Proceedings of the Royal Society of Edinburgh, **142A**, 239–271, *2012*

A relaxation result in the framework of structured deformations in a bounded variation setting

Margarida Baía, **José Matias** and **Pedro Miguel Santos**
CAMGSD, Departamento de Matemática, Instituto Superior Tecnico, Av. Rovisco Pais, 1049-001 Lisbon, Portugal (mbaia@math.ist.utl.pt; jmatias@math.ist.utl.pt; pmsantos@math.ist.utl.pt)

(MS received 6 September 2010; accepted 2 May 2011)

We obtain an integral representation of an energy for structured deformations of continua in the space of functions of bounded variation, as a first step to the study of asymptotic models for thin defective crystalline structures, where phenomena as slips, vacancies and dislocations prevent the effectiveness of classical theories.

1. Introduction

The theory of first-order structured deformations introduced by Del Piero and Owen [17] forms a basis for addressing a large variety of problems in continuum mechanics, where geometrical changes can be associated to smooth-classical deformations, piecewise deformations and more complex deformations for which an analysis at the macroscopic and microscopic levels is required. Specifically, this theory can be applied to problems relating to plasticity, crystals with defects, liquid crystals and material composites.

From a variational point of view, the problem of assigning a free energy to a body that undergoes a structured deformation was first studied by Choksi and Fonseca [13] in the context of functions of special bounded variation (SBV) (the main notation and concepts used throughout this work are given in § 2). This energy was defined as the most effective way to build up the deformation using sequences of approximating simple deformations in SBV. More precisely, Choksi and Fonseca defined a structured deformation in an open and bounded set $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, as a pair (g, G) where the macroscopic deformation g is an element of $\mathrm{SBV}(\Omega; \mathbb{R}^d)$ and G (deformation without disarrangements) is an integrable tensor field in Ω. Using a Lusin-type result of Alberti [1] (see theorem 2.5) it was proven in [13] that, given such a pair, there exist deformations u_n in $\mathrm{SBV}(\Omega; \mathbb{R}^d)$ with

$$u_n \xrightarrow{L^1} g \quad \text{and} \quad \nabla u_n \overset{\mathcal{M}(\Omega)}{\rightharpoonup} G. \tag{1.1}$$

As a result, the energy associated to (g, G), $I(g, G)$ was defined as the relaxation with respect to the topology given in (1.1) of the functional

$$\mathcal{E}(u) = \int_\Omega \mathcal{W}(\nabla u)\,\mathrm{d}x + \int_{S_u} \Psi([u], \nu_u)\,\mathrm{d}\mathcal{H}^{N-1}, \quad u \in \mathrm{SBV}(\Omega; \mathbb{R}^d),$$

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

for appropriate bulk and interfacial densities $\mathcal{W}$ and Ψ, that is,

$$I(g,G) := \inf_{\{u_n\}\subset \mathrm{SBV}(\Omega;\mathbb{R}^d)} \Big\{ \liminf_{n\to\infty} \mathcal{E}(u_n),\ u_n \xrightarrow{L^1} g,\ \nabla u_n \overset{\mathcal{M}(\Omega)}{\rightharpoonup} G \Big\}. \tag{1.2}$$

The main objective in [13] was then to characterize $I(g,G)$ through an integral representation formula. The principal difference of this characterization from previous integral representation results for similar relaxed energies, where relaxation is taken with respect to the L^1 (BV-weak) topology [4, 7–10, 20, 21] is the fact that gradients of approximating sequences $\{u_n\}$ in (1.1) are constrained to converge to the given function G (not necessarily ∇g). In this case, if $\nabla u_n \to G$ in L^1, the difference $G - Dg$ is achieved by the limit of singular measures since $Du_n \to Dg$ in the sense of distributions. Moreover, the Hausdorff measure of the jump set of u_n tends necessarily to infinity, otherwise theorem 2.1 of Ambrosio [3] asserts that $G = \nabla g$ almost everywhere.

In the context of defective crystals, as mentioned in [13], (1.2) can be interpreted as a way of realizing the deformed crystal by piecing together elastic crystals on an increasingly fine scale. We refer to Choksi *et al*. [14], where the framework introduced in [13] has been used to predict, for simple models, the origins and main characteristics of phenomena such as fracturing, yielding and hysterisis, with applications to single defective crystals (see [14, § 4.3]).

Our objective is to generalize the relaxation result derived in [13] to the full BV setting for a class of second-order energies suitable for the study of equilibrium configurations of thin defective crystalline structures, as addressed by Matias and Santos [26]. As noted in [26], the energy considered in [13] is not appropriate for this study since some control in the second-order derivatives of the deformation is needed to avoid geometrical obstacles in the thin film limit.

To present our main result, theorem 4.1, whose complete statement is given in § 4, we denote by $\mathrm{BV}^2(\Omega;\mathbb{R}^d)$ ($\mathrm{SBV}^2(\Omega;\mathbb{R}^d)$), $d \geqslant 1$, the space of functions $u \in \mathrm{BV}(\Omega;\mathbb{R}^d)$ such that $\nabla u \in \mathrm{BV}(\Omega;\mathbb{R}^{d\times N})$ (respectively, for $\mathrm{SBV}^2(\Omega;\mathbb{R}^d)$) and we start by introducing the space of generalized structured deformations

$$\mathrm{GSD}(\Omega;\mathbb{R}^d) := \mathrm{BV}^2(\Omega;\mathbb{R}^d) \times \mathrm{BV}(\Omega;\mathbb{R}^{d\times N}).$$

For any $(g,G) \in \mathrm{GSD}(\Omega;\mathbb{R}^d)$ we consider the relaxed energy

$$I(g,G) = \inf_{\{u_n\}\subset \mathrm{SBV}^2(\Omega;\mathbb{R}^d)} \Big\{ \liminf_{n\to\infty} E(u_n),\ u_n \xrightarrow{L^1} g,\ \nabla u_n \xrightarrow{L^1} G \Big\}, \tag{1.3}$$

where

$$E(u) = \int_\Omega W(\nabla u, \nabla^2 u)\,\mathrm{d}x + \int_{S_u} \Psi_1([u],\nu_u)\,\mathrm{d}\mathcal{H}^{N-1} + \int_{S_{\nabla u}} \Psi_2([\nabla u],\nu_{\nabla u})\,\mathrm{d}\mathcal{H}^{N-1}$$

for $u \in \mathrm{SBV}^2(\Omega;\mathbb{R}^d)$, and the functions W, Ψ_1 and Ψ_2 satisfy the hypotheses (H_1)–(H_7) introduced in § 3. Under hypotheses (H_1)–(H_7), theorem 4.1 asserts that there exist bulk and interfacial densities W_1, W_2, γ_1, γ_2 (see (3.2)–(3.5)) such that, for all $(g,G) \in \mathrm{GSD}(\Omega;\mathbb{R}^d)$,

$$I(g,G) = \int_{\Omega} (W_1(G-\nabla g) + W_2(G,\nabla G))\,\mathrm{d}x + \int_{S_g} \gamma_1([g],\nu_g)\,\mathrm{d}\mathcal{H}^{N-1} + \int_{S_G} \gamma_2(G^+,G^-,\nu_G)\,\mathrm{d}\mathcal{H}^{N-1} + \int_{\Omega} W_1\biggl(-\frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|}\biggr)\,\mathrm{d}|D^c g| + \int_{\Omega} W_2^{\infty}\biggl(G,\frac{\mathrm{d}D^c G}{\mathrm{d}|D^c G|}\biggr)\,\mathrm{d}|D^c G| \tag{1.4}$$

where W_2^{∞}, as usual, denotes the *recession function* of W_2 in the second variable, that is,

$$W_2^{\infty}(A,B) = \limsup_{t\to\infty} \frac{W(A,tB)}{t}, \quad A \in \mathbb{R}^{d\times N},\ B \in \mathbb{R}^{d\times N\times N}.$$

To show (1.4) we start by deriving a similar relaxation result in the SBV setting: theorem 3.2. Although this characterization could have been derived directly for the whole energy $I(g,G)$ using localization and blow-up methods, Alberti's theorem (theorem 2.5) allows us to divide this energy into two first-order relaxed energies $I_1(g,G)$ and $I_2(G)$, making our arguments more concise. The effect of the structured deformation is captured in the first energy $I_1(g,G)$ through the limit energy density W_1 that depends on $G-\nabla g$ (deformation due to disarrangements at the microscopic level). The full BV characterization (1.4) then follows from theorem 3.2 together with a sequential characterization of the energy $I_1(g,G)$ (see lemma 4.3), Reshetnyak's theorem (see theorem 2.2) and Alberti's rank-one theorem [2].

We finish this introduction by referring the interested reader to Carriero *et al.* (see [11,12] and the references therein) for other second-order variational problems arising from some models in image segmentation and in continuum mechanics.

The overall plan of this work in the ensuing sections will be as follows. Section 2 gives the main notation and results used throughout. Section 3 is devoted to a proof of the SBV counterpart of our main result, theorem 3.2. A proof of theorem 4.1 is obtained in § 4.

2. Preliminaries

The purpose of this section is to set some notation and to give a brief overview of the concepts and main results that are used in subsequent sections. All of these results are stated without proof since they can be readily found in the references given below.

2.1. Notation

Throughout the text, $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, will denote an open bounded set, and we will use the following notation.

- $\mathcal{A}(\Omega)$ is the family of all open subsets of Ω.
- $\mathcal{M}(\Omega)$ is the set of finite Radon measures on Ω.
- $\|\mu\|$ stands for the total variation of a measure $\mu \in \mathcal{M}(\Omega)$.

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

- S^{N-1} stands for the unit sphere in $\mathbb{R}^N$.
- e_i denotes the ith element of the canonical basis of $\mathbb{R}^N$ for $i = 1, \dots, N$.
- Q denotes the unit cube centred at the origin with one side orthogonal to e_N.
- $Q(x, \delta)$ denotes a cube centred at $x \in \Omega$ with side length δ and with one side orthogonal to e_N.
- $Q_\nu(x, \delta)$ is the cube centred at $x \in \Omega$ with side length δ and with one side orthogonal to $\nu \in S^{N-1}$.
- $Q_\nu := Q_\nu(0, 1)$.
- $\mathbb{R}^{d \times N \times N}$ is the set of real tensors of order $d \times N \times N$, $d \geqslant 1$.
- C represents a generic constant,
- $\lim_{n,m} := \lim_n \lim_m$, while $\lim_{m,n} := \lim_m \lim_n$.

2.2. Measure theory

We start by recalling a generalization of the Besicovitch differentiation theorem due to [4].

THEOREM 2.1. *If λ and μ are Radon measures in $\Omega, \mu \geqslant 0$, then there exists a Borel set $E \subset \Omega$ such that $\mu(E) = 0$ and, for every $x \in \operatorname{supp} \mu \setminus E$,*

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}(x) := \lim_{\epsilon \to 0} \frac{\lambda(x + \epsilon C)}{\mu(x + \epsilon C)}$$

exists and is finite whenever C is a bounded, convex, open set containing the origin.

We also recall Reshetnyak's theorem on weak convergence of vector measures (see [27]; see also [6]).

THEOREM 2.2. *Let μ and μ_n be $\mathbb{R}^d$-valued finite Radon measures in Ω such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in Ω and such that $\|\mu_n\|(\Omega) \to \|\mu\|(\Omega)$. Then*

$$\lim_{n \to \infty} \int_\Omega f\left(x, \frac{\mu_n}{\|\mu_n\|}(x)\right) \mathrm{d}\|\mu_n\|(x) = \int_\Omega f\left(x, \frac{\mu}{\|\mu\|}(x)\right) \mathrm{d}\|\mu\|(x)$$

for every continuous and bounded function $f\colon \Omega \times S^{d-1} \to \mathbb{R}$.

2.3. BV-functions

In this section we briefly summarize some facts on functions of functions of bounded variation to be used later. We refer to the interested reader to [6, 18, 19, 23, 28] for a detailed description of this subject.

A function $u \in L^1(\Omega; \mathbb{R}^d)$ is said to be of *bounded variation*, and we write $u \in \mathrm{BV}(\Omega; \mathbb{R}^d)$ if all its first distributional derivatives $D_j u_i \in \mathcal{M}(\Omega)$ for $i = 1, \dots, d$

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

and $j = 1, \dots, N$. The matrix-valued measure whose entries are $D_j u_i$ is denoted by Du. The space $\mathrm{BV}(\Omega;\mathbb{R}^d)$ is a Banach space when endowed with the norm

$$\|u\|_{\mathrm{BV}} = \|u\|_{L^1} + \|Du\|(\Omega).$$

By the Lebesgue decomposition theorem, Du can be split into the sum of two mutually singular measures $D^{\mathrm{a}}u$ and $D^{\mathrm{s}}u$ (the absolutely continuous part and singular part, respectively, of Du with respect to the Lebesgue measure $\mathcal{L}^N$). We denote by ∇u the Radon–Nikodým derivative of $D^{\mathrm{a}}u$ with respect to $\mathcal{L}^N$, so that we can write

$$Du = \nabla u \mathcal{L}^N \lfloor \Omega + D^{\mathrm{s}}u.$$

Let Ω_u be the set of points where the approximate limit of u exists, i.e. $x \in \Omega$ such that there exist $z \in \mathbb{R}^N$ with

$$\lim_{\varepsilon\to 0} \fint_{Q(x,\varepsilon)} |u(y) - z| \,\mathrm{d}y = 0.$$

If $x \in \Omega_u$ and $z = u(x)$, we say that u is *approximately continuous* at x (or that x is a Lebesgue point of u). The function u is approximately continuous $\mathcal{L}^N$-a.e. $x \in \Omega_u$ and

$$\mathcal{L}^N(\Omega \setminus \Omega_u) = 0. \tag{2.1}$$

Let S_u be the *jump set* of this function, i.e. the set of points $x \in \Omega \setminus \Omega_u$ for which there exists $a, b \in \mathbb{R}^N$ and a unit vector $\nu \in S^{N-1}$, normal to S_u at x, such that $a \neq b$ and

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in Q_\nu(x,\varepsilon)\colon (y-x)\cdot\nu > 0\}} |u(y) - a| \,\mathrm{d}y = 0 \tag{2.2}$$

and

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in Q_\nu(x,\varepsilon)\colon (y-x)\cdot\nu < 0\}} |u(y) - b| \,\mathrm{d}y = 0. \tag{2.3}$$

The triple (a, b, ν) uniquely determined by (2.2) and (2.3) up to permutation of (a, b) and a change of sign of ν is denoted by $(u^+(x), u^-(x), \nu_u(x))$.

