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Computation of critical groups in elliptic boundary-value problems where the asymptotic limits may not exist

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We compute critical groups in semilinear elliptic boundary-value problems in which the nonlinear term may fail to have asymptotic limits at zero and at infinity. As applications, we prove several new existence results.

1. Introduction

Consider the semilinear elliptic boundary-value problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and f is a Carathéodory function on $\Omega \times \mathbb{R}$ that satisfies

$$f(x,0) = 0, \qquad x \in \Omega, \tag{1.2}$$

$$|f(x,t)| \leqslant C(|t|^{p-1} + 1), \quad x \in \Omega, \quad t \in \mathbb{R} \tag{1.3}$$

for some $p < 2n/(n-2)$. As is well known, solutions of (1.1) are the critical points of the C^1 functional

$$G(u) = \int_\Omega |\nabla u|^2 - 2F(x,u), \quad u \in H_0^1(\Omega), \tag{1.4}$$

where

$$F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s.$$

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Since $f(x,0) \equiv 0$, we have the trivial solution $u(x) \equiv 0$, and computing the critical groups of G at zero and at infinity may yield non-trivial solutions (see, for example, [2] or [8]). These critical groups depend mainly upon the behaviour of the nonlinearity f near zero and infinity, respectively.

When G is C^2, it is well known that the critical groups of a non-degenerate critical point are completely determined by its Morse index. It was observed in [17–19] that even in some degenerate cases where G has a local linking near zero, $C_*(G,0)$ can be computed explicitly via the shifting theorem. For Landesman–Lazer-type problems, $C_*(G,0)$ and $C_*(G,\infty)$ were computed in [2, 5, 6] when G is only C^1, but assuming that the limits $\lim_{t\to 0} f(x,t)/t$ and $\lim_{|t|\to\infty} f(x,t)/t$ exist. In [3, 4, 9, 10, 12], $C_*(G,0)$ were computed for problems with a jumping nonlinearity at zero, i.e. assuming only that the one-sided limits $\lim_{t\to 0^\pm} f(x,t)/t$ exist. Similarly, $C_*(G,\infty)$ were computed in [11] for some resonance problems with a jumping nonlinearity at infinity, i.e. when $\lim_{t\to+\infty} f(x,t)/t$ and $\lim_{t\to-\infty} f(x,t)/t$ are different.

In the present paper we compute $C_*(G,0)$ and $C_*(G,\infty)$ without even assuming that the one-sided limits exist. We assume that f satisfies

$$|f(x,t_1) - f(x,t_2)| \leqslant C(|t_1|^{p-2} + |t_2|^{p-2} + 1)|t_1 - t_2|, \quad t_1, t_2 \in \mathbb{R} \tag{1.5}$$

for some $p \in (2, 2n/(n-2))$, so that G is of class C^{2-0}. Let $\lambda_1 < \lambda_2 < \dots$ denote the distinct Dirichlet eigenvalues of $-\Delta$ on Ω. Our computations include the following as special cases.

Proposition 1.1. *If*

$$\lambda_l \leqslant \frac{f(x,t)}{t} \leqslant \lambda_{l+1}, \quad 0 < |t| \leqslant \delta, \tag{1.6}$$

for some $\delta > 0$, then $C_q(G,0) = \delta_{qd_l}\mathcal{G}$, where d_l is the sum of the multiplicities of $\lambda_1, \dots, \lambda_l$ and $\mathcal{G}$ is the coefficient group.

Proposition 1.2. *If*

$$\lambda_l + \varepsilon \leqslant \frac{f(x,t)}{t} \leqslant \lambda_{l+1} - \varepsilon, \quad |t| \geqslant M \tag{1.7}$$

for some $\varepsilon, M > 0$, then G satisfies the Palais–Smale compactness condition (PS) and $C_q(G,\infty) = \delta_{qd_l}\mathcal{G}$.

