$. But, by Proposition 6.1, $\mathrm{id}_{\overline{A}}(\alpha) = \mathrm{id}_K(\alpha)$. On the other hand, applying Theorem 4.7 and Proposition 2.3, we have

$$\mathrm{id}_K(\alpha) \le \mathrm{d}_K(L)\mathrm{id}_L(\alpha) \le \mathrm{d}_A(B)\mathrm{id}_B(\alpha).$$

Hence, $\mathrm{id}_A(\alpha) \le \mu_A(\overline{A})\mathrm{d}_A(B)\mathrm{id}_B(\alpha)$. By Lemma 3.1, we are done. $\square$

Remark 6.6. The ring $\overline{A}[C]$ appears in the proof of Theorem 6.5. A natural question is whether this ring is the tensor product $\overline{A} \otimes_A C$. Observe that indeed there is a natural surjective morphism of rings $\overline{A} \otimes_A C \to \overline{A}[C]$. However this morphism is not necessarily an isomorphism. For instance, take $A = k[t^2, t^3]$ and $C = \overline{A}$, where $\overline{A} = k[t]$. So $\overline{A}[C] = \overline{A} = k[t]$. One can check that $\overline{A} \otimes_A C$ is not a domain. Indeed, write $\overline{A} = A[X]/I$, with $I = (X^2 - t^2, t^2 X - t^3, t^3 X - t^4)$. Therefore $\overline{A} \otimes_A C = A[X, Y]/H$, where $H = (X^2 - t^2, t^2 X - t^3, t^3 X - t^4, Y^2 - t^2, t^2 Y - t^3, t^3 Y - t^4)$. Note that $X^2 - Y^2$ is in H, but neither $X - Y$ nor $X + Y$ are in H. Hence $\overline{A} \otimes_A C$ is not a domain and cannot be isomorphic to $\overline{A}[C] = k[T]$, which is a domain.

As an immediate consequence of Theorem 6.5, we get the sub-multiplicativity of the integral degree with respect to $A \subset B$ when A is integrally closed.

Corollary 6.7. *Let $A \subset B$ and $B \subset C$ be two integral ring extensions. Suppose that A is an integrally closed domain. Then*

$$\mathrm{d}_A(C) \le \mathrm{d}_A(B)\mathrm{d}_B(C).$$

However, in the non-integrally closed case, this formula may fail already for noetherian domains of dimension 1, as shown below. To see this, we take advantage of an example due to Dedekind of a non-monogenic number field L. Concretely, we consider B as the ring of integers of L and find A and C such that $\mathrm{d}_A(C) > \mathrm{d}_A(B)\mathrm{d}_B(C)$.

Example 6.8. Let γ_1 be a root of the irreducible polynomial $T^3 - T^2 - 2T - 8 \in \mathbb{Q}[T]$. Let $L = \mathbb{Q}(\gamma_1)$. Let B be the integral closure of $\mathbb{Z}$ in L (i.e., the ring of integers of L). Then:

(a) B is a free $\mathbb{Z}$-module with basis $\{1, \gamma_1, \gamma_2\}$, where $\gamma_2 = (\gamma_1^2 + \gamma_1)/2$;

(b) $\mathrm{d}_{\mathbb{Q}}(L) = \mathrm{d}_{\mathbb{Z}}(B) = \mu_{\mathbb{Z}}(B) = 3$ and the extension $\mathbb{Z} \subset B$ is not simple (L is non-monogenic).

Let $A = \langle 1, 2\gamma_1, 2\gamma_2 \rangle_{\mathbb{Z}} = \{a + b\gamma_1 + c\gamma_2 \in B \mid a, b, c \in \mathbb{Z}, b \equiv c \equiv 0 \pmod 2\}$. Then:

(c) A is a free $\mathbb{Z}$-module and an integral domain with field of fractions $K = Q(A) = L$;

(d) B is the integral closure of A in L, $\mathrm{d}_A(B) = 2$ and $\mu_A(B) = 3$.

