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# Translates of Functions on the Heisenberg Group and the HRT Conjecture

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*Abstract.* We prove that the HRT (Heil, Ramanathan, and Topiwala) Conjecture is equivalent to the conjecture that co-central translates of square-integrable functions on the Heisenberg group are linearly independent.

## 1 Preliminaries

Given $x, y \in \mathbb{R}$, define unitary operators $T_x$ and $M_y$ on $L^2(\mathbb{R})$ by

$$T_x\phi(t) = \phi(t - x), \quad M_y\phi(t) = e^{2\pi ity}\phi(t), \quad \phi \in L^2(\mathbb{R}).$$

Let $\Gamma$ be a countable subset of $\mathbb{R}^2$ and let $\phi \in L^2(\mathbb{R})$. The time-frequency system

$$\mathcal{G}(\phi, \Gamma) = \{M_y T_x \phi : (x, y) \in \Gamma\}$$

is called a Gabor system. There is an extensive literature devoted to Gabor systems that are orthonormal bases, frames, and Riesz bases of $L^2(\mathbb{R})$; for expositions of the basic theory and examples, see [4, 10].

A fundamental open question is whether or not a Gabor system $\mathcal{G}(\phi, \Gamma)$ is necessarily linearly independent in the vector space $L^2(\mathbb{R})$. It is not known whether there is a nonzero vector $\phi$ and a finite set $\mathcal{F} \subset \mathbb{R}^2$ such that $\mathcal{G}(\phi, \mathcal{F})$ is linearly dependent. By comparison with time-scale systems, the existence of a scaling function in multiresolution analysis shows that there are functions $\varphi \in L^2(\mathbb{R})$, and finite sets of translations and *dilations*, for which the resulting system is linearly dependent. Thus, the apparent independence of time-frequency systems is a bit surprising and motivates the so-called HRT conjecture, which first appeared in the literature about twenty years ago in the paper by Chris Heil, Jay Ramanathan, and Pankaj Topiwala [13].

***Conjecture 1.1*** (The HRT conjecture) *Let $\phi \in L^2(\mathbb{R}), \phi \neq 0$, and let $\mathcal{F}$ be a finite subset of $\mathbb{R}^2$. Then the set $\mathcal{G}(\phi, \mathcal{F})$ is linearly independent in $L^2(\mathbb{R})$.*

Partial results on the HRT conjecture are numerous and varied; a sampling is [1–3, 11, 12, 14, 20]. Generally, partial results show that $\mathcal{G}(\phi, \mathcal{F})$ is linearly independent under various conditions on $\phi$, or on $\mathcal{F}$. For example, in [15], Linnell proves that for nonzero $\phi \in L^2(\mathbb{R})$, $\mathcal{G}(\phi, \mathcal{F})$ is linearly independent when $\mathcal{F}$ is a subset of a full-rank lattice in the time-frequency plane. A detailed compilation of examples and partial results for the HRT conjecture is found in [14]. The purpose of this paper is to establish the

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relationship between the HRT Conjecture and translation systems on the Heisenberg group. Our main result provides an affirmative answer to a question asked at the HRT workshop at Saint Louis University in 2016 [18].

Observe that

$$T_x M_y = e^{-2\pi i x y} M_y T_x \tag{1.1}$$

holds for all $x, y \in \mathbb{R}$. This means that the family $\{M_y T_x : (x, y) \in \mathbb{R}^2\}$ of time-frequency operators generates a three-dimensional subgroup $\mathbb{H}$ of the group of all unitary operators on $L^2(\mathbb{R})$. Let $\mathbb{T} = \{\tau \in \mathbb{C} : |\tau| = 1\}$. Then the Heisenberg group $\mathbb{H}$ consists of all operators of the form $\tau M_y T_x$, where $\tau \in \mathbb{T}$, $y, x \in \mathbb{R}$. Letting

$$(\tau, y, x) = \tau M_y T_x, \quad \tau \in \mathbb{T}, (x, y) \in \mathbb{R}^2;$$

the group product in $\mathbb{H}$ is given by

$$(\tau_1, y_1, x_1)(\tau_2, y_2, x_2) = (\tau_1 \tau_2 e^{-2\pi i x_1 y_2}, y_1 + y_2, x_1 + x_2).$$