Proposition 1.3. *If*

$$\lambda_l + \varepsilon \leqslant \frac{f(x,t)}{t} \leqslant \lambda_{l+1}, \quad 2F(x,t) \leqslant (\lambda_{l+1} - \varepsilon)t^2, \quad |t| \geqslant M, \tag{1.8}$$

for some $\varepsilon, M > 0$, then G satisfies (PS) and $C_q(G,\infty) = \delta_{qd_l}\mathcal{G}$. If

$$\lambda_l \leqslant \frac{f(x,t)}{t} \leqslant \lambda_{l+1} - \varepsilon, \quad 2F(x,t) \geqslant (\lambda_l + \varepsilon)t^2, \quad |t| \geqslant M, \tag{1.9}$$

then G satisfies (PS) and $C_{d_l}(G,\infty) \neq 0$.

Note that (1.6) characterizes (1.1) as double resonant between two consecutive eigenvalues near zero, and (1.8), (1.9) characterize (1.1) as resonant from one side at infinity. Proposition 1.1 improves some results in [13], and proposition 1.2 extends a result in [2], where it was required that $\lim_{|t|\to\infty} f(x,t)/t$ exist and be in $(\lambda_l, \lambda_{l+1})$.

Some immediate applications are as follows.

THEOREM 1.4. *If*

$$\lambda_l \leqslant \frac{f(x,t)}{t} \leqslant \lambda_{l+1}, \quad 0 < |t| \leqslant \delta, \tag{1.10}$$

$$\lambda_m \leqslant \frac{f(x,t)}{t} \leqslant \lambda_{m+1} - \varepsilon, \quad 2F(x,t) \geqslant (\lambda_m + \varepsilon)t^2, \quad |t| \geqslant M \tag{1.11}$$

for some $\delta, \varepsilon, M > 0$ and $l \neq m$, then (1.1) has a non-trivial solution.

THEOREM 1.5. *If*

$$\lambda_l t^2 \leqslant 2F(x,t) \leqslant \lambda_{l+1} t^2, \qquad |t| \leqslant \delta, \tag{1.12}$$

$$\lambda_m + \varepsilon \leqslant \frac{f(x,t)}{t} \leqslant \lambda_{m+1} - \varepsilon, \quad |t| \geqslant M \tag{1.13}$$

for some $\delta, \varepsilon, M > 0$ and $l \neq m$, then (1.1) has a non-trivial solution.

THEOREM 1.6. *If*

$$\lambda_l t^2 \leqslant 2F(x,t) \leqslant \lambda_{l+1} t^2, \quad |t| \leqslant \delta, \tag{1.14}$$

$$\lambda_m + \varepsilon \leqslant \frac{f(x,t)}{t} \leqslant \lambda_{m+1}, \quad 2F(x,t) \leqslant (\lambda_{m+1} - \varepsilon)t^2, \quad |t| \geqslant M \tag{1.15}$$

for some $\delta, \varepsilon, M > 0$ and $l \neq m$, then (1.1) has a non-trivial solution.

We will carry out our critical group computations in §§ 2 and 3 and give more existence theorems for (1.1) in § 4.

2. Critical groups at zero

Throughout this section we assume that 0 is an isolated critical point of G in order to ensure that $C_*(G, 0)$ are defined.

Set $A_l = I - \lambda_l(-\Delta)^{-1}$, let N_{l-1}, $E(\lambda_l)$, M_l denote the negative, zero and positive subspaces of A_l, respectively, and for $a, b \in \mathbb{R}$, let

$$I(u, a, b) = \int_\Omega |\nabla u|^2 - a(u^-)^2 - b(u^+)^2, \tag{2.1}$$

$$\gamma_l(a) = \sup_{\substack{v \in N_l \\ \|v^+\|_{L^2} = 1}} I(v, a, 0), \qquad \Gamma_l(a) = \inf_{\substack{w \in M_l \\ \|w^+\|_{L^2} = 1}} I(w, a, 0), \tag{2.2}$$

where $u^\pm(x) = \max\{\pm u(x), 0\}$. The functions γ_l and Γ_l were introduced in [15], where it was shown that they are continuous, decreasing and satisfy

$$\gamma_l(\lambda_l) = \lambda_l, \qquad \Gamma_l(\lambda_{l+1}) = \lambda_{l+1}, \qquad \Gamma_l \leqslant \gamma_{l+1}. \tag{2.3}$$