Let $C = B[\alpha]$, where α is a root of $p(T) = T^2 + \gamma_1 T + (1 + \gamma_2) \in B[T]$. Then:

(e) $B \subset C$ is an integral extension with $\mathrm{d}_B(C) = 2$ and $\mathrm{d}_A(C) = 6$.

In particular, $\mathrm{d}_A(B)\mathrm{d}_B(C) < \mathrm{d}_A(C) < \mathrm{d}_A(\overline{A})\mathrm{d}_A(B)\mathrm{d}_B(C)$.

Proof. By Corollary 5.7, $\mathbb{Z} \subset B$ is free and $\mathrm{d}_{\mathbb{Q}}(L) = \mathrm{d}_{\mathbb{Z}}(B) = \mu_{\mathbb{Z}}(B)$. Moreover, since $\gamma_1 \in B$ with $\mathrm{id}_{\mathbb{Z}}(\gamma_1) = 3$, then $\mathrm{d}_{\mathbb{Z}}(B) \ge 3$. The proof that $\{1, \gamma_1, \gamma_2\}$ is a free $\mathbb{Z}$-basis of B and that $\mathbb{Z} \subset B$ is not simple is due to Dedekind (see, e.g., [**16**, p. 64]). This proves (a) and (b).

Note that, from the equalities

$$\gamma_1^2 = -\gamma_1 + 2\gamma_2, \qquad \gamma_2^2 = 6 + 2\gamma_1 + 3\gamma_2 \quad \text{and} \quad \gamma_1\gamma_2 = 4 + 2\gamma_2,$$

the product in B can be immediately computed in terms of its $\mathbb{Z}$-basis $\{1, \gamma_1, \gamma_2\}$.

Clearly $\{1, 2\gamma_1, 2\gamma_2\}$ are $\mathbb{Z}$-linearly independent. One can easily check that A is a ring and that $x^2 + x \in A$, for every $x \in B$. Hence, $A \subset B$ is an integral extension with $\mathrm{d}_A(B) = 2$. Moreover, the field of fractions of A is $K = Q(A) = L$, and the integral closure of A in K is B. Observe that $\mu_A(B) \le \mu_{\mathbb{Z}}(B) = 3$. Below we will see that $\mu_A(B) = 3$.

Now let us prove that $\mathrm{d}_B(C) = 2$. One readily checks that the discriminant $\Delta = -\gamma_1^2 - 2\gamma_1 - 4$ of $p(T)$ has norm $N_{L/\mathbb{Q}}(\Delta) = -16$. Since -16 is not a square in $\mathbb{Q}$, then Δ cannot be a square in L. Therefore $p(T)$ is irreducible over L and $\mathrm{d}_B(C) = 2$.

Let $h(T) \in A[T]$ be a minimal degree polynomial of α over A. Since $p(T)$ is the irreducible polynomial of α over L, it follows that $h(T) = p(T)q(T)$, for some $q(T) \in L[T]$. Moreover, $q(T)$ must necessarily belong to $B[T]$, because B is integrally closed in L

(see, e.g., [**1**, Chapter V, §1.3, Proposition 11]). Therefore, $q(T)$ is a monic polynomial in $B[T]$ such that $p(T)q(T) \in A[T]$. An easy computation shows that this implies that $\deg(q(T)) \geq 4$ (note that the existence of such a polynomial $q(T) = T^n + b_1 T^{n-1} + \cdots + b_{n-1}T + b_n$ is equivalent to the solvability in $\mathbb{Z}$ modulo 2 of a certain system of linear equations with coefficients in $\mathbb{Z}$, in the unknowns $a_{ij} \in \mathbb{Z}$, where $b_i = a_{i,1} + a_{i,2}\gamma_1 + a_{i,3}\gamma_2$). Thus, $\mathrm{id}_A(\alpha) = \deg(h(T)) \geq 6$. By Theorem 3.2,

$$6 \leq \mathrm{id}_A(\alpha) \leq \mathrm{d}_A(C) \leq \mu_A(B)\mathrm{d}_B(C) \leq 6.$$