If $u \in \mathrm{BV}(\Omega)$, it is well known that S_u is countably $N-1$ rectifiable, i.e.

$$S_u = \bigcup_{n=1}^{\infty} K_n \cup E,$$

where $\mathcal{H}^{N-1}(E) = 0$ and K_n are compact subsets of C^1-hypersurfaces. Furthermore, $\mathcal{H}^{N-1}((\Omega \setminus \Omega_u) \setminus S_u) = 0$ and the following decomposition holds:

$$Du = \nabla u \mathcal{L}^N \lfloor \Omega + [u] \otimes \nu_u \mathcal{H}^{N-1} \lfloor S_u + D^c u,$$

where $[u] := u^+ - u^-$ and $D^c u$ is the Cantor part of the measure Du, i.e. $D^c u = D^{\mathrm{s}}u \lfloor (\Omega_u)$.

If Ω is an open and bounded set with Lipschitz boundary, then the outer unit normal to $\partial\Omega$ (denoted by ν) exists $\mathcal{H}^{N-1}$ almost everywhere and the trace for functions in $\mathrm{BV}(\Omega;\mathbb{R}^d)$ is defined.

Next we recall some useful results on BV functions used in what follows.

THEOREM 2.3 (approximate differentiability). *If $u \in \mathrm{BV}(\Omega;\mathbb{R}^d)$, then, for $\mathcal{L}^N$-a.e. $x \in \Omega$*

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^N}\left\{\int_{Q(x,\epsilon)}|u(y)-u(x)-\nabla u(x)\cdot(y-x)|^{N/(N-1)}\,\mathrm{d}y\right\}^{(N-1)/N}=0.$$

LEMMA 2.4. *Let $u \in \mathrm{BV}(\Omega;\mathbb{R}^d)$. There exist piecewise constant functions u_n such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ and*

$$\|Du\|(\Omega)=\lim_{n\to\infty}\|Du_n\|(\Omega)=\lim_{n\to\infty}\int_{S_{u_n}}|[u_n](x)|\,\mathrm{d}H^{N-1}(x).$$

The space of *special functions of bounded variation* introduced in [16] for problems arising from pattern recognition and the mathematical theory of liquid crystals, $\mathrm{SBV}(\Omega;\mathbb{R}^d)$, is the space of functions $u \in \mathrm{BV}(\Omega;\mathbb{R}^d)$ such that $D^c u = 0$, i.e. for which

$$Du = \nabla u\mathcal{L}^N + [u]\otimes\nu_u\mathcal{H}^{N-1}\lfloor S_u.$$

The next result is a Lusin-type theorem for gradients due to [1] and is essential for our arguments.

THEOREM 2.5. *Given $f \in L^1(\Omega;\mathbb{R}^{d\times N})$, there exists $u \in \mathrm{SBV}(\Omega;\mathbb{R}^d)$ and a Borel function $g\colon \Omega \to \mathbb{R}^{d\times N}$ such that*

$$Du = f\mathcal{L}^N + g\mathcal{H}^{N-1}\lfloor S_u,$$

$$\int_{S_u}|g|\,\mathrm{d}\mathcal{H}^{N-1}\leqslant C\|f\|_{L^1(\Omega;\mathbb{R}^{d\times N})}.$$

REMARK 2.6. From the proof of theorem 2.5, it also follows that

$$\|u\|_{L^1(\Omega)}\leqslant C\|f\|_{L^1(\Omega;\mathbb{R}^{d\times N})}.$$

The following technical result is a simplified version of lemma 4.3 of [25].

LEMMA 2.7. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $A \in \mathbb{R}^{d\times N}$. Then there exists $u \in \mathrm{SBV}(\Omega;\mathbb{R}^d)$ such that $u_{\lfloor\partial Q} = 0$ and $\nabla u = A$ almost everywhere in Ω. In addition,*

$$|D^{\mathrm{s}}u|(\Omega)\leqslant C(N)|A|\,|\Omega|.$$

Following [11, 12], we define, as presented in § 1,

$$\mathrm{SBV}^2(\Omega;\mathbb{R}^d)=\{v\in\mathrm{SBV}(\Omega;\mathbb{R}^d),\ \nabla v\in\mathrm{SBV}(\Omega;\mathbb{R}^{d\times N})\}.$$

If $u \in \mathrm{SBV}^2(\Omega;\mathbb{R}^d)$, we use the notation $\nabla^2 u = \nabla(\nabla u)$, that is, $\nabla^2 u$ is the absolutely continuous part of $D(\nabla u)$ with respect to the Lebesgue measure. We will also define

$$\mathrm{BV}^2(\Omega;\mathbb{R}^d)=\{v\in\mathrm{BV}(\Omega;\mathbb{R}^d),\ \nabla v\in\mathrm{BV}(\Omega;\mathbb{R}^{d\times N})\}.$$

3. Integral representation in SBV

The aim of this section is to derive an integral representation of the relaxed functional energy defined in (1.3),

$$I(g,G) = \inf_{\{u_n\}\subset \mathrm{SBV}^2(\Omega;\mathbb{R}^d)} \Big\{ \liminf_{n\to\infty} E(u_n),\ u_n \xrightarrow{L^1} g,\ \nabla u_n \xrightarrow{L^1} G \Big\},$$

$$E(u) = \int_\Omega W(\nabla u, \nabla^2 u)\,\mathrm{d}x + \int_{S_u} \Psi_1([u],\nu_u)\,\mathrm{d}\mathcal{H}^{N-1} + \int_{S_{\nabla u}} \Psi_2([\nabla u],\nu_{\nabla u})\,\mathrm{d}\mathcal{H}^{N-1}, \quad u \in \mathrm{SBV}^2(\Omega;\mathbb{R}^d),$$

for $(g,G) \in \mathrm{SD}(\Omega;\mathbb{R}^d)$, where

$$\mathrm{SD}(\Omega;\mathbb{R}^d) := \mathrm{SBV}^2(\Omega;\mathbb{R}^d) \times \mathrm{SBV}(\Omega;\mathbb{R}^{d\times N}).$$

In a similar way to the approach used in [13], we assume the density functions W, Ψ_1, Ψ_2 to satisfy the following conditions.

(H_1) There exists $C > 0$ such that

$$\frac{1}{C}|B| - C \leqslant W(A,B) \leqslant C(1+|B|)$$

for all $A \in \mathbb{R}^{d\times N}$ and $B \in \mathbb{R}^{d\times N\times N}$.

(H_2) There exists $C > 0$ such that

$$|W(A_1,B_1) - W(A_2,B_2)| \leqslant C(|A_1 - A_2| + |B_1 - B_2|)$$

for all $A_i \in \mathbb{R}^{d\times N}$ and $B_i \in \mathbb{R}^{d\times N\times N}$, $i = 1, 2$.

(H_3) There exists $0 < \alpha < 1$ and $L > 0$ such that

$$\left| W^\infty(A,B) - \frac{W(A,tB)}{t} \right| \leqslant \frac{C}{t^\alpha}$$

for all $t > L$, $A \in \mathbb{R}^{d\times N}$, $B \in \mathbb{R}^{d\times N\times N}$ with $|B| = 1$.

(H_4) There exist $c_1 > 0$ and $C_1 > 0$ such that

$$c_1|\lambda| \leqslant \Psi_1(\lambda,\nu) \leqslant C_1|\lambda|$$

for all $\lambda \in \mathbb{R}^d$ and $\nu \in S^{N-1}$.

(H_5) There exist $c_2 > 0$ and $C_2 > 0$ such that

$$c_2|\Lambda| \leqslant \Psi_2(\Lambda,\nu) \leqslant C_2|\Lambda|$$

for all $\nu \in S^{N-1}$ and $\Lambda \in \mathbb{R}^{d\times N}$.

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

(H_6) (Homogeneity of degree one.)

$$\Psi_1(t\lambda, \nu) = t\Psi_1(\lambda, \nu), \qquad \Psi_2(t\Lambda, \nu) = t\Psi_2(\Lambda, \nu)$$

for all $\nu \in S^{N-1}$, $\lambda \in \mathbb{R}^d$, $\Lambda \in \mathbb{R}^{d\times N}$ and $t > 0$.

(H_7) (Sub-additivity.)

$$\Psi_1(\lambda_1 + \lambda_2, \nu) \leqslant \Psi_1(\lambda_1, \nu) + \Psi_1(\lambda_2, \nu),$$
$$\Psi_2(\Lambda_1 + \Lambda_2, \nu) \leqslant \Psi_2(\Lambda_1, \nu) + \Psi_2(\Lambda_2, \nu)$$

for all $\nu \in S^{N-1}$, $\lambda_i \in \mathbb{R}^d$, $\Lambda_i \in \mathbb{R}^{d\times N}$, $i = 1, 2$.

We observe that, as already mentioned in [13], the coercivity hypotheses above and the homogeneity condition (H_6) are of technical order and can be relaxed (see [13, remark 3.3]). In addition, the linear growth and subadditivity requirements on the interfacial densities could be weakened by assuming a Lipschitz continuity property instead.

Remark 3.1. (i) In what follows, we extend Ψ_i, $i = 1, 2$, as homogeneous functions of degree one in the second variable to all of $\mathbb{R}^N$.

(ii) We note that the class of functions over which the infimum in the definition of $I(g, G)$ is taken is non-empty and, in addition, under hypotheses (H_1), (H_4) and (H_5), the energy $I(g, G) < \infty$. More precisely, there exists $C > 0$ such that

$$I(g, G) \leqslant C\biggl[\int_\Omega (1 + |\nabla g| + |G| + |\nabla G|)\,\mathrm{d}x + \|D^s g\|(\Omega) + \|D^s G\|(\Omega)\biggr] \tag{3.1}$$

for all $(g, G) \in \mathrm{SD}(\Omega; \mathbb{R}^d)$.

Furthermore, given $(g, G) \in \mathrm{SD}(\Omega; \mathbb{R}^d)$ from theorem 2.5, there exists $h \in \mathrm{SBV}(\Omega; \mathbb{R}^N)$ such that

$$\nabla h = G, \quad \mathcal{L}^N \text{ a.e. in } \Omega$$

and

$$\|D^s h\|(\Omega) \leqslant C_1 \|G\|_{L^1}.$$

Moreover, by lemma 2.4, there exists a sequence $\{\bar{u}_n\}$ of piecewise constant functions such that

$$\bar{u}_n \xrightarrow[n\to\infty]{L^1} g - h, \qquad \|D^s \bar{u}_n\|(\Omega) \xrightarrow[n\to\infty]{} \|Dg - Dh\|(\Omega).$$

Now define $u_n \in \mathrm{SBV}(\Omega; \mathbb{R}^N)$ as

$$u_n := \bar{u}_n + h.$$

Clearly, $\nabla u_n(x) = G(x)$ for $\mathcal{L}^N$-almost every $x \in \Omega$ and $u_n \to g$ in L^1. Thus, by hypotheses (H_1), (H_4) and (H_5),

$$I(g, G) \leqslant \liminf_{n\to\infty} \biggl\{\int_\Omega W(\nabla u_n, \nabla^2 u_n)\,\mathrm{d}x + \int_{S_{u_n}} \Psi_1([u_n](x), \nu(u_n(x)))\,\mathrm{d}\mathcal{H}^{N-1} + \int_{S_{\nabla u_n}} \Psi_2([\nabla u_n](x), \nu(\nabla u_n(x)))\,\mathrm{d}\mathcal{H}^{N-1}\biggr\}$$

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

$$\leqslant C \liminf_{n\to\infty} \left\{ \int_\Omega W(G,\nabla G)\,\mathrm{d}x + \|D^s u_n\|(\Omega) + \|D^s \nabla u_n\|(\Omega) \right\}$$

$$\leqslant C \left[\int_\Omega (1 + |\nabla g| + |G| + |\nabla G|)\,\mathrm{d}x + \|D^s g\|(\Omega) + \|D^s G\|(\Omega) \right].$$

The main result of this section is stated as follows.

THEOREM 3.2. *For all $(g,G) \in \mathrm{SD}(\Omega;\mathbb{R}^d)$, under hypotheses* (H_1)–(H_7) *we have that*

$$I(g,G) = \int_\Omega (W_1(G-\nabla g) + W_2(G,\nabla G))\,\mathrm{d}x\mathcal{H}^{N-1} + \int_{S_g} \gamma_1([g],\nu_g)\,\mathrm{d} + \int_{S_G} \gamma_2(G^+,G^-,\nu_G)\,\mathrm{d}\mathcal{H}^{N-1},$$

where, given $A, B, \Lambda, \Gamma \in \mathbb{R}^{d\times N}$, $C \in \mathbb{R}^{d\times N\times N}$, $\lambda \in \mathbb{R}^d$ and $\nu \in \mathcal{S}^{N-1}$,

$$W_1(A) = \inf_{u\in \mathrm{SBV}^2(Q;\mathbb{R}^d)} \left\{ \int_{S_u} \Psi_1([u],\nu_u)\,\mathrm{d}\mathcal{H}^{N-1},\ u|_{\partial Q} = 0,\ \nabla u = A \text{ a.e. in } Q \right\}, \tag{3.2}$$

$$W_2(B,C) = \inf_{v\in \mathrm{SBV}(Q;\mathbb{R}^{d\times N})} \left\{ \int_Q W(B,\nabla v(x))\,\mathrm{d}x + \int_{S_v} \Psi_2([v],\nu(v))\,\mathrm{d}\mathcal{H}^{N-1},\ v|_{\partial Q}(x) = Cx \right\}, \tag{3.3}$$

$$\gamma_1(\lambda,\nu) = \inf_{u\in \mathrm{SBV}^2(Q_\nu;\mathbb{R}^d)} \left\{ \int_{S_u} \Psi_1([u],\nu_u)\,\mathrm{d}\mathcal{H}^{N-1},\ u|_{\partial Q_\nu} = \gamma_{(\lambda,\nu)},\ \nabla u = 0 \text{ a.e. in } Q_\nu \right\}, \tag{3.4}$$

$$\gamma_{(\lambda,\nu)}(x) := \begin{cases} \lambda & \text{if } x\cdot\nu > 0, \\ 0 & \text{if } x\cdot\nu < 0, \end{cases}$$

$$\gamma_2(\Lambda,\Gamma,\nu) = \inf_{v\in \mathrm{SBV}(Q_\nu;\mathbb{R}^{d\times N})} \left\{ \int_{Q_\nu} W^\infty(v,\nabla v)\,\mathrm{d}x + \int_{S_v} \Psi_2([v],\nu(v))\,\mathrm{d}\mathcal{H}^{N-1},\ v|_{\partial Q_\nu} = \gamma_{(\Lambda,\Gamma,\nu)} \right\}, \tag{3.5}$$

$$\gamma_{(\Lambda,\Gamma,\nu)}(x) := \begin{cases} \Lambda & \text{if } x\cdot\nu > 0, \\ \Gamma & \text{if } x\cdot\nu < 0. \end{cases}$$

REMARK 3.3. We observe that, as a consequence of hypothesis (H_6), the function W_1 is homogeneous of degree one. Moreover, it is easy to check that the recession

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

function of W_2 in the second variable is given by

$$W_2^\infty(B,C) = \inf_{v\in \mathrm{SBV}(Q;\mathbb{R}^{d\times N})} \biggl\{ \int_Q W^\infty(B,\nabla v(x))\,\mathrm{d}x + \int_{S_v} \Psi_2([v],\nu(v))\,\mathrm{d}\mathcal{H}^{N-1},\ v|_{\partial Q}(x) = Cx \biggr\}$$

for all $(B,C) \in \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N\times N}$.

