SUNSPOTS AND MAGNETOHYDRODYNAMIC FLOWS*

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ABSTRACT

Theoretical techniques are developed to study compressible, steady-state, magnetically aligned gas flows in sunspot regions. The flows are adiabatic and occur in a known streamline configuration. The non-linear parabolic partial differential equation describing the flow reduces to an ordinary linear differential equation. The solutions are briefly discussed.

1. Introduction

This paper develops some theoretical tools necessary to study compressible, steady-state, magnetically aligned gas flows in sunspots. The motions in the spot region take place along magnetic field lines, i.e.,

$$\mathbf{v}(\mathbf{r}) = \alpha(\mathbf{r})\, \mathbf{H}(\mathbf{r}), \tag{1.1}$$

where $\mathbf{v}$ is the velocity of a gaseous element, α a function of position, and $\mathbf{H}$ the magnetic field.

We specify, at the outset, the form of the magnetic field. We impose no restriction of irrotationality or incompressibility. Such constraints immediately define the streamline configuration. From the known streamline configuration we deduce the velocity $\mathbf{v}(\mathbf{r})$, mass density $\varrho(\mathbf{r})$, and pressure $p(\mathbf{r})$ distributions through the flow region by solutions of the MHD equations.

The magnetic field, which possesses cylindrical symmetry, must satisfy

$$\nabla \cdot \mathbf{H} = 0, \quad \frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu}(\mathbf{v} \times \mathbf{H}) = 0. \tag{1.2}$$

Specifically, we employ the dipole-like field of Menzel and Shore (1966) as given by

$$\mathbf{H}(r, z) = H_r(r, z)\, \mathbf{e}_r + H_z(r, z)\, \mathbf{e}_z, \tag{1.3}$$

where

$$H_r(r, z) = 3\, Mrza\, (a^2 + r^2 + z^2)^{-5/2}, \tag{1.4}$$

$$H_z(r, z) = Ma\,(2\,a^2 + 2\,z^2 - r^2)(a^2 + r^2 + z^2)^{-5/2}. \tag{1.5}$$

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The vector $\mathbf{e}_z$ is a vertical unit vector directed upward from the solar surface and $\mathbf{e}_r$ is the horizontal unit vector. In addition, we shall assume that the gas is perfectly conducting and obeys the adiabatic relation

$$p/p_0 = (\rho/\rho_0)^{\gamma}, \tag{1.6}$$

where γ is the ratio of specific heats.

2. The Formulation of the Problem

We need two additional equations to determine the steady-state flow: the equations of momentum transport and of continuity. The first of these is

$$\nabla\left(\frac{v^2}{2} + \phi + \int \frac{\mathrm{d}p}{\rho}\right) = \mathbf{v} \times (\nabla \times \mathbf{v}) - \frac{1}{4\pi\rho}\, \mathbf{H} \times (\nabla \times \mathbf{H}), \tag{2.1}$$

where ϕ is the gravitational potential. Using the Equations (1.1)–(1.5) and taking the scalar product of (2.1) with $\mathbf{H}$, we get

$$\frac{\alpha^3}{2}(\mathbf{H}\cdot\nabla H^2) + \alpha(\mathbf{H}\cdot\nabla\phi) + (\mathbf{H}\cdot\nabla\alpha)(\alpha^2 H^2 - K\gamma\rho^{\gamma-1}) = 0, \tag{2.2}$$

where $K = p_0 \varrho_0^{-\gamma}$. The continuity equation leads to the result

$$\nabla\cdot\rho\mathbf{v} = \nabla\cdot\rho\alpha\mathbf{H} = \mathbf{H}\cdot\nabla\rho\alpha = 0. \tag{2.3}$$

This equation tells us that the gradient of $\varrho\alpha$ is perpendicular to the magnetic field. In other words, $\varrho\alpha$ is constant along the stream lines.

We now seek solutions of (2.2) where the proportionality function α can be separated into a product of three functions

$$\alpha(\mathbf{r}) = E(\xi)\, G(\rho)\, S(H), \tag{2.4}$$

where $G(\varrho) = (\varrho/\varrho_0)^{(\gamma-1)/2}$, $S(H) = (H/H_0)^{-1}$ and H_0 is the value of the magnetic at the point p_0, ϱ_0. The form of $E(\xi)$ will be demonstrated shortly. Substitution of (2.4) into (2.2) leads to