Note that

$$I(v, a, \gamma_l(a)) \leqslant 0, \quad v \in N_l, \quad I(w, a, \Gamma_l(a)) \geqslant 0, \quad w \in M_l. \tag{2.4}$$

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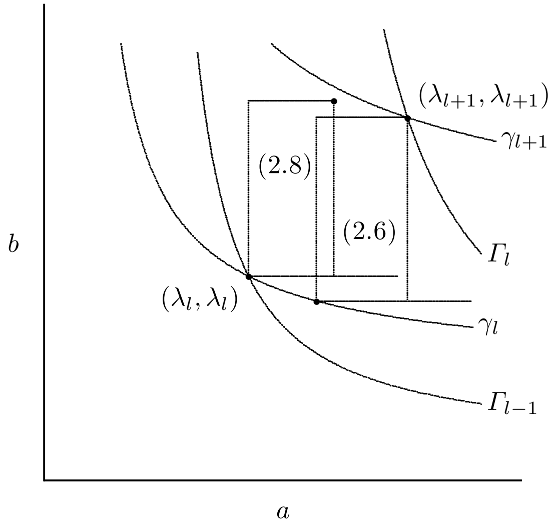

Figure 1.

It was shown in [15] that if $b > \gamma_l(a)$ (respectively, $b < \Gamma_l(a)$), then there is an $\varepsilon > 0$ such that

$$I(v, a, b) \leqslant -\varepsilon\|v\|^2, \quad v \in N_l \quad (\text{respectively, } I(w, a, b) \geqslant \varepsilon\|w\|^2, \quad w \in M_l). \tag{2.5}$$

We shall use these properties of γ_l and Γ_l in our critical group computations.

PROPOSITION 2.1. *If*

$$a(t^-)^2 + \gamma_l(a)(t^+)^2 \leqslant f(x, t)t \leqslant \lambda_{l+1}t^2, \quad |t| \leqslant \delta, \tag{2.6}$$

for some $a \in \mathbb{R}$ *and* $\delta > 0$, *then*

$$C_q(G, 0) = \delta_{qd_l}\mathcal{G}, \tag{2.7}$$

where $d_l = \dim N_l$. *The same conclusion holds if*

$$\lambda_l t^2 \leqslant f(x, t)t \leqslant a(t^-)^2 + b(t^+)^2, \quad |t| \leqslant \delta, \tag{2.8}$$

for some $b < \Gamma_l(a)$.

Conditions (2.6) and (2.8) are illustrated in the figure 1. Proposition 1.1 follows by taking $a = \lambda_l$ in (2.6) and using $\gamma_l(\lambda_l) = \lambda_l$.

Proof of proposition 2.1. Set

$$\tilde{f}(x, t) = f(x, t) - \lambda_{l+1}t, \tag{2.9}$$

write

$$G(u) = (A_{l+1}u, u) - 2\int_\Omega \tilde{F}(x, u), \tag{2.10}$$

where

$$\tilde{F}(x, t) = \int_0^t \tilde{f}(x, s)\,\mathrm{d}s,$$

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and for $u = v + y + w \in N_l \oplus E(\lambda_{l+1}) \oplus M_{l+1}$, set $\hat{u} = -v + y + w$. By (2.6),

$$0 \leqslant -\tilde{f}(x,t)t \leqslant \lambda_{l+1}t^2 - a(t^-)^2 - \gamma_l(a)(t^+)^2, \quad |t| \leqslant \delta, \tag{2.11}$$

so

$$\tilde{f}(x,u)\hat{u} = -\frac{\tilde{f}(x,u)}{u}[v^2 - (y+w)^2] \tag{2.12}$$

$$\leqslant \begin{cases} 0 & \text{if } u\hat{u} \geqslant 0, \\ (\lambda_{l+1} - a)v^2 & \text{if } u < 0,\ \hat{u} > 0, \\ (\lambda_{l+1} - \gamma_l(a))v^2 & \text{if } u > 0,\ \hat{u} < 0 \end{cases} \tag{2.13}$$

$$\leqslant \lambda_{l+1}v^2 - a(v^-)^2 - \gamma_l(a)(v^+)^2, \quad |u| \leqslant \delta. \tag{2.14}$$

