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BOOK REVIEW

Mathematical Geophysics: An Introduction to Rotating Fluids and the Navier-Stokes Equations. By J.-Y. Chemin, B. Desjardins, I. Gallagher & E. Grenier. Oxford University Press, 2006. 272 pp. ISBN 0 19 857133 X. £ 47.
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It is very likely that a good number of *JFM* readers will be misled by this book's title. The title will suggest to some that it is focused on topics like those found in Joe Pedlosky's *Geophysical Fluid Dynamics* or perhaps Adrian Gill's *Atmosphere-Ocean Dynamics*. The reader will quickly discern that this is not the case – but this does not mean that the title is inaccurate. Rather, *Mathematical Geophysics: An Introduction to Rotating Fluids and the Navier-Stokes Equations* is concerned with the rigorous analysis of the Navier–Stokes equations, facing up to fundamental issues such as existence, uniqueness and regularity of solutions.

More specifically, this book describes developments in the study of quantitative and qualitative features of solutions in the presence of rotation. In that regard it complements classic monographs on modern mathematical analysis of incompressible (and largely non-rotating) viscous flows such as those by Constantin & Foias (1988) and Temam (1995) while adding to those works in terms of topics of interest for applications in geophysical fluid dynamics. After a brief (just 14 page) summary of the physical motivation for studying fluid dynamics in rotating frames in Part I, *Mathematical Geophysics* contains a complete review of the classical theory of weak and strong solutions in the 70 pages comprising Part II, and then a thorough presentation of recent developments for the analysis of rotating flows in the 130 pages of Part III. The concluding 30 pages in Part IV, entitled 'Perspectives', contain discussions of very recent developments on related systems as well as unsolved problems.

This kind of mathematical analysis is not normal fare for many *JFM* readers although it can be argued that more of it should be. From the mathematical point of view, the Navier–Stokes equations on many domains with a variety of applied forces and a selection of initial data lead to globally well-posed problems. By this we mean that the solutions are unique flows given by smooth functions of space and time. The situation is generally satisfactory for two-dimensional problems where only technical restrictions on data and driving are necessary to ensure that solutions are well-behaved. In three spatial dimensions, however, the combinations of forces and boundary and initial conditions that are known to produce smooth unique solutions are all quite mild, corresponding in essence to weak driving and small initial data, i.e. small Grashof and Reynolds numbers. For strong driving or large initial data in three dimensions we only know that such unique and smooth 'strong' solutions exist locally, i.e. for a finite (and in practice embarassingly small) interval of time that shrinks as the Grashof and Reynolds numbers grow. Beyond that time all that is known to exist are so-called 'weak' solutions, finite kinetic energy flows that satisfy the Navier–Stokes equations in the sense of distributions but that are not (known to be) smooth enough to ensure uniqueness or even to satisfy the basic power balance.

Many flows of interest for applications, notably many that are regarded as turbulent flows, are not characterized by small Grashof or Reynolds numbers and the long-time behaviour of solutions is of central concern. In such cases the true nature – the actual regularity and uniqueness – of solutions to the Navier–Stokes equations remains a grand challenge for the mathematical community. Indeed, a satisfactory answer to the question of whether or not all solutions of the three-dimensional Navier–Stokes equations remain smooth and unique is worth $\$10^6$ from the Clay Mathematics Institute.† This question is not only of interest for abstract analysis. From a practical point of view, for example, uniqueness and regularity of solutions are necessary to establish convergence and *a priori* error estimates for numerical schemes. From a fundamental physics point of view, a loss of regularity in solutions would signal a breakdown of the separation-of-scales assumption that goes into the Navier–Stokes equations' derivation from the underlying Boltzmann equation. Loss of uniqueness of solutions to the hydrodynamic equations violates the principles of Newtonian determinism and, perhaps more importantly, obviates the Navier–Stokes equations' utility as a predictive theory.

Despite these challenges a wide variety of useful results concerning weak solutions, results that apply as well to unique smooth solutions *if* they exist, can be obtained by careful analysis. The most obvious care that must be taken is to keep in mind just how much regularity is available, i.e. what function spaces the solutions reside in, so that no 'illegal' calculations are performed and no unjustified conclusions are drawn from merely formal manipulation of the equations. This approach may be unfamiliar to many *JFM* readers but the presentation in *Mathematical Geophysics*, and especially the full recounting of the classical results in Part II, is very well written. In this regard I can recommend Part II as an excellent introduction to mathematical fluid dynamics and the theory of weak solutions to the Navier–Stokes equations. Part II alone is suitable as the basis for an advanced (postgraduate level) course or for self-instruction by appropriately mathematically minded applied scientists.

What takes *Mathematical Geophysics* beyond other mathematical monographs and secures its place within the modern literature is its treatment of strongly rotating flows. *JFM* readers will be familiar with the Taylor–Proudman Theorem asserting that flows in a rapidly rotating frame lose their dependence on the direction along the axis of rotation. This effective 'two-dimensionalization' of three-dimensional flows has profound effects on the fluid dynamics, of course, but it also has profound effects on the analysis. Because two-dimensional flows are known to be better behaved than three-dimensional flows, rotation can have a mollifying effect on potential problems and, given sufficiently strong rotation, correspondingly stronger rigorous results can be derived. That rotation may aid in the analysis of the three-dimensional Navier–Stokes equations is thus not unexpected, but rigorous developments along these lines are only relatively recent, dating to very late last century (Babin, Mahalov & Nicolaenko 1997). This is understandable given the singular nature of the vanishing Rossby and/or Ekman number limits and the above-mentioned difficulties in the proper mathematical handling of the three-dimensional Navier–Stokes equations. Moreover, just as strong rotation brings new physics into the picture (e.g. Poincaré and Rossby waves) it also brings new tools into play for the rigorous analysis of solutions (e.g. Strichartz estimates for dispersive systems).

Mathematical Geophysics is recommended for those who are interested in modern methods of rigorous analysis for the fundamental equations of incompressible fluid mechanics. Besides providing an excellent overview of the technical aspects of

† See the website ⟨http://www.claymath.org/millennium/Navier-Stokes_Equations/⟩.

Navier–Stokes analysis, the authors have done an admirable job of explaining the physical effects – or at least the physical intuition – behind the analysis of rotating flows. An appropriate background in partial differential equations and functional analysis, i.e. a background typically possessed by mathematical physicists or analysis-oriented applied mathematicians, is necessary to appreciate everything that the monograph has to offer. However, less well-prepared but adequately dedicated readers will gain an understanding of the motivations and methods of this community of mathematical researchers. A word of warning: definitions must be carefully heeded and readers cannot assume that familar words always correspond to the most familar concepts. For example when the authors say that a solution is 'stable' they can mean either that it exhibits continuous dependence on initial data and/or parameters or that perturbations decay. There is no ambiguity, though, and a careful reading will leave no doubt.

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