Let $\mathbb{H}$ have the usual topology as the product $\mathbb{T} \times \mathbb{R}^2$. The center of the group $\mathbb{H}$ is $\{(\tau, 0, 0) : \tau \in \mathbb{T}\}$. By a slight abuse of notation, we denote the center of $\mathbb{H}$ by $\mathbb{T}$, and we can write $\tau = (\tau, 0, 0)$. The topological space $\mathbb{H}$ has a natural differentiable structure with respect to which we define the vector spaces of smooth functions and compactly supported smooth functions as $C^\infty(\mathbb{H})$, and $C_c^\infty(\mathbb{H}) = C_c(\mathbb{H}) \cap C^\infty(\mathbb{H})$, respectively. Moreover, the Schwartz space $\mathcal{S}(\mathbb{H})$ is defined by

$$\mathcal{S}(\mathbb{H}) = \{F \in C^\infty(\mathbb{H}) : (y, x) \longmapsto F(\tau, y, x) \in \mathcal{S}(\mathbb{R}^2), \forall \tau \in \mathbb{T}\},$$

where $\mathcal{S}(\mathbb{R}^2)$ is the Schwartz space on $\mathbb{R}^2$. Let $\mathbb{T}$ have the measure $d\tau$ given by $\int_{\mathbb{T}} g(\tau) d\tau = \int_0^1 g(e^{2\pi i t}) dt$, and let $\mathbb{H}$ have the measure given by

$$\int_{\mathbb{H}} F = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{T}} F(\tau, y, x) d\tau dy dx, \quad F \in C_c(\mathbb{H}).$$

Thus, for each $1 \leq p < \infty$, we have the space $L^p(\mathbb{H})$ containing $C_c^\infty(\mathbb{H})$ and $\mathcal{S}(\mathbb{H})$ as dense subspaces. We can also write $h = (\tau, y, x)$ and $dh = d\tau dy dx$.

The partial Fourier transform $\wedge_1$ of $F \in \mathcal{S}(\mathbb{H})$ at $k \in \mathbb{Z}$ is defined by

$$\wedge_1 F(k, y, x) = \int_{\mathbb{T}} F(\tau, y, x) \tau^k d\tau.$$

By extension, we have the unitary isomorphism $\wedge_1 : L^2(\mathbb{H}) \to L^2(\mathbb{Z} \times \mathbb{R}^2)$. In particular,

$$\|F\|^2 = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^2} |\wedge_1 F(k, y, x)|^2 dy dx, \quad F \in L^2(\mathbb{H}). \tag{1.2}$$

For $F \in \mathcal{S}(\mathbb{H})$, define $\widehat{F}_k \in \mathcal{S}(\mathbb{H})$ by $\widehat{F}_k(\tau, y, x) = \wedge_1 F(k, y, x) \cdot \tau^{-k}$.

***Lemma 1.2*** *The map $P_k : F \mapsto \widehat{F}_k$ extends to an orthogonal projection $P_k : L^2(\mathbb{H}) \to L^2(\mathbb{H})_k$ where $L^2(\mathbb{H})_k$ is the closed subspace defined by*

$$L^2(\mathbb{H})_k = \{F \in L^2(\mathbb{H}) : \Lambda_1 F(j, y, x) = 0, j \neq k\}.$$

*Moreover, the spaces $L^2(\mathbb{H})_k$ are pairwise orthogonal and $L^2(\mathbb{H}) = \oplus_{k \in \mathbb{Z}} L^2(\mathbb{H})_k$.*

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**Proof** By (1.2), for $F \in \mathcal{S}(\mathbb{H})$,

$$\|P_k F\|^2 = \int_{\mathbb{H}} \left|\wedge_1 F(k,y,x)\tau^{-k}\right|^2 dh = \int_{\mathbb{R}^2}\int_{\mathbb{T}} \left|\wedge_1 F(k,y,x)\right|^2 d\tau dy dx$$
$$= \int_{\mathbb{R}^2} \left|\wedge_1 F(k,y,x)\right|^2 dy dx \le \|F\|^2, \tag{1.3}$$

so $P_k$ extends to a linear contraction on $L^2(\mathbb{H})$. The fact that $P_k$ is an orthogonal projection follows from the following calculations. First, for $k \in \mathbb{Z}$,

$$\langle P_k F, G\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} \wedge_1 F(k,y,x)\overline{\wedge_1 G}(k,y,x) dy dx = \langle F, P_k G\rangle,$$

and secondly, for all $j,k \in \mathbb{Z}$,

$$\langle P_k F, P_j F\rangle = \int_{\mathbb{H}} \widehat{F}_k \overline{\widehat{F}}_j = \int_{\mathbb{R}^2}\int_{\mathbb{T}} \tau^{j-k} \wedge_1 F(k,y,x)\overline{\wedge_1 F}(j,y,x) d\tau dy dx$$
$$= \begin{cases} \|\widehat{F}_k\|^2, & \text{if } j = k, \\ 0, & \text{if } j \ne k. \end{cases}$$

