The Journal of Symbolic Logic
Volume 86, Number 3, September 2021

VAUGHT'S CONJECTURE FOR ALMOST CHAINABLE THEORIES

MILOŠ S. KURILIĆ

Abstract. A structure $\mathbb{Y}$ of a relational language L is called *almost chainable* iff there are a finite set $F \subset Y$ and a linear order $<$ on the set $Y \setminus F$ such that for each partial automorphism φ (i.e., local automorphism, in Fraïssé's terminology) of the linear order $\langle Y \setminus F, < \rangle$ the mapping $\mathrm{id}_F \cup \varphi$ is a partial automorphism of $\mathbb{Y}$. By theorems of Fraïssé and Pouzet, an infinite structure $\mathbb{Y}$ is almost chainable iff the profile of $\mathbb{Y}$ is bounded; namely, iff there is a positive integer m such that $\mathbb{Y}$ has $\leq m$ non-isomorphic substructures of size n, for each positive integer n. A complete first order L-theory $\mathcal{T}$ having infinite models is called almost chainable iff all models of $\mathcal{T}$ are almost chainable and it is shown that the last condition is equivalent to the existence of one countable almost chainable model of $\mathcal{T}$. In addition, it is proved that an almost chainable theory has either one or continuum many non-isomorphic countable models and, thus, the Vaught conjecture is confirmed for almost chainable theories.

§1. Introduction. In this article we confirm Vaught's conjecture for almost chainable theories, extending the result of [7], which concerns the smaller class of monomorphic theories. We recall that the Vaught conjecture is related to the number $I(\mathcal{T}, \omega)$ of non-isomorphic countable models of a countable complete first order theory $\mathcal{T}$. In 1959 Robert Vaught [14] asked if there is a theory $\mathcal{T}$ such that the equality $I(\mathcal{T}, \omega) = \omega_1$ is provable without the use of the continuum hypothesis; since then, the implication $I(\mathcal{T}, \omega) > \omega \Rightarrow I(\mathcal{T}, \omega) = \mathfrak{c}$ is known as Vaught's conjecture.

The rich history of the investigation related to that (still unresolved) conjecture includes a long list of results confirming the conjecture in particular classes of theories (see, for example, the introduction and references of [9]) and, on the other hand, intriguing results concerning the consequences of the existence of counterexamples and the properties of (potential) counterexamples (see, e.g., [1]).

The results of this paper are built on the fundament consisting of two (groups of) results. The first one is the basic Rubin's paper [11] from 1974 in which the Vaught conjecture is confirmed for theories of linear orders with unary predicates; we will use the following result of Rubin (see Theorem 6.12 of [11]).

Theorem 1.1 (Rubin). *If $\mathcal{T}$ is a complete theory of a linear order with a finite set of unary predicates, then $I(\mathcal{T}, \omega) \in \{1, \mathfrak{c}\}$.*

The second group of results is a part of the fundamental work concerning combinatorial properties of relational structures collected in the book of Roland Fraïssé [2]. We will use Fraïssé's results related to almost chainable structures, as well as a theorem of Gibson, Pouzet and Woodrow from [4], describing all linear

Received August 18, 2019.
2020 *Mathematics Subject Classification*. 03C15, 03C50, 03C35, 06A05.
Key words and phrases. Vaught's conjecture, almost chainable structure, almost chainable theory.

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0022-4812/21/8603-0006
DOI:10.1017/jsl.2021.60

https://doi.org/10.1017/jsl.2021.60 Published online by Cambridge University Press

orders which chain an almost chainable structure over a fixed finite set, which is derived from similar results obtained independently by Frasnay [3] and by Hodges et al. [6]. These results are presented in Section 2.

In Section 3 we show that a complete theory $\mathcal{T}$ with infinite models has a countable almost chainable model iff all models of $\mathcal{T}$ are almost chainable and, so, establish the notion of an almost chainable theory. In Section 4 we prove that for each complete almost chainable theory $\mathcal{T}$ having infinite models we have $I(\mathcal{T}, \omega) \in \{1, \mathfrak{c}\}$ and, thus, confirm the Vaught conjecture for such theories.

The results of this paper can be regarded as extensions of the existing results concerning Vaught's conjecture in several ways. First, the role of definability and interpretability in extensions of classes for which Vaught's conjecture is already confirmed is well known (see [12] or [5]); here, taking a specific class $\mathcal{K}$ of colored linear orders, we confirm the conjecture for the theories of structures which are Σ_0-definable in structures from $\mathcal{K}$. More precisely, if $\mathbb{X} = \langle X, \lhd, \{a_0\}, \dots, \{a_{n-1}\}\rangle$ is a linear order with n unary predicates, where $a_0 < \dots < a_{n-1}$ are the first n elements of $\mathbb{X}$, and if $L_n = \langle R, U_0, \dots, U_{n-1}\rangle$ is the corresponding language, then each quantifier-free L_n-formula $\varphi(v_0, \dots, v_{m-1})$ defines an m-ary almost chainable relation on the set X. (In this way we obtain "finite modifications" of several relevant monomorphic relations; e.g., betweenness, cyclic-order and dihedral relation, see [2].) More generally, if $L = \langle R_i : i \in I\rangle$ is any relational language and $\varphi_i(v_0, \dots, v_{\mathrm{ar}(R_i)-1})$, for $i \in I$, are Σ_0-formulas of L_n defining the relations $R_i^{\mathbb{Y}} \subset X^{\mathrm{ar}(R_i)}$, then the structure $\mathbb{Y} := \langle X, \langle R_i^{\mathbb{Y}} : i \in I\rangle\rangle$ is almost chainable and $I(\mathrm{Th}(\mathbb{Y}), \omega) \in \{1, \mathfrak{c}\}$. So we obtain a large jungle of theories satisfying Vaught's conjecture; some simple examples are given in Section 5.

Second, it is known that Vaught's conjecture is equivalent to its restriction to (the theories of) partial orders (see [5, p. 231]) and, for example, it is confirmed for the theories of trees (Steel [13]) and reticles, that is, the posets which do not embed the four element poset N (Schmerl [12]). Here we note that simple infinite almost chainable posets are obtained taking a finite poset $\mathbb{P} = \langle P, <\rangle$ and replacing some element $p \in P$ by an infinite linear order $\mathbb{L}$. By Theorem 4.1, the theory of the resulting poset will be ω-categorical iff $\mathrm{Th}(\mathbb{L})$ has that property; otherwise it will have continuum many non-isomorphic models. We note that in this way we can construct posets of any finite Dushnik–Miller dimension.

Third, one combinatorial invariant of a complete theory $\mathcal{T}$ is its profile function: $\varphi_{\mathcal{T}}(n)$ is the number of non-isomorphic n-element substructures of models of $\mathcal{T}$. So, in [7] Vaught's conjecture is confirmed for monomorphic theories; namely, the theories having the minimal profile ($\varphi_{\mathcal{T}}(n) = 1$, for all $n \in \mathbb{N}$); here we extend that result for the theories with bounded profile ($\varphi_{\mathcal{T}} \leq m$, for some $m \in \mathbb{N}$).

§2. Preliminaries. Almost chainable structures.. Throughout the paper we assume that $L = \langle R_i : i \in I\rangle$ is a relational language, where $\mathrm{ar}(R_i) = n_i \in \mathbb{N}$, for $i \in I$. If Y is a non-empty set and $\mathcal{T} \subset \mathrm{Sent}_L$ an L-theory, then $\mathrm{Mod}_L^{\mathcal{T}}(Y)$ (resp. $\mathrm{Mod}_L(Y)$; $\mathrm{Mod}_L^{\mathcal{T}}$) will denote the set of all models of $\mathcal{T}$ with domain Y (resp. the set of all L-structures with domain Y; the class of all models of $\mathcal{T}$). Let $\mathbb{Y} = \langle Y, \langle R_i^{\mathbb{Y}} : i \in I\rangle\rangle$ be an L-structure. For a non-empty set $H \subset Y$, $\mathbb{H} := \langle H, \langle R_i^{\mathbb{Y}} \restriction H : i \in I\rangle\rangle$ is the corresponding *substructure* of $\mathbb{Y}$. If $J \subset I$, then $L_J := \langle R_i : i \in J\rangle$ is the corresponding *reduction* of L and $\mathbb{Y}|L_J := \langle Y, \langle R_i^{\mathbb{Y}} : i \in J\rangle\rangle$ the corresponding

https://doi.org/10.1017/jsl.2021.60 Published online by Cambridge University Press

reduct of $\mathbb{Y}$. By $[\mathbb{Y}]$ we will denote the class of all L-structures being isomorphic to $\mathbb{Y}$ (the isomorphism type of $\mathbb{Y}$).

If $\mathbb{X} = \langle X, < \rangle$ is a linear order, then $\mathbb{X}^*$ will denote its reverse, $\langle X, <^{-1} \rangle$. By LO_X we denote the set of all linear orders on the set X.

We recall the notions and concepts introduced by Fraïssé which will be used in this paper and fix a convenient notation. For $n \in \mathbb{N}$, by $\mathrm{Age}_n(\mathbb{Y})$ we denote the collection $\{[\mathbb{H}] : H \in [Y]^n\}$ of isomorphism types of n-element substructures of $\mathbb{Y}$ (or, equivalently, $\mathrm{Age}_n(\mathbb{Y}) = \{\mathbb{H} \in \mathrm{Mod}_L(n) : \mathbb{H} \hookrightarrow \mathbb{Y}\}/\cong$). The *age* of $\mathbb{Y}$ is the collection $\mathrm{Age}(\mathbb{Y}) := \bigcup_{n \in \mathbb{N}} \mathrm{Age}_n(\mathbb{Y})$. The function $\varphi_{\mathbb{Y}}$ with the domain $\mathbb{N}$ defined by $\varphi_{\mathbb{Y}}(n) = |\mathrm{Age}_n(\mathbb{Y})|$, for all $n \in \mathbb{N}$, is the *profile* of $\mathbb{Y}$.

By $\mathrm{Pa}(\mathbb{Y})$ we denote the set of all partial automorphisms of $\mathbb{Y}$ (isomorphisms between substructures of $\mathbb{Y}$, or, in Fraïssé's terminology, *local automorphisms*). The L-structure $\mathbb{Y}$ is *freely interpretable* in an L'-structure $\mathbb{X}$ having the same domain iff $\mathrm{Pa}(\mathbb{X}) \subset \mathrm{Pa}(\mathbb{Y})$. We will say that $\mathbb{Y}$ is *simply definable* in $\mathbb{X}$ iff each relation $R_i^{\mathbb{Y}}$ is definable by a quantifier free L'-formula in the structure $\mathbb{X}$.

Almost chainable structures. Let $\mathbb{Y} \in \mathrm{Mod}_L(Y)$, $F \in [Y]^{<\omega}$ and $< \in LO_{Y \setminus F}$. Following Fraïssé (see [2, p. 294]), the structure $\mathbb{Y}$ is called $(F, <)$-*chainable* iff

$$\forall \varphi \in \mathrm{Pa}(\langle Y \setminus F, < \rangle) \quad \mathrm{id}_F \cup \varphi \in \mathrm{Pa}(\mathbb{Y}). \tag{1}$$

The structure $\mathbb{Y}$ is called F-*chainable* if it is $(F, <)$-chainable for some linear order $<$ on $Y \setminus F$. $\mathbb{Y}$ is called *almost chainable* if it is F-chainable for some $F \in [Y]^{<\omega}$.

The following four statements are proved in [2] for $|L| = 1$ and have straightforward generalizations for arbitrary relational language L. So, these results of Fraïssé are cited and used in the paper in such, more general, form.