$$\begin{aligned}
&\mathbf{H}\cdot\nabla(\mathbf{H}\cdot\nabla\xi) + \frac{(\mathbf{H}\cdot\nabla\xi)^2}{\mu\xi} \\
&\quad\times\left\{\left(\frac{4}{\gamma+1}\right)\frac{\xi E'}{E} + \frac{K\gamma}{E^2}\left[1 + \frac{2(\gamma-1)}{(\gamma+1)}\frac{\xi E'}{E}\right] - 1 + \frac{\mu[\mathbf{H}\cdot\nabla(\xi E'/E)]}{(\mathbf{H}\cdot\nabla\xi)(E'/E)}\right\} \\
&\quad + (\mathbf{H}\cdot\nabla\xi)\left[\frac{2h}{H^2}\frac{(\gamma-1)}{(\gamma+1)} - \frac{\mathbf{H}\cdot\nabla\mathscr{G}}{\mathscr{G}}\right] \\
&\quad + \frac{(\gamma+1-2\mu)}{4\mu H^2}\left(\frac{E}{E'}\right)\left[\mathbf{H}\cdot\nabla h - h\left(\frac{\mathbf{H}\cdot\nabla\mathscr{G}}{\mathscr{G}}\right) - \frac{2h^2}{(\gamma+1)\, H^2}\right] = 0,
\end{aligned} \tag{2.5}$$

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where

$$E' = \mathrm{d}E/\mathrm{d}\xi\,, \quad h = \mathbf{H}\cdot\nabla H^2\,, \quad \mu = 1 - K\gamma/E^2\,, \quad \text{and} \quad \mathscr{G} = \mathbf{H}\cdot\nabla\phi\,.$$

Equation (2.5) constitutes a second-order, non-linear, parabolic partial differential equation for $\xi(\mathbf{r})$. Convergent and stable numerical solutions to partial differential equations of this type depend upon the form of the variable coefficients as well as upon the nature of the integration scheme and the boundary conditions. Well-behaved numerical solutions for Equation (2.5) readily follow if we set the coefficient of $(\mathbf{H}\cdot\nabla\xi)^2$ equal to zero.

This assumption not only eliminates the second term from (2.5) but also leads to an ordinary differential equation for E as a function of ξ,

$$\frac{E''}{E'} - \frac{E'}{E}\left[\frac{(3\gamma - 1)}{\gamma + 1} - 2\left(1 - \frac{K\gamma}{E^2}\right)^{-1}\right] = 0\,. \tag{2.6}$$

Equation (2.6) integrates directly to yield

$$C_1\xi + C_2 = E^{(4/(\gamma+1))}\left[\frac{(\gamma - 1)}{2} + \frac{K\gamma}{E^2}\right], \tag{2.7}$$

where C_1, C_2 are constants. Substituting (2.7) into (2.5), we find the equation for ξ

$$\begin{aligned}\mathbf{H}\cdot\nabla(\mathbf{H}\cdot\nabla\xi) + (\mathbf{H}\cdot\nabla\xi)\left[\frac{2(\gamma - 1)}{(\gamma + 1)}\frac{h}{H^2} - \frac{\mathbf{H}\cdot\nabla\mathscr{G}}{\mathscr{G}}\right]& \\ + \frac{1}{H^2}\frac{(\gamma - 1)}{(\gamma + 1)}\left(\xi + \frac{C_2}{C_1}\right)\left[\mathbf{H}\cdot\nabla h - h\frac{(\mathbf{H}\cdot\nabla\mathscr{G})}{\mathscr{G}} - \frac{2h^2}{H^2(\gamma + 1)}\right] &= 0\,.\end{aligned} \tag{2.8}$$

Let us expand (2.8) in cylindrical coordinates into the canonical form (Koshlyakov *et al.*, 1964)

$$A(\eta, \lambda)\,\xi_{\eta\eta} + B(\eta, \lambda)\,\xi_\lambda + C(\eta, \lambda)\,\xi_\eta + D(\eta, \lambda)\,\xi = 0\,, \tag{2.9}$$

by means of the transformation equations

$$\lambda = \lambda(r, z)\,, \quad \eta = \eta(r, z)\,, \tag{2.10}$$

where

$$\xi_\eta = \partial\xi/\partial\eta\,, \quad \xi_{\eta\eta} = \partial^2\xi/\partial\eta^2\,, \quad \xi_\lambda = \partial\xi/\partial\lambda\,.$$

A parabolic differential equation has only one family of characteristic curves and is derived by integration of the equation of characteristics

$$H_r\,\mathrm{d}z - H_z\,\mathrm{d}r = 0\,. \tag{2.11}$$

The integration of (2.11) yields

$$\lambda(r, z) = r^2(a^2 + z^2 + r^2)^{-3/2} = \text{constant}\,, \tag{2.12}$$

and provides us with the first of our set of transformation equations (2.10). The

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second relation $\eta(\mathbf{r}, z)$ is chosen independently. Let us select

$$\eta(r, z) = r. \tag{2.13}$$

The transformation is valid everywhere, provided its Jacobian,

$$\frac{\partial(\lambda, \eta)}{\partial(r, z)} = \frac{\partial\lambda}{\partial z} \times \frac{\partial\eta}{\partial r} - \frac{\partial\lambda}{\partial r} \times \frac{\partial\eta}{\partial z}, \tag{2.14}$$

is not zero.