Referring to (1.2) and (1.3), we obtain

$$\|F\|^2 = \sum_{k\in\mathbb{Z}} \|P_k F\|^2,$$

and the lemma follows. ■

We make the natural identifications: $\mathbb{H}/\mathbb{T} = \mathbb{R}^2$ via $(\tau,y,x)\mathbb{T} = \mathbb{T}(\tau,y,x) = (y,x)$, $C_c^\infty(\mathbb{H}/\mathbb{T}) = C_c^\infty(\mathbb{R}^2)$, and $\mathcal{S}(\mathbb{H}/\mathbb{T}) = \mathcal{S}(\mathbb{R}^2)$. For each $h = (\tau,y,x) \in \mathbb{H}$, we say that the point $(y,x) \in \mathbb{R}^2$ is the projection of $h$. For $F \in \mathcal{S}(\mathbb{H})$, observe that $P_0 F \in \mathcal{S}(\mathbb{R}^2)$: $P_0 F(\tau,y,x) = \wedge_1 F(0,y,x) = \int_{\mathbb{T}} F(\tau,y,x) d\tau$. Accordingly, we identify $L^2(\mathbb{H})_0$ with $L^2(\mathbb{R}^2)$ via $P_0$, and put

$$Q = 1 - P_0 \colon L^2(\mathbb{H}) \longrightarrow L^2(\mathbb{R}^2)^\perp = \oplus_{k\ne 0} L^2(\mathbb{H})_k.$$

We can also write $\mathcal{S}(\mathbb{H})_k = P_k(\mathcal{S}(\mathbb{H}))$.

## 2 Translation Systems in $L^2(\mathbb{H})$

For $h \in \mathbb{H}$, define left and right translations (unitary) operators $L_h$ and $R_h$ on $L^2(\mathbb{H})$ by

$$L_h F = F(h^{-1}\cdot), \quad R_h F = F(\cdot h)$$

for $F \in \mathcal{S}(\mathbb{H})$. Observe that for all $h_1, h_2 \in \mathbb{H}$,

$$L_{h_1} R_{h_2} = R_{h_2} L_{h_1}. \tag{2.1}$$

With $h_0 = (\tau_0, y_0, x_0)$, we compute that $h_0^{-1} = (\tau_0^{-1} e^{-2\pi i x_0 y_0}, -y_0, -x_0)$, and we find that for each $k \in \mathbb{Z}$,

$$\left(P_k(L_{h_0}F)\right)(\tau,y,x) = \wedge_1 F(k, y-y_0, x-x_0)\left(\tau\tau_0^{-1} e^{2\pi i x_0(y-y_0)}\right)^{-k}$$
$$= \left(L_{h_0} P_k F\right)(\tau,y,x)$$

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and similarly

$$\begin{aligned}\big(P_k(R_{h_0}F)\big)(\tau,y,x) &= \wedge_1 F(k,y+y_0,x+x_0)\tau_0^{-k}e^{2\pi ikxy_0}\\ &= \big(R_{h_0}P_kF\big)(\tau,y,x).\end{aligned}$$

In particular, when $P_0F(\tau,y,x) = \wedge_1 F(0,y,x)$ is identified with an element of $L^2(\mathbb{R}^2)$, then left translation by $L_{h_0}$ is just translation by the projection $(y_0,x_0)$ of $h_0$ in $L^2(\mathbb{R}^2)$:

$$P_0(L_{h_0}F)(\tau,y,x) = L_{h_0}P_0F(\tau,y,x) = \wedge_1 F(0,y-y_0,x-x_0).$$

This shows that

(a) each of the subspaces $L^2(\mathbb{H})_k$ of $L^2(\mathbb{H})$, and their orthogonal complements, are left and right translation invariant subspaces of $L^2(\mathbb{H})$, and
(b) in the subspace $L^2(\mathbb{H})_0 = L^2(\mathbb{R}^2)$, translation by $h = (\tau,y,x)$ is just translation by its projection $(y,x)$.