Generally, if Y is a set, $F \in [Y]^n$, $< \in LO_{Y \setminus F}$ and $F = \{a_0, \dots, a_{n-1}\}$ is an enumeration of the elements of the set F, we introduce the auxiliary language $L_n := \langle R, U_0, \dots, U_{n-1} \rangle$, consisting of new relational symbols, where $\mathrm{ar}(R) = 2$ and $\mathrm{ar}(U_j) = 1$, for $j < n$, and define the linear order $\lhd$ on the set Y and the L_n-structure (in fact, the linear order with n unary predicates) $\mathbb{X}$ by

(L1) $\lhd \restriction (Y \setminus F) = <$,

(L2) $\langle Y, \lhd \rangle = \{a_0\} + \dots + \{a_{n-1}\} + (Y \setminus F)$, and

(L3) $\mathbb{X} := \langle Y, \lhd, \{a_0\}, \dots, \{a_{n-1}\} \rangle$.

FACT 2.1. *Let $\mathbb{Y}$ be an L-structure, $F = \{a_0, \dots, a_{n-1}\} \in [Y]^n$ and $< \in LO_{Y \setminus F}$. If $\lhd$ and $\mathbb{X}$ are defined by (L1)–(L3), then the following conditions are equivalent*:

(a) *$\mathbb{Y}$ is $(F, <)$-chainable*,
(b) *$\mathbb{Y}$ is freely interpretable in $\mathbb{X}$, (that is, $\mathrm{Pa}(\mathbb{X}) \subset \mathrm{Pa}(\mathbb{Y})$), and*
(c) *$\mathbb{Y}$ is simply definable in $\mathbb{X}$.*

PROOF. (a) $\Rightarrow$ (b). Let (1) hold and $f \in \mathrm{Pa}(\mathbb{X})$. Then, since f preserves U_j's, for each $y \in \mathrm{dom}\, f$ and each $j < n$ we have: $y = a_j$ iff $f(y) = a_j$. So, if $a_j \in \mathrm{dom}\, f$, then $f(a_j) = a_j$ and, hence, $f \restriction (F \cap \mathrm{dom}\, f) = \mathrm{id}_{F \cap \mathrm{dom}\, f}$. In addition, if $y \in (\mathrm{dom}\, f) \setminus F$, then $f(y) \notin F$ and, hence, $f[(\mathrm{dom}\, f) \setminus F] \subset Y \setminus F$. So, by (L1), $\varphi := f \restriction ((\mathrm{dom}\, f) \setminus F) \in \mathrm{Pa}(\langle Y \setminus F, < \rangle)$ and by (1) we have $g := \mathrm{id}_F \cup \varphi \in \mathrm{Pa}(\mathbb{Y})$. Finally, since $f = g \restriction \mathrm{dom}\, f$, it follows that $f \in \mathrm{Pa}(\mathbb{Y})$.

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(b) $\Rightarrow$ (c). Let $\mathrm{Pa}(\mathbb{X}) \subset \mathrm{Pa}(\mathbb{Y})$ and $i \in I$. Let $\mathrm{Lit}_{L_n}(v_0, \dots, v_{n_i-1})$ be the set of all L_n-literals η such that $\mathrm{Fv}(\eta) \subset \{v_0, \dots, v_{n_i-1}\}$. For $\bar{y} = \langle y_0, \dots, y_{n_i-1}\rangle \in Y^{n_i}$ let

$$\Phi_{\bar{y}}(\bar{v}) := \Big\{\eta \in \mathrm{Lit}_{L_n}(\bar{v}) : \mathbb{X} \models \eta[\bar{y}]\Big\}.$$

Since $|L_n| < \omega$ we have $|\mathrm{Lit}_{L_n}(\bar{v})| < \omega$, so the relation $\sim \subset Y^{n_i} \times Y^{n_i}$ defined by $\bar{x} \sim \bar{y}$ iff $\Phi_{\bar{x}}(\bar{v}) = \Phi_{\bar{y}}(\bar{v})$ is an equivalence relation with finitely many equivalence classes, say $[\bar{y}_k]$, $k < m$. Let $\varepsilon_{\bar{y}_k}(\bar{v}) := \bigwedge \Phi_{\bar{y}_k}(\bar{v})$, for $k < m$, and let

$$\varphi_i(\bar{v}) := \bigvee \Big\{\varepsilon_{\bar{y}_k}(\bar{v}) : k < m \wedge \exists \bar{x} \in R_i^{\mathbb{Y}}\ \mathbb{X} \models \varepsilon_{\bar{y}_k}[\bar{x}]\Big\}. \tag{2}$$

We prove that

$$\forall \bar{y} \in Y^{n_i}\ \Big(\bar{y} \in R_i^{\mathbb{Y}} \Leftrightarrow \mathbb{X} \models \varphi_i[\bar{y}]\Big). \tag{3}$$

Let $\bar{y} \in R_i^{\mathbb{Y}}$ and $k < m$, where $\bar{y} \sim \bar{y}_k$. Then $\mathbb{X} \models \varepsilon_{\bar{y}_k}[\bar{y}]$ and $\varepsilon_{\bar{y}_k}(\bar{v})$ appears in $\varphi_i(\bar{v})$. Since $\mathbb{X} \models \varepsilon_{\bar{y}_k}[\bar{y}]$ we have $\mathbb{X} \models \varphi_i[\bar{y}]$ as well.

If $\mathbb{X} \models \varphi_i[\bar{y}]$, there are $k < m$ and $\bar{x} \in R_i^{\mathbb{Y}}$ such that $\mathbb{X} \models \varepsilon_{\bar{y}_k}[\bar{x}]$ and $\mathbb{X} \models \varepsilon_{\bar{y}_k}[\bar{y}]$. Thus $\bar{x} \sim \bar{y}$ and, hence, $p := \{\langle x_r, y_r\rangle : r < n_i\} \in \mathrm{Pa}(\mathbb{X}) \subset \mathrm{Pa}(\mathbb{Y}) \subset \mathrm{Pa}(\langle Y, R_i^{\mathbb{Y}}\rangle)$. Thus, since $\bar{x} \in R_i^{\mathbb{Y}} \restriction \{x_r : r < n_i\}$, we have $\bar{y} = p\bar{x} \in R_i^{\mathbb{Y}} \restriction \{y_r : r < n_i\}$ and, hence, $\bar{y} \in R_i^{\mathbb{Y}}$.

(c) $\Rightarrow$ (a). For $i \in I$, let $\varphi_i(v_0, \dots, v_{n_i-1}) \in \mathrm{Form}_{L_n}$ be a Σ_0-formula satisfying (3). For a proof of (1) we take $\varphi \in \mathrm{Pa}(\langle Y \setminus F, <\rangle)$ and show that $f := \mathrm{id}_F \cup \varphi \in \mathrm{Pa}(\mathbb{Y})$. By (L1) we have $\varphi \in \mathrm{Pa}(\langle Y, \lhd\rangle)$; by (L2), $f \in \mathrm{Pa}(\langle Y, \lhd\rangle)$ and, since $f(a_j) = a_j$, for all $j < n$, we obtain $f \in \mathrm{Pa}(\mathbb{X})$. So, for $K := \mathrm{dom}\, f$ and $H := \mathrm{ran}\, f$, denoting by $\mathbb{K}$ and $\mathbb{H}$ the corresponding substructures of $\mathbb{X}$, we have $f \in \mathrm{Iso}(\mathbb{K}, \mathbb{H})$. Now, for $i \in I$ and $\bar{y} \in K^{n_i}$ we have $\bar{y} \in R_i^{\mathbb{Y}}$ iff (by (3)) $\mathbb{X} \models \varphi_i[\bar{y}]$ iff (since φ_i is a Σ_0-formula) $\mathbb{K} \models \varphi_i[\bar{y}]$ iff (since $f \in \mathrm{Iso}(\mathbb{K}, \mathbb{H})$) $\mathbb{H} \models \varphi_i[f\,\bar{y}]$ iff (since φ_i is a Σ_0-formula) $\mathbb{X} \models \varphi_i[f\,\bar{y}]$ iff (by (3)) $f\,\bar{y} \in R_i^{\mathbb{Y}}$. So $f \in \mathrm{Pa}(\langle Y, R_i^{\mathbb{Y}}\rangle)$, for all $i \in I$; thus $f \in \mathrm{Pa}(\mathbb{Y})$ and (1) is true. ⊣

FACT 2.2. *If $\mathbb{Y}$ is an infinite almost chainable L-structure, then there is a minimal finite set $F \subset Y$ such that $\mathbb{Y}$ is F-chainable* (*the kernel of* $\mathbb{Y}$, *in notation* $\mathrm{Ker}(\mathbb{Y})$).

PROOF. For $|L| = 1$, this is 10.9.3 of [2, p. 296]. But the proof of 10.9.3 as well as the proofs of propositions (1), (2) and (3) of 10.9.2, which are used in the proof of 10.9.3 have straightforward generalizations for arbitrary relational language L. We note that the Coherence lemma (2.4.1 of [2, p. 50]) used in the proof of 10.9.2(2) works if, in particular, the language is finite and $\mathcal{I} = [X]^{<\omega}$ for some set X. ⊣

FACT 2.3 (Pouzet). *An infinite L-structure $\mathbb{Y}$ is almost chainable iff its profile is bounded* (*that is, there is a positive integer m such that $\varphi_{\mathbb{Y}}(n) \leq m$, for each $n \in \mathbb{N}$*). (*See* [8, Theorem 1.2 in p. 317] *or* [2, p. 297], *for finite L.*)

FACT 2.4. *Let $\mathbb{Y}$ and $\mathbb{Z}$ be L-structures. If $\mathbb{Y}$ is almost chainable and* $\mathrm{Age}(\mathbb{Z}) \subset \mathrm{Age}(\mathbb{Y})$, *then $\mathbb{Z}$ is almost chainable and* $|\mathrm{Ker}(\mathbb{Z})| \leq |\mathrm{Ker}(\mathbb{Y})|$.

PROOF. For $|L| = 1$, this is Lemma 10.9.6 of [2, p. 297], which has a straightforward generalization for arbitrary relational language L. We note that 10.1.4 of [2, p. 275], which is used in the proof 10.9.6 holds for (in the notation of [2]) R and R' of arbitrary signature and for S' of finite signature. ⊣

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If $\mathbb{Y} \in \mathrm{Mod}_L(Y)$ is an infinite $(F, <)$-chainable structure, then the set

$$\mathcal{L}^F_{\mathbb{Y}} := \Big\{\langle Y \setminus F, \lhd\rangle : \lhd \in LO_{Y\setminus F} \text{ and } \mathbb{Y} \text{ is } (F, \lhd)\text{-chainable}\Big\} \tag{4}$$

is a non-empty set of linear orders and it is easy to see that $\langle Y \setminus F, \lhd\rangle \in \mathcal{L}^F_{\mathbb{Y}}$ iff $\langle Y \setminus F, \lhd\rangle^* \in \mathcal{L}^F_{\mathbb{Y}}$. Theorem 9 of [4] gives the following description of the set $\mathcal{L}^F_{\mathbb{Y}}$.

THEOREM 2.5 (Gibson, Pouzet and Woodrow). *If $\mathbb{Y} \in \mathrm{Mod}_L(Y)$ is an infinite $(F, <)$-chainable L-structure and $\mathbb{L} := \langle Y \setminus F, <\rangle$, then one of the following holds*

(I) $\mathcal{L}^F_{\mathbb{Y}} = LO_{Y\setminus F}$, *that is, each linear order $\lhd$ on $Y \setminus F$ chains $\mathbb{Y}$ over F,*

(II) $\mathcal{L}^F_{\mathbb{Y}} = \bigcup_{\mathbb{L}=\mathbb{I}+\mathbb{F}} \Big\{\mathbb{F} + \mathbb{I}, \mathbb{I}^* + \mathbb{F}^*\Big\}$, *and*

(III) *There are finite subsets K and H of $Y \setminus F$ such that $\mathbb{L} = \mathbb{K} + \mathbb{M} + \mathbb{H}$ and*
$$\mathcal{L}^F_{\mathbb{Y}} = \bigcup_{\substack{\lhd_K \in LO_K \\ \lhd_H \in LO_H}} \Big\{\langle K, \lhd_K\rangle + \mathbb{M} + \langle H, \lhd_H\rangle, \langle H, \lhd_H\rangle^* + \mathbb{M}^* + \langle K, \lhd_K\rangle^*\Big\}.$$

§3. Almost chainable theories. A complete theory $\mathcal{T} \subset \mathrm{Sent}_L$ will be called *almost chainable* iff each model $\mathbb{Y}$ of $\mathcal{T}$ is almost chainable and this notion is established by the following theorem.