The proof of theorem 3.2 is an immediate consequence of lemma 3.4, proposition 3.5 and theorem 3.6.

LEMMA 3.4. *Under hypotheses* (H_4) *and* (H_7)*, for all* $(g,G) \in \mathrm{SD}(\Omega;\mathbb{R}^d)$*,*

$$I(g,G) = I_1(g,G) + I_2(G), \tag{3.6}$$

where

$$I_1(g,G) = \inf_{\{u_n\}\subset \mathrm{SBV}^2(\Omega;\mathbb{R}^d)} \biggl\{ \liminf_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n],\nu(u_n))\,\mathrm{d}\mathcal{H}^{N-1},\ u_n \xrightarrow{L^1} g,\ \nabla u_n \xrightarrow{L^1} G \biggr\}$$

and

$$I_2(G) = \inf_{\{v_n\}\subset \mathrm{SBV}(\Omega;\mathbb{R}^{d\times N})} \biggl\{ \liminf_{n\to\infty} \int_\Omega W(v_n,\nabla v_n)\,\mathrm{d}x + \int_{S_{v_n}} \Psi_2([v_n],\nu(v_n))\,\mathrm{d}\mathcal{H}^{N-1},\ v_n \xrightarrow{L^1} G \biggr\}.$$

Proof. In fact, it is immediate to see that

$$I_1(g,G) + I_2(G) \leqslant I(g,G).$$

On the other hand, let $u_n \in \mathrm{SBV}^2(\Omega;\mathbb{R}^d)$ with $u_n \to g$ in L^1 and $\nabla u_n \to G$ in L^1 be such that

$$I_1(g,G) = \lim_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n],\nu(u_n))\,\mathrm{d}\mathcal{H}^{N-1}$$

and let $v_n \in \mathrm{SBV}(\Omega;\mathbb{R}^{d\times N})$ with $v_n \to G$ in L^1 be such that

$$I_2(G) = \lim_{n\to\infty} \biggl[\int_\Omega W(v_n,\nabla v_n)\,\mathrm{d}x + \int_{S_{v_n}} \Psi_2([v_n],\nu(v_n))\,\mathrm{d}\mathcal{H}^{N-1} \biggr].$$

By theorem 2.5, let $h_n \in \mathrm{SBV}(\Omega;\mathbb{R}^d)$ be such that $\nabla h_n = v_n - \nabla u_n$, and by lemma 2.4, let $\tilde{h}_n$ be a piecewise constant function with $\|\tilde{h}_n - h_n\|_{L^1} < 1/n$ and $|\|D\tilde{h}_n\|(\Omega) - \|Dh_n\|(\Omega)| < 1/n$. Then the sequence

$$w_n = u_n + h_n - \tilde{h}_n$$

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

is admissible for $I(g,G)$ and

$$\begin{aligned} I(g,G) &\leqslant \liminf_{n\to\infty}\bigg[\int_\Omega W(\nabla w_n,\nabla^2 w_n)\,\mathrm{d}x + \int_{S_{w_n}} \Psi_1([w_n],\nu(w_n))\,\mathrm{d}\mathcal{H}^{N-1} \\ &\qquad + \int_{S_{\nabla w_n}} \Psi_2([\nabla w_n],\nu(\nabla w_n))\,\mathrm{d}\mathcal{H}^{N-1}\bigg] \\ &\leqslant \lim_{n\to\infty}\int_{S_{u_n}} \Psi_1([u_n],\nu(u_n))\,\mathrm{d}\mathcal{H}^{N-1} \\ &\quad + \limsup_{n\to\infty}\int_{S_{h_n}} \Psi_1([h_n],\nu(h_n))\,\mathrm{d}\mathcal{H}^{N-1} \\ &\quad + \limsup_{n\to\infty}\int_{S_{\tilde h_n}} \Psi_1([\tilde h_n],\nu(\tilde h_n))\,\mathrm{d}\mathcal{H}^{N-1} \\ &\quad + \lim_{n\to\infty}\bigg[\int_\Omega W(v_n,\nabla v_n)\,\mathrm{d}x + \int_{S_{v_n}} \Psi_2([v_n],\nu(v_n))\,\mathrm{d}\mathcal{H}^{N-1}\bigg] \\ &\leqslant I_1(g,G) + I_2(G) + C\int_\Omega |v_n - \nabla u_n|\,\mathrm{d}x \end{aligned}$$

by conditions (H_4) and (H_7), theorem 2.5 and lemma 2.4. Inequality $I_1(g,G) + I_2(G) \geqslant I(g,G)$ follows by letting $n\to\infty$ since $v_n \to G$ and $\nabla u_n \to G$ in L^1. □

PROPOSITION 3.5. *For all* $G \in \mathrm{SBV}(\Omega;\mathbb{R}^{d\times N})$, *under hypotheses* ($H_1$)–($H_3$) *and* ($H_5$),

$$I_2(G) = \int_\Omega W_2(G,\nabla G)\,\mathrm{d}x + \int_{S_G} \gamma_2(G^+,G^-,\nu_G)\,\mathrm{d}\mathcal{H}^{N-1}.$$

Proof. The proof is a consequence of theorem 4.2.2 in [8]. □

THEOREM 3.6. *Under hypotheses* (H_4), (H_6) *and* (H_7), *for all* $(g,G) \in \mathrm{SD}(\Omega;\mathbb{R}^d)$,

$$I_1(g,G) = \int_\Omega W_1(G-\nabla g)\,\mathrm{d}x + \int_{S_g} \gamma_1([g],\nu_g)\,\mathrm{d}\mathcal{H}^{N-1}. \tag{3.7}$$

The proof of theorem 3.6 will be divided into three parts (§§ 3.1–3.3). Specifically, in § 3.1 we will introduce a local version of $I_1(g,G)$ defined on $\mathcal{A}(\Omega)$ and show that $I_1(g,G,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure absolutely continuous with respect to $\mathcal{L}^N + \mathcal{H}^{N-1} \lfloor S_g$. In §§ 3.2 and 3.3, we will prove that, for all $A \in \mathcal{A}(\Omega)$,

$$I_1(g,G,A) = \int_A W_1(G-\nabla g)\,\mathrm{d}x + \int_{A\cap S(g)} \gamma_1([g],\nu_g)\,\mathrm{d}\mathcal{H}^{N-1}$$

from which, taking $A=\Omega$, equality (3.7) follows, completing the proof of theorem 3.6.

3.1. Localization

We start by localizing $I_1(g,G)$, i.e. we define, for $A \in \mathcal{A}(\Omega)$,

$$I_1(g,G,A) := \inf_{\{u_n\}\subset \mathrm{SBV}^2(A;\mathbb{R}^d)} \bigg\{ \liminf_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n],\nu(u_n))\,\mathrm{d}\mathcal{H}^{N-1}, \\ u_n \xrightarrow{L^1(A;\mathbb{R}^d)} g,\ \nabla u_n \xrightarrow{L^1(A;\mathbb{R}^{d\times N})} G \bigg\}.$$

Our objective in this subsection is to show that $I_1(g,G,\cdot)\lfloor\mathcal{A}(\Omega)$ is a Radon measure.

Remark 3.7. Following the argument used in remark 3.1, it is easy to see that there exists $C>0$ such that, for all $A\in\mathcal{A}(\Omega)$,

$$I_1(g,G,A) \leqslant C\bigg[\int_A (|\nabla g| + |G|)\,\mathrm{d}x + \|D^s g\|(A)\bigg]. \tag{3.8}$$

The following lemma shows that $I_1(g,G,\cdot)$ is nested-subadditive.

Lemma 3.8. *Let $A,B,C\in\mathcal{A}(\Omega)$ with $A\subset\subset B\subset C$. Then*

$$I_1(g,G,C) \leqslant I_1(g,G,B) + I_1(g,G,C\setminus\bar{A}). \tag{3.9}$$

Proof. Let $u_n \in \mathrm{SBV}^2(B;\mathbb{R}^d)$ and $v_n\in\mathrm{SBV}^2(C\setminus\bar{A};\mathbb{R}^d)$ be two sequences such that $u_n\to g$ in $L^1(B;\mathbb{R}^d)$, $\nabla u_n \to G$ in $L^1(B;\mathbb{R}^{d\times N})$, $v_n\to g$ in $L^1(C\setminus\bar{A};\mathbb{R}^d)$, $\nabla v_n\to G$ in $L^1(C\setminus\bar{A};\mathbb{R}^{d\times N})$, and such that, in addition,

$$I_1(g,G,B) = \lim_{n\to\infty}\int_{S_{u_n}} \Psi_1([u_n],\nu(u_n))\,\mathrm{d}\mathcal{H}^{N-1}$$

and

$$I_1(g,G,C\setminus\bar{A}) = \lim_{n\to\infty}\int_{S_{v_n}} \Psi_1([v_n],\nu(v_n))\,\mathrm{d}\mathcal{H}^{N-1}.$$

Note that

$$u_n - v_n \to 0 \quad \text{in } L^1(B\cap(C\setminus\bar{A});\mathbb{R}^d) \tag{3.10}$$

and

$$\nabla u_n - \nabla v_n \to 0 \quad \text{in } L^1(B\cap(C\setminus\bar{A});\mathbb{R}^{d\times N}).$$

For $\delta>0$, define

$$A_\delta := \{x\in B,\ \operatorname{dist}(x,A)<\delta\}.$$

Let $d(x) := \operatorname{dist}(x,A), x\in C$. Since the distance function to a fixed set is Lipschitz continuous (see [28, exercise 1.1]), we can apply the change of variables formula (see [18, theorem 2, § 3.4.3]) to obtain

$$\int_{A_\delta\setminus\bar{A}} |u_n - v_n| Jd(x)\,\mathrm{d}x = \int_0^\delta \bigg[\int_{d^{-1}(y)} |u_n - v_n|\,\mathrm{d}\mathcal{H}^{N-1}(x)\bigg]\,\mathrm{d}y$$

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

and, as $Jd(\cdot)$ is bounded and (3.10) holds, then, for almost every $\rho \in [0, \delta]$, it follows that

$$\lim_{n\to\infty} \int_{d^{-1}(\rho)} |u_n - v_n| \, \mathrm{d}\mathcal{H}^{N-1} = \lim_{n\to\infty} \int_{\partial A_\rho} |u_n - v_n| \, \mathrm{d}\mathcal{H}^{N-1} = 0. \tag{3.11}$$

Fix ρ_0 such that (3.11) holds. We observe that A_{ρ_0} is a set with locally Lipschitz boundary since it is a level set of a Lipschitz function (see, for example, [22]). Hence, we can consider u_n, v_n, ∇u_n, ∇v_n on ∂A_{ρ_0} in the sense of traces and define

$$w_n = \begin{cases} u_n & \text{in } \bar{A}_{\rho_0}, \\ v_n & \text{in } C \setminus \bar{A}_{\rho_0}. \end{cases}$$

Then

$$I_1(g, G, C) \leqslant \liminf_{n\to\infty} \int_{S_{w_n}} \Psi_1([w_n], \nu(w_n)) \, \mathrm{d}H^{N-1}$$

and, using (H_4) and (3.10), we obtain (3.9). □

THEOREM 3.9. *Assume that hypothesis* (H_4) *holds. Then $I_1(g, G, \cdot) \lfloor \mathcal{A}(\Omega)$ is a Radon measure absolutely continuous with respect to $\mathcal{L}^N + \mathcal{H}^{N-1} \lfloor S_g$.*

Proof. Let $u_n \in \mathrm{SBV}^2(\Omega; \mathbb{R}^d)$ be such that $u_n \to g$ in $L^1(\Omega; \mathbb{R}^d)$, $\nabla u_n \to G$ in $L^1(\Omega; \mathbb{R}^{d\times N})$ and

$$I_1(g, G, \Omega) = \lim_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n], \nu(u_n)) \, \mathrm{d}\mathcal{H}^{N-1},$$

and define, for all Borel sets $B \subset \mathbb{R}^N$,

$$\mu_n(B) := \int_{S_{u_n} \cap B} \Psi_1([u_n], \nu(u_n)) \, \mathrm{d}\mathcal{H}^{N-1}.$$