Let $\mathcal{E}$ be a finite subset of $\mathbb{H}$. For $F \neq 0$ in $L^2(\mathbb{H})$, consider the system of left translates

$$\mathcal{L}(F,\mathcal{E}) = \{L_hF : h \in \mathcal{E}\} \subset L^2(\mathbb{H}).$$

There are two ways to see that such a system is not always linearly independent in $L^2(\mathbb{H})$. First, let $\tau$ be any $p$-th root of unity in $\mathbb{T}$, and let $F \in L^2(\mathbb{H})$, $F \neq 0$. Then

$$G = \sum_{k=1}^{p} L_{\tau^k}F$$

satisfies $L_\tau G = G$. Thus, there are non-zero functions $F \in L^2(\mathbb{H})$ with dependent $\mathbb{T}$-translates. Second, suppose that $h_2 = \tau h_1$ for some $\tau \in \mathbb{T}$, and let $F \in L^2(\mathbb{R}^2)$. Then $L_{h_1}F = L_{h_2}F$. In light of this observation, we will only consider subsets $\mathcal{E}$ whose projections in $\mathbb{R}^2$ are distinct, that is, sets whose elements satisfy $h \neq h' \implies \mathbb{T}h \neq \mathbb{T}h'$. In this case, we say that $\mathcal{E}$ is *co-central*.

***Conjecture 2.1*** (The Heisenberg Translate Conjecture (HT)) *Let $\mathcal{E}$ be a finite co-central subset of $\mathbb{H}$ and let $F \in L^2(\mathbb{H})$ be non-zero. Then $\mathcal{L}(F,\mathcal{E})$ is linearly independent.*

For a subset $\mathcal{K}$ of $L^2(\mathbb{H})$, consider the partial Heisenberg Translate Conjecture:

HT($\mathcal{K}$): For each $F \in \mathcal{K}$, and for $\mathcal{E}$ a finite co-central subset of $\mathbb{H}$, the system $\mathcal{L}(F,\mathcal{E})$ is linearly independent.

Observe that for any closed, left translation-invariant subspace $\mathcal{K}$ of $L^2(\mathbb{H})$, the HT Conjecture is true if and only if both HT($\mathcal{K}$) and HT($\mathcal{K}^\perp$) are true. More generally, if $L^2(\mathbb{H})$ is an orthogonal direct sum of closed left-invariant subspaces $\mathcal{K}_k$, then the HT Conjecture is true if and only if HT($\mathcal{K}_k$) is true for all $k$.

It is easy to see that HT($L^2(\mathbb{H})_0$) is true. Let $\mathcal{E}$ be a finite co-central subset of $\mathbb{H}$, and let $F \in L^2(\mathbb{H})_0$, $F \neq 0$. Let $\mathcal{T}$ be the projection of $\mathcal{E}$ in $\mathbb{R}^2$, and identify $L^2(\mathbb{H})_0$ with $L^2(\mathbb{R}^2)$ as above. By (a) and (b), the system $\mathcal{L}(F,\mathcal{E})$ is just the translation system corresponding to $F$ and $\mathcal{T}$. But every translation system in $L^2(\mathbb{R}^2)$ is linearly independent [7, Theorem 1.2], or [12, Theorem 9.18]. We conclude that

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***Conclusion 2.2*** *The Heisenberg Translate Conjecture is true if and only if* $\mathrm{HT}(L^2(\mathbb{H})_0^\perp)$ *is true.*

## 3 Irreducible Unitary Representations of $\mathbb{H}$

Just as the Fourier transform changes translations to modulations in $L^2(\mathbb{R}^n)$, the operator-valued Plancherel transform for a sufficiently nice locally compact group changes both left and right translations into compositions by irreducible unitary representations.

Recall that a unitary representation $\pi$ of $\mathbb{H}$ is a homomorphism of $\mathbb{H}$ into the group of unitary operators on some non-zero Hilbert space $\mathcal{H}$, which is strongly continuous: $h_n \to h$ implies $\pi(h_n)v \to \pi(h)v$ for each $v \in \mathcal{H}$. A unitary representation $\pi$ acting in $\mathcal{H}$ is *irreducible* means that every non-zero vector $v$ is *cyclic* for $\pi$: $\{\pi(h)v : h \in \mathbb{H}\}$ is dense in $\mathcal{H}$ for each $v \in \mathcal{H}$. Equivalently, the only non-zero closed subspace that is invariant under all $\pi(h), h \in \mathbb{H}$ is $\mathcal{H}$.

The following lemma allows one to find a list $\pi_k, k \in \mathbb{Z}\backslash\{0\}$, of irreducible unitary representations of $\mathbb{H}$, acting in the Hilbert space $L^2(\mathbb{R})$. The proof of the lemma is a consequence of the well-known characterization of closed translation invariant subspaces of $L^2(\mathbb{R})$, and is left to the reader.