THEOREM 3.1. *If $\mathcal{T}$ is a complete L-theory with infinite models, then the following conditions are equivalent*:

(*a*) *All models of $\mathcal{T}$ are almost chainable,*

(*b*) *$\mathcal{T}$ has an almost chainable model, and*

(*c*) *$\mathcal{T}$ has a countable almost chainable model.*

If (*a*) *is true, then there is $n \in \omega$ such that $|\mathrm{Ker}(\mathbb{Y})| = n$, for each model $\mathbb{Y}$ of $\mathcal{T}$.*

A proof of the theorem is given at the end of the section.

CLAIM 3.2. *Let $\mathcal{K}$ be a finite family of non-isomorphic L-structures of size $n \in \mathbb{N}$. Then we have*

(*a*) *For each finite set $J \subset I$ there is an L_J-sentence $\psi_{\mathcal{K},J}$ such that for each $\mathbb{Y} \in \mathrm{Mod}_L$ we have*: $\mathbb{Y} \models \psi_{\mathcal{K},J}$ *iff* $\{\mathbb{H}|L_J : H \in [Y]^n\}/\cong\, = \{[\mathbb{K}|L_J] : \mathbb{K} \in \mathcal{K}\}$ *and*

(*b*) *For the first-order theory $\mathcal{T}_{\mathcal{K}} := \{\psi_{\mathcal{K},J} : J \in [I]^{<\omega}\}$ and each $\mathbb{Y} \in \mathrm{Mod}_L$ we have*: $\mathbb{Y} \models \mathcal{T}_{\mathcal{K}}$ *iff* $\mathrm{Age}_n(\mathbb{Y}) = \{[\mathbb{K}] : \mathbb{K} \in \mathcal{K}\}$.

PROOF. First, without loss of generality we can assume that the domain of each structure $\mathbb{K} \in \mathcal{K}$ is the same set, say K. Let $K = \{x_0, \dots, x_{n-1}\}$ be an enumeration of its elements and $\bar{x} := \langle x_0, \dots, x_{n-1}\rangle$.

(a) For a structure $\mathbb{K} \in \mathcal{K}$, let $\alpha_{\mathbb{K},J}(\bar{v}) := \bigwedge\{\eta \in \mathrm{Lit}_{L_J}(\bar{v}) : \mathbb{K} \models \eta[\bar{x}]\}$, where $\mathrm{Lit}_{L_J}(\bar{v})$ is the set of all literals of L_J with variables in the set $\{v_0, \dots, v_{n-1}\}$. Then for $\mathbb{Y} \in \mathrm{Mod}_L$, $\bar{y} \in Y^n$ and $H := \{y_0, \dots, y_{n-1}\}$, we have $\mathbb{Y} \models \alpha_{\mathbb{K},J}[\bar{y}]$ iff $\{\langle x_k, y_k\rangle : k < n\}$ is an isomorphism from $\mathbb{K}|L_J$ onto $\mathbb{H}|L_J$. If $\pi \in \mathrm{Sym}(n)$ and $\alpha^{\pi}_{\mathbb{K},J}(\bar{v})$ is the formula obtained from $\alpha_{\mathbb{K},J}$ by replacement of v_k by $v_{\pi(k)}$, for all $k < n$, then $\mathbb{Y} \models \alpha^{\pi}_{\mathbb{K},J}[\bar{y}]$ iff $\mathbb{Y} \models \alpha_{\mathbb{K},J}[y_{\pi(0)}, \dots, y_{\pi(n-1)}]$ iff $p_\pi := \{\langle x_k, y_{\pi(k)}\rangle : k < n\}$ is an isomorphism from $\mathbb{K}|L_J$ onto $\mathbb{H}|L_J$. So, for the formula $\varphi_{\mathbb{K},J}(\bar{v}) := \bigvee_{\pi\in\mathrm{Sym}(n)} \alpha^{\pi}_{\mathbb{K},J}(\bar{v})$ we have $\mathbb{Y} \models \varphi_{\mathbb{K},J}[\bar{y}]$ iff $\mathbb{H}|L_J \cong \mathbb{K}|L_J$, and the equivalence in (a) is true for the formula

$$\psi_{\mathcal{K},J} := \bigwedge_{\mathbb{K}\in\mathcal{K}} \exists\bar{v}\, \varphi_{\mathbb{K},J}(\bar{v}) \wedge \forall\bar{v}\, \Big((\bigwedge_{k<l<n} v_k \neq v_l) \Rightarrow \bigvee_{\mathbb{K}\in\mathcal{K}} \varphi_{\mathbb{K},J}(\bar{v})\Big). \tag{5}$$

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(b) Let $\mathbb{Y} \models \mathcal{T}_{\mathcal{K}}$. Suppose that $H = \{y_0, \dots, y_{n-1}\} \in [Y]^n$ and that $\mathbb{H} \ncong \mathbb{K}$, for all $\mathbb{K} \in \mathcal{K}$. Then for each $\mathbb{K} \in \mathcal{K}$ and each $\pi \in \mathrm{Sym}(n)$ we have $p_{\mathbb{K},\pi} := \{\langle x_k, y_{\pi(k)}\rangle : k < n\} \notin \mathrm{Iso}(\mathbb{K}, \mathbb{H})$ and, since $p_{\mathbb{K},\pi} : K \to H$ is a bijection, there is $i_{\mathbb{K},\pi} \in I$ such that $p_{\mathbb{K},\pi} \notin \mathrm{Iso}(\langle K, R^{\mathbb{K}}_{i_{\mathbb{K},\pi}}\rangle, \langle H, R^{\mathbb{H}}_{i_{\mathbb{K},\pi}}\rangle)$.

Since $J := \{i_{\mathbb{K},\pi} : \mathbb{K} \in \mathcal{K} \wedge \pi \in \mathrm{Sym}(n)\} \in [I]^{<\omega}$ and $\mathbb{Y} \models \psi_{\mathcal{K},J}$, by (a) there are $\mathbb{K}_0 \in \mathcal{K}$ and $\pi_0 \in \mathrm{Sym}(n)$ such that $p_{\mathbb{K}_0,\pi_0} \in \mathrm{Iso}(\mathbb{K}_0|L_J, \mathbb{H}|L_J)$, which implies that $p_{\mathbb{K}_0,\pi_0} \in \mathrm{Iso}(\langle K_0, R^{\mathbb{K}_0}_{i_{\mathbb{K}_0,\pi_0}}\rangle, \langle H, R^{\mathbb{H}}_{i_{\mathbb{K}_0,\pi_0}}\rangle)$ and we have a contradiction. So we have proved

$$\forall H \in [Y]^n \ \ \exists \mathbb{K} \in \mathcal{K} \ \ \mathbb{H} \cong \mathbb{K}, \tag{6}$$

that is, $\mathrm{Age}_n(\mathbb{Y}) \subset \{[\mathbb{K}] : \mathbb{K} \in \mathcal{K}\}$. Concerning the inclusion "$\supset$," suppose that for some $\mathbb{K}_0 \in \mathcal{K}$

$$\forall H \in [Y]^n \ \ \mathbb{H} \ncong \mathbb{K}_0 \tag{7}$$

and let $\mathcal{K} = \{\mathbb{K}_0, \dots, \mathbb{K}_{s-1}\}$ be an enumeration. Then, for each $0 < r < s$ and $\pi \in \mathrm{Sym}(n)$, since $\mathbb{K}_0 \ncong \mathbb{K}_r$, we have $p_\pi := \{\langle x_k, x_{\pi(k)}\rangle : k < n\} \notin \mathrm{Iso}(\mathbb{K}_0, \mathbb{K}_r)$ and, hence, there is $i_{r,\pi} \in I$ such that

$$p_\pi \notin \mathrm{Iso}(\langle K, R^{\mathbb{K}_0}_{i_{r,\pi}}\rangle, \langle K, R^{\mathbb{K}_r}_{i_{r,\pi}}\rangle). \tag{8}$$

Now $J := \{i_{r,\pi} : 0 < r < s \wedge \pi \in \mathrm{Sym}(n)\} \in [I]^{<\omega}$ and, since $\mathbb{Y} \models \psi_{\mathcal{K},J}$, there is $H \in [Y]^n$ such that $\mathbb{H}|L_J \cong \mathbb{K}_0|L_J$. By (6) and (7) there is $r > 0$ such that $\mathbb{H} \cong \mathbb{K}_r$, which implies $\mathbb{H}|L_J \cong \mathbb{K}_r|L_J$ and, hence, $\mathbb{K}_0|L_J \cong \mathbb{K}_r|L_J$. Thus, there is $\pi \in \mathrm{Sym}(n)$ such that $p_\pi \in \mathrm{Iso}(\mathbb{K}_0|L_J, \mathbb{K}_r|L_J)$, which in particular gives $p_\pi \in \mathrm{Iso}(\langle K, R^{\mathbb{K}_0}_{i_{r,\pi}}\rangle, \langle K, R^{\mathbb{K}_r}_{i_{r,\pi}}\rangle)$, but this contradicts (8). So for each $\mathbb{K} \in \mathcal{K}$ there is $H \in [Y]^n$ such that $\mathbb{H} \cong \mathbb{K}$; that is, $\{[\mathbb{K}] : \mathbb{K} \in \mathcal{K}\} \subset \mathrm{Age}_n(\mathbb{Y})$.

Conversely, if $\mathrm{Age}_n(\mathbb{Y}) = \{[\mathbb{K}] : \mathbb{K} \in \mathcal{K}\}$ and $J \in [I]^{<\omega}$, then for each $H \in [Y]^n$ there is $\mathbb{K} \in \mathcal{K}$ such that $\mathbb{H} \cong \mathbb{K}$; so $\mathbb{H}|L_J \cong \mathbb{K}|L_J$ and, hence, $[\mathbb{H}|L_J] \cong [\mathbb{K}|L_J]$. In addition, for each $\mathbb{K} \in \mathcal{K}$ there is $H \in [Y]^n$ such that $\mathbb{H} \cong \mathbb{K}$ so $\mathbb{H}|L_J \cong \mathbb{K}|L_J$ again and, by (a), $\mathbb{Y} \models \psi_{\mathcal{K},J}$. Thus we have $\mathbb{Y} \models \mathcal{T}_{\mathcal{K}}$. ⊣

Claim 3.3. *Let $\mathbb{Y}$ be an infinite L-structure and $|\mathrm{Age}_n(\mathbb{Y})| < \omega$, for each $n \in \mathbb{N}$. Then for the theory $\mathcal{T}_{\mathrm{Age}(\mathbb{Y})} := \bigcup_{n\in\mathbb{N}} \mathcal{T}_{\mathrm{Age}_n(\mathbb{Y})}$ and any L-structure $\mathbb{Z}$ we have*

(a) $\mathcal{T}_{\mathrm{Age}(\mathbb{Y})} \subset \mathrm{Th}(\mathbb{Y})$;
(b) $\mathbb{Z} \models \mathcal{T}_{\mathrm{Age}(\mathbb{Y})}$ *iff* $\mathrm{Age}(\mathbb{Z}) = \mathrm{Age}(\mathbb{Y})$; *and*
(c) *If* $\mathbb{Z} \models \mathrm{Th}(\mathbb{Y})$, *then* $\mathrm{Age}(\mathbb{Z}) = \mathrm{Age}(\mathbb{Y})$.

Proof. (a) By Claim 3.2(b), for $n \in \mathbb{N}$ we have $\mathbb{Y} \models \mathcal{T}_{\mathrm{Age}_n(\mathbb{Y})}$ so $\mathcal{T}_{\mathrm{Age}_n(\mathbb{Y})} \subset \mathrm{Th}(\mathbb{Y})$.