By (H_4), the sequence of non-negative Radon measures $\{\mu_n\}$ is uniformly bounded in $\mathcal{M}(\mathbb{R}^N)$ and thus, passing to a subsequence if necessary, we conclude that

$$\mu_n \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\mathbb{R}^N).$$

Let us show that, for all $V \in \mathcal{A}(\Omega)$,

$$\mu(V) = I_1(g, G, V). \tag{3.12}$$

Given $V \in \mathcal{A}(\Omega)$, let $\epsilon > 0$ and take $W \subset\subset V$ such that $\mu(V \setminus W) < \epsilon$. It follows that

$$\begin{aligned} \mu(V) &\leqslant \mu(W) + \epsilon \\ &= \mu(\Omega) - \mu(\Omega \setminus W) + \epsilon \\ &\leqslant I_1(g, G, \Omega) - I_1(g, G, \Omega \setminus \bar{W}) + \epsilon \\ &\leqslant I_1(g, G, V) + \epsilon, \end{aligned}$$

where we have used the equality $\mu(\Omega) = \mu(\bar{\Omega})$ and lemma 3.8. Thus, letting $\epsilon \to 0$, we obtain

$$\mu(V) \leqslant I_1(g, G, V). \tag{3.13}$$

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

Now let us see the reverse inequality. Define, for $A \in \mathcal{A}(\Omega)$,

$$\lambda(A) := \int_A (|\nabla g| + |G|)\,\mathrm{d}x + \|D^s g\|(A). \tag{3.14}$$

Let $K \subset\subset V$ be a compact set such that $\lambda(V \setminus K) < \epsilon$, and choose an open set W such that $K \subset\subset W \subset\subset V$. Again using lemma 3.8, (3.14) and (3.8),

$$\begin{aligned} I_1(g, G, V) &\leqslant I_1(g, G, W) + I_1(g, G, V \setminus K) \\ &\leqslant \mu(\bar{W}) + C\lambda(V \setminus K) \\ &\leqslant \mu(V) + C\epsilon, \end{aligned}$$

which, together with (3.13), yields (3.12) by letting $\epsilon \to 0$. □

3.2. Lower bound

The objective of this part is to show that, for all $A \in \mathcal{A}(\Omega)$,

$$I_1(g, G, A) \geqslant \int_A W_1(G(x) - \nabla g(x))\,\mathrm{d}x + \int_{A \cap S(g)} \gamma_1([g], \nu_g)\,\mathrm{d}\mathcal{H}^{N-1}. \tag{3.15}$$

To prove inequality (3.15), let $u_n \in \mathrm{SBV}^2(\Omega; \mathbb{R}^N)$ be such that

$$u_n \xrightarrow[n\to\infty]{L^1} g, \qquad \nabla u_n \xrightarrow[n\to\infty]{L^1} G,$$
$$\lim_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n], \nu(u_n))\,\mathrm{d}\mathcal{H}^{N-1} < \infty.$$

Define μ_n as

$$\mu_n(B) = \int_{B \cap S(u_n)} \Psi_1([u_n], \nu(u_n))\,\mathrm{d}\mathcal{H}^{N-1}$$

for all Borel sets $B \subset \Omega$. Since, by the hypotheses on Ψ_1, the sequence of Radon measures $\{\mu_n\}$ is bounded, then there exists (up to a subsequence) $\mu \in \mathcal{M}(\Omega)$ with $\mu_n \overset{*}{\rightharpoonup} \mu \mathrm{in} \mathcal{M}(\Omega)$. We now show that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^N}(x_0) \geqslant W_1(G(x_0) - \nabla g(x_0)) \tag{3.16}$$

for $\mathcal{L}^N$-almost every $x_0 \in \Omega$ and that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{N-1} \lfloor S_g}(x_0) \geqslant \gamma_1([g](x_0), \nu_g(x_0)) \tag{3.17}$$

for $\mathcal{H}^{N-1} \lfloor S_g$-almost every $x_0 \in \Omega$.

Proof of equation (3.16). Let $x_0 \in \Omega$ be a point of approximate differentiability of g and of approximately continuity of G (see theorem 2.3 and equality (2.1)) and such that $\mathrm{d}\mu/\mathrm{d}\mathcal{L}^N(x_0)$ exists. Let $\{\delta_k\} \to 0$ be such that $\mu(\partial Q(x_0, \delta_k)) = 0$. Then

$\lim_{n\to\infty}\mu_n(Q(x_0,\delta_k)) = \mu(Q(x_0,\delta_k))$ and

$$\begin{aligned}
&\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^N}(x_0)\\
&\quad= \lim_{k\to\infty}\frac{\mu(Q(x_0,\delta_k))}{\mathcal{L}^N(Q(x_0,\delta_k))}\\
&\quad= \lim_{k,n\to\infty}\frac{1}{\delta_k^N}\int_{S_{u_n}\cap Q(x_0,\delta_k)} \Psi_1([u_n],\nu(u_n))\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad= \lim_{k,n\to\infty}\frac{1}{\delta_k}\int_{Q\cap\{y\colon x_0+\delta_k y\in S_{u_n}\}} \Psi_1([u_n](x_0+\delta_k y),\nu(u_n)(x_0+\delta_k y))\,\mathrm{d}\mathcal{H}^{N-1}(y).
\end{aligned}$$

Defining

$$v_{n,k}(y) := \frac{u_n(x_0+\delta_k y)-g(x_0)}{\delta_k},\quad y\in Q,$$

we have that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^N}(x_0) = \lim_{k,n\to\infty}\int_{Q\cap S_{v_{n,k}}} \Psi_1([v_{n,k}],\nu(v_{n,k}))\,\mathrm{d}\mathcal{H}^{N-1}. \tag{3.18}$$

As x_0 is a point of approximate differentiability of g, then

$$v_{n,k} \xrightarrow[k,n\to\infty]{L^1} \nabla g(x_0)(\cdot)$$

since

$$\begin{aligned}
\lim_{n\to\infty}\int_Q |v_{n,k}(y)-\nabla g(x_0)y|\,\mathrm{d}y
&= \lim_{n\to\infty}\int_Q\left|\frac{u_n(x_0+\delta_k y)-g(x_0)}{\delta_k}-\nabla g(x_0)y\right|\mathrm{d}y\\
&= \int_Q\left|\frac{g(x_0+\delta_k y)-g(x_0)}{\delta_k}-\nabla g(x_0)y\right|\mathrm{d}y\\
&= \frac{1}{\delta_k^N}\int_{Q(x_0,\delta_k)}\left|\frac{g(z)-g(x_0)-\nabla g(x_0)(z-x_0)}{\delta_k}\right|\mathrm{d}y
\end{aligned}$$

by a change of variables. Similarly, as x_0 is also an approximately continuity point of G,

$$\nabla v_{k,n} \xrightarrow[k,n\to\infty]{L^1} G(x_0). \tag{3.19}$$

We now change the sequence $\{v_{n,k}\}$ to comply in (3.18) with the definition of W_1 (see (3.2)). We start by setting

$$w_{n,k}(y) = v_{n,k}(y) - \nabla g(x_0)y,\quad y\in Q,$$

and, following the argument used in (3.11), we choose $r_{n,k}\in\,]0,1[$ such that

$$r_{n,k} \xrightarrow[k,n\to\infty]{} 1$$

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

and

$$\int_{\partial Q(0,r_{n,k})} |w_{n,k}| \, \mathrm{d}\mathcal{H}^{N-1} \xrightarrow[k,n\to\infty]{} 0. \tag{3.20}$$

By theorem 2.5, let $\rho_{n,k}$ be such that

$$\nabla \rho_{n,k}(y) = G(x_0) - \nabla v_{n,k}(y), \quad y \in Q,$$

and define

$$z_{n,k} := w_{n,k} + \rho_{n,k} \quad \text{in } Q(0, r_{n,k}).$$

Note that

$$\nabla z_{n,k} = G(x_0) - \nabla g(x_0) \quad \text{in } Q(0, r_{n,k}).$$

In addition, by (3.19), $\nabla \rho_{n,k} \xrightarrow[k,n\to\infty]{} 0$, and then, by theorem 2.5,

$$|D^{\mathrm{s}} \rho_{n,k}|(Q(0, r_{n,k})) \xrightarrow[k,n\to\infty]{} 0. \tag{3.21}$$

Thus, by the continuity of the trace with respect to the intermediate topology (see [6, proposition 3.88]), it follows that

$$\int_{\partial Q(0,r_{n,k})} |\rho_{n,k}| \, \mathrm{d}\mathcal{H}^{N-1} \xrightarrow[k,n\to\infty]{} 0. \tag{3.22}$$

Applying lemma 2.7 in $Q \setminus (Q(0, r_{n,k}))$, let $\{\eta_{n,k}\}$ be a sequence of functions such that

$$\nabla \eta_{n,k}(y) = G(x_0) - \nabla g(x_0) \quad \text{in } Q \setminus (Q(0, r_{n,k})), \tag{3.23}$$

$$\eta_{n,k} = 0 \quad \text{on } \partial(Q \setminus (Q(0, r_{n,k}))) \tag{3.24}$$

and

$$|D^{\mathrm{s}} \eta_{n,k}|(Q \setminus Q(0, r_{n,k})) \leqslant C(N)|Q \setminus Q(0, r_{n,k})|. \tag{3.25}$$

Then the sequence

$$\tilde{z}_{n,k}(y) := \begin{cases} z_{n,k}(y), & y \in Q(0, r_{n,k}), \\ \eta_{n,k}(y), & y \in Q \setminus (Q(0, r_{n,k})) \end{cases}$$

is admissible for $W^1(G(x_0) - \nabla g(x_0))$, and, in addition by, (H$_4$), (H$_7$) and (3.24), we have, for any n and k, that

$$\begin{aligned} &\int_{Q\cap S_{\tilde{z}_{n,k}}} \Psi_1([\tilde{z}_{n,k}], \nu(\tilde{z}_{n,k})) \, \mathrm{d}\mathcal{H}^{N-1} \\ &\quad \leqslant \int_{Q(0,r_{n,k})\cap S_{z_{n,k}}} \Psi_1([z_{n,k}], \nu(z_{n,k})) \, \mathrm{d}\mathcal{H}^{N-1} \\ &\qquad + C\biggl[\int_{\partial Q(0,r_{n,k})} |z_{n,k}| \, \mathrm{d}\mathcal{H}^{N-1} + \int_{[Q\setminus Q(0,r_{n,k})]\cap S_{\eta_{n,k}}} |[\eta_{n,k}]| \, \mathrm{d}\mathcal{H}^{N-1}\biggr] \end{aligned}$$

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

$$\leqslant \int_{Q(0,r_{n,k})\cap S_{v_{n,k}}} \Psi_1([v_{n,k}],\nu(v_{n,k}))\,\mathrm{d}\mathcal{H}^{N-1} + C\biggl[\int_{\partial Q(0,r_{n,k})} |w_{n,k}|\,\mathrm{d}\mathcal{H}^{N-1} + \int_{\partial Q(0,r_{n,k})} |\rho_{n,k}|\,\mathrm{d}\mathcal{H}^{N-1} + \int_{Q(0,r_{n,k})\cap S_{\rho_{n,k}}} |[\rho_{n,k}]|\,\mathrm{d}\mathcal{H}^{N-1} + \int_{[Q\setminus Q(0,r_{n,k})]\cap S_{\eta_{n,k}}} |[\eta_{n,k}]|\,\mathrm{d}\mathcal{H}^{N-1}\biggr].$$

Therefore, by (3.20)–(3.22) and (3.25), we obtain that

$$\liminf_{k,n\to\infty} \int_{Q\cap S_{\tilde{z}_{n,k}}} \Psi_1([\tilde{z}_{n,k}],\nu(\tilde{z}_{n,k}))\,\mathrm{d}\mathcal{H}^{N-1} \leqslant \lim_{k,n\to\infty} \int_{Q\cap S_{v_{n,k}}} \Psi_1([v_{n,k}],\nu(v_{n,k}))\,\mathrm{d}\mathcal{H}^{N-1},$$

which, together with (3.18), implies (3.16). □

Proof of equation (3.17). Let $x_0 \in S_g$ be such that $\mathrm{d}\mu/\mathrm{d}\mathcal{H}^{N-1} \lfloor S_g(x_0)$ exists,

$$\lim_{\delta\to 0} \frac{\mathcal{H}^{N-1}(S_g \cap Q_\nu(x_0,\delta))}{\delta^{N-1}} = 1, \tag{3.26}$$

$$\lim_{\delta\to 0} \frac{1}{\delta^{N-1}} \int_{Q_\nu(x_0,\delta)} |G(x)|\,\mathrm{d}x = 0, \tag{3.27}$$

where $\nu \equiv \nu_g(x_0)$. We note that, for $\mathcal{H}^{N-1}$-a.e. $x_0 \in S_g$, all the conditions above hold (see [6] for (3.26)). Equality (3.27) holds by (3.26) and the fact that

$$\frac{\mathrm{d}|G|\mathcal{L}^N}{\mathrm{d}\mathcal{H}^{N-1} \lfloor S_g} = 0.$$

Let $\{\delta_k\} \to 0$ be such that $\mu(\partial Q_\nu(x_0,\delta_k)) = 0$. Then,

$$\lim_{n\to\infty} \mu_n(Q_\nu(x_0,\delta_k)) = \mu(Q_\nu(x_0,\delta_k))$$

and

$$\begin{aligned}
&\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{N-1} \lfloor S_g}(x_0) \\
&\quad = \lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{\mathcal{H}^{N-1}(S_g \cap Q_\nu(x_0,\delta_k))}\mu_n(Q_\nu(x_0,\delta_k)) \\
&\quad = \lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{\mathcal{H}^{N-1}(S_g \cap Q_\nu(x_0,\delta_k))} \int_{S_{u_n}\cap Q_\nu(x_0,\delta)} \Psi_1([u_n],\nu(u_n))\,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad = \lim_{k\to\infty}\lim_{n\to\infty} \frac{\delta_k^{N-1}}{\mathcal{H}^{N-1}(S_g \cap Q_\nu(x_0,\delta_k))} \\
&\qquad\quad \times \int_{Q_\nu\cap\{y\colon x_0+\delta_k y\in S_{u_n}\}} \Psi_1([u_n](x_0+\delta_k y),\nu(u_n)(x_0+\delta_k y))\,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad = \lim_{k\to\infty}\lim_{n\to\infty} \int_{Q_\nu\cap\{y\colon x_0+\delta_k y\in S_{u_n}\}} \Psi_1([u_n](x_0+\delta_k y),\nu(u_n)(x_0+\delta_k y))\,\mathrm{d}\mathcal{H}^{N-1}
\end{aligned}$$

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

by (3.26). Defining

$$w^1_{n,k}(y) = u_n(x_0 + \delta_k y) - g^-(x_0), \quad y \in Q_\nu,$$

it follows that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{N-1} \lfloor S_g}(x_0) = \lim_{k\to\infty} \lim_{n\to\infty} \int_{Q_\nu \cap S_{w^1_{n,k}}} \Psi_1([w^1_{n,k}](y), \nu(w^1_{n,k})(y))\,\mathrm{d}\mathcal{H}^{N-1}.$$

By (2.2) and (2.3), we have

$$w^1_{n,k} \xrightarrow[n,k\to\infty]{L^1} \gamma_{([g](x_0),\nu)},$$

and, in addition, by (3.27),

$$\nabla w^1_{n,k} \xrightarrow[n,k\to\infty]{L^1} 0$$

since, for all k,

$$\nabla u_n(x_0 + \delta_k \cdot) \xrightarrow[n\to\infty]{} G(x_0 + \delta_k \cdot).$$

Using theorem 2.5, and following the arguments of lemma 3.8, we note that it is possible to modify $w^1_{n,k}$ so that $\nabla w^1_{n,k} = 0$ and $w^1_{n,k}|_{\partial Q_\nu} = \gamma_{([g](x_0),\nu)}$. Thus, by definition of γ_1 (see (3.4)), inequality (3.17) holds. □

As a consequence of (3.16) and (3.17), we now derive (3.15).