***Lemma 3.1*** *Let* $\mathcal{I} \subset L^2(\mathbb{R})$ *be a closed, non-zero subspace that is invariant under the action of the group of unitary operators generated by* $T_x$ *and* $M_y$*, where* $x, y \in \mathbb{R}$*. Then* $\mathcal{I} = L^2(\mathbb{R})$.

Observe that $\mathbb{H}$ is a subgroup of the group $\mathcal{U}(L^2(\mathbb{R}))$ of unitary operators on $L^2(\mathbb{R})$, and $h_n \to h$ in $\mathbb{H}$ implies $h_n\phi \to h\phi$ in $L^2(\mathbb{R})$. Thus, the inclusion map $\pi_1 : \mathbb{H} \hookrightarrow \mathcal{U}(L^2(\mathbb{R}))$ is a unitary representation of $\mathbb{H}$, and Lemma 3.1 shows that $\pi_1$ is irreducible. Slightly more generally, let $k \in \mathbb{Z}\backslash\{0\}$, and for each $(z, y, x) \in \mathbb{H}$, put

$$\pi_k(\tau, y, x) = \tau^k M_{ky} T_x.$$

The relation (1.1) shows that $\pi_k$ is a homomorphism of $\mathbb{H}$ into the unitary group $\mathcal{U}(L^2(\mathbb{R}))$, and as with $\pi_1$, $\pi_k$ is an irreducible unitary representation of $\mathbb{H}$. It is also easy to check that if $k_1 \neq k_2$, then $\pi_{k_1}$ and $\pi_{k_2}$ are not equivalent.

It turns out that up to a natural notion of equivalence, the $\pi_k, k \in \mathbb{Z}\backslash\{0\}$, are the only irreducible unitary representations of $\mathbb{H}$ acting in $L^2(\mathbb{R})$. The proof of this fact uses two fundamental results: the Stone-von Neumann Theorem and Schur's Lemma. We omit the proofs here, as we do not need this result in what follows. A self-contained source for this material is the text [5].

The relation between $\pi_k$ and time-frequency systems in $L^2(\mathbb{R})$ is simple.

***Lemma 3.2*** *Let* $\mathcal{E}$ *be a finite co-central subset of* $\mathbb{H}$*, and let* $k \in \mathbb{Z}\backslash\{0\}$*. Write* $\mathcal{E} = \{h_1, \ldots, h_N\}$ *with* $h_i = (\tau_i, y_i, x_i)$*, and put* $\mathcal{F}_k = \{(ky_1, x_1), \ldots, (ky_N, x_N)\} \subset \mathbb{R}^2$*. Then for each* $\phi \in L^2(\mathbb{R})$*,* $\mathcal{G}(\phi, \mathcal{F}_k)$ *is independent if and only if* $\pi_k(\mathcal{E})\phi$ *is independent.*

**Proof** By definition of $\pi_k$, for each $1 \le i \le N$, the element $\pi_k(h_i)\phi$ of the system $\pi_k(\mathcal{E})\phi$ is a non-zero complex multiple of the element $M_{ky_i} T_{x_i}\phi$ of $\mathcal{G}(\phi, \mathcal{F}_k)$. ∎

We next show that each element of $L^2(\mathbb{H})_0^{\perp} \subset L^2(\mathbb{H})$ has a type of operator-valued Fourier series. Let $F \in \mathcal{S}(\mathbb{H})$; for each $k \in \mathbb{Z}\setminus\{0\}$, define a sesquilinear form $s_{F,k}$ on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ by

$$s_{F,k}\colon (\phi,\psi) \longmapsto \int_{\mathbb{H}} F(h)\langle \pi_k(h)\phi,\psi\rangle \, dh.$$

Since $F$ is integrable on $\mathbb{H}$, the form $s_{F,k}$ is bounded:

$$|s_{F,k}(\phi,\psi)| \le \|F\|_1 \|\phi\|_2 \|\psi\|_2.$$

and hence defines a bounded linear operator $\pi_k(F)$ on $L^2(\mathbb{R})$ by

$$s_{F,k}(\phi,\psi) = \langle \pi_k(F)\phi,\psi\rangle.$$

In fact, $\pi_k(F)$ is an integral operator. A straightforward calculation shows

$$s_{F,k}(\phi,\psi) = \int_{\mathbb{R}}\int_{\mathbb{R}} \wedge_{1,2}F(k,kt,t-x)\phi(x)\overline{\psi}(t)dxdt,$$

where $\wedge_{1,2}F = \wedge_2\wedge_1 F$ is the Fourier transform of $F$ in the variables $\tau$ and $y$. Regarding the Fourier transform $\wedge_2$ in $y$ as a unitary map on $L^2(\mathbb{R}^2)$, we get