(b) $\mathbb{Z} \models \mathcal{T}_{\mathrm{Age}(\mathbb{Y})}$ iff $\mathbb{Z} \models \mathcal{T}_{\mathrm{Age}_n(\mathbb{Y})}$, for all $n \in \mathbb{N}$; iff (by Claim 3.2(b)) $\mathrm{Age}_n(\mathbb{Z}) = \mathrm{Age}_n(\mathbb{Y})$, for all $n \in \mathbb{N}$; iff $\mathrm{Age}(\mathbb{Z}) = \mathrm{Age}(\mathbb{Y})$.

(c) If $\mathbb{Z} \models \mathrm{Th}(\mathbb{Y})$, then by (a) $\mathbb{Z} \models \mathcal{T}_{\mathrm{Age}(\mathbb{Y})}$ and by (b) $\mathrm{Age}(\mathbb{Z}) = \mathrm{Age}(\mathbb{Y})$. ⊣

Claim 3.4. *If $\mathbb{Y}$ is an infinite almost chainable L-structure and $\mathbb{Z} \models \mathrm{Th}(\mathbb{Y})$, then $\mathrm{Age}(\mathbb{Z}) = \mathrm{Age}(\mathbb{Y})$, the structure $\mathbb{Z}$ is almost chainable and $|\mathrm{Ker}(\mathbb{Z})| = |\mathrm{Ker}(\mathbb{Y})|$.*

Proof. By Fact 2.3 we have $|\mathrm{Age}_n(\mathbb{Y})| < \omega$, for all $n \in \mathbb{N}$. So, by Claim 3.3(c) we have $\mathrm{Age}(\mathbb{Z}) = \mathrm{Age}(\mathbb{Y})$ and, by Fact 2.4, the structure $\mathbb{Z}$ is almost chainable and $|\mathrm{Ker}(\mathbb{Z})| = |\mathrm{Ker}(\mathbb{Y})|$. ⊣

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CLAIM 3.5. *If $\mathcal{T}$ is a complete almost chainable L-theory with infinite models and $|I| > \omega$, then $\mathcal{T}$ has a countable model and there are a countable language $L_J \subset L$ and a complete almost chainable L_J-theory $\mathcal{T}_J$ such that*

$$\left|\mathrm{Mod}_L^{\mathcal{T}}(\omega)/\cong\right| = \left|\mathrm{Mod}_{L_J}^{\mathcal{T}_J}(\omega)/\cong\right|. \tag{9}$$

PROOF. Let $\mathbb{Y} = \langle Y, \langle R_i^{\mathbb{Y}} : i \in I\rangle\rangle \in \mathrm{Mod}_L^{\mathcal{T}}$. By Fact 2.1, there are a finite set $F = \{a_0, \dots, a_{n-1}\} \subset Y$, a linear order $\lhd \mathrm{LO}_Y$ and an L_n-structure $\mathbb{X}$ satisfying (L1)–(L3) and for each $i \in I$ there is a quantifier-free formula $\varphi_i(v_0, \dots, v_{n_i-1})$ such that

$$\forall \bar{y} \in Y^{n_i} \ \left(\bar{y} \in R_i^{\mathbb{Y}} \Leftrightarrow \mathbb{X} \models \varphi_i[\bar{y}]\right). \tag{10}$$

Since there are countably many L_n-formulas, there is a partition $I = \bigcup_{j\in J} I_j$, where $|J| \le \omega$, such that, picking $i_j \in I_j$, for all $j \in J$, we have $R_i^{\mathbb{Y}} = R_{i_j}^{\mathbb{Y}}$, for all $i \in I_j$. So, for the L-sentences $\eta_{i,j} := \forall \bar{v}\ (R_i(\bar{v}) \Leftrightarrow R_{i_j}(\bar{v}))$, where $j \in J$ and $i \in I_j$, we have $\mathcal{T}_\eta := \bigcup_{j\in J}\{\eta_{i,j} : i \in I_j\} \subset \mathrm{Th}_L(\mathbb{Y}) = \mathcal{T}$. Now, $L_J := \langle R_{i_j} : j \in J\rangle \subset L$ and, using recursion, to each L-formula φ we adjoin an L_J-formula φ_J in the following way: $(v_k = v_l)_J := v_k = v_l$; $(R_i(v_{k_0}, \dots, v_{k_{n_i-1}}))_J := R_{i_j}(v_{k_0}, \dots, v_{k_{n_i-1}})$, for all $i \in I_j$; $(\neg\varphi)_J := \neg\varphi_J$; $(\varphi \wedge \psi)_J := \varphi_J \wedge \psi_J$ and $(\forall v\ \varphi)_J := \forall v\ \varphi_J$. A simple induction proves that

$$\forall \mathbb{Z} \in \mathrm{Mod}_L^{\mathcal{T}_\eta} \ \ \forall \varphi(\bar{v}) \in \mathrm{Form}_L\, \forall \bar{z} \in Z \ \left(\mathbb{Z} \models \varphi[\bar{z}] \Leftrightarrow \mathbb{Z}|L_J \models \varphi_J[\bar{z}]\right). \tag{11}$$

We prove that, in addition, for each $\mathbb{Z}_1, \mathbb{Z}_2 \in \mathrm{Mod}_L^{\mathcal{T}_\eta}$ we have

$$\mathbb{Z}_1 \cong \mathbb{Z}_2 \Leftrightarrow \mathbb{Z}_1|L_J \cong \mathbb{Z}_2|L_J \quad \text{and} \quad \mathbb{Z}_1 \equiv_L \mathbb{Z}_2 \Leftrightarrow \mathbb{Z}_1|L_J \equiv_{L_J} \mathbb{Z}_2|L_J. \tag{12}$$

The first claim is true since $\mathrm{Iso}(\mathbb{Z}_1, \mathbb{Z}_2) = \mathrm{Iso}(\mathbb{Z}_1|L_J, \mathbb{Z}_2|L_J)$. For the second, suppose that $\mathbb{Z}_1 \equiv_L \mathbb{Z}_2$ and $\mathbb{Z}_1|L_J \models \psi$, where $\psi \in \mathrm{Sent}_{L_J}$. Then $\psi \in \mathrm{Sent}_L$ and $\mathbb{Z}_1 \models \psi$, which gives $\mathbb{Z}_2 \models \psi$ so, by (11), $\mathbb{Z}_2|L_J \models \psi$, because $\psi_J = \psi$. Conversely, suppose that $\mathbb{Z}_1|L_J \equiv_{L_J} \mathbb{Z}_2|L_J$ and $\mathbb{Z}_1 \models \varphi$, where $\varphi \in \mathrm{Sent}_L$. Then, by (11), $\mathbb{Z}_1|L_J \models \varphi_J$ and, hence, $\mathbb{Z}_2|L_J \models \varphi_J$ so, by (11), $\mathbb{Z}_2 \models \varphi$.

Let $\mathcal{T}_J := \mathrm{Th}_{L_J}(\mathbb{Y}|L_J)$. If $\mathbb{Z} \in \mathrm{Mod}_L^{\mathcal{T}}$, that is, $\mathbb{Z} \equiv_L \mathbb{Y}$, then by (12) we have $\mathbb{Z}|L_J \equiv_{L_J} \mathbb{Y}|L_J$, which means that $\mathbb{Z}|L_J \in \mathrm{Mod}_{L_J}^{\mathcal{T}_J}$. So we obtain the mapping $\Lambda : \mathrm{Mod}_L^{\mathcal{T}} \to \mathrm{Mod}_{L_J}^{\mathcal{T}_J}$, where $\Lambda(\mathbb{Z}) = \mathbb{Z}|L_J$, for all $\mathbb{Z} \in \mathrm{Mod}_L^{\mathcal{T}}$, which is an injection, because $\mathcal{T}_\eta \subset \mathcal{T}$.

If $\mathbb{A} \in \mathrm{Mod}_{L_J}^{\mathcal{T}_J}$, then $\mathbb{Z} = \langle A, \langle R_i^{\mathbb{Z}} : i \in I\rangle\rangle \in \mathrm{Mod}_L^{\mathcal{T}_\eta}$, where $R_i^{\mathbb{Z}} = R_{i_j}^{\mathbb{A}}$, for $j \in J$ and $i \in I_j$. Now $\mathbb{Z}|L_J = \mathbb{A} \equiv_{L_J} \mathbb{Y}|L_J$ and, by (12), $\mathbb{Z} \equiv_L \mathbb{Y}$, that is, $\mathbb{Z} \in \mathrm{Mod}_L^{\mathcal{T}}$ and Λ is a surjection. Since the mapping Λ preserves cardinalities of structures, we have $\Lambda[\mathrm{Mod}_L^{\mathcal{T}}(\omega)] = \mathrm{Mod}_{L_J}^{\mathcal{T}_J}(\omega)$. By the Löwenheim–Skolem theorem there is $\mathbb{A} \in \mathrm{Mod}_{L_J}^{\mathcal{T}_J}(\omega)$ and $\Lambda^{-1}(\mathbb{A})$ is a countable model of $\mathcal{T}$.

By (12), the mapping Λ preserves the isomorphism relation and (9) is true. By (10) the reduct $\mathbb{Y}|L_J$ is simply definable in $\mathbb{X}$ and, by Fact 2.1, it is almost chainable. By Claim 3.4, the theory $\mathcal{T}_J = \mathrm{Th}_{L_J}(\mathbb{Y}|L_J)$ is almost chainable. ⊣

PROOF OF THEOREM 3.1. The implication (a) ⇒ (c) follows from Claim 3.5, the implication (c) ⇒ (b) is trivial and (b) ⇒ (a) follows from Claim 3.4. ⊣

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REMARK 3.6. We note that for finite languages the implication (b) $\Rightarrow$ (a) in Theorem 3.1 follows from Pouzet's result (our Fact 2.3).

§4. Vaught's Conjecture. In this section we confirm Vaught's Conjecture for almost chainable theories. More precisely, the whole section is devoted to a proof of the following statement.

THEOREM 4.1. *If $\mathcal{T}$ is a complete almost chainable theory having infinite models, then $I(\mathcal{T},\omega) \in \{1, \mathfrak{c}\}$. In addition, the theory $\mathcal{T}$ is ω-categorical iff it has a countable model which is chained by an ω-categorical linear order over its kernel.*

So, let $\mathcal{T}$ be a complete almost chainable L-theory having infinite models. By Theorem 3.1, there is $n \in \omega$ such that each model of $\mathcal{T}$ has the kernel of size n and, by Claim 3.5, w.l.o.g. we suppose that $|L| \leq \omega$, which gives $\mathrm{Mod}_L^{\mathcal{T}}(\omega) \neq \emptyset$. As above, let L_n denote the language $\langle R, U_0, \dots U_{n-1}\rangle$, where R is a binary and U_j's are unary symbols. In the sequel, for $\mathbb{Y} \in \mathrm{Mod}_L(\omega)$, by $[\mathbb{Y}]$ we denote the set $\{\mathbb{Y}' \in \mathrm{Mod}_L(\omega) : \mathbb{Y}' \cong \mathbb{Y}\}$ and similarly for the structures from $\mathrm{Mod}_{L_n}(\omega)$.

Following the architecture of the proof of the corresponding statement from [7] we divide the proof into two subsections. In "Preliminaries" we take an arbitrary countable model $\mathbb{Y}_0$ of $\mathcal{T}$ and a linear order with n unary predicates $\mathbb{X}_0$ such that $\mathrm{Pa}(\mathbb{X}_0) \subset \mathrm{Pa}(\mathbb{Y}_0)$ (see Figure 1) and describe the cardinal argument which will be used in our proof. In "Proof," distinguishing some cases, taking convenient structures $\mathbb{Y}_0$ and $\mathbb{X}_0$ and using that cardinal argument, we prove Theorem 4.1.