Proof of equation (3.15). Denote by μ_a the absolutely continuous part of μ with respect to the Lebesgue measure and denote by μ^s_g the absolutely continuous part of μ with respect to $H^{N-1} \lfloor S_g$. Since μ is a positive measure, we conclude that

$$\begin{aligned}\liminf_{n\to\infty} \mu_n(A) &\geqslant \mu(A)\\ &\geqslant \int_A \mu_a(x)\,\mathrm{d}x + \int_{A\cap S_g} \mu^s_g(x)\,\mathrm{d}\mathcal{H}^{N-1}(x)\\ &\geqslant \int_A W_1(G(x) - \nabla g(x))\,\mathrm{d}x + \int_{A\cap S_g} \gamma_1([g], \nu_g)\,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

Taking the infimum over all sequences

$$u_n \in \mathrm{SBV}^2(\Omega; \mathbb{R}^d), \qquad u_n \xrightarrow[n\to\infty]{L^1} g, \qquad \nabla u_n \xrightarrow[n\to\infty]{L^1} G,$$

inequality (3.15) holds. □

3.3. Upper bound

Our objective here is to show that, for all $A \in \mathcal{A}(\Omega)$,

$$I_1(g, G, A) \leqslant \int_A W_1(G(x) - \nabla g(x))\,\mathrm{d}x + \int_{A\cap S(g)} \gamma_1([g], \nu_g)\,\mathrm{d}\mathcal{H}^{N-1}.$$

For this purpose, it is enough to prove that

$$\frac{\mathrm{d}I_1(g,G,\cdot)}{\mathrm{d}\mathcal{L}^N}(x_0) \leqslant W_1(G(x_0)-\nabla g(x_0)), \qquad \mathcal{L}^N\text{-a.e. } x_0 \in \Omega, \tag{3.28}$$

$$\frac{\mathrm{d}I_1(g,G,\cdot)}{\mathrm{d}\mathcal{H}^{N-1}\lfloor S_g}(x_0) \leqslant \gamma_1([g](x_0),\nu(g)(x_0)), \quad \mathcal{H}^{N-1}\text{-a.e. } x_0 \in S_g. \tag{3.29}$$

We start with two auxiliary results.

Proposition 3.10. *Let Ψ_1 satisfy* (H$_4$) *and* (H$_7$)*. Then constants $C_1, C_2 > 0$ exist such that*

$$|\gamma_1(\lambda,\nu)-\gamma_1(\lambda',\nu)| \leqslant C_1|\lambda-\lambda'|, \quad \forall \lambda,\lambda' \in \mathbb{R}^d, \tag{3.30}$$

$$|W_1(A)-W_1(B)| \leqslant C_2|A-B|, \quad \forall A,B \in \mathbb{R}^{d\times N}. \tag{3.31}$$

Moreover, γ_1 is upper semicontinuous with respect to ν.

Proof. We show (3.31) and we refer to the proof of proposition 4.3 in [13] for the remainder of the statement. We start by showing that

$$W_1(A) \leqslant W_1(B) + C_1|B-A|, \quad \forall A,B \in \mathbb{R}^{d\times N}.$$

Fixing $\epsilon > 0$, let $u \in \mathrm{SBV}(Q;\mathbb{R}^d)$ be such that $u|_{\partial Q} = 0, \nabla u = A$ and

$$\epsilon + W_1(A) \geqslant \int_{S_u \cap Q} \Psi_1([u],\nu_u)\,\mathrm{d}\mathcal{H}^{N-1}.$$

Now let $v \in \mathrm{SBV}(Q;\mathbb{R}^d)$ be such that $v|_{\partial Q} = 0$, $\nabla v = B - A$ and $|D^s v|(Q) \leqslant C|B-A|$ (see lemma 2.7), and set $w = u + v$. Then, by (H$_4$) and (H$_7$),

$$\begin{aligned} W_1(B) &\leqslant \int_{S_w\cap Q} \Psi_1([w],\nu(w))\,\mathrm{d}\mathcal{H}^{N-1} \\ &\leqslant \left\{ \int_{S_u\cap Q} \Psi_1([u],\nu_u)\,\mathrm{d}\mathcal{H}^{N-1} + \int_{S_v\cap Q} \Psi_1([v],\nu(v))\,\mathrm{d}\mathcal{H}^{N-1} \right\} \\ &\leqslant W_1(A) + \epsilon + C_1|B-A|. \end{aligned}$$

The reverse inequality is proved in a similar way. □

The following proposition easily follows from a diagonalization argument.

Proposition 3.11. *Let $(g_n, G_n) \in \mathrm{SD}(\Omega;\mathbb{R}^d)$ be such that*

$$g_n \xrightarrow[n\to\infty]{L^1} g \quad \textit{and} \quad G_n \xrightarrow[n\to\infty]{L^1} G.$$

Then

$$I_1(g,G) \leqslant \liminf_{n\to\infty} I_1(g_n,G_n).$$

We now show that inequalities (3.28) and (3.29) hold.

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

Proof of (3.28). Let x_0 be a point of approximate continuity for G and ∇g, that is, such that

$$\frac{1}{\delta^N}\left\{\int_{Q(x_0,\delta)} |G(x)-G(x_0)| + |\nabla g(x) - \nabla g(x_0)|\,\mathrm{d}x\right\} \xrightarrow[\delta\to 0]{} 0. \tag{3.32}$$

Let $\epsilon > 0$ and consider $u \in \mathrm{SBV}^2(\Omega;\mathbb{R}^N)$ such that

$$W_1(G(x_0) - \nabla g(x_0)) + \epsilon \geqslant \int_{Q\cap Su} \Psi_1([u],\nu_u)\,\mathrm{d}\mathcal{H}^{N-1}, \tag{3.33}$$

$u|_{\partial Q} = 0$ and $\nabla u(x) = G(x_0) - \nabla g(x_0)$ for a.e. $x \in Q$. Extend u by periodicity to all of $\mathbb{R}^N$ and define, for $n \in \mathbb{N}$ and $\delta > 0$,

$$u_{n,\delta}(x) = \frac{\delta}{n} u\biggl(\frac{n(x-x_0)}{\delta}\biggr).$$

Given $\delta > 0$, apply theorem 2.5 and let $\rho_\delta \in \mathrm{SBV}^2(Q(x_0,\delta);\mathbb{R}^N)$ be a function such that

$$\nabla\rho_\delta(x) = G(x) - G(x_0) + \nabla g(x_0) - \nabla g(x) \tag{3.34}$$

$\mathcal{L}^N$-a.e. $x \in Q(x_0,\delta)$ and satisfying

$$\|D\rho_\delta\|(Q(x_0,\delta)) \leqslant C(N)\int_{Q(x_0,\delta)} |G(x) - G(x_0)| + |\nabla g(x_0) - \nabla g(x)|\,\mathrm{d}x.$$

Note that, by (3.32),

$$\frac{\|D\rho_\delta\|(Q(x_0,\delta))}{\delta^N} \xrightarrow[\delta\to 0]{} 0. \tag{3.35}$$

In addition, using lemma 2.4, define a sequence of piecewise constant functions $\rho_{n,\delta}$ such that, for all $\delta > 0$,

$$\rho_{n,\delta} \xrightarrow[n\to\infty]{L^1} -\rho_\delta \quad \text{and} \quad \|D\rho_{n,\delta}\|(Q(x_0,\delta)) \xrightarrow[n\to\infty]{} \|D\rho_\delta\|(Q(x_0,\delta)). \tag{3.36}$$

Now define

$$w_{n,\delta}(x) := g(x) + u_{n,\delta}(x) + \rho_\delta(x) + \rho_{n,\delta}(x), \quad x \in Q(x_0,\delta).$$

Clearly,

$$w_{n,\delta} \in \mathrm{SBV}^2(Q(x_0,\delta);\mathbb{R}^d), \qquad w_{n,\delta} \xrightarrow[n\to\infty]{L^1} g, \qquad \nabla w_{n,\delta} \xrightarrow[n\to\infty]{L^1} G.$$

For each $\delta > 0$, the sequence $w_{n,\delta}$ is admissible for I_1 and

$$\frac{\mathrm{d}I_1(g,G,\cdot)}{\mathrm{d}\mathcal{L}^N}(x_0) = \lim_{\delta\to 0}\frac{I_1(g,G,Q(x_0,\delta))}{\delta^N}.$$

Then

$$\frac{\mathrm{d}I_1(g,G,\cdot)}{\mathrm{d}\mathcal{L}^N}(x_0) \leqslant \liminf_{\delta\to 0}\liminf_{n\to\infty}\left\{\frac{1}{\delta^N}\int_{S_{w_{n,\delta}}\cap Q(x_0,\delta)} \psi_1([w_{n,\delta}],\nu(w_{n,\delta}))\,\mathrm{d}\mathcal{H}^{N-1}\right\}$$

and, by (H_7),

$$\frac{\mathrm{d}I_1(g,G,\cdot)}{\mathrm{d}\mathcal{L}^N}(x_0)$$
$$\leqslant \liminf_{\delta\to 0}\liminf_{n\to\infty}\biggl\{\frac{1}{\delta^N}\int_{S_g\cap Q(x_0,\delta)}\Psi_1([g],\nu_g)\,\mathrm{d}\mathcal{H}^{N-1}$$
$$+\frac{1}{\delta^N}\int_{\{x_0+(\delta/n)S_u\}\cap Q(x_0,\delta)}\Psi_1\biggl(\frac{\delta}{n}[u]\biggl(\frac{n(x-x_0)}{\delta}\biggr),\nu_u\biggl(\frac{n(x-x_0)}{\delta}\biggr)\biggr)\,\mathrm{d}\mathcal{H}^{N-1}$$
$$+\frac{1}{\delta^N}\int_{S_{\rho_\delta}\cap Q(x_0,\delta)}\Psi_1([\rho_\delta],\nu(\rho_\delta))\,\mathrm{d}\mathcal{H}^{N-1}$$
$$+\frac{1}{\delta^N}\int_{S_{\rho_{n,\delta}}\cap Q(x_0,\delta)}\Psi_1([\rho_{n,\delta}],\nu(\rho_{n,\delta}))\,\mathrm{d}\mathcal{H}^{N-1}\biggr\}.$$

By (H_4), we observe that

$$\frac{1}{\delta^N}\int_{S_g\cap Q(x_0,\delta)}\Psi_1([g],\nu_g)\,\mathrm{d}\mathcal{H}^{N-1}\xrightarrow[\delta\to 0]{}0$$

since

$$\frac{\mathrm{d}|D^s g|}{\mathrm{d}\mathcal{L}^N}(x_0)=0.$$

Moreover,

$$\frac{1}{\delta^N}\int_{S_{\rho_\delta}\cap Q(x_0,\delta)}\Psi_1([\rho_\delta],\nu(\rho_\delta))\,\mathrm{d}\mathcal{H}^{N-1}\xrightarrow[\delta\to 0]{}0$$

and

$$\lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{\delta^N}\int_{S_{\rho_{n,\delta}}\cap Q(x_0,\delta)}\Psi_1([\rho_{n,\delta}],\nu(\rho_{n,\delta}))\,\mathrm{d}\mathcal{H}^{N-1}=0$$

by (H_4), (3.35) and (3.36). Finally, changing variables, we obtain that

$$\frac{1}{\delta^N}\int_{\{x_0+(\delta/n)S_u\}\cap Q(x_0,\delta)}\Psi_1\biggl(\frac{\delta}{n}[u]\biggl(\frac{n(x-x_0)}{\delta}\biggr),\nu_u\biggl(\frac{n(x-x_0)}{\delta}\biggr)\biggr)\,\mathrm{d}\mathcal{H}^{N-1}$$
$$=\int_{Q\cap S_u}\Psi_1([u],\nu_u)\,\mathrm{d}\mathcal{H}^{N-1},$$

from where

$$\frac{\mathrm{d}I_1(g,G,\cdot)}{\mathrm{d}\mathcal{L}^N}(x_0)\leqslant\int_{Q\cap S_u}\Psi_1([u],\nu_u)\,\mathrm{d}\mathcal{H}^{N-1}.$$

As a consequence, letting $\varepsilon\to 0$ in (3.33), inequality (3.28) follows. □

Proof of equation (3.29). Following an argument in [5], we note that it suffices to prove (3.29) for $g=\lambda\chi_E$ with $\lambda\in\mathbb{R}$ and where χ_E is the characteristic function of a set of finite perimeter E.