$$\begin{aligned}(3.1)\qquad \int_{\mathbb{R}}\int_{\mathbb{R}} |\wedge_{1,2} F(k,kt,t-x)|^2 dxdt &= \frac{1}{|k|}\int_{\mathbb{R}}\int_{\mathbb{R}} |\wedge_{1,2} F(k,t,x)|^2 dxdt \\ &= \frac{1}{|k|}\int_{\mathbb{R}}\int_{\mathbb{R}} |\wedge_1 F(k,y,x)|^2 dydt < \infty,\end{aligned}$$

so the theory of integral operators says that $\pi_k(F)$ is a Hilbert-Schmidt operator given by

$$\big(\pi_k(F)\phi\big)(t) = \int_{\mathbb{R}} K_k^F(t,x)\phi(x)dx,$$

where $K_k^F(t,x) = \wedge_{1,2}F(k,kt,t-x)$ and the Hilbert–Schmidt norm of $\pi_k(F)$ is given by (3.1). Note that if $P_kF = 0$, that is, if $F \in L^2(\mathbb{H})_k^{\perp}$, then $\pi_k(F) = 0$. In particular, $\pi_k(F) = 0$ holds for all $F \in L^2(\mathbb{H})_0$. Finally, it is easy to check that

$$(3.2)\qquad \pi_k(L_hF) = \pi_k(h)\pi_k(F), \quad \pi_k(R_hF) = \pi_k(F)\pi_k(h)^{-1}.$$

These relations are what is meant by the statement at the beginning of this section; the operators $\pi_k(F)$ are values of the Plancherel transform of $F$, and left (resp. right) translation of $F$ by $h \in \mathbb{H}$ converts into the composition of $\pi_k(F)$ on the left (resp. on the right) by the unitary operator $\pi_k(h)$ (resp. $\pi_k(h)^{-1}$.) A bit more is said below about the Plancherel transform.

The representations $\pi_k$ are related to the spaces $L^2(\mathbb{H})_k$ as follows.

***Lemma 3.3*** *Let $F \in \mathcal{S}(\mathbb{H})$. Then $\pi_k(F) = \pi_k(P_kF)$. Moreover,*

$$\|\pi_k(F)\|_{\mathbb{HS}}^2 = \frac{1}{|k|}\|P_kF\|^2.$$

**Proof** Observe that $\wedge_1 P_kF(k,y,x) = \wedge_1 F(k,y,x)$, so

$$K_K^{P_kF}(t,x) = \wedge_2\big(\wedge_1 P_kF\big)(k,kt,t-x) = \wedge_2\big(\wedge_1 F\big)(k,kt,t-x) = K_K^F(t,x),$$

and hence $\pi_k(F) = \pi_k(P_kF)$. The second part is a consequence of (3.1). ∎

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Denote the Hilbert space of Hilbert–Schmidt operators on $L^2(\mathbb{R})$ by $\mathbf{HS}(L^2(\mathbb{R}))$. In light of Lemmas 1.2 and 3.3, the following proposition is almost immediate.

***Proposition 3.4*** *The map $F \mapsto (|k|^{1/2}\pi_k(F))_{k\in\mathbb{Z}\setminus\{0\}}$, $F \in L^2(\mathbb{H})_0^{\perp} \cap \mathcal{S}(\mathbb{H})$, extends to a unitary isomorphism*

$$L^2(\mathbb{H})_0^{\perp} \longrightarrow \bigoplus_{k\in\mathbb{Z}\setminus\{0\}} \mathbf{HS}\big(L^2(\mathbb{R})\big).$$

**Proof** Given $F \in L^2(\mathbb{H})_0^{\perp}$, write $F = \sum_{k\neq 0} P_k F$. By Lemma 3.3, $\|P_k F\|_2^2 = \||k|^{1/2}\pi_k(F)\|_{\mathbf{HS}}^2$. ∎

The Plancherel transform for the group $\mathbb{H}$, also called the group Fourier transform, derives from the decomposition $L^2(\mathbb{H}) = \oplus_k L^2(\mathbb{H})_k$, and is described as follows. Start with $F \in \mathcal{S}(\mathbb{H})$ and write $F = F_0 + QF$, with $F_0 \in L^2(\mathbb{H})_0$, and $QF \in L^2(\mathbb{H})_0^{\perp}$. Identifying $F_0$ with a function on $\mathbb{R}^2$, the Plancherel transform sends $F_0$ to its Euclidean Fourier transform, and sends $QF$ to the operator field $(\pi_k(F))_{k\neq 0}$. (The Plancherel measure $\mu$ is then defined on $\mathbb{Z}\setminus\{0\}$ by $\mu(\{k\}) = |k|$.)