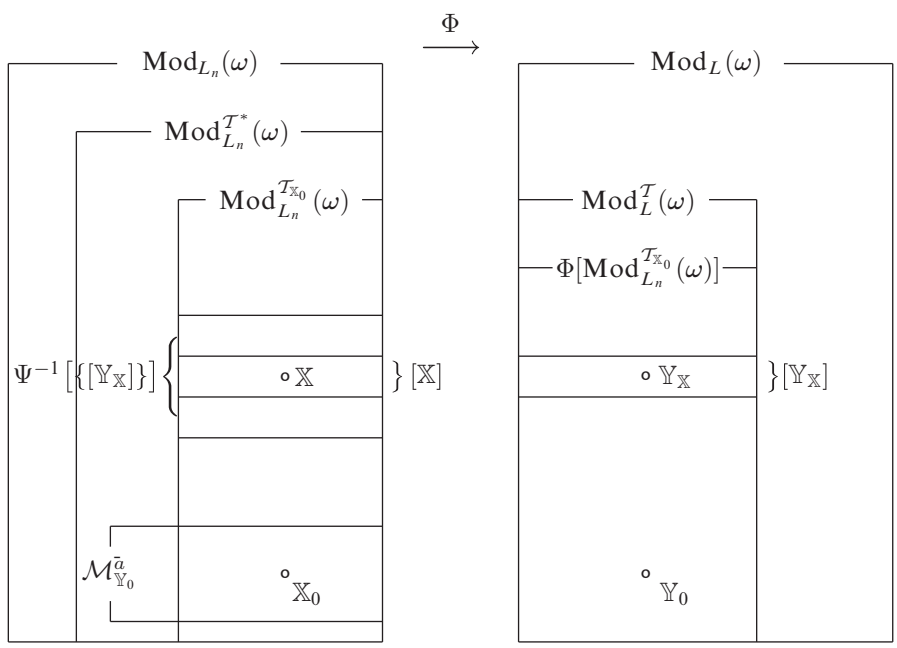

FIGURE 1. The mappings Φ and Ψ.

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4.1. Preliminaries. For convenience, let $\Delta_n := \{\langle x_0, \dots, x_{n-1}\rangle \in \omega^n : \bigvee_{k<l<n} x_k = x_l\}$ and, for an n-tuple $\bar{a} := \langle a_0, \dots, a_{n-1}\rangle \in \omega^n$, let us define $F_{\bar{a}} := \{a_0, \dots, a_{n-1}\}$.

We fix a model $\mathbb{Y}_0 = \langle \omega, \langle R_i^{\mathbb{Y}_0} : i \in I\rangle\rangle \in \mathrm{Mod}_L^{\mathcal{T}}(\omega)$ and an enumeration of its kernel, $\mathrm{Ker}(\mathbb{Y}_0) = \{a_0, \dots, a_{n-1}\}$. By Fact 2.1 there is a linear order $\prec \in LO_\omega$ such that, defining $\bar{a} = \langle a_0, \dots, a_{n-1}\rangle$ and $\mathbb{X}_0 := \langle \omega, \prec, \{a_0\}, \dots, \{a_{n-1}\}\rangle$,

$$\langle \omega, \prec\rangle = \{a_0\} + \dots + \{a_{n-1}\} + (\omega \setminus F_{\bar{a}}) \quad \text{and} \quad \mathrm{Pa}(\mathbb{X}_0) \subset \mathrm{Pa}(\mathbb{Y}_0).$$

Thus, the structure $\mathbb{Y}_0$ is $(F_{\bar{a}}, \prec\restriction(\omega \setminus F_{\bar{a}}))$-chainable. Let $\mathcal{T}_{\mathbb{X}_0}$ denote the complete theory of $\mathbb{X}_0$, $\mathrm{Th}_{L_n}(\mathbb{X}_0)$. The structure $\mathbb{X}_0$ has the following properties expressible by first order sentences of the language L_n:

(i) The interpretation of R is a linear order,
(ii) The interpretations of the relations U_k, $k < n$, are different singletons,
(iii) These singletons are ordered as the indices of U_k's (that is, the L_n-sentence $\bigwedge_{k<l<n} \forall u, v\ (U_k(u) \wedge U_l(v) \Rightarrow R(u, v))$ is true in $\mathbb{X}_0$), and
(iv) The union of these singletons is an initial segment of the linear order; that is $\mathbb{X}_0 \models \forall u, v\ ((U_{n-1}(u) \wedge \bigwedge_{k<n} \neg U_k(v)) \Rightarrow R(u, v))$.

So, if $\mathcal{T}^*$ is the set of the L_n-sentences expressing (i)–(iv), then $\mathcal{T}^* \subset \mathcal{T}_{\mathbb{X}_0}$ and

$$\mathbb{X}_0 \in \mathcal{M}_{\mathbb{Y}_0}^{\bar{a}} := \Big\{\langle \omega, \lhd, \{a_0\}, \dots, \{a_{n-1}\}\rangle \in \mathrm{Mod}_{L_n}^{\mathcal{T}^*}(\omega) : \mathbb{Y}_0 \text{ is } (F_{\bar{a}}, \lhd\restriction(\omega \setminus F_{\bar{a}}))\text{-chainable}\Big\}.$$

By Fact 2.1 the structure $\mathbb{Y}_0$ is simply definable in the L_n-structure $\mathbb{X}_0$. Thus, for each $i \in I$ there is a quantifier free L_n-formula $\varphi_i(v_0, \dots, v_{n_i-1})$ such that

$$\forall \bar{x} \in \omega^{n_i}\ \Big(\bar{x} \in R_i^{\mathbb{Y}_0} \Leftrightarrow \mathbb{X}_0 \models \varphi_i[\bar{x}]\Big). \tag{13}$$

Generally speaking, using the L_n-formulas φ_i, $i \in I$, to each L_n-structure $\mathbb{X} \in \mathrm{Mod}_{L_n}(\omega)$ we can adjoin the L-structure $\mathbb{Y}_{\mathbb{X}} := \langle \omega, \langle R_i^{\mathbb{Y}_{\mathbb{X}}} : i \in I\rangle\rangle \in \mathrm{Mod}_L(\omega)$, where, for each $i \in I$, the relation $R_i^{\mathbb{Y}_{\mathbb{X}}}$ is defined in the structure $\mathbb{X}$ by the formula φ_i, that is,

$$\forall \bar{x} \in \omega^{n_i}\ \Big(\bar{x} \in R_i^{\mathbb{Y}_{\mathbb{X}}} \Leftrightarrow \mathbb{X} \models \varphi_i[\bar{x}]\Big). \tag{14}$$

Claim 4.2. *For each structure* $\mathbb{Y}_0 \in \mathrm{Mod}_L^{\mathcal{T}}(\omega)$, *each enumeration* $\mathrm{Ker}(\mathbb{Y}_0) = \{a_0, \dots, a_{n-1}\}$ *each structure* $\mathbb{X}_0 \in \mathcal{M}_{\mathbb{Y}_0}^{\bar{a}}$ *and each choice of formulas* φ_i, $i \in I$, *satisfying* (13), *defining* $\mathbb{Y}_{\mathbb{X}}$ *by* (14), *for* $\mathbb{X} \in \mathrm{Mod}_{L_n}(\omega)$, *we have*

(a) *The mapping* $\Phi : \mathrm{Mod}_{L_n}(\omega) \to \mathrm{Mod}_L(\omega)$, *defined by* $\Phi(\mathbb{X}) = \mathbb{Y}_{\mathbb{X}}$, *for each* $\mathbb{X} \in \mathrm{Mod}_{L_n}(\omega)$, *preserves elementary equivalence and isomorphism*; *moreover*, $\mathrm{Iso}(\mathbb{X}_1, \mathbb{X}_2) \subset \mathrm{Iso}(\mathbb{Y}_{\mathbb{X}_1}, \mathbb{Y}_{\mathbb{X}_2})$, *for all* $\mathbb{X}_1, \mathbb{X}_2 \in \mathrm{Mod}_{L_n}(\omega)$ *and*
(b) *The mapping* $\Psi : \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}}}(\omega)/\cong \to \mathrm{Mod}_L^{\mathcal{T}}(\omega)/\cong$, *given by* $\Psi([\mathbb{X}]) = [\mathbb{Y}_{\mathbb{X}}]$, *for all* $[\mathbb{X}] \in \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)/\cong$, *is well defined.*

Proof. (a) By recursion on the construction of L-formulas to each L-formula $\varphi(\bar{v})$ we adjoin an L_n-formula $\varphi^*(\bar{v})$ in the following way: $(v_k = v_l)^* := v_k = v_l$, $R_i(v_{k_0}, \dots, v_{k_{n_i-1}})^* := \varphi_i(v_{k_0}, \dots, v_{k_{n_i-1}})$ (replacement of v_j by v_{k_j} in φ_i), $(\neg\varphi)^* :=$

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$\neg\varphi^*, (\varphi \wedge \psi)^* := \varphi^* \wedge \psi^*$ and $(\exists v_k\ \varphi)^* := \exists v_k\ \varphi^*$. A routine induction shows that, writing $\bar{v}$ instead of $v_0, \dots, v_{n-1}$, we have (see [5, p. 216])

$$\forall \mathbb{X} \in \mathrm{Mod}_{L_n}(\omega)\ \ \forall \varphi(\bar{v}) \in \mathrm{Form}_L\ \forall \bar{x} \in \omega^n \Big(\mathbb{X} \models \varphi^*[\bar{x}] \Leftrightarrow \mathbb{Y}_{\mathbb{X}} \models \varphi[\bar{x}]\Big). \tag{15}$$

Let $\mathbb{X}_1, \mathbb{X}_2 \in \mathrm{Mod}_{L_n}(\omega)$. If $\mathbb{X}_1 \equiv \mathbb{X}_2$, then for an L-sentence φ we have: $\mathbb{Y}_{\mathbb{X}_1} \models \varphi$ iff $\mathbb{X}_1 \models \varphi^*$ (by (15)) iff $\mathbb{X}_2 \models \varphi^*$ (since $\mathbb{X}_1 \equiv \mathbb{X}_2$) iff $\mathbb{Y}_{\mathbb{X}_2} \models \varphi$ (by (15) again). So, $\mathbb{Y}_{\mathbb{X}_1} \equiv \mathbb{Y}_{\mathbb{X}_2}$ and the mapping Φ preserves elementary equivalence.

If $f : \mathbb{X}_1 \to \mathbb{X}_2$ is an isomorphism, then by (14) and since isomorphisms preserve all formulas in both directions, for each $i \in I$ and $\bar{x} \in \omega^{n_i}$ we have: $\bar{x} \in R_i^{\mathbb{Y}_{\mathbb{X}_1}}$ iff $\mathbb{X}_1 \models \varphi_i[\bar{x}]$ iff $\mathbb{X}_2 \models \varphi_i[f\bar{x}]$ iff $f\bar{x} \in R_i^{\mathbb{Y}_{\mathbb{X}_2}}$. Thus $f \in \mathrm{Iso}(\mathbb{Y}_{\mathbb{X}_1}, \mathbb{Y}_{\mathbb{X}_2})$.

(b) For $\mathbb{X} \in \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$ we have $\mathbb{X} \equiv \mathbb{X}_0$, which, by (a), (13) and (14), implies that $\Phi(\mathbb{X}) = \mathbb{Y}_{\mathbb{X}} \equiv \mathbb{Y}_{\mathbb{X}_0} = \mathbb{Y}_0$. So, since $\mathbb{Y}_0 \models \mathcal{T}$, we have $\Phi(\mathbb{X}) \in \mathrm{Mod}_L^{\mathcal{T}}(\omega)$ and, thus,

$$\Phi[\mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)] \subset \mathrm{Mod}_L^{\mathcal{T}}(\omega). \tag{16}$$

Assuming that $\mathbb{X}_1, \mathbb{X}_2 \in \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$ and $\mathbb{X}_1 \cong \mathbb{X}_2$, by (a) we have $\mathbb{Y}_{\mathbb{X}_1} \cong \mathbb{Y}_{\mathbb{X}_2}$, that is $[\mathbb{Y}_{\mathbb{X}_1}] = [\mathbb{Y}_{\mathbb{X}_2}]$. So, the mapping Ψ is well defined. $\dashv$

Thus, by Claim 4.2(b), if $I(\mathcal{T}_{\mathbb{X}_0}, \omega) = \mathfrak{c}$, then for a proof that $I(\mathcal{T}, \omega) = \mathfrak{c}$ it is sufficient to show that the mapping Ψ is at-most-countable-to-one, which will be true if for each $\mathbb{X} \in \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$ we have $|\Psi^{-1}[\{[\mathbb{Y}_{\mathbb{X}}]\}] \leq \omega$. We note that, by Example 4.2 of [7], it is possible that $|\Psi^{-1}[\{[\mathbb{Y}_{\mathbb{X}}]\}] = \omega$.