Hence

$$\int_{|u|\leqslant\delta} \tilde{f}(x,u)\hat{u} \leqslant \int_\Omega \lambda_{l+1}v^2 - a(v^-)^2 - \gamma_l(a)(v^+)^2. \tag{2.15}$$

On the other hand, there is a $\rho > 0$ such that

$$\|v\| \leqslant \rho \quad \Rightarrow \quad |v(x)| \leqslant \tfrac{1}{3}\delta, \tag{2.16}$$

$$\|y\| \leqslant \rho \quad \Rightarrow \quad |y(x)| \leqslant \tfrac{1}{3}\delta, \tag{2.17}$$

since N_l and $E(\lambda_{l+1})$ are finite dimensional. Suppose that $\|u\| \leqslant \rho$ and $|u(x)| > \delta$. Then

$$|u(x)| \leqslant |w(x)| + |y(x)| + |v(x)| \leqslant |w(x)| + \tfrac{2}{3}\delta, \tag{2.18}$$

so

$$|u(x)|,\ |\hat{u}(x)| < 3|w(x)|. \tag{2.19}$$

Thus

$$\int_{|u|>\delta} |\tilde{f}(x,u)\hat{u}| \leqslant C\int_{|u|>\delta} |u|^{p-1}|\hat{u}| \leqslant C\int_{|u|>\delta} |w|^p \leqslant C\|w\|^p \tag{2.20}$$

by (1.5).

Now consider the homotopy

$$G_t(u) = (1-t)G(u) + t(-\|v\|^2 + \|y\|^2 + \|w\|^2), \quad t \in [0,1]. \tag{2.21}$$

We have

$$\tfrac{1}{2}(G_t'(u), \hat{u}) = (1-t)\left[(A_{l+1}w, w) - (A_{l+1}v, v) - \int_\Omega \tilde{f}(x,u)\hat{u}\right] + t\|\hat{u}\|^2 \tag{2.22}$$

$$\geqslant (1-t)\left[\left(1 - \frac{\lambda_{l+1}}{\lambda_{l+2}}\right)\|w\|^2 - C\|w\|^p - I(v,a,\gamma_l(a))\right] + t\|u\|^2 \tag{2.23}$$

by (2.15) and (2.20). Since $p > 2$ and $I(v,a,\gamma_l(a)) \leqslant 0$, it follows that 0 is the only critical point of G_t in $B_\rho(0)$ if ρ is sufficiently small, so

$$C_q(G,0) = C_q(G_0,0) \cong C_q(G_1,0) = \delta_{qd_l}\mathcal{G} \tag{2.24}$$

by the homotopy invariance of critical groups.

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If (2.8) holds, then we take

$$\tilde{f}(x,t) = f(x,t) - \lambda_l t, \tag{2.25}$$

so that

$$G(u) = (A_l u, u) - 2\int_{\Omega} \tilde{F}(x,u), \tag{2.26}$$

and take $\hat{u} = -v - y + w$ for $u = v + y + w \in N_{l-1} \oplus E(\lambda_l) \oplus M_l$. Now

$$0 \leqslant \tilde{f}(x,t)t \leqslant a(t^-)^2 + b(t^+)^2 - \lambda_l t^2, \quad |t| \leqslant \delta, \tag{2.27}$$

so

$$\tilde{f}(x,u)\hat{u} = \frac{\tilde{f}(x,u)}{u}[w^2 - (v+y)^2] \tag{2.28}$$

$$\leqslant \begin{cases} 0 & \text{if } u\hat{u} \leqslant 0, \\ (a-\lambda_l)w^2 & \text{if } u, \hat{u} < 0, \\ (b-\lambda_l)w^2 & \text{if } u, \hat{u} > 0 \end{cases} \tag{2.29}$$

$$\leqslant a(w^-)^2 + b(w^+)^2 - \lambda_l w^2, \quad |u| \leqslant \delta. \tag{2.30}$$

Now setting

$$G_t(u) = (1-t)G(u) + t(-\|v\|^2 - \|y\|^2 + \|w\|^2), \quad t \in [0,1], \tag{2.31}$$

we see that

$$\tfrac{1}{2}(G_t'(u), \hat{u}) \geqslant (1-t)\left[I(w,a,b) - C\|w\|^p + \left(\frac{\lambda_l}{\lambda_{l-1}} - 1\right)\|v\|^2\right] + t\|u\|^2. \tag{2.32}$$

Using (2.5) to estimate $I(w,a,b)$, the conclusion follows as before. □

We get the following weaker result if we replace (2.6) and (2.8) by their integrated versions.