One can also see that the map defined in Proposition 3.4 gives a Fourier series-type expansion of $F \in L^2(\mathbb{H})_0^{\perp}$. By the polarization identity, for $F_1, F_2 \in L^2(\mathbb{H})_0^{\perp}$, we have

$$\langle F_1, F_2\rangle = \sum_{k\neq 0} |k| \operatorname{trace}\big(\pi_k(F_1)\pi_k(F_2)^*\big).$$

Define the involution on $\mathcal{S}(\mathbb{H})$ by $F^*(h) = \overline{F}(h^{-1})$, and the convolution on $\mathcal{S}(\mathbb{H})$ by

$$F_1 * F_2(h) = \int_{\mathbb{H}} F_1(g)F_2(g^{-1}h)\,dg.$$

Then with $e = (1,0,0)$, $(F_1 * F_2^*)(e) = \langle F_1, F_2\rangle$, and hence,

$$(F_1 * F_2^*)(e) = \sum_{k\neq 0} |k| \operatorname{trace}\big(\pi_k(F_1)\pi_k(F_2)^*\big).$$

By the factorization theorem of Dixmier–Malliavin [6], every function $F \in C_c^\infty(\mathbb{H})$ is a finite sum of functions of the form $F_1 * F_2$. It follows that for $F \in C_c^\infty(\mathbb{H})$, $\pi_k(F)$ is actually trace-class, and we have

$$F(e) = \sum_{k\neq 0} |k| \operatorname{trace}\big(\pi_k(F)\big),$$

and hence for all $h \in \mathbb{H}$,

$$F(h) = (R_h F)(e) = \sum_{k\neq 0} |k| \operatorname{trace}\big(\pi_k(F)\pi_k(h)^{-1}\big).$$

Thus, Proposition 3.4 can be regarded as giving a Fourier series-type expansion of $F$.

## 4 The Heisenberg Translate Conjecture and the HRT Conjecture

We have seen that the partial Heisenberg Translate Conjecture $\mathrm{HT}(L^2(\mathbb{H})_0)$ for the subspace $L^2(\mathbb{H})_0$ is true. We now exhibit two more subsets $\mathcal{K}$ for which $\mathrm{HT}(\mathcal{K})$ is true.

Recall that a unitary representation $\pi$ of $\mathbb{H}$ acting in a Hilbert space $\mathcal{H}$ is irreducible if every non-zero element in $\mathcal{H}$ is cyclic for $\pi$. When a unitary representation is not

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irreducible it can still admit a cyclic element. An element $v \in \mathcal{H}$ such that the linear span of $\{\pi(h)v : h \in \mathbb{H}\}$ is dense in $\mathcal{H}$. In this case, the representation is said to be a *cyclic representation*. Even though the right regular representation is not irreducible, it is cyclic.

***Proposition 4.1*** *The right regular representation of $\mathbb{H}$ is a cyclic representation.*

For a proof of Proposition 4.1, we refer the interested reader to a paper of Losert and Rindler [17], which gives a construction of a cyclic element for the regular representation of any first countable locally compact group. A non-constructive proof of Proposition 4.1 can also be found in [9].

Denote by $\mathcal{C}$ the subset of all cyclic elements for the right regular representation of $\mathbb{H}$.

***Proposition 4.2*** *The statement* $\mathrm{HT}(\mathcal{C})$ *is true.*

**Proof** Let $\mathcal{E} = \{h_1, \dots, h_N\}$ be co-central, and let $A$ be an operator belonging to the span of $\{L_h : h \in \mathcal{E}\}$. Let $F \in \mathcal{C}$, $F \neq 0$. We must show that if $AF = 0$, then $A = 0$. But if $AF = 0$, then (2.1) shows that $AG = 0$ for all $G \in \mathcal{R} = \operatorname{span}_{\mathbb{C}}\{R_h F : h \in \mathbb{H}\}$. Since $\mathcal{R}$ is dense, the result follows. ∎

Next we show an example of how Proposition 3.4 can be used together with known partial results for the HRT Conjecture. Fix $x_0 \in \mathbb{R}$ and put $\mathbb{H}(x \le x_0) = \{(\tau, y, x) : x \le x_0\}$. Let $L^2(\mathbb{H}(x \le x_0))$ be the subspace of all $F \in L^2(\mathbb{H})$ with essential support contained in $\mathbb{H}(x \le x_0)$.