Now, let $\mathbb{X} \in \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$. Then we have $\mathbb{X} \models \mathcal{T}^*$ and, hence, there is $\bar{b} := \langle b_0, \dots, b_{n-1}\rangle \in \omega^n \setminus \Delta_n$ such that $\mathbb{X} = \langle \omega, \prec_{\mathbb{X}}, \{b_0\}, \dots, \{b_{n-1}\}\rangle$ and $\mathbb{Y}_{\mathbb{X}}$ is definable in $\mathbb{X}$ by (14). So, by Fact 2.1, the structure $\mathbb{Y}_{\mathbb{X}}$ is $(F_{\bar{b}}, \prec_{\mathbb{X}} \restriction (\omega \setminus F_{\bar{b}}))$-chainable and (see (4))

$$\mathbb{L}_{\mathbb{X}} := \langle \omega \setminus F_{\bar{b}}, \prec_{\mathbb{X}} \restriction (\omega \setminus F_{\bar{b}})\rangle \in \mathcal{L}_{\mathbb{Y}_{\mathbb{X}}}^{F_{\bar{b}}}. \tag{17}$$

For an n-tuple $\bar{c} := \langle c_0, \dots, c_{n-1}\rangle \in \omega^n \setminus \Delta_n$ let us define

$$\mathcal{M}_{\mathbb{Y}_{\mathbb{X}}}^{\bar{c}} := \Big\{\langle \omega, \prec, \{c_0\}, \dots, \{c_{n-1}\}\rangle \in \mathrm{Mod}_{L_n}^{\mathcal{T}^*}(\omega) : \mathbb{Y}_{\mathbb{X}} \text{ is } (F_{\bar{c}}, \prec \restriction (\omega \setminus F_{\bar{c}}))\text{-chainable}\Big\}, \tag{18}$$

$$\mathcal{M}_{\mathbb{Y}_{\mathbb{X}}} := \textstyle\bigcup_{\bar{c} \in \omega^n \setminus \Delta_n} \mathcal{M}_{\mathbb{Y}_{\mathbb{X}}}^{\bar{c}}. \tag{19}$$

Thus, $\mathbb{X} \in \mathcal{M}_{\mathbb{Y}_{\mathbb{X}}}^{\bar{b}} \subset \mathcal{M}_{\mathbb{Y}_{\mathbb{X}}}$. For $\mathcal{M} \subset \mathrm{Mod}_{L_n}^{\mathcal{T}^*}(\omega)$, let $\mathcal{M}^{\cong} := \{[\mathbb{A}] : \mathbb{A} \in \mathcal{M}\}$.

Claim 4.3. *For each structure* $\mathbb{X} \in \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$ *we have*

$$\Psi^{-1}\Big[\{[\mathbb{Y}_{\mathbb{X}}]\}\Big] \subset \mathcal{M}_{\mathbb{Y}_{\mathbb{X}}}^{\cong} \cap \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)/\cong.$$

Proof. Let $\mathbb{X} \in \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$. Then, by (16) we have $\mathbb{Y}_{\mathbb{X}} \in \mathrm{Mod}_L^{\mathcal{T}}(\omega)$; so $\Psi([\mathbb{X}]) = [\mathbb{Y}_{\mathbb{X}}] \in \mathrm{Mod}_L^{\mathcal{T}}(\omega)/\cong$ and $\Psi^{-1}[\{[\mathbb{Y}_{\mathbb{X}}]\}] \subset \mathrm{dom}(\Psi) = \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)/\cong$.

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Let $[\mathbb{X}_1] \in \Psi^{-1}[\{[\mathbb{Y}_\mathbb{X}]\}]$. Then $[\mathbb{X}_1] \in \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)/\cong$ and, since the set $\mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$ is closed under $\cong$, we have $\mathbb{X}_1 \in \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$. This implies that $\mathbb{X}_1 \models \mathcal{T}^*$ and, hence, $\mathbb{X}_1 = \langle \omega, \prec_{\mathbb{X}_1}, \{c_0\}, \dots, \{c_{n-1}\}\rangle$, for some $\bar{c} := \langle c_0, \dots, c_{n-1}\rangle \in \omega^n \setminus \Delta_n$. Since $\mathbb{X}_1 \in \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$, by (14) for $i \in I$ we have

$$\forall \bar{x} \in \omega^{n_i} \ \ (\bar{x} \in R_i^{\mathbb{Y}_{\mathbb{X}_1}} \Leftrightarrow \mathbb{X}_1 \models \varphi_i[\bar{x}]). \tag{20}$$

Also we have $[\mathbb{Y}_{\mathbb{X}_1}] = \Psi([\mathbb{X}_1]) = [\mathbb{Y}_\mathbb{X}]$, so there is $f \in \mathrm{Iso}(\mathbb{Y}_\mathbb{X}, \mathbb{Y}_{\mathbb{X}_1})$ and we prove that $[\mathbb{X}_1] \in \mathcal{M}_{\mathbb{Y}_\mathbb{X}}^{\cong}$.

Clearly, $\mathbb{X}_2 := \langle \omega, f^{-1}[\prec_{\mathbb{X}_1}], \{f^{-1}(c_0)\}, \dots, \{f^{-1}(c_{n-1})\}\rangle \cong \mathbb{X}_1$ and $f \in \mathrm{Iso}(\mathbb{X}_2, \mathbb{X}_1)$. For $i \in I$ and $\bar{x} \in \omega^{n_i}$ we have $\bar{x} \in R_i^{\mathbb{Y}_\mathbb{X}}$ iff $f\bar{x} \in R_i^{\mathbb{Y}_{\mathbb{X}_1}}$ (since $f \in \mathrm{Iso}(\mathbb{Y}_\mathbb{X}, \mathbb{Y}_{\mathbb{X}_1})$), iff $\mathbb{X}_1 \models \varphi_i[f\bar{x}]$ (by 20), iff $\mathbb{X}_2 \models \varphi_i[\bar{x}]$ (since $f \in \mathrm{Iso}(\mathbb{X}_2, \mathbb{X}_1)$). Thus $\bar{x} \in R_i^{\mathbb{Y}_\mathbb{X}}$ iff $\mathbb{X}_2 \models \varphi_i[\bar{x}]$, for all $\bar{x} \in \omega^{n_i}$, so, by Fact 2.1, the structure $\mathbb{Y}_\mathbb{X}$ is $(F_{f^{-1}\bar{c}}, f^{-1}[\prec_{\mathbb{X}_1}] \restriction (\omega \setminus F_{f^{-1}\bar{c}}))$-chainable. Now, $\mathbb{X}_2 \in \mathcal{M}_{\mathbb{Y}_\mathbb{X}}^{f^{-1}\bar{c}}$ and, hence, $[\mathbb{X}_1] = [\mathbb{X}_2] \in (\mathcal{M}_{\mathbb{Y}_\mathbb{X}}^{f^{-1}\bar{c}})^{\cong} \subset \mathcal{M}_{\mathbb{Y}_\mathbb{X}}^{\cong}$. Thus $\Psi^{-1}[\{[\mathbb{Y}_\mathbb{X}]\}] \subset \mathcal{M}_{\mathbb{Y}_\mathbb{X}}^{\cong}$. ⊣

The following folklore statement will be used in our case analysis as well.

Claim 4.4. *If some structure $\mathbb{Y} \in \mathrm{Mod}_L^{\mathcal{T}}(\omega)$ is simply definable in an ω-categorical structure $\mathbb{X}$ with domain ω, then $\mathbb{Y}$ is an ω-categorical structure and $I(\mathcal{T}, \omega) = 1$.*

Proof. By the theorem of Engeler, Ryll-Nardzewski and Svenonius (see [5, p. 341]), the automorphism group of $\mathbb{X}$ is oligomorphic; that is, for each $n \in \mathbb{N}$ we have $|\omega^n/{\sim_{\mathbb{X},n}}| < \omega$, where $\bar{x} \sim_{\mathbb{X},n} \bar{y}$ iff $f\bar{x} = \bar{y}$, for some $f \in \mathrm{Aut}(\mathbb{X})$.

As in Claim 4.2(a) we prove that $\mathrm{Aut}(\mathbb{X}) \subset \mathrm{Aut}(\mathbb{Y})$, which implies that for $n \in \mathbb{N}$ and each $\bar{x}, \bar{y} \in \omega^n$ we have $\bar{x} \sim_{\mathbb{X},n} \bar{y} \Rightarrow \bar{x} \sim_{\mathbb{Y},n} \bar{y}$. Thus $|\omega^n/{\sim_{\mathbb{Y},n}}| \leq |\omega^n/{\sim_{\mathbb{X},n}}| < \omega$, for all $n \in \mathbb{N}$, and, since $|L| \leq \omega$, using the same theorem we conclude that $\mathbb{Y}$ is an ω-categorical L-structure. ⊣

4.2. Proof. First we prove that $|\mathrm{Mod}_L^{\mathcal{T}}(\omega)/\cong| \in \{1, \mathfrak{c}\}$, using definitions and notation from "Preliminaries" and distinguishing the following cases.

Case A. *There exist a structure $\mathbb{Y}_0 \in \mathrm{Mod}_L^{\mathcal{T}}(\omega)$, an enumeration of its kernel, $\mathrm{Ker}(\mathbb{Y}_0) = \{a_0, \dots, a_{n-1}\}$, and a structure $\mathbb{X}_0 \in \mathcal{M}_{\mathbb{Y}_0}^{\bar{a}}$ such that the theory $\mathcal{T}_{\mathbb{X}_0}$ is ω-categorical.* Then by Fact 2.1 the structure $\mathbb{Y}_0$ is simply definable in $\mathbb{X}_0$ and by Claim 4.4 we have $I(\mathcal{T}, \omega) = 1$.

In particular, Case A appears if there is a structure $\mathbb{Y} \in \mathrm{Mod}_L^{\mathcal{T}}(\omega)$ satisfying condition (i) of Theorem 2.5: $\mathbb{Y}$ is F-chainable and $\mathcal{L}_\mathbb{Y}^F = LO_{\omega\setminus F}$. Then, taking an enumeration $F = \{a_0, \dots, a_{n-1}\}$, the relations $R_i^\mathbb{Y}$ of the structure $\mathbb{Y}$ are definable in the structure $\mathbb{X} := \langle \omega, \{a_0\}, \dots, \{a_{n-1}\}\rangle$ of the unary language $L' := \langle U_0, \dots, U_{n-1}\rangle$ by quantifier free L'-formulas and, since the structure $\mathbb{X}$ is ω-categorical, $\mathbb{Y}$ is ω-categorical as well; so, $I(\mathcal{T}, \omega) = 1$ again. We note that such structures are called *finitist* by Fraïssé, see [2, p. 292].

Case B. *For each structure $\mathbb{Y}_0 \in \mathrm{Mod}_L^{\mathcal{T}}(\omega)$, each enumeration of its kernel $\mathrm{Ker}(\mathbb{Y}_0) = \{a_0, \dots, a_{n-1}\}$ and each structure $\mathbb{X}_0 \in \mathcal{M}_{\mathbb{Y}_0}^{\bar{a}}$, the theory $\mathcal{T}_{\mathbb{X}_0}$ is not ω-categorical; so, by Theorem 1.1, $|\mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)/\cong| = \mathfrak{c}$, for all $\mathbb{X}_0 \in \mathcal{M}_{\mathbb{Y}_0}^{\bar{a}}$.*

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Then, by the remark from Case A concerning condition (i) of Theorem 2.5, we have

$$\forall \mathbb{Y} \in \mathrm{Mod}_L^{\mathcal{T}}(\omega) \;\; \mathcal{L}_{\mathbb{Y}}^{\mathrm{Ker}(\mathbb{Y})} \neq LO_{\omega \setminus \mathrm{Ker}(\mathbb{Y})}, \tag{21}$$

and we prove that $|\mathrm{Mod}_L^{\mathcal{T}}(\omega)/\cong| = \mathfrak{c}$, distinguishing the following two subcases.