Proposition 2.2. *If*

$$\underline{a}(t^-)^2 + \gamma_l(\underline{a})(t^+)^2 \leqslant 2F(x,t) \leqslant \overline{a}(t^-)^2 + \overline{b}(t^+)^2, \quad |t| \leqslant \delta, \tag{2.33}$$

for some $\underline{a} \in \mathbb{R}$, $\overline{b} < \Gamma_l(\overline{a})$ *and* $\delta > 0$, *then*

$$C_{d_l}(G,0) \neq 0. \tag{2.34}$$

The same conclusion holds if $\overline{a} = \overline{b} = \lambda_{l+1}$.

Proof. We will show that G has a local linking near zero with respect to the splitting $H_0^1(\Omega) = N_l \oplus M_l$, i.e.

$$G(v) \leqslant 0, \quad v \in N_l, \quad \|v\| \leqslant \rho, \tag{2.35}$$

$$G(w) > 0, \quad w \in M_l, \quad 0 < \|w\| \leqslant \rho, \tag{2.36}$$

for sufficiently small ρ. By [7], equation (2.34) follows from this.

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Since N_l is finite dimensional,

$$G(v) \leqslant I(v, \underline{a}, \gamma_l(\underline{a})) \leqslant 0 \tag{2.37}$$

for $v \in N_l$ with $\|v\|$ sufficiently small. If $\overline{b} < \Gamma_l(\overline{a})$, then, for $w \in M_l$,

$$G(w) \geqslant \int_{\Omega} |\nabla w|^2 - \int_{|w|\leqslant\delta} \overline{a}(w^-)^2 + \overline{b}(w^+)^2 - 2\int_{|w|>\delta} |F(x,w)| \tag{2.38}$$

$$\geqslant I(w, \overline{a}, \overline{b}) - C\int_{|w|>\delta} |w|^p \tag{2.39}$$

$$\geqslant \varepsilon\|w\|^2 - C\|w\|^p \tag{2.40}$$

for some $\varepsilon > 0$, so (2.36) also holds in this case. We refer the reader to [16] for the proof of (2.36) when $\overline{a} = \overline{b} = \lambda_{l+1}$. □

3. Critical groups at infinity

In this section we compute $C_*(G, \infty)$ under the corresponding assumptions at infinity, using the following homotopy invariance theorem for critical groups at infinity from [14] (the proof is included here for the convenience of the reader).

THEOREM 3.1. *Let $G_t, t \in [0,1]$ be a family of C^1 functionals defined on a Hilbert space H, which satisfy (PS), such that G'_t, $\partial_t G_t$ are locally Lipschitz continuous. If there are $a \in \mathbb{R}$, $\delta > 0$ such that*

$$G_t(u) \leqslant a \quad \Rightarrow \quad \|G'_t(u)\| \geqslant \delta \quad \forall t, \tag{3.1}$$

then

$$C_*(G_0, \infty) \cong C_*(G_1, \infty). \tag{3.2}$$

In particular, equation (3.2) holds if there is an $R > 0$ such that

$$\inf_{t\in[0,1],\ \|u\|>R} \|G'_t(u)\| > 0, \qquad \inf_{t\in[0,1],\ \|u\|\leqslant R} G_t(u) > -\infty. \tag{3.3}$$