The Jacobian is zero only for points on the z-axis and on the $z=0$ plane. When the differential Equation (2.9) is transformed by means of the relations (2.12), (2.13), we have the result

$$\begin{aligned} H_r^2 \xi_{\eta\eta} &+ \left\{ H_r \left[\frac{2(\gamma-1)}{(\gamma+1)} \frac{h}{H^2} - \frac{\mathbf{H}\cdot\nabla\mathscr{G}}{\mathscr{G}} \right] + (\mathbf{H}\cdot\nabla)\, H_r \right\} \xi_\eta \\ &+ \frac{h}{H^2} \frac{(\gamma-1)}{(\gamma+1)} \left[\frac{\mathbf{H}\cdot\nabla h}{h} - h \frac{(\mathbf{H}\cdot\nabla\mathscr{G})}{\mathscr{G}} - \frac{2h}{H^2(\gamma+1)} \right] \xi = 0, \end{aligned} \tag{2.15}$$

where $C_2=0$. The coefficients of ξ_η, $\xi_{\eta\eta}$ and ξ are now functions of the new variables λ, η. The differential equation upon which the solution to our problem depends has been reduced to an ordinary, linear, second-order differential equation that can be solved by standard numerical methods.

3. The Solution on the z-axis and on the $z=0$ Plane

The integration over η in Equation (2.15) is equivalent to an integration over r. We first deduce $\xi(\mathbf{r})$ on the z-axis. From these values we compute $\xi(\mathbf{r})$ away from the axis by the numerical integration of (2.15). Since Equation (2.15) is not valid on the axis, we must return to our original expression, (2.5). This equation reduces to

$$\begin{aligned} (a^2+z^2)^2\, \xi_{zz} &- \frac{12(\gamma-1)}{(\gamma+1)} z(a^2+z^2)\, \xi_z \\ &+ \frac{6(\gamma-1)}{(\gamma+1)^2} [z^2(7\gamma-5) - a^2(\gamma+1)]\, \xi = 0, \end{aligned} \tag{3.1}$$

and can be either integrated numerically or transformed into an equation whose solution is a Gaussian hypergeometric function. In the neighborhood of the origin (0, 0), Equation (3.1) leads to

$$a^2 \xi_{zz} - \frac{6(\gamma-1)}{\gamma+1} \xi = 0, \tag{3.2}$$

whose general solution is

$$\xi(z) = \xi_+ e^{+mz} + \xi_- e^{-mz}, \tag{3.3}$$

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where $m = \frac{1}{a}\sqrt{\left(6\frac{(\gamma-1)}{(\gamma+1)}\right)}$, and $\xi_{+,-}$ are constants.

Similarly, the form of the solution in the plane can be derived. In this instance we find it necessary to use a transformation of variables. The result is

$$\xi/\xi_0 = \left\{\frac{(r^2/a^2)^2\,[(r^2/a^2)-2]^4\,(|(r^2/a^2)-2|)}{[(r^2/a^2)+4]}\right\}^{(\gamma-1)/(\gamma+1)} \tag{3.4}$$

where ξ_0 is the value of ξ at some reference point in the sunspot region. Near the origin, ξ/ξ_0 approaches zero. As a result, from (3.3),

$$\lim_{z\to 0} \xi = \xi_+ + \xi_- = 0\,, \tag{3.5}$$

and $\xi_- = -\xi_+$. Hence, the solution for $\xi(\mathbf{r})$ along the z-axis in the neighborhood of the origin is of the form $\xi(z) = \xi_+(e^{mz} - e^{-mz})$.

4. The Velocity Fields

The velocity is given by

$$v(\mathbf{r}) = \alpha H = E[\xi(\mathbf{r})]\,[\rho(\mathbf{r})]^{(\gamma-1)/2}\,. \tag{4.1}$$

In addition to $E[\xi(\mathbf{r})]$, we must specify the mass density, $\varrho(\mathbf{r})$, throughout the flow regime. The mass density is computed in the following manner. Along a streamline

$$\rho\alpha = [\rho(\mathbf{r})]^{(\gamma+1)/2}\,E[\xi(\mathbf{r})]/H(\mathbf{r}) = \text{constant}\,. \tag{4.2}$$

For our magnetic field all streamlines intercept the $z=0$ plane. Since we know the variation of E and H in the $z=0$ plane, we need only compute $\varrho(r, z=0)$ in order to evaluate the constant in (4.2) for any given streamline. After we evaluate the constant, we use (4.2) to compute $\varrho(\mathbf{r})$ along streamlines extending anywhere in the sunspot region. We determine $\varrho(r, z=0)$ from Equation (2.2). After some manipulation we find

$$\rho(r, z=0)/\rho_0 = -\left[\frac{(\gamma-1)\,\mathcal{G}}{C_1\xi_z}\,(E/E_0)^{-2/(\gamma+1)}\right]_{z=0}, \tag{4.3}$$

where $E_0 = E(\xi_0)$. The term $\xi_z(r, z=0)$ appearing in Equation (4.3) is calculated from $\xi(\mathbf{r})$.

Finally, once we have deduced the mass density, we can derive the pressure and temperature distributions from the polytropic condition (1.6).

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