Hence $\mathrm{d}_A(C) = 6$ and $\mu_A(B) = 3$. $\qquad\qquad\square$

Remark 6.9. It is not possible to construct a similar example with B having rank 2 over $\mathbb{Z}$, because $\mathrm{d}_A(B) \leq \mu_A(B) \leq \mathrm{rank}_\mathbb{Z}(B) = 2$ implies $\mathrm{d}_A(B) = \mu_A(B)$ and then, by Corollary 3.4, $\mathrm{d}_A(B) \leq \mathrm{d}_A(B)\mathrm{d}_B(C)$.

7. Upper-semicontinuity

Recall that $A \subset B$ is an integral ring extension of integral domains, and $K = Q(A)$ and $L = Q(B)$ are their fields of fractions. Let $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$ be defined by $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p})$. In this section we study the upper-semicontinuity of d, that is, whether or not,

$$\mathrm{d}^{-1}([n, +\infty)) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{d}(\mathfrak{p}) \geq n\}$$

is a closed set for every $n \geq 1$. There are two cases in which upper-semicontinuity follows easily from our previous results.

Proposition 7.1. *Let $A \subset B$ be an integral ring extension. Then $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$, defined by $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p})$, is upper-semicontinuous in any of the following cases:*

(a) *$A \subset B$ is simple;*

(b) *$A \subset B$ has minimal integral degree (e.g., $A \subset B$ is projective finite and $K \subset L$ is simple; or A is integrally closed).*

Proof. If $A \subset B = A[b]$ is simple, then $A_\mathfrak{p} \subset B_\mathfrak{p} = A_\mathfrak{p}[b/1]$ is simple too, for every $\mathfrak{p} \in \mathrm{Spec}(A)$. By Proposition 2.3, it follows that $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) = \mu_{A_\mathfrak{p}}(B_\mathfrak{p})$. But the minimal number of generators is known to be an upper-semicontinuous function (see, e.g., [**10**, Chapter IV, §2, Corollary 2.6]). This shows case (a). By Proposition 2.3(e), $\mathrm{d}_K(L) \leq \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \leq \mathrm{d}_A(B)$, for every $\mathfrak{p} \in \mathrm{Spec}(A)$. In case (b), that is, if $\mathrm{d}_K(L) = \mathrm{d}_A(B)$, then $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) = \mathrm{d}_K(L)$ is constant, and thus upper-semicontinuous. $\qquad\square$

A possible way to weaken the integrally closed hypothesis is to shrink the conductor $\mathcal{C} = (A : \overline{A})$ of A in its integral closure $\overline{A}$. A first thought would be to suppose that $\mathcal{C}$ is of maximal height. However, with some extra assumptions on A, e.g., A local Cohen–Macaulay, analytically unramified and A not integrally closed, one can prove that the conductor must have height 1 (see, e.g., [**4**, Exercise 12.6]). In this sense, it seems appropriate to start by considering the case when $\dim A = 1$.

Theorem 7.2. *Let $A \subset B$ be an integral ring extension. Suppose that A is a noetherian domain of dimension 1 and with finite integral closure (e.g., A is a Nagata ring). Then $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$, defined by $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p})$, is upper-semicontinuous.*