Proof. Let $\eta(t)u$ be the flow generated by

$$\dot{\eta} = -\frac{\partial_t G_t(\eta)}{\|G'_t(\eta)\|^2} G'_t(\eta), \quad t > 0, \quad \eta(0) = u \in G_0^a. \tag{3.4}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} G_t(\eta(t)u) = (G'_t(\eta), \dot{\eta}) + \partial_t G_t(\eta) = 0, \tag{3.5}$$

so

$$G_t(\eta(t)u) = G_0(u). \tag{3.6}$$

In particular, $G_t(\eta) \leqslant a$ and hence this flow exists by (3.1). It can be reversed by replacing G_t with G_{1-t} in (3.4). Thus $\eta(1)$ is a homeomorphism of G_0^a onto G_1^a, so

$$C_*(G_0, \infty) = H_*(H, G_0^a) \cong H_*(H, G_1^a) = C_*(G_1, \infty). \tag{3.7}$$

□

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First we consider the following 'non-resonance' case.

PROPOSITION 3.2. *If*

$$a(t^-)^2 + b(t^+)^2 \leqslant f(x,t)t \leqslant (\lambda_{l+1} - \varepsilon)t^2, \quad |t| \geqslant M, \tag{3.8}$$

for some $b > \gamma_l(a)$ *and* $\varepsilon, M > 0$, *then* G *satisfies (PS) and*

$$C_q(G,\infty) = \delta_{qd_l}\mathcal{G}. \tag{3.9}$$

The same conclusion holds if

$$(\lambda_l + \varepsilon)t^2 \leqslant f(x,t)t \leqslant a(t^-)^2 + b(t^+)^2, \quad |t| \geqslant M, \tag{3.10}$$

for some $b < \Gamma_l(a)$.

Proof of proposition 3.2. Set

$$\tilde{f}(x,t) = f(x,t) - (\lambda_{l+1} - \varepsilon)t, \tag{3.11}$$

write

$$G(u) = \int_\Omega |\nabla u|^2 - (\lambda_{l+1} - \varepsilon)u^2 - 2\tilde{F}(x,u), \tag{3.12}$$

and set $\hat{u} = -v + w$ for $u = v + w \in N_l \oplus M_l$. Then

$$0 \leqslant -\tilde{f}(x,t)t \leqslant (\lambda_{l+1} - \varepsilon)t^2 - a(t^-)^2 - b(t^+)^2, \quad |t| \geqslant M, \tag{3.13}$$

by (3.8), so

$$\tilde{f}(x,u)\hat{u} = -\frac{\tilde{f}(x,u)}{u}(v^2 - w^2) \tag{3.14}$$

$$\leqslant \begin{cases} 0 & \text{if } u\hat{u} \geqslant 0, \\ (\lambda_{l+1} - \varepsilon - a)v^2 & \text{if } u < 0,\ \hat{u} > 0, \\ (\lambda_{l+1} - \varepsilon - b)v^2 & \text{if } u > 0,\ \hat{u} < 0 \end{cases} \tag{3.15}$$

$$\leqslant (\lambda_{l+1} - \varepsilon)v^2 - a(v^-)^2 - b(v^+)^2, \quad |u| \geqslant M, \tag{3.16}$$

and hence

$$\int_{|u|\geqslant M} \tilde{f}(x,u)\hat{u} \leqslant \int_\Omega (\lambda_{l+1} - \varepsilon)v^2 - a(v^-)^2 - b(v^+)^2. \tag{3.17}$$

On the other hand,

$$\int_{|u|<M} |\tilde{f}(x,u)\hat{u}| \leqslant C\|\hat{u}\|. \tag{3.18}$$

So setting

$$G_t(u) = (1-t)G(u) + t(-\|v\|^2 + \|w\|^2), \quad t \in [0,1], \tag{3.19}$$

we see that

$$\tfrac{1}{2}(G_t'(u), \hat{u}) \geqslant (1-t)\left[\frac{\varepsilon}{\lambda_{l+1}}\|w\|^2 - I(v,a,b) - C\|\hat{u}\|\right] + t\|\hat{u}\|^2. \tag{3.20}$$

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Estimating $I(v,a,b)$ by (2.5), it follows that

$$\inf_{t\in[0,1],\,\|u\|>R} \|G_t'(u)\| > 0 \tag{3.21}$$

for sufficiently large R. Hence G_t satisfies (PS) for each t, and

$$C_q(G,\infty) = C_q(G_0,\infty) \cong C_q(G_1,\infty) = \delta_{qd_l}\mathcal{G} \tag{3.22}$$

by lemma 3.1. The proof when (3.10) holds is similar and is omitted. □

We have the following weaker result in the 'resonance' case.