SUBCASE B1. *There exist a structure* $\mathbb{Y}_0 \in \mathrm{Mod}_L^{\mathcal{T}}(\omega)$, *an enumeration of its kernel*, $\mathrm{Ker}(\mathbb{Y}_0) = \{a_0, \dots, a_{n-1}\}$, *and a structure* $\mathbb{X}_0 \in \mathcal{M}_{\mathbb{Y}_0}^{\bar{a}}$ *such that the linear order* $\mathbb{L}_{\mathbb{X}_0} := \langle \omega \setminus F_{\bar{a}}, \prec_{\mathbb{X}_0} \restriction (\omega \setminus F_{\bar{a}}) \rangle \in \mathcal{L}_{\mathbb{Y}_0}^{F_{\bar{a}}}$ *has at least one end-point*.

Then we take such $\mathbb{Y}_0$, $\bar{a}$ and $\mathbb{X}_0$ and notice that $\mathbb{X}_0 \models \mathcal{T}^*$ and that the mentioned property of $\mathbb{L}_{\mathbb{X}_0}$ gives a first order property of $\mathbb{X}_0$. Namely, $\mathbb{X}_0 \models \theta_0 \vee \theta_1$, where

$$\theta_0 := \exists v \, \forall u \; (U_{n-1}(u) \Rightarrow R(u,v) \wedge \neg \exists w \; (R(u,w) \wedge R(w,v))), \tag{22}$$

$$\theta_1 := \exists v \, \forall u \; (\neg u = v \Rightarrow R(u,v)). \tag{23}$$

Now we have $|\mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)/\cong| = \mathfrak{c}$ and, by Claim 4.2(b), for a proof that $|\mathrm{Mod}_L^{\mathcal{T}}(\omega)/\cong| = \mathfrak{c}$ it is sufficient to show that the mapping Ψ is at-most-countable-to-one. This will follow from the following claim and Claim 4.3.

CLAIM 4.5. $\left| \mathcal{M}_{\mathbb{Y}_{\mathbb{X}}}^{\cong} \cap \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)/\cong \right| \leq \omega$, *for all* $\mathbb{X} \in \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$.

PROOF. Let $\mathbb{X} \in \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$. By (19) it is sufficient to show that for each $\bar{c} \in \omega^n \setminus \Delta_n$ we have

$$\left| (\mathcal{M}_{\mathbb{Y}_{\mathbb{X}}}^{\bar{c}})^{\cong} \cap \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)/\cong \right| \leq \omega. \tag{24}$$

Let $\mathbb{X}_1 = \langle \omega, \prec_{\mathbb{X}_1}, \{c_0\}, \dots, \{c_{n-1}\} \rangle \in \mathcal{M}_{\mathbb{Y}_{\mathbb{X}}}^{\bar{c}}$. Then by (18) and (4) we have

$$\mathbb{L}_{\mathbb{X}_1} := \langle \omega \setminus F_{\bar{c}}, \prec_{\mathbb{X}_1} \restriction (\omega \setminus F_{\bar{c}}) \rangle \in \mathcal{L}_{\mathbb{Y}_{\mathbb{X}}}^{F_{\bar{c}}}.$$

First, if the set $\mathcal{L}_{\mathbb{Y}_{\mathbb{X}}}^{F_{\bar{c}}}$ satisfies condition (iii) of Theorem 2.5, then we have $(\mathcal{L}_{\mathbb{Y}_{\mathbb{X}}}^{F_{\bar{c}}})^{\cong} = \{[\mathbb{L}_{\mathbb{X}_1}], [\mathbb{L}_{\mathbb{X}_1}^*]\}$ (because all "$\mathbb{K} + \mathbb{M} + \mathbb{H}$-sums" are isomorphic and all "$\mathbb{H}^* + \mathbb{M}^* + \mathbb{K}^*$-sums" are isomorphic). Thus each structure $\mathbb{X}_2 \in \mathcal{M}_{\mathbb{Y}_{\mathbb{X}}}^{\bar{c}}$ consists of the initial part, $\{c_0\} + \dots + \{c_{n-1}\}$, labeled by the unary relations $U_j^{\mathbb{X}_2} = \{c_j\}$, $j < n$, and a final part, which is either isomorphic to the linear order $\mathbb{L}_{\mathbb{X}_1}$ or to its reverse, $\mathbb{L}_{\mathbb{X}_1}^*$. So we have $|(\mathcal{M}_{\mathbb{Y}_{\mathbb{X}}}^{\bar{c}})^{\cong}| \leq 2$ and (24) is true.

Otherwise, by (21) and Theorem 2.5, $\mathcal{L}_{\mathbb{Y}_{\mathbb{X}}}^{F_{\bar{c}}} = \bigcup_{\mathbb{L}_{\mathbb{X}_1} = \mathbb{I} + \mathbb{F}} \{\mathbb{F} + \mathbb{I}, \mathbb{I}^* + \mathbb{F}^*\}$.

Let $\mathbb{X}_2 = \langle \omega, \prec_{\mathbb{X}_2}, \{c_0\}, \dots, \{c_{n-1}\} \rangle \in \mathcal{M}_{\mathbb{Y}_{\mathbb{X}}}^{\bar{c}} \cap \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$. Then $\mathbb{L}_{\mathbb{X}_2} := \langle \omega \setminus F_{\bar{c}}, \prec_{\mathbb{X}_2} \restriction (\omega \setminus F_{\bar{c}}) \rangle \in \mathcal{L}_{\mathbb{Y}_{\mathbb{X}}}^{F_{\bar{c}}}$ and, hence, there is a cut $\{\mathbb{I}, \mathbb{F}\}$ in $\mathbb{L}_{\mathbb{X}_1}$ (i.e. a decomposition $\mathbb{L}_{\mathbb{X}_1} = \mathbb{I} + \mathbb{F}$) such that $\mathbb{L}_{\mathbb{X}_2} = \mathbb{F} + \mathbb{I}$ or $\mathbb{L}_{\mathbb{X}_2} = \mathbb{I}^* + \mathbb{F}^*$. Suppose that $\mathbb{I}, \mathbb{F} \neq \emptyset$, that $\mathbb{I}$ does not have a largest element and that $\mathbb{F}$ does not have a smallest element. Then $\mathbb{F} + \mathbb{I}$ and $\mathbb{I}^* + \mathbb{F}^*$ are linear orders without end points. But, since $\mathbb{X}_2 \in \mathrm{Mod}_{L_n}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$

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we have $\mathbb{X}_2 \models \theta_0 \vee \theta_1$ and, hence, the linear order $\mathbb{L}_{\mathbb{X}_2}$ must have at least one end-point, which gives a contradiction.

So, for each $\mathbb{X}_2 \in \mathcal{M}^{\bar{c}}_{\mathbb{Y}_\mathbb{X}} \cap \mathrm{Mod}^{\mathcal{T}_{\mathbb{X}_0}}_{L_n}(\omega)$ we have $\mathbb{L}_{\mathbb{X}_2} = \mathbb{F} + \mathbb{I}$ or $\mathbb{L}_{\mathbb{X}_2} = \mathbb{I}^* + \mathbb{F}^*$, where $\mathbb{I}$ has a largest element or $\mathbb{F}$ has a smallest element. Since such cuts $\{\mathbb{I}, \mathbb{F}\}$ in $\mathbb{L}_{\mathbb{X}_1}$ are defined by the elements of the set $\omega \setminus F_{\bar{c}}$, there are countably many of them. Thus $|\mathcal{M}^{\bar{c}}_{\mathbb{Y}_\mathbb{X}} \cap \mathrm{Mod}^{\mathcal{T}_{\mathbb{X}_0}}_{L_n}(\omega)| = \omega$, which implies (24), since each class from the set $(\mathcal{M}^{\bar{c}}_{\mathbb{Y}_\mathbb{X}})^{\cong} \cap \mathrm{Mod}^{\mathcal{T}_{\mathbb{X}_0}}_{L_n}(\omega)/\cong$ has a representative in $\mathcal{M}^{\bar{c}}_{\mathbb{Y}_\mathbb{X}} \cap \mathrm{Mod}^{\mathcal{T}_{\mathbb{X}_0}}_{L_n}(\omega)$. ⊣

Subcase B2. *For each structure $\mathbb{Y}_0 \in \mathrm{Mod}^{\mathcal{T}}_L(\omega)$, each enumeration of its kernel, $\mathrm{Ker}(\mathbb{Y}_0) = \{a_0, \dots, a_{n-1}\}$, and each structure $\mathbb{X}_0 \in \mathcal{M}^{\bar{a}}_{\mathbb{Y}_0}$, the linear order $\mathbb{L}_{\mathbb{X}_0} := \langle \omega \setminus F_{\bar{a}}, \prec_{\mathbb{X}_0} \restriction (\omega \setminus F_{\bar{a}})\rangle \in \mathcal{L}^{F_{\bar{a}}}_{\mathbb{Y}_0}$ is a linear order without end points.*

Then we fix arbitrary $\mathbb{Y}_0 \in \mathrm{Mod}^{\mathcal{T}}_L(\omega)$ and $\mathbb{X}_0 \in \mathcal{M}^{\bar{a}}_{\mathbb{Y}_0}$, where $\mathrm{Ker}(\mathbb{Y}_0) = F_{\bar{a}}$.

Again we have $|\mathrm{Mod}^{\mathcal{T}_{\mathbb{X}_0}}_{L_n}(\omega)/\cong| = \mathfrak{c}$ and, as in Subcase B1, the equality $|\mathrm{Mod}^{\mathcal{T}}_L(\omega)/\cong| = \mathfrak{c}$ will follow from Claims 4.2(b), 4.3 and the next claim.

Claim 4.6. $\left|\mathcal{M}^{\cong}_{\mathbb{Y}_\mathbb{X}} \cap \mathrm{Mod}^{\mathcal{T}_{\mathbb{X}_0}}_{L_n}(\omega)/\cong\right| \le \omega$, *for all* $\mathbb{X} \in \mathrm{Mod}^{\mathcal{T}_{\mathbb{X}_0}}_{L_n}(\omega)$.

Proof. Let $\mathbb{X} \in \mathrm{Mod}^{\mathcal{T}_{\mathbb{X}_0}}_{L_n}(\omega)$. By (19) it is sufficient to show that for each $\bar{c} \in \omega^n \setminus \Delta_n$ we have $|(\mathcal{M}^{\bar{c}}_{\mathbb{Y}_\mathbb{X}})^{\cong}| \le 2$.

Let $\mathbb{X}_1 = \langle \omega, \prec_{\mathbb{X}_1}, \{c_0\}, \dots, \{c_{n-1}\}\rangle \in \mathcal{M}^{\bar{c}}_{\mathbb{Y}_\mathbb{X}}$. Then, by our assumption, $\mathbb{L}_{\mathbb{X}_1} = \langle \omega \setminus F_{\bar{c}}, \prec_{\mathbb{X}_1} \restriction (\omega \setminus F_{\bar{c}})\rangle$ is a linear order without end points and $\mathbb{L}_{\mathbb{X}_1} \in \mathcal{L}^{F_{\bar{c}}}_{\mathbb{Y}_\mathbb{X}}$.