Proof. If A is integrally closed, the result follows from Proposition 7.1. Thus we can suppose that A is not integrally closed. Since $\overline{A}$ is finitely generated as an A-module, then $\mathcal{C} = (A : \overline{A}) \neq 0$. Since A is a one-dimensional domain, $\mathcal{C}$ has height 1 and any prime ideal $\mathfrak{p}$ containing $\mathcal{C}$ must be minimal over it. Therefore, the closed set $V(\mathcal{C})$ coincides with the set of minimal primes over $\mathcal{C}$, so it is finite. Note that, for any $\mathfrak{p} \in \mathrm{Spec}(A)$, $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \geq \mathrm{d}_K(L)$ (see Proposition 2.3). Moreover, if $\mathfrak{p} \notin V(\mathcal{C})$, then $A_\mathfrak{p} = \overline{A_\mathfrak{p}}$ and $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_K(L)$ (see Proposition 6.1). Now, take $n \geq 1$. If $n > \mathrm{d}_K(L)$, then $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{d}(\mathfrak{p}) \geq n\} \subseteq V(\mathcal{C})$ is a finite set, hence a closed set. If $n \leq \mathrm{d}_K(L)$, then $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{d}(\mathfrak{p}) \geq n\} = \mathrm{Spec}(A)$. Thus, for every $n \geq 1$, $\mathrm{d}^{-1}([n, +\infty))$ is a closed set and $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$ is upper-semicontinuous. $\square$

Remark 7.3. Note that the proof of Theorem 7.2 only uses that $V(\mathcal{C})$ is a finite set of $\mathrm{Spec}(A)$. For instance, it also holds if A is a noetherian local domain of dimension 2 and with finite integral closure $\overline{A}$. Another example where it would work would be the following: let A be the coordinate ring of a reduced and irreducible variety V over a field of characteristic zero. Then the conductor $\mathcal{C}$ contains the Jacobian ideal J. Now J defines the singular locus of V, so if we suppose that V has only isolated singularities, then J is of dimension zero, so $\mathcal{C}$ is of dimension zero also. Hence $V(\mathcal{C})$ is finite (see [**4**, Theorem 4.4.9] and [**19**, Corollary 6.4.1]).

If we skip the condition that $\overline{A}$ be finitely generated, the result may fail. The following example is inspired by [**17**, Example 1.4] (see also [**3**, Example 6.6]).

Example 7.4. There exists a noetherian domain A of dimension 1 with $\mathrm{d}_A(\overline{A}) = 2$, but $\mu_A(\overline{A}) = \infty$, and such that $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$, defined by $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(\overline{A_\mathfrak{p}})$, is not upper-semicontinuous.

Proof. Let $t_1, t_2, \ldots, t_n, \ldots$ be infinitely many indeterminates over a field k. Let

$$R = k[t_1^2, t_1^3, t_2^2, t_2^3, \ldots] \subset D = k[t_1, t_2, \ldots].$$

Clearly $\overline{R} = D$. Note that for $f \in D = k[t_1, t_2, \ldots]$, $f \in R$ if and only if every monomial $\lambda t_1^{i_1} \cdots t_r^{i_r}$ of f has each $i_j = 0$ or $i_j \geq 2$.

For every $n \geq 1$, let $\mathfrak{q}_n = (t_n^2, t_n^3)R$, which is a prime ideal of R of height 1. Note that for $f \in R$, $f \in \mathfrak{q}_n$ if and only if every monomial $\lambda t_1^{i_1} \cdots t_r^{i_r}$ of f has $i_n \geq 2$. It follows that $t_n \notin R_{\mathfrak{q}_n}$, because if $t_n = a/b$, $a, b \in R$ and $b \notin \mathfrak{q}_n$, then every monomial of $a = bt_n$ has each $i_j = 0$ or $i_j \geq 2$, so has $i_n \geq 2$. Therefore, t_n appears in each monomial of b, but since $b \in R$, the exponent of t_n in each monomial of b must be at least 2, so $b \in \mathfrak{q}_n$, a contradiction.

Now, set $R \subset D_n = k[t_1, \ldots, t_{n-1}, t_n^2, t_n^3, t_{n+1}, \ldots] \subset D$ and $S_n = R \setminus \mathfrak{q}_n$, a multiplicatively closed subset of R. Clearly $R_{\mathfrak{q}_n} = S_n^{-1}D_n$.

Claim. Let I be an ideal of R such that $I \subseteq \cup_{n \geq 1}\mathfrak{q}_n$. Then I is contained in some $\mathfrak{q}_j$.