PROPOSITION 3.3. *If*

$$a(t^-)^2 + b(t^+)^2 \leqslant f(x,t)t \leqslant \lambda_{l+1}t^2, \quad 2F(x,t) \leqslant (\lambda_{l+1}-\varepsilon)t^2, \quad |t| \geqslant M, \tag{3.23}$$

for some $b > \gamma_l(a)$ and $\varepsilon, M > 0$, then G satisfies (PS) and

$$C_q(G,\infty) = \delta_{qd_l}\mathcal{G}. \tag{3.24}$$

If

$$\lambda_l t^2 \leqslant f(x,t)t \leqslant a(t^-)^2 + b(t^+)^2, \quad 2F(x,t) \geqslant (\lambda_l+\varepsilon)t^2, \quad |t| \geqslant M, \tag{3.25}$$

for some $b < \Gamma_l(a)$, then G satisfies (PS) and

$$C_{d_l}(G,\infty) \neq 0. \tag{3.26}$$

Proof of proposition 3.3. We show (3.24) by applying lemma 3.1 to

$$G_t(u) = (1-t)G(u) + t(-\|v\|^2 + \|y\|^2 + \|w\|^2), \tag{3.27}$$

where $u = v + y + w \in N_l \oplus E(\lambda_{l+1}) \oplus M_{l+1}$. We claim that if $G_{t_j}'(u_j) \to 0$ and $\rho_j = \|u_j\| \to \infty$, then $G_{t_j}(u_j) \to \infty$ for a subsequence, which implies both (PS) and (3.1). To see this, let

$$\tilde{u}_j = \frac{u_j}{\rho_j} = \tilde{v}_j + \tilde{y}_j + \tilde{w}_j.$$

Setting $\hat{u}_j = -v_j + y_j + w_j$ and

$$\tilde{f}(x,t) = f(x,t) - \lambda_{l+1}t, \tag{3.28}$$

an argument similar to the one in the proof of proposition 3.2 shows that

$$\int_\Omega \tilde{f}(x,u_j)\hat{u}_j \leqslant \int_\Omega \lambda_{l+1}v_j^2 - a(v_j^-)^2 - b(v_j^+)^2 + C\rho_j. \tag{3.29}$$

Thus

$$\begin{aligned} o(1)\rho_j &= \tfrac{1}{2}(G_{t_j}'(u_j), \hat{u}_j) \\ &\geqslant (1-t_j)((A_{l+1}w_j, w_j) - I(v_j,a,b) - C\rho_j) + t_j\rho_j^2 \\ &\geqslant (1-t_j)\rho_j^2\biggl[\biggl(1 - \frac{\lambda_{l+1}}{\lambda_{l+2}}\biggr)\|\tilde{w}_j\|^2 + \varepsilon'\|\tilde{v}_j\|^2\biggr] - C\rho_j + t_j\rho_j^2 \end{aligned} \tag{3.30}$$

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for some $\varepsilon' > 0$, and it follows that $t_j \to 0$, $\tilde{v}_j, \tilde{w}_j \to 0$. Since $\|\tilde{u}_j\| = 1$, then $\tilde{y}_j \to \tilde{y} \neq 0$ for a subsequence. Hence

$$\begin{aligned} G_{t_j}(u_j) &\geqslant (1-t_j)\int_\Omega [|\nabla u_j|^2 - (\lambda_{l+1}-\varepsilon)u_j^2] - C - t_j\|v_j\|^2 \\ &\geqslant (1-t_j)\rho_j^2[\varepsilon\|\tilde{y}_j\|_{L^2}^2 - C(\|\tilde{w}_j\|^2 + \|\tilde{v}_j\|^2)] - C \to \infty \end{aligned} \tag{3.31}$$

by (3.23).