(i) We will start by addressing the case where E *is a polyhedron*. Let $x_0\in S_g$ be such that

$$\lim_{\delta\to 0}\frac{1}{\delta^{N-1}}\int_{Q_\nu(x_0,\delta)}|G(x)|\,\mathrm{d}x=0, \tag{3.37}$$

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

where $\nu \equiv \nu_g(x_0)$. Given $\epsilon > 0$, let $u \in \mathrm{SBV}(Q_\nu;\mathbb{R}^d)$ be such that $\nabla u = 0$, $u|_{\partial Q_\nu} = \gamma_{(\lambda,\nu)}$ and

$$\gamma_1(\lambda,\nu) + \epsilon \geqslant \int_{Q_\nu} \Psi_1([u],\nu_u)\,\mathrm{d}\mathcal{H}^{N-1} \tag{3.38}$$

(see (3.4)). For sufficiently small $\delta > 0$,

$$D^n_\nu(x_0,\delta) := Q_\nu(x_0,\delta) \cap \left\{x\colon \frac{|(x-x_0)\cdot\nu|}{\delta} < \frac{1}{2n}\right\},$$
$$Q^+_\nu(x_0,\delta) \ = Q_\nu(x_0,\delta) \cap \left\{x\colon \frac{(x-x_0)\cdot\nu}{\delta} > 0\right\}$$

and $Q^-_\nu(x_0,\delta)$ in an analogous way. Now set

$$u_{n,\delta}(x) = \begin{cases} \lambda, & x \in Q^+_\nu(x_0,\delta) \setminus D^n_\nu(x_0,\delta), \\ u\left(\dfrac{n(x-x_0)}{\delta}\right), & x \in D^n_\nu(x_0,\delta), \\ 0, & x \in Q^-_\nu(x_0,\delta) \setminus D^n_\nu(x_0,\delta), \end{cases}$$

where u has been extended by periodicity to all of $\mathbb{R}^N$. Note that, for $x \in D^n_\nu(x_0,\delta)$, we have that $n/\delta|(x-x^0)\cdot\nu| < \frac{1}{2}$. Clearly,

$$u_{n,\delta} \xrightarrow[n\to\infty,\delta\to 0]{L^1} \tilde{\gamma}_{(\lambda,\nu)},$$

where, for $x \in Q_\nu(x_0,\delta)$,

$$\tilde{\gamma}_{(\lambda,\nu)}(x) := \begin{cases} \lambda & \text{if } x\cdot\nu > 0, \\ 0 & \text{if } x\cdot\nu < 0. \end{cases}$$

By theorem 2.5, there exists $\zeta_\delta \in \mathrm{SBV}(Q_\nu(x_0,\delta);\mathbb{R}^d)$ such that $\nabla\zeta_\delta = G$ and

$$|D^s\zeta_\delta|(Q_\nu(x_0,\delta)) \leqslant C\|G\|_{L^1(Q_\nu(x_0,\delta);\mathbb{R}^{d\times N})}. \tag{3.39}$$

Moreover, by lemma 2.4, there exists a sequence $\zeta_{n,\delta}$ of piecewise constant functions defined on $Q_\nu(x_0,\delta)$ such that

$$\zeta_{n,\delta} \xrightarrow[n\to\infty]{L^1} \zeta_\delta$$

and

$$|D\zeta_{n,\delta}|(Q_\nu(x_0,\delta)) \xrightarrow[n\to\infty]{} |D\zeta_\delta|(Q_\nu(x_0,\delta)).$$

Set

$$w_{n,\delta} = u_{n,\delta} + \zeta_\delta - \zeta_{n,\delta}.$$

Clearly, $w_{n,\delta}$ is admissible for $I_1(g,G,Q_\nu(x_0,\delta))$. Therefore,

$$\begin{aligned}
&\frac{\mathrm{d}I_1(g,G,\cdot)}{\mathrm{d}\mathcal{H}^{N-1}\lfloor S_g}(x_0)\\
&\quad=\lim_{\delta\to 0}\frac{1}{\delta^{N-1}}I_1(g,G,Q_\nu(x_0,\delta))\\
&\quad\leqslant \lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{\delta^{N-1}}\bigg\{\int_{Q_\nu(x_0,\delta)\cap S_{w_{n,\delta}}}\Psi_1([w_{n,\delta}](x),\nu(w_{n,\delta})(x))\,\mathrm{d}\mathcal{H}^{N-1}\bigg\}\\
&\quad=\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{\delta^{N-1}}\bigg\{\int_{Q_\nu(x_0,\delta)\cap S_{u_{n,\delta}}}\Psi_1([u_{n,\delta}](x),\nu(u_{n,\delta})(x))\,\mathrm{d}\mathcal{H}^{N-1}\\
&\qquad\qquad+\int_{Q_\nu(x_0,\delta)\cap S_{\zeta_\delta}}\Psi_1([\zeta_\delta](x),\nu(\zeta_\delta)(x))\,\mathrm{d}\mathcal{H}^{N-1}\\
&\qquad\qquad+\int_{Q_\nu(x_0,\delta)\cap S_{\zeta_{n,\delta}}}\Psi_1([\zeta_{n,\delta}](x),\nu(\zeta_{n,\delta})(x))\,\mathrm{d}\mathcal{H}^{N-1}\bigg\}\\
&\quad=: J_1+J_2+J_3.
\end{aligned}$$

The terms J_2 and J_3 go to zero due to (H_4), (3.39), (3.3) and (3.37). Moreover,

$$\begin{aligned}
J_1&=\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{\delta^{N-1}}\int_{Q_\nu(x_0,\delta)\cap S_{u_{n,\delta}}}\Psi_1([u_{n,\delta}](x),\nu(u_{n,\delta})(x))\,\mathrm{d}\mathcal{H}^{N-1}\\
&=\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{\delta^{N-1}}\\
&\quad\times\int_{D_\nu^n(x_0,\delta)\cap\{x:\,(n(x-x_0)/\delta)\in S_u\}}\Psi_1\bigg([u]\bigg(\frac{n(x-x_0)}{\delta}\bigg),\nu_u\bigg(\frac{n(x-x_0)}{\delta}\bigg)\bigg)\,\mathrm{d}\mathcal{H}^{N-1}\\
&=\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n^{N-1}}\int_{nQ_\nu\cap\{y:\,|y\cdot\nu|<1/2\}\cap S_u}\Psi_1([u](y),\nu_u(y))\,\mathrm{d}\mathcal{H}^{N-1}(y)\\
&=\int_{Q_\nu\cap S_u}\Psi_1([u](y),\nu_u(y))\,\mathrm{d}\mathcal{H}^{N-1}(y).
\end{aligned}$$

Thus

$$\frac{\mathrm{d}I_1(g,G,\cdot)}{\mathrm{d}\mathcal{H}^{N-1}\lfloor S_g}(x_0)\leqslant\int_{Q_\nu\cap S_u}\Psi_1([u](y),\nu_u(y))\,\mathrm{d}\mathcal{H}^{N-1}(y)$$

and, consequently, (3.29) follows by letting $\epsilon\to 0$ in (3.38).

(ii) Now let E be a general set of finite perimeter and let $g=\lambda\chi_E$, $\lambda\in\mathbb{R}$. Consider E_n a sequence of polyhedra such that (see [15])

$$\begin{aligned}
\mathrm{per}(E_n)&\xrightarrow[n\to\infty]{}\mathrm{per}(E),\\
\mathcal{L}^N(E_n\Delta E)&\xrightarrow[n\to\infty]{}0,
\end{aligned}\tag{3.40}$$

and

$$\chi_{E_n}\xrightarrow[n\to\infty]{L^1}\chi_E.\tag{3.41}$$

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

By proposition 3.10 and [7, proposition 3.6] we obtain a sequence of functions $\gamma_1^m\colon \mathbb{R}^N \to [0,\infty)$ that are continuous, homogeneous of degree one and satisfy

$$\gamma_1(\lambda, y) \leqslant \gamma_1^m(y) \leqslant C|y|, \quad \forall y \in \mathbb{R}^N, \tag{3.42}$$

$$\gamma_1(\lambda, y) = \inf_m \gamma_1^m(y), \tag{3.43}$$

where $\gamma_1(\lambda,\cdot)$ has been extended as an homogeneous function of degree one to all of $\mathbb{R}^N$. Let

$$g_n = \lambda\chi_{E_n}. \tag{3.44}$$

By (3.41) it is clear that

$$g_n \xrightarrow[n\to\infty]{L^1} g.$$

Given $A \in \mathcal{A}(\Omega)$, from the previous case and proposition 3.11, we have that

$$\begin{aligned} I_1(g,G,A) &\leqslant \liminf_{n\to\infty} I_1(g_n,G,A) \\ &\leqslant \liminf_{n\to\infty} \left\{ \int_A W_1(G-\nabla g_n)\,\mathrm{d}x + \int_{S_{g_n}\cap A} \gamma_1([g_n],\nu(g_n))\,\mathrm{d}\mathcal{H}^{N-1} \right\} \\ &= \liminf_{n\to\infty} \left\{ \int_A W_1(G)\,\mathrm{d}x + \int_{S_{g_n}\cap A} \gamma_1([g_n],\nu(g_n))\,\mathrm{d}\mathcal{H}^{N-1} \right\} \\ &\leqslant C\int_A |G|\,\mathrm{d}x + \lim_{n\to\infty}\int_{S_{g_n}\cap A} \gamma_1([g_n],\nu(g_n))\,\mathrm{d}\mathcal{H}^{N-1}, \end{aligned} \tag{3.45}$$

where, in the last inequality, we have used (3.31) together with the fact that $W_1(0) = 0$. For fixed m by (3.45), the definition of g_n, (3.43) and theorem 2.2, it follows that

$$\begin{aligned} I_1(g,G,A) &\leqslant C\int_A |G|\,\mathrm{d}x + \lim_{n\to\infty}\int_{\partial E_n\cap A} \gamma_1^m(\nu(g_n))\,\mathrm{d}\mathcal{H}^{N-1} \\ &\leqslant C\int_A |G|\,\mathrm{d}x + \int_{\partial E\cap A} \gamma_1^m(\nu_g)\,\mathrm{d}\mathcal{H}^{N-1}. \end{aligned}$$

Now, letting $m\to\infty$ and using the monotone convergence theorem, we obtain

$$I_1(g,G,A) \leqslant C\int_A |G|\,\mathrm{d}x + \int_{S_g\cap A} \gamma_1(\lambda,\nu_g)\,\mathrm{d}\mathcal{H}^{N-1}. \tag{3.46}$$

Consider x_0 satisfying (3.37). Then, from (3.46), we immediately conclude that, for g defined by (3.44),

$$\begin{aligned} \frac{\mathrm{d}I_1(g,G,\cdot)}{\mathrm{d}\mathcal{H}^{N-1}\lfloor S_g}(x_0) &= \lim_{\delta\to 0}\frac{1}{\delta^{N-1}} I_1(g,G,Q_\nu(x_0,\delta)) \\ &\leqslant \gamma_1([g](x_0),\nu_g(x_0)). \end{aligned}$$

□

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

4. Integral representation in BV

This section is devoted to the characterization of the energy $I(g,G)$ (see (1.3)) in the full BV setting for $(g,G) \in \mathrm{GSD}(\Omega;\mathbb{R}^d) := \mathrm{BV}^2(\Omega;\mathbb{R}^d) \times \mathrm{BV}(\Omega;\mathbb{R}^{d\times N})$. We refer the reader to theorem 3.2 for the hypotheses and notation used throughout.

THEOREM 4.1. *Let* $(g,G) \in \mathrm{GSD}(\Omega;\mathbb{R}^d)$. *Under hypotheses* (H_1)–(H_7),

$$\begin{aligned} I(g,G) = {} & \int_\Omega (W_1(G-\nabla g) + W_2(G,\nabla G))\,\mathrm{d}x + \int_{S_g} \gamma_1([g],\nu_g)\,\mathrm{d}\mathcal{H}^{N-1} \\ & + \int_{S_G} \gamma_2(G^+,G^-,\nu_G)\,\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega W_1\biggl(-\frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|}\biggr)\,\mathrm{d}|D^c g| \\ & + \int_\Omega W_2^\infty\biggl(G,\frac{\mathrm{d}D^c G}{\mathrm{d}|D^c G|}\biggr)\,\mathrm{d}|D^c G|. \end{aligned} \tag{4.1}$$

To prove theorem 4.1, we start by deriving two auxiliary results.

PROPOSITION 4.2. *Let* $\nu \in S^{N-1}$ *and define, for all* $C \in \mathbb{R}^{d\times N}$,

$$\begin{aligned} \tilde{W}_1(C) = \inf\biggl\{ & \int_{Q_\nu} W_1(C-\nabla v(x))\,\mathrm{d}x + \int_{Q_\nu\cap S_v} \gamma_1([v],\nu(v))\,\mathrm{d}\mathcal{H}^{N-1}, \\ & v \in \mathrm{SBV}^2(Q_\nu;\mathbb{R}^d),\ v|_{\partial Q_\nu}(x) = b(x\cdot\nu), \\ & b \in \mathrm{SBV}^2([-\tfrac{1}{2},\tfrac{1}{2}];\mathbb{R}^d),\ \tfrac{1}{2}b = b(-\tfrac{1}{2})\biggr\}. \end{aligned}$$

Then $\tilde{W}_1(C) = W_1(C)$.