If I is contained in a finite union of $\mathfrak{q}_i$, using the ordinary prime avoidance lemma, we are done. Suppose that I is not contained in any finite union of $\mathfrak{q}_i$ and let us reach a contradiction. Take $f \in I$, $f \neq 0$. Then $f \in k[t_1, \ldots, t_n]$ for some $n \geq 1$ and f is in a finite number of $\mathfrak{q}_i$, corresponding to the variables t_i that appear in every single monomial of f. We can suppose that $f \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$, for some $1 \leq r \leq n$, and $f \notin \mathfrak{q}_i$, for $i > r$. Since $I \not\subset \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_r$, there exists $g \in I$ such that $g \notin \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_r$. Let $h = t_s^2 g \in I$, where $s > n$, so that f and h have no common monomials. Since $\mathfrak{q}_i$ are prime, then $h = t_s^2 g \notin \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_r$. Since $f, h \in I \subseteq \cup_{n \geq 1} \mathfrak{q}_n$, then $f + h \in \mathfrak{q}_m$, for some $m \geq 1$. But since $f \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ and $h \notin \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_r$, then necessarily $m > r$. Thus $f + h \in \mathfrak{q}_m$, where $m > r$. But since f and h have no common monomials, this implies that every monomial of f must contain t_m^2, so $f \in \mathfrak{q}_m$, a contradiction. Hence $I \subseteq \mathfrak{q}_j$ for some j and the Claim is proved. (An alternative proof would follow from [**18**, Proposition 2.5], provided that k is uncountable.)

Let $S = R \setminus \cup_{n \geq 1} \mathfrak{q}_n$, a multiplicatively closed subset of R. Let $A = S^{-1}R$ and $\mathfrak{p}_n = S^{-1}\mathfrak{q}_n$. If Q is a prime ideal of R such that $Q \subseteq \cup_{n \geq 1} \mathfrak{q}_n$, then, by the Claim above, $Q \subseteq \mathfrak{q}_j$, for some $n \geq 1$. In particular, $\mathrm{Spec}(A) = \{(0)\} \cup \{\mathfrak{p}_n \mid n \geq 1\}$, where each $\mathfrak{p}_n$ is finitely generated. Therefore A is a one-dimensional noetherian domain.

For every $n \geq 1$, $A_{\mathfrak{p}_n} = (S^{-1}R)_{S^{-1}\mathfrak{q}_n} = R_{\mathfrak{q}_n} = S_n^{-1}D_n$. Moreover, $t_n = t_n^3/t_n^2$ is in the field of fractions of $A_{\mathfrak{p}_n}$ and $t_n^2 \in A_{\mathfrak{p}_n}$, that is, t_n is integral over $A_{\mathfrak{p}_n}$. Thus

$$A_{\mathfrak{p}_n}[t_n] = (S_n^{-1}D_n)[t_n] = S_n^{-1}D \quad \text{and} \quad \overline{A_{\mathfrak{p}_n}} = \overline{A_{\mathfrak{p}_n}[t_n]} = \overline{S_n^{-1}D} = S_n^{-1}(\overline{D}) = S_n^{-1}D.$$

Hence $\overline{A_{\mathfrak{p}_n}} = A_{\mathfrak{p}_n}[t_n]$. Recall that $t_n \notin R_{\mathfrak{q}_n} = A_{\mathfrak{p}_n}$ and $\mathrm{d}_{A_{\mathfrak{p}_n}}(t_n) \leq 2$. By Proposition 2.3, $\mathrm{d}_{A_{\mathfrak{p}_n}}(\overline{A_{\mathfrak{p}_n}}) = \mathrm{d}_{A_{\mathfrak{p}_n}}(A_{\mathfrak{p}_n}[t_n]) = 2$.