If (3.25) holds, a similar argument gives (PS). Since

$$(\lambda_l + \varepsilon)t^2 - C \leqslant 2F(x,t) \leqslant a(t^-)^2 + b(t^+)^2 + C \tag{3.32}$$

and $b < \Gamma_l(a)$,

$$G(v) \leqslant -\varepsilon\|v\|_{L^2}^2 + C \to -\infty \quad \text{as } \|v\| \to \infty, \quad v \in N_l, \tag{3.33}$$

$$G(w) \geqslant I(w,a,b) - C \geqslant -C, \quad w \in M_l, \tag{3.34}$$

so (3.26) follows from proposition 3.8 of [1]. □

4. Applications

The following existence theorems for problem (1.1) are immediate consequences of the propositions of §§ 2 and 3, and include theorems 1.4–1.6 as special cases.

THEOREM 4.1. *Assume that*

$$\underline{a}_0(t^-)^2 + \underline{b}_0(t^+)^2 \leqslant f(x,t)t \leqslant \overline{a}_0(t^-)^2 + \overline{b}_0(t^+)^2, \quad |t| \leqslant \delta, \tag{4.1}$$

$$\lambda_m t^2 \leqslant f(x,t)t \leqslant a(t^-)^2 + b(t^+)^2, \quad 2F(x,t) \geqslant (\lambda_m + \varepsilon)t^2, \quad |t| \geqslant M, \tag{4.2}$$

for some $b < \Gamma_m(a)$, $\delta, \varepsilon, M > 0$ and $l \neq m$. Then (1.1) has a non-trivial solution in each of the following cases.

(i) $\underline{b}_0 = \gamma_l(\underline{a}_0)$, $\overline{a}_0 = \overline{b}_0 = \lambda_{l+1}$.

(ii) $\underline{a}_0 = \underline{b}_0 = \lambda_l$, $\overline{b}_0 < \Gamma_l(\overline{a}_0)$.

Proof. By (4.2) and proposition 3.3, $C_{d_m}(G,\infty) \neq 0$, whereas $C_{d_m}(G,0) = 0$ by (4.1), proposition 2.1 and the assumption that $l \neq m$, so it follows that G must have a non-trivial critical point. □

Similarly, we have the following result.

THEOREM 4.2. *Assume that*

$$\underline{a}_0(t^-)^2 + \gamma_l(\underline{a}_0)(t^+)^2 \leqslant 2F(x,t) \leqslant \overline{a}_0(t^-)^2 + \overline{b}_0(t^+)^2, \quad |t| \leqslant \delta, \tag{4.3}$$

$$\underline{a}(t^-)^2 + \underline{b}(t^+)^2 \leqslant f(x,t)t \leqslant \overline{a}(t^-)^2 + \overline{b}(t^+)^2, \quad |t| \geqslant M, \tag{4.4}$$

for some $\overline{b}_0 < \Gamma_l(\overline{a}_0)$ or $\overline{a}_0 = \overline{b}_0 = \lambda_{l+1}$, $\delta, M > 0$ and $l \neq m$. Then (1.1) has a non-trivial solution in each of the following cases.

(i) $\underline{b} > \gamma_m(\underline{a})$, $\overline{a} = \overline{b} < \lambda_{m+1}$.

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(ii) $\underline{a} = \underline{b} > \lambda_m$, $\overline{b} < \Gamma_m(\overline{a})$.

THEOREM 4.3. *Assume that*

$$\underline{a}_0(t^-)^2 + \gamma_l(\underline{a}_0)(t^+)^2 \leqslant 2F(x,t) \leqslant \overline{a}_0(t^-)^2 + \overline{b}_0(t^+)^2, \quad |t| \leqslant \delta, \tag{4.5}$$

$$a(t^-)^2 + b(t^+)^2 \leqslant f(x,t)t \leqslant \lambda_{m+1}t^2, \quad 2F(x,t) \leqslant (\lambda_{m+1} - \varepsilon)t^2, \quad |t| \geqslant M, \tag{4.6}$$

for some $\overline{b}_0 < \Gamma_l(\overline{a}_0)$ *or* $\overline{a}_0 = \overline{b}_0 = \lambda_{l+1}$, $b > \gamma_m(a)$, $\delta, \varepsilon, M > 0$ *and* $l \neq m$. *Then (1.1) has a non-trivial solution.*

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