Proof. Clearly, $\tilde{W}_1(C) \leqslant W_1(C)$. Let us prove the reverse inequality. Fix $\epsilon > 0$ and let $v \in \mathrm{SBV}^2(Q_\nu;\mathbb{R}^d)$ with $v|_{\partial Q_\nu}(x) = b(x\cdot\nu)$ for some $b \in \mathrm{SBV}^2([-\frac{1}{2},\frac{1}{2}];\mathbb{R}^d)$, with $\frac{1}{2}b = b(-\frac{1}{2})$ be such that

$$\tilde{W}_1(C) \geqslant \int_{Q_\nu} W_1(C-\nabla v(x))\,\mathrm{d}x + \int_{Q_\nu\cap S_v} \gamma_1([v],\nu(v))\,\mathrm{d}\mathcal{H}^{N-1} - \epsilon. \tag{4.2}$$

Extend v by periodicity to all of $\mathbb{R}^N$ and define

$$w_n(y) = \frac{v(ny)}{n} - Cy, y \in Q_\nu.$$

By theorem 3.2 it follows that

$$\begin{aligned} I_1(w_n,0,Q_\nu) &= \int_{Q_\nu} W_1(C-\nabla v(ny))\,\mathrm{d}y \\ &\quad + \frac{1}{n}\int_{Q_\nu\cap\{y\colon ny\in S_v\}} \gamma_1([v](ny),\nu(v)(ny))\,\mathrm{d}\mathcal{H}^{N-1} \\ &= \int_{Q_\nu} W_1(C-\nabla v(ny))\,\mathrm{d}y + \frac{1}{n^N}\int_{nQ_\nu\cap S_v} \gamma_1([v](y),\nu(v)(y))\,\mathrm{d}\mathcal{H}^{N-1} \\ &= \int_{Q_\nu} W_1(C-\nabla v(ny))\,\mathrm{d}y + \int_{Q_\nu\cap S_v} \gamma_1([v](y),\nu(v)(y))\,\mathrm{d}\mathcal{H}^{N-1}. \end{aligned}$$

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

Therefore, by the Riemann–Lebesgue lemma,

$$\lim_{n\to\infty} I_1(w_n, 0, Q_\nu) = \int_{Q_\nu} W_1(C - \nabla v(y))\,\mathrm{d}y + \int_{Q_\nu\cap S_v} \gamma_1([v](y), \nu(v)(y))\,\mathrm{d}\mathcal{H}^{N-1},$$

and, consequently, by proposition 3.11,

$$I_1(-C(\cdot), 0, Q_\nu) \leqslant \int_{Q_\nu} W_1(C - \nabla v(y))\,\mathrm{d}y + \int_{Q_\nu\cap S_v} \gamma_1([v](y), \nu(v)(y))\,\mathrm{d}\mathcal{H}^{N-1}.$$

Since

$$I_1(-C(\cdot), 0, Q_\nu) = \int_{Q_\nu} W_1(C)\,\mathrm{d}y = W_1(C),$$

then, by (4.2),

$$\tilde{W}_1(C) \geqslant W_1(C) - \epsilon$$

and the result follows by letting $\varepsilon \to 0$. □

LEMMA 4.3. *Let $(g, G) \in \mathrm{GSD}(\Omega; \mathbb{R}^d)$. For all $A \in \mathcal{A}(\Omega)$, define*

$$\tilde{I}_1(g, G, A) = \inf_{\substack{\{g_n\}\subset \mathrm{SBV}^2(A;\mathbb{R}^d)\\ \{G_n\}\subset \mathrm{SBV}(A;\mathbb{R}^{d\times N})}} \Big\{ \liminf_{n\to\infty} I_1(g_n, G_n, A), g_n \xrightarrow[n\to\infty]{L^1} g,\ G_n \xrightarrow[n\to\infty]{L^1} G \Big\}.$$

Then $\tilde{I}_1(g, G, A) = I_1(g, G, A)$.

Proof. Let $(g_n, \nabla g_n) \in \mathrm{SD}(\Omega; \mathbb{R}^d)$ with $g_n \xrightarrow[n\to\infty]{L^1} g$ and $\nabla g_n \xrightarrow[n\to\infty]{L^1} G$. Then

$$I_1(g_n, \nabla g_n, A) \leqslant \int_{S_{g_n}\cap A} \psi_1([g_n], \nu(g_n))\,\mathrm{d}\mathcal{H}^{N-1}$$

for all $n \in \mathbb{N}$. Hence,

$$\liminf_{n\to\infty} \int_{S_{g_n}\cap A} \psi_1([g_n], \nu(g_n))\,\mathrm{d}\mathcal{H}^{N-1} \geqslant \liminf_{n\to\infty} I_1(g_n, \nabla g_n, A) \geqslant \tilde{I}_1(g, G, A).$$

By the arbitrariness of the sequence $\{g_n\}$, it follows that

$$I_1(g, G, A) \geqslant \tilde{I}_1(g, G, A).$$

To show that the reverse inequality is true, let

$$\tilde{I}_1(g, G, A) = \liminf_{n\to\infty} I_1(g_n, G_n, A)$$

with

$$g_n \in \mathrm{SBV}^2(A; \mathbb{R}^d), \qquad G_n \in \mathrm{SBV}(A; \mathbb{R}^{d\times N}), \qquad g_n \xrightarrow[n\to\infty]{L^1} g, \qquad G_n \xrightarrow[n\to\infty]{L^1} G.$$

For each $n \in \mathbb{N}$, let $u_n \in \mathrm{SBV}^2(A; \mathbb{R}^d)$ be such that

$$I_1(g_n, G_n, A) + \frac{1}{n} \geqslant \int_{S_{u_n}} \psi_1([u_n], \nu(u_n))\,\mathrm{d}\mathcal{H}^{N-1},$$

$$|u_n - g_n|_{L_1} \leqslant \frac{1}{n},$$

$$|\nabla u_n - G_n|_{L_1} \leqslant \frac{1}{n}.$$

Therefore,

$$\begin{aligned}\tilde{I}_1(g,G,A) &= \liminf_{n\to\infty} I_1(g_n,G_n,A)\\ &\geqslant \liminf_{n\to\infty} \int_{S_{u_n}} \psi_1([u_n],\nu(u_n))\,\mathrm{d}\mathcal{H}^{N-1}\\ &\geqslant I_1(g,G,A).\end{aligned}$$

□

We proceed now to the proof of theorem 4.1.

Proof of theorem 4.1. Given $(g,G)\in \mathrm{GSD}(\Omega;\mathbb{R}^d)$ by lemma 3.4 (which still holds), we can decompose

$$I(g,G) = I_1(g,G) + I_2(G),$$

where

$$I_1(g,G) = \inf_{\{u_n\}\subset \mathrm{SBV}^2(\Omega;\mathbb{R}^d)} \left\{ \liminf_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n],\nu(u_n))\,\mathrm{d}\mathcal{H}^{N-1},\ u_n \xrightarrow{L^1} g,\ \nabla u_n \xrightarrow{L^1} G \right\} \tag{4.3}$$

and

$$I_2(G) = \inf_{\{v_n\}\subset \mathrm{SBV}(\Omega;\mathbb{R}^{d\times N})} \left\{ \liminf_{n\to\infty} \int_\Omega W(v_n,\nabla v_n)\,\mathrm{d}x + \int_{S_{v_n}} \Psi_2([v_n],\nu(v_n))\,\mathrm{d}\mathcal{H}^{N-1},\ v_n \xrightarrow{L^1} G \right\}. \tag{4.4}$$

As in the proof of theorem 3.2, by theorem 4.2.2 of [8] we have that

$$\begin{aligned} I_2(G) = \int_\Omega W_2(G,\nabla G)\,\mathrm{d}x &+ \int_{S_G\cap\Omega} \gamma_2(G^+,G^-,\nu_G)\,\mathrm{d}\mathcal{H}^{N-1}\\ &+ \int_\Omega W_2^\infty\left(G,\frac{\mathrm{d}D^cG}{\mathrm{d}|D^cG|}\right)\mathrm{d}|D^cG|,\end{aligned}$$

and, hence, to prove our claim (4.1), it is enough to show that

$$\begin{aligned} I_1(g,G) = \int_\Omega W_1(G-\nabla g)\,\mathrm{d}x &+ \int_{S_g} \gamma_1([g],\nu_g)\,\mathrm{d}\mathcal{H}^{N-1}\\ &+ \int_\Omega W_1\left(-\frac{\mathrm{d}D^cg}{\mathrm{d}|D^cg|}\right)\mathrm{d}|D^cg|. \end{aligned} \tag{4.5}$$

We divide the argument into four steps.

Step 1 (localization). As in § 3.1, we can see that $I_1(g,G,\cdot)\lfloor \mathcal{A}(\Omega)$ is an absolutely continuous Radon measure with respect to $\mathcal{L}^N + \mathcal{H}^{N-1}\lfloor S_g + d|D^cg|$.

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

Step 2 (upper bound for the density energy derivation with respect to the Cantor part $D^c g$ of Dg). Let us prove that, for $|D^c g|$-a.e. $x_0 \in \Omega$,

$$\frac{\mathrm{d}I_1(g,G,\cdot)}{\mathrm{d}|D^c g|}(x_0) \leqslant W_1\Big(-\frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|}(x_0)\Big). \tag{4.6}$$

Let g_n be a sequence of regular functions such that

$$g_n \xrightarrow[n\to\infty]{L^1} g, \qquad \|Dg_n\|(\Omega) \xrightarrow[n\to\infty]{} \|Dg\|(\Omega)$$

and, in addition, consider

$$G_n \in \mathrm{SBV}(\Omega;\mathbb{R}^{d\times N}) \quad \text{with } G_n \xrightarrow[n\to\infty]{L^1} G.$$

Given $A \in \mathcal{A}(\Omega)$ by theorem 3.2 (see (3.7)), lemma 4.3 and proposition 3.10, we obtain that

$$\begin{aligned} I_1(g,G,A) &\leqslant \liminf_{n\to\infty} I_1(g_n,G_n,A) \\ &= \liminf_{n\to\infty} \int_A W_1(G_n(x) - \nabla g_n(x))\,\mathrm{d}x \\ &\leqslant C\int_A |G(x)|\,\mathrm{d}x + \lim_{n\to\infty}\int_A W_1(-\nabla g_n(x))\,\mathrm{d}x \\ &= C\int_A |G(x)|\,\mathrm{d}x + \int_A W_1\Big(-\frac{\mathrm{d}Dg}{\mathrm{d}|Dg|}\Big)\,\mathrm{d}|Dg|, \end{aligned} \tag{4.7}$$

where the last equality follows by theorem 2.2 since W_1 is Lipschitz continuous and homogeneous of degree one.

Let $x_0 \in \operatorname{supp}|D^c g|$ such that $\mathrm{d}I_1(g,G,\cdot)/\mathrm{d}|D^c g|(x_0)$ exists and

$$W_1\Big(-\frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|}(x_0)\Big) = \lim_{\delta\to 0}\frac{1}{|D^c g|(Q(x_0,\delta))}\int_{Q(x_0,\delta)} W_1\Big(-\frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|}\Big)\,\mathrm{d}|D^c g|.$$

Then,

$$\begin{aligned} \frac{\mathrm{d}I_1(g,G,\cdot)}{\mathrm{d}|D^c g|}(x_0) &= \lim_{\delta\to 0}\frac{I_1(g,G,Q(x_0,\delta))}{|D^c g|(Q(x_0,\delta))} \\ &\leqslant W_1\Big(-\frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|}(x_0)\Big) \end{aligned}$$

by (4.7) so that (4.6) holds.

Step 3 (lower bound for the density energy derivation with respect to the Cantor part $D^c g$ of Dg). Let us prove that, for $|D^c g|$-a.e. $x_0 \in \Omega$,

$$\frac{\mathrm{d}I_1(g,G,\cdot)}{\mathrm{d}|D^c g|}(x_0) \geqslant W_1\Big(-\frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|}(x_0)\Big). \tag{4.8}$$

Let $x_0 \in \operatorname{supp}|D^c g|$ be such that

$$\frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|}(x_0) = a_{x_0} \otimes \nu_{g(x_0)}$$

for some $a_{x_0} \equiv a \in \mathbb{R}^d$ and $\nu_{g(x_0)} \equiv \nu \in S^{N-1}$ (see Alberti's rank one theorem [2]) and with

$$\frac{\mathrm{d}|G|\mathcal{L}^N}{\mathrm{d}|D^c g|}(x_0) = 0. \tag{4.9}$$

Thus, showing (4.8) is equivalent to showing that

$$\frac{\mathrm{d}I_1(g, G, \cdot)}{\mathrm{d}|D^c g|}(x_0) \geqslant W_1(-a \otimes \nu). \tag{4.10}$$

To prove (4.10), let $(g_n, G_n) \in \mathrm{SBV}^2(\Omega; \mathbb{R}^d) \times \mathrm{SBV}(\Omega; \mathbb{R}^{d\times N})$ be a sequence with

$$g_n \xrightarrow[n\to\infty]{L^1(\Omega;\mathbb{R}^d)} g \quad \text{and} \quad G_n \xrightarrow[n\to\infty]{L^1(\Omega;\mathbb{R}^{d\times N})} G,$$

and fix $\delta > 0$. Note that, by proposition 3.10,

$$\lim_{n\to\infty} [I_1(g_n, G_n, Q_\nu(x_0, \delta)) - I_1(g_n, G, Q_\nu(x_0, \delta))] = 0$$

and, by theorem 3.2 and (4.3), we have that

$$\begin{aligned} I_1(g_n, G, Q_\nu(x_0, \delta)) &= \int_{Q_\nu(x_0,\delta)} W_1(G - \nabla g_n)\,\mathrm{d}x + \int_{Q_\nu(x_0,\delta)\cap S_{g_n}} \gamma_1([g_n], \nu(g_n))\,\mathrm{d}\mathcal{H}^{N-1} \\ &= \delta^N \int_{Q_\nu} W_1(G(x_0 + \delta y) - \nabla g_n(x_0 + \delta y))\,\mathrm{d}y \\ &\quad + \delta^{N-1} \int_{Q_\nu \cap \{y : x_0 + \delta y \in S_{g_n}\}} \gamma_1([g_n](x_0 + \delta y), \nu(g_n)(x_0 + \delta y))\,\mathrm{d}\mathcal{H}^{N-1}. \end{aligned}$$