Consider the integral extension $A \subset \overline{A}$ and $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$, defined by $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_{\mathfrak{p}}}(\overline{A_{\mathfrak{p}}}) = \mathrm{d}_{A_{\mathfrak{p}}}(\overline{A_{\mathfrak{p}}})$. We have just shown that, for every $n \geq 1$, $\mathrm{d}(\mathfrak{p}_n) = \mathrm{d}_{A_{\mathfrak{p}_n}}(\overline{A_{\mathfrak{p}_n}}) = 2$. On the other hand, $\mathrm{d}((0)) = \mathrm{d}_{Q(A)}(Q(\overline{A})) = 1$ because $Q(A) = Q(\overline{A})$. Therefore, $\mathrm{d}^{-1}([2, +\infty)) = \mathrm{Spec}(A) \setminus \{(0)\}$, which is not a closed set. Indeed, suppose that $\mathrm{Spec}(A) \setminus \{(0)\} = V(I)$, for some non-zero ideal I. Since A is a one-dimensional noetherian domain, I has height 1 and $V(I)$ is the finite set of associated primes to I. However, $\mathrm{Spec}(A) \setminus \{(0)\} = \mathrm{Max}(A)$, which is infinite, a contradiction. So $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$ is not upper-semicontinuous. $\qquad\square$

Remark 7.5. Contrary to the upper-semicontinuity, sub-multiplicativity does not work for one-dimensional noetherian domains with finite integral closure. See Example 6.8, where A was a noetherian domain of dimension 1 and with finite integral closure.

8. Final comments

We finish the paper by mentioning some points that we think would be worth clarifying. To simplify, suppose that $A \subset B$ and $B \subset C$ are two finite ring extensions, where, as always, A and B are two integral domains, and K and L are their fields of fractions, respectively.

(1) We have shown that $A \subset B$ of maximal integral degree implies sub-multiplicativity (cf. Corollary 3.4). Does the same work for minimal integral degree?

(2) We have shown that $A \subset B$ of minimal integral degree implies upper-semicontinuity (cf. Proposition 7.1). Does the same work for maximal integral degree?

(3) We have shown that $A \subset B$ free and $K \subset L$ simple implies $\mathrm{d}_K(L) = \mathrm{d}_A(B)$ (Corollary 5.3). Can we omit the hypothesis $K \subset L$ simple? In other words, does $[L : K] = \mu_A(B)$ imply $\mathrm{d}_K(L) = \mathrm{d}_A(B)$? If so, we would have a 'down-to-up rigidity' in the diagram of Notation 2.4. Note that the 'up-to-down rigidity' is not true (see, e.g., Example 6.4).

(4) Does the condition $\mathrm{d}_A(B) = \mu_A(B)$ localize? In particular, does $\mathrm{d}_A(B) = \mu_A(B)$ imply $\mathrm{d}_K(L) = [L : K]$? That would imply a 'right-to-left rigidity' in the diagram of Notation 2.4. If A is integrally closed, the answer is affirmative. Note that Examples 2.6 and 6.8 affirm that the 'left-to-right rigidity' is not true.

(5) It would be interesting to study the sub-multiplicativity and upper-semicontinuity properties for the specific case of affine domains A and B.

(6) Can one replace $\mu_A(\overline{A})$ by $\mathrm{d}_A(\overline{A})$ in the inequality $\mathrm{d}_A(C) \leq \mu_A(\overline{A})\mathrm{d}_A(B)\mathrm{d}_B(C)$ of Theorem 6.5?

(7) Is the integral degree upper-semicontinuous for Nagata rings of dimension greater than 1?

(8) Is there any clear relationship between $\mathrm{d}_A(B)$ and the pair of numbers $\mathrm{d}_{A/\mathfrak{p}}(B/\mathfrak{p}B)$ and $\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p})$? An affirmative answer could be useful in recursive arguments.

(9) Upper-semicontinuity does not imply sub-multiplicativity. We wonder to what extent sub-multiplicativity could imply upper-semicontinuity.

Liam O'Carroll, our friend and coauthor, died aged 72 years, of illness, on October 25, 2017. We greatly miss him.

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