Suppose that the set $\mathcal{L}^{F_{\bar{c}}}_{\mathbb{Y}_\mathbb{X}}$ satisfies condition (ii) of Theorem 2.5; that is, $\mathcal{L}^{F_{\bar{c}}}_{\mathbb{Y}_\mathbb{X}} = \bigcup_{\mathbb{L}_{\mathbb{X}_1} = \mathbb{I}+\mathbb{F}} \{\mathbb{F} + \mathbb{I}, \mathbb{I}^* + \mathbb{F}^*\}$. Then, taking an arbitrary $x \in \omega \setminus F_{\bar{c}}$ we have $\mathbb{L}_{\mathbb{X}_1} = (-\infty, x)_{\mathbb{L}_{\mathbb{X}_1}} + [x, \infty)_{\mathbb{L}_{\mathbb{X}_1}} =: \mathbb{I} + \mathbb{F}$; and, since $\mathbb{L}_{\mathbb{X}_1}$ is a linear order without end points, $\mathbb{I}, \mathbb{F} \neq \emptyset$. Let $\mathbb{X}_2 = \langle \omega, \prec_{\mathbb{X}_2}, \{c_0\}, \dots, \{c_{n-1}\}\rangle$, where $\prec_{\mathbb{X}_2}$ is the linear order on ω such that $\langle \omega, \prec_{\mathbb{X}_2}\rangle = \{c_0\} + \dots + \{c_{n-1}\} + \mathbb{F} + \mathbb{I}$. Then the structure $\mathbb{Y}_\mathbb{X}$ is $(F_{\bar{c}}, \prec_{\mathbb{X}_2} \restriction (\omega \setminus F_{\bar{c}}))$-chainable and, hence, $\mathbb{X}_2 \in \mathcal{M}^{\bar{c}}_{\mathbb{Y}_\mathbb{X}}$. But the linear order $\mathbb{L}_{\mathbb{X}_2} := \langle \omega \setminus F_{\bar{c}}, \prec_{\mathbb{X}_2} \restriction (\omega \setminus F_{\bar{c}})\rangle = \mathbb{F} + \mathbb{I}$ has a smallest element, which contradicts the assumption of Subcase B2.

Thus there are finite sets $K, H \subset \omega \setminus F_{\bar{c}}$ such that $\mathbb{L}_{\mathbb{X}_1} = \mathbb{K} + \mathbb{M} + \mathbb{H}$ and

$$\mathcal{L}^{F_{\bar{c}}}_{\mathbb{Y}_\mathbb{X}} = \bigcup_{\substack{\lhd_K \in LO_K \\ \lhd_H \in LO_H}} \left\{\langle K, \lhd_K\rangle + \mathbb{M} + \langle H, \lhd_H\rangle, \langle H, \lhd_H\rangle^* + \mathbb{M}^* + \langle K, \lhd_K\rangle^*\right\}.$$

In addition, since each element of $\mathcal{L}^{F_{\bar{c}}}_{\mathbb{Y}_\mathbb{X}}$ is a linear order without end points, we have $K = H = \emptyset$ and, hence, $\mathbb{M} = \mathbb{L}_{\mathbb{X}_1}$ and $\mathcal{L}^{F_{\bar{c}}}_{\mathbb{Y}_\mathbb{X}} = \{\mathbb{L}_{\mathbb{X}_1}, \mathbb{L}^*_{\mathbb{X}_1}\}$. Thus each structure $\mathbb{X}_2 \in \mathcal{M}^{\bar{c}}_{\mathbb{Y}_\mathbb{X}}$ consists of the initial part, $\{c_0\} + \dots + \{c_{n-1}\}$, labeled by the unary relations $U^{\mathbb{X}_2}_j = \{c_j\}$, $j < n$, and a final part, which is either isomorphic to the linear order $\mathbb{L}_{\mathbb{X}_1}$ or to its reverse, $\mathbb{L}^*_{\mathbb{X}_1}$. So we have $|(\mathcal{M}^{\bar{c}}_{\mathbb{Y}_\mathbb{X}})^{\cong}| \le 2$. ⊣

Finally we prove the second part of Theorem 4.1. By our analysis, the theory $\mathcal{T}$ is ω-categorical iff Case A appears; so, we have to prove that the L_n-structure

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$\mathbb{X}_0 = \langle \omega, \prec, \{a_0\}, \dots, \{a_{n-1}\}\rangle$ is ω-categorical iff $\mathbb{L}_{\mathbb{X}_0} := \langle \omega \setminus F_{\bar{a}}, \prec\restriction (\omega \setminus F_{\bar{a}})\rangle$ is an ω-categorical linear order. Since $\mathrm{Aut}(\mathbb{X}_0) = \{\mathrm{id}_{F_{\bar{a}}} \cup f : f \in \mathrm{Aut}(\mathbb{L}_{\mathbb{X}_0})\}$, for $n \in \mathbb{N}$ and $\bar{x}, \bar{y} \in (\omega \setminus F_{\bar{a}})^n$ we have $\bar{x} \sim_{\mathbb{L}_{\mathbb{X}_0}} \bar{y} \Leftrightarrow \bar{x} \sim_{\mathbb{X}_0} \bar{y}$, which implies that $|(\omega \setminus F_{\bar{a}})^n / \sim_{\mathbb{L}_{\mathbb{X}_0}}| \leq |\omega^n / \sim_{\mathbb{X}_0}|$. So, if $\mathbb{X}_0$ is ω-categorical, then $\mathbb{L}_{\mathbb{X}_0}$ is ω-categorical (by the theorem of Engeler, Ryll-Nardzewski and Svenonius).

On the other hand, if $\mathbb{L}_{\mathbb{X}_0}$ is ω-categorical, then the linear order $\langle \omega, \prec\rangle \cong n + \mathbb{L}_{\mathbb{X}_0}$ is ω-categorical (see Rosenstein's theorem, [10, p. 299]) and, since $\mathrm{Aut}(\mathbb{X}_0) = \mathrm{Aut}(\langle \omega, \prec\rangle)$, $\mathbb{X}_0$ is ω-categorical too. □

§5. Comments and examples. By Fact 2.3 and Theorems 3.1 and 4.1, if a complete theory $\mathcal{T}$ has an infinite model $\mathbb{Y}$ with bounded profile, that is $\varphi_{\mathbb{Y}} \leq m$, for some $m \in \mathbb{N}$, then $I(\mathcal{T}, \omega) \in \{1, \mathfrak{c}\}$. In [4], Gibson et al. have considered the extension of the profile function to infinite cardinals and proved that if $\mathbb{Y}$ is an L-structure such that $\varphi_{\mathbb{Y}}(\mu) < \omega$, for some infinite cardinal $\mu \leq |Y|$, then $\mathbb{Y}$ is almost chainable (see Theorem 22 of [4]). So, by Theorems 3.1 and 4.1 we have

Corollary 5.1. *If a complete theory $\mathcal{T}$ has a model $\mathbb{Y}$ such that $\varphi_{\mathbb{Y}}(\mu) < \omega$, for some infinite cardinal $\mu \leq |Y|$, then $I(\mathcal{T}, \omega) \in \{1, \mathfrak{c}\}$.*

We note that Example 4.1 of [7] shows that if $\mathcal{T}$ is the complete theory of the ternary betwenness relation defined on the rational line with doubled integers, then $\mathcal{T}$ is a complete monomorphic (and, hence, almost chainable) theory which is unstable, does not have definable Skolem functions and has a countable model which is not bi-interpretable with any linear order. By Theorem 4.1 we have $I(\mathcal{T}, \omega) = \mathfrak{c}$.

Further, Example 4.2 of [7] shows that if $\mathcal{T}$ is the complete theory of the ternary cyclic-order relation defined on the discrete linear order $\omega + \omega^*$, then the mapping Ψ defined in Claim 4.2(b) is not finite-to-one. Here we have $I(\mathcal{T}, \omega) = \mathfrak{c}$ again.

Finally, we give three simple examples of ternary almost chainable structures which are not monomorphic.

Example 5.2. Let $L = \langle R_0\rangle$, $\mathrm{ar}(R_0) = 3$ and let $\mathbb{X} = \langle \omega, <_\omega, \{0\}, \{1\}, \{2\}\rangle$, where $<_\omega$ is the usual strict linear order on ω. Then we have $\mathbb{X} \in \mathrm{Mod}_{L_3}(\omega)$, where $L_3 = \langle <, U_0, U_1, U_2\rangle$, and the L_3-formula

$$\begin{aligned}\varphi_0(u, v, w) := {} & (v_0 < v_1 < v_2) \vee (v_1 < v_2 < v_0) \vee (v_2 < v_0 < v_1) \\ & \vee \ \Big(U_0(v_0) \wedge U_0(v_1) \wedge \textstyle\bigwedge_{j<3} \neg U_j(v_3)\Big)\end{aligned}$$

defines the L-structure $\mathbb{Y} = \langle \omega, R_0^{\mathbb{Y}}\rangle$ in $\mathbb{X}$. Clearly, $R_0^{\mathbb{Y}}$ is the union of the cyclic-order relation defined on the ordinal $\langle \omega, <_\omega\rangle$ and the set $\{\langle 0, 0, z\rangle : 2 <_\omega z\}$. Thus, for example, $\langle 70, 80, 90\rangle, \langle 80, 90, 0\rangle, \langle 90, 1, 3\rangle, \langle 0, 0, 5\rangle, \langle 0, 0, 90\rangle \in R_0^{\mathbb{Y}}$. Since $\mathrm{Ker}(\mathbb{Y}) = \{y \in Y : \mathrm{Age}(Y \setminus \{y\}) \subsetneq \mathrm{Age}(\mathbb{Y})\}$ (see [8, p. 300]), it is easy to check that $\mathrm{Ker}(\mathbb{Y}) = 3 = \{0, 1, 2\}$ (for example, for $K := \{0, 1, 2, 3\}$ we have $\mathbb{K} \not\leftrightarrow Y \setminus \{1\}$). Here we have case (iii) of Theorem 2.5 and $I(\mathrm{Th}(\mathbb{Y}), \omega) = \mathfrak{c}$.

Example 5.3. If in Example 5.2 we take the L_3-formula $\varphi_0'(v_0, v_1, v_2)$ given by

$$\begin{aligned}&\Big(\textstyle\bigwedge_{i,j<3} \neg U_j(v_i) \wedge ((v_0 < v_1 < v_2) \vee (v_1 < v_2 < v_0) \vee (v_2 < v_0 < v_1))\Big) \\ &\vee \ \Big(U_0(v_0) \wedge U_0(v_1) \wedge \textstyle\bigwedge_{j<3} \neg U_j(v_3)\Big),\end{aligned}$$

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we obtain the ternary almost chainable structure $\mathbb{Y}' = \langle \omega, R_0^{\mathbb{Y}'} \rangle$. Now, the relation $R_0^{\mathbb{Y}'}$ is the union of the cyclic-order relation defined on the linear order $\langle \omega \setminus 3, <_\omega \rangle$ and the set $\{\langle 0, 0, z \rangle : z \in \omega \setminus 3\}$. Here we have case (ii) of Theorem 2.5 and, by Theorem 4.1, the equality $I(\mathrm{Th}(\mathbb{Y}'), \omega) = \mathfrak{c}$ holds again.

Example 5.4. If in Example 5.2 we take $\varphi_0''(v_0, v_1, v_2) := U_0(v_0) \wedge U_0(v_1) \wedge \bigwedge_{j<3} \neg U_j(v_3)$, we obtain $\mathbb{Y}'' = \langle \omega, R_0^{\mathbb{Y}''} \rangle$, where $R_0^{\mathbb{Y}''} = \{\langle 0, 0, z \rangle : z \in \omega \setminus 3\}$. Now we have case (i) of Theorem 2.5 and, by Theorem 4.1, $I(\mathrm{Th}(\mathbb{Y}''), \omega) = 1$.

Acknowledgements. The author acknowledges financial support of the Ministry of Education, Science and Technological Development of the Republic of Serbia (Grant No. 451-03-68/2020-14/200125).

The author would like to thank the referee for constructive suggestions.

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DEPARTMENT OF MATHEMATICS AND INFORMATICS
FACULTY OF SCIENCES, UNIVERSITY OF NOVI SAD
TRG DOSITEJA OBRADOVIĆA 4, NOVI SAD 21000, SERBIA
E-mail: milos@dmi.uns.ac.rs

https://doi.org/10.1017/jsl.2021.60 Published online by Cambridge University Press