Defining

$$t_\delta = \frac{|D^c g|(Q_\nu(x_0, \delta))}{\delta^N}, \tag{4.11}$$

we can write that

$$\begin{aligned} \frac{I_1(g_n, G_n, Q_\nu(x_0, \delta))}{|D^c g|(Q_\nu(x_0, \delta))} &= \frac{1}{t_\delta} \int_{Q_\nu} W_1(G(x_0 + \delta y) - \nabla g_n(x_0 + \delta y))\,\mathrm{d}y \\ &\quad + \frac{1}{\delta t_\delta} \int_{Q_\nu \cap \{y : x_0 + \delta y \in S_{g_n}\}} \gamma_1([g_n](x_0 + \delta y), \nu(g_n)(x_0 + \delta y))\,\mathrm{d}\mathcal{H}^{N-1} \end{aligned}$$

and setting

$$w_{n,\delta}(y) = \frac{g_n(x_0 + \delta y) - \int_{Q_\nu} g_n(x_0 + \delta y)\,\mathrm{d}y}{\delta t_\delta}, \quad y \in Q_\nu,$$

we derive that

$$\begin{aligned} \frac{I_1(g_n, G_n, Q_\nu(x_0, \delta))}{|D^c g|(Q_\nu(x_0, \delta))} = \frac{1}{t_\delta} \int_{Q_\nu} & W_1(G(x_0 + \delta y) - t_\delta \nabla w_{n,\delta}(y))\,\mathrm{d}y \\ & + \int_{Q_\nu \cap S_{w_{n,\delta}}} \gamma_1([w_{n,\delta}](y), \nu(w_{n,\delta})(y))\,\mathrm{d}\mathcal{H}^{N-1}. \end{aligned}$$

Consequently,

$$\liminf_{\delta\to 0}\liminf_{n\to\infty}\frac{I_1(g_n,G_n,Q_\nu(x_0,\delta))}{|D^c g|(Q_\nu(x_0,\delta))} = \liminf_{\delta\to 0}\liminf_{n\to\infty}\biggl[\frac{1}{t_\delta}\int_{Q_\nu} W_1(G(x_0+\delta y)-t_\delta\nabla w_{n,\delta}(y))\,\mathrm{d}y + \int_{Q_\nu\cap S_{w_{n,\delta}}}\gamma_1([w_{n,\delta}](y),\nu(w_{n,\delta})(y))\,\mathrm{d}\mathcal{H}^{N-1}\biggr].$$

Using Alberti's result on the blow-up of the Cantor part (see [2], [4, theorem 2.3] and [24, lemma 5.1]), there exists a non-decreasing function $\zeta\in \mathrm{BV}[-\frac{1}{2},\frac{1}{2}]$ such that

$$\zeta(\tfrac{1}{2})-\zeta(-\tfrac{1}{2})=1,\qquad \int_{-1/2}^{1/2}\zeta(s)\,\mathrm{d}s=0 \tag{4.12}$$

and

$$\lim_{\delta\to 0}\lim_{n\to\infty}\int_{Q_\nu}|w_{n,\delta}(y)-a\zeta(y\cdot\nu)|\,\mathrm{d}y=0.$$

Therefore, passing to a diagonalizing sequence $w_k\equiv w_{n(k),\delta(k)}$, setting $\delta_k=\delta(k)$ and using the homogeneity property of W_1, we have that

$$\liminf_{\delta\to 0}\liminf_{n\to\infty}\frac{I_1(g_n,G_n,Q_\nu(x_0,\delta))}{|D^c g|(Q_\nu(x_0,\delta))} = \lim_{k\to\infty}\biggl[\int_{Q_\nu} W_1\biggl(\frac{G(x_0+\delta_k y)}{t_{\delta_k}}-\nabla w_k(y)\biggr)\,\mathrm{d}y + \int_{Q_\nu\cap S_{w_k}}\gamma_1([w_k](y),\nu(w_k)(y))\,\mathrm{d}\mathcal{H}^{N-1}\biggr].$$

Now set

$$v_k(y)=\frac{a(\rho_k*\zeta)(y.\nu)}{c_k},\quad y\in Q_\nu, \tag{4.13}$$

where ρ_k denotes the standard mollifier sequence, and

$$c_k=(\rho_k*\zeta)(\tfrac{1}{2})-(\rho_k*\zeta)(-\tfrac{1}{2}).$$

It is clear, by (4.12), that $c_k\to 1$ as $k\to\infty$. Since

$$w_k-v_k\xrightarrow[k\to\infty]{L^1}0$$

with a similar argument to the one used in lemma 3.8, we can assume that $w_k\lfloor_{\partial Q_\nu}= v_k\lfloor_{\partial Q_\nu}$. Thus, defining

$$\bar{w}_k(y)=w_k(y)-(a\otimes\nu)y, y\in Q_\nu,$$

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

we have that

$$\liminf_{\delta\to 0}\liminf_{n\to\infty}\frac{I_1(g_n,G_n,Q_\nu(x_0,\delta))}{|D^c g|(Q_\nu(x_0,\delta))} = \lim_{k\to\infty}\int_{Q_\nu}\biggl(W_1\biggl(\frac{G(x_0+\delta_k y)}{t_{\delta_k}}-\nabla\bar{w}_k(y)\biggr)-(a\otimes\nu)\biggr)\,\mathrm{d}y + \int_{Q_\nu\cap S_{\bar{w}_k}}\gamma_1([\bar{w}_k](y),\nu(\bar{w}_k)(y))\,\mathrm{d}\mathcal{H}^{N-1}.$$

Since, by proposition 3.10 and (4.11),

$$\begin{aligned}\limsup_{k\to\infty}\int_{Q_\nu}\biggl|W_1\biggl(\frac{G(x_0+\delta_k y)}{t_{\delta_k}}-\nabla\bar{w}_k(y)-(a\otimes\nu)\biggr)-W_1(-\nabla\bar{w}_k(y)-(a\otimes\nu))\biggr|\,\mathrm{d}y \\ \leqslant \limsup_{k\to\infty} C\int_{Q_\nu}\biggl|\frac{G(x_0+\delta_k y)}{t_{\delta_k}}\biggr|\,\mathrm{d}y \\ = \limsup_{k\to\infty}\frac{C}{t_{\delta_k}}\int_{Q_\nu}\Bigl|G(x_0+\delta_k y)\Bigr|\,\mathrm{d}y \\ = \lim_{k\to\infty}\frac{C}{t_{\delta_k}\delta_k^N}\int_{Q_\nu(x_0,\delta_k)}|G(x)|\,\mathrm{d}x \\ = 0 \end{aligned}\tag{4.14}$$

by (4.9). From (4.13), it is easy to see that, for each $k\in\mathbb{N}$, the function $\bar{w}_k(y)$ is admissible for $\tilde{W}_1$. Therefore, from (4.14) and proposition 4.2, we conclude that

$$\begin{aligned}\liminf_{\delta\to 0}\liminf_{n\to\infty}\frac{I_1(g_n,G_n,Q_\nu(x_0,\delta))}{|D^c g|(Q_\nu(x_0,\delta))} &\geqslant \lim_{k\to\infty}\int_{Q_\nu}\Bigl(W_1(-\nabla\bar{w}_k(y))-(a\otimes\nu)\Bigr)\,\mathrm{d}y \\ &\quad+\int_{Q\cap S_{\bar{w}_k}}\gamma_1([\bar{w}_k](y),\nu(\bar{w}_k)(y))\,\mathrm{d}\mathcal{H}^{N-1} \\ &\geqslant \tilde{W}_1(-a\otimes\nu)=W_1(-a\otimes\nu).\end{aligned}$$

Finally, the lower bound (4.10) follows from the arbitrariness of the considered sequence

$$(g_n,G_n)\in \mathrm{SBV}^2(\Omega;\mathbb{R}^d)\times\mathrm{SBV}(\Omega;\mathbb{R}^{d\times N})$$

and from the characterization of I_1 given in lemma 4.3.

Step 4. We remark that, as in theorem 3.2,

$$\frac{\mathrm{d}I_1(g,G,\cdot)}{\mathrm{d}\mathcal{L}^N}(x_0)=W_1(G(x_0)-\nabla g(x_0)),\quad \mathcal{L}^N\text{-a.e. } x_0\in\Omega$$

and

$$\frac{\mathrm{d}I_1(g,G,\cdot)}{\mathrm{d}\mathcal{H}^{N-1}\lfloor S_g}(x_0)=\gamma_1([g](x_0),\nu(g)(x_0)),\quad \mathcal{H}^{N-1}\text{-a.e. } x_0\in S_g.$$

In fact, the proof of the upper bounds can be obtained in a similar way to (4.6). To do so, it is sufficient to choose sequences

$$g_n \xrightarrow[n\to\infty]{L^1} g \quad \text{with } |Dg_n|(\Omega)\xrightarrow[n\to\infty]{}|Dg|(\Omega),$$

https://doi.org/10.1017/S0308210510001460 Published online by Cambridge University Press

which are regular functions for the case of

$$\frac{\mathrm{d}I_1(g, G, \cdot)}{\mathrm{d}\mathcal{L}^N}$$

and piecewise constant functions to address

$$\frac{\mathrm{d}I_1(g, G, \cdot)}{\mathrm{d}\mathcal{H}^{N-1} \lfloor S_g}.$$

Since both W_1 and γ_1 are homogeneous functions of degree one, the result follows from lemma 4.3 and theorem 2.2. It is also easy to check that the lower bounds hold since the proof of their counterparts in theorem 3.2 is still valid in the BV setting.

As a consequence of steps 1–4,

$$I_1(g, G, A) = \int_A W_1(G - \nabla g)\,\mathrm{d}x + \int_{A\cap S_g} \gamma_1([g], \nu_g)\,\mathrm{d}\mathcal{H}^{N-1} + \int_A W_1\biggl(-\frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|}\biggr)\,\mathrm{d}|D^c g| \tag{4.15}$$

for all $A \in \mathcal{A}(\Omega)$, from which, taking $A = \Omega$, equality (4.5) follows, completing the proof of theorem 4.1. □

Acknowledgements

The authors thank Irene Fonseca and David Owen for their fruitful discussions on the model. This work was partly supported by the Fundação para a Ciência e a Tecnologia (FCT/Portugal) by Project no. UTA-CMU/MAT/0005/2009.

References

1 G. Alberti. A Lusin type theorem for gradients. *J. Funct. Analysis* **100** (1991), 110–118.

2 G. Alberti. Rank-one property for derivatives of functions with bounded variation. *Proc. R. Soc. Edinb.* A **123** (1993), 237–274.

3 L. Ambrosio. A compactness theorem for a special class of functions of bounded variation. *Boll. UMI* B **3** (1989), 857–881.

4 L. Ambrosio and G. Dal Maso. On the representation in $\mathrm{BV}(\Omega; \mathbb{R}^m)$ of quasi-convex integrals. *J. Funct. Analysis* **109** (1992), 76–97.

5 L. Ambrosio, S. Mortola and V. M. Tortorelli. Functionals with linear growth defined on vector valued *BV* functions. *J. Math. Pures Appl.* **70** (1991), 269–323.

6 L. Ambrosio, N. Fusco and D. Pallara. *Functions of bounded variation and free discontinuity problems* (Oxford University Press, 2000).

7 A. Barroso, G. Bouchitté, G. Buttazzo and I. Fonseca. Relaxation of bulk and interfacial energies. *Arch. Ration. Mech. Analysis* **135** (1996), 107–173.

8 G. Bouchitté, I. Fonseca and L. Mascarenhas. A global method for relaxation. *Arch. Ration. Mech. Analysis* **145** (1998), 51–98.

9 G. Bouchitté, I. Fonseca, G. Leoni and L. Mascarenhas. A global method for relaxation in $W^{1,p}$ and in SBV_p. *Arch. Ration. Mech. Analysis* **165** (2002), 187–242.

10 A. Braides and A. Coscia. The interaction between bulk energy and surface energy in multiple integrals. *Proc. R. Soc. Edinb.* A **124** (1994), 737–756.

11 M. Carriero, A. Leaci and F. Tomarelli. A second order model in image segmentation: Blake and Zisserman functional. *Prog. Nonlin. Diff. Eqns* **25** (1996), 57–72.

12 M. Carriero, A. Leaci and F. Tomarelli. Second order variational problems with free discontinuity and free gradient discontinuity. In *Calculus of variations: topics from the mathematical heritage of E. De Giorgi*, Quaderni di Matematica, vol. 14, pp. 135–186 (Caserta: Seconda Universita di Napoli, 2004).

13 R. Choksi and I. Fonseca. Bulk and interfacial energies for structured deformations of continua. *Arch. Ration. Mech. Analysis* **138** (1997), 37–103.

14 R. Choksi, G. Del Piero, I. Fonseca and D. Owen. Structured deformations as energy minimizers in models of fracture and hysteresis. *Math. Mech. Solids* **4** (1999), 321–356.

15 E. De Giorgi. Su una teoria generale della missura $(r-1)$-dimensionale in uno spazio a r dimensioni. *Annali Mat. Pura Appl.* **36** (1954), 191–213.

16 E. De Giorgi and L. Ambrosio. Un nuovo tipo di funzionale del calcolo delle variazioni. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **82** (1988), 199–210.

17 G. Del Piero and D. Owen. Structured deformations of continua. *Arch. Ration. Mech. Analysis* **124** (1993), 99–155.

18 L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*, Studies in Advanced Mathematics (CRC Press, 1992).

19 H. Federer. *Geometric measure theory* (Springer, 1969).

20 I. Fonseca and S. Müller. Quasi-convex integrands and lower semicontinuity in L^1. *SIAM J. Math. Analysis* **23** (1992), 1081–1098.

21 I. Fonseca and S. Müller. Relaxation of quasiconvex functionals in $\mathrm{BV}(\Omega, R^p)$ for integrands $f(x, u, \nabla u)$. *Arch. Ration. Mech. Analysis* **123** (1993), 1–49.

22 R. Gariepy and W. D. Pepe. On the level sets of a distance function on a Minkowski space. *Proc. Am. Math. Soc.* **31** (1972), 255–259.

23 E. Giusti. *Minimal surfaces and functions of bounded variation* (Birkhäuser, 1984).

24 J. C. Larsen. Quasiconvexification in $W^{1,1}$ and optimal jump microstructure in BV relaxation. *SIAM J. Math. Analysis* **29** (1998), 823–848.

25 J. Matias. Differential inclusions in $\mathrm{SBV}_0(\Omega)$ and applications to the calculus of variations. *J. Convex Analysis* **14** (2007), 465–477.

26 J. Matias and P. M. Santos. A dimension reduction result in the framework of structured deformations. In preparation.

27 Y. G. Reshetnyak. Weak convergence of completely additive vector functions on a set. *Sb. Math. J.* **9** (1968), 1039–1045.

28 W. Ziemer. *Weakly differentiable functions* (Springer, 1989).

(*Issued 6 April 2012*)