***Proposition 4.3*** *The statement* $\mathrm{HT}(L^2(\mathbb{H}(x \le x_0)))$ *is true.*

**Proof** Let $F \in L^2(\mathbb{H}(-\infty, x_0))$, $F \neq 0$, and let $\mathcal{E} \subset \mathbb{H}$ be finite and co-central. Without loss of generality, we can assume that $F \in L^2(\mathbb{H})_0^{\perp}$. Since $F \neq 0$, there exists $k \in \mathbb{Z}\backslash\{0\}$ such that $\pi_k(F) \neq 0$. Since $C_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, there exists also $\phi \in C_c(\mathbb{R})$ such that $\pi_k(F)\phi \neq 0$. Choose $a \in \mathbb{R}$ such that $\operatorname{supp}(\phi) \subset [-a, a]$. We claim that $\pi_k(F)\phi$ is necessarily supported on a half-line in $L^2(\mathbb{R})$. First, observe that $F \in L^2(\mathbb{H}(x \le x_0))$ implies $\operatorname{supp}(\wedge_{1,2}F(k, \cdot, \cdot)) \subset \mathbb{R} \times (-\infty, x_0]$. For each $t \in \mathbb{R}$, put

$$M_{k,t}(x) = K_k^F(t, x)\phi(x) = \wedge_{1,2}F(k, kt, t - x)\phi(x).$$

It follows that

$$\operatorname{supp}(M_{k,t}) \subset [-a, a] \cap [t - x_0, +\infty).$$

Hence, $\pi_k(F)\phi(t) \neq 0$ implies $M_{k,t} \neq 0$, implies $[-a, a] \cap [t - x_0, +\infty) \neq \varnothing$, implies $t < a + x_0$, proving the claim.

Next, let $\mathcal{F}_k$ be the finite subset of $\mathbb{R}^2$ associated with $\mathcal{E}$ as in Lemma 3.2. By [13, Proposition 3], $\mathcal{G}(\pi_k(F)\phi, \mathcal{F}_k)$ is linearly independent, so by Lemma 3.2, $\pi_k(\mathcal{E})\pi_k(F)\phi$ is linearly independent. But by (3.2),

$$\pi_k(\mathcal{E})\pi_k(F)\phi = \pi_k\Big(\mathcal{L}(F, \mathcal{E})\Big)\phi,$$

and hence $\mathcal{L}(F, \mathcal{E})$ must be independent. ∎

The following is immediate.

***Corollary 4.4*** *The* HT *Conjecture holds for all non-zero elements of* $L^2(\mathbb{H})$ *with bounded essential support.*

***Proposition 4.5*** *Let* $A: L^2(\mathbb{H}) \to L^2(\mathbb{H})$ *be linear injective operator with a bounded inverse, and let* $\mathcal{K}$ *be a subset of* $L^2(\mathbb{H})$. *Assume that* $AL_h = L_h A$ *holds for all* $h \in \mathbb{H}$. *If* $\mathrm{HT}(\mathcal{K})$ *is true, then* $\mathrm{HT}(A(\mathcal{K}))$ *is true.*

**Proof** Suppose that $\mathrm{HT}(\mathcal{K})$ is true. Let $F \in A(\mathcal{K})$ and let $\mathcal{E} \subset \mathbb{H}$ be finite and co-central. Then $A^{-1}: A(L^2(\mathbb{H})) \to L^2(\mathbb{H})$ is linear and $A^{-1}F \in \mathcal{K}$ so $\mathcal{L}(A^{-1}F, \mathcal{E})$ is linearly independent; hence, $\mathcal{L}(F, \mathcal{E}) = A(\mathcal{L}(A^{-1}F, \mathcal{E}))$ is linearly independent. ∎

Finally, we turn to the equivalence of the HRT Conjecture with the HT Conjecture.

***Theorem 4.6*** *The* HRT *Conjecture is true if and only if the Heisenberg Translate Conjecture is true.*

We begin with a proof of a standard result; see also [5, 8, 19].

***Lemma 4.7*** *Fix* $k \in \mathbb{Z}\backslash\{0\}$ *and let* $f, g \in L^2(\mathbb{R})$. *Then the function* $F_{g,f}: h \mapsto \langle g, \pi_k(h) f\rangle$ *is continuous and square-integrable on* $\mathbb{H}$.

**Proof** The fact that $F_{g,f}$ is continuous is a consequence of the strong continuity of the representation $\pi_k$. To see that $F_{g,f}$ is square-integrable on $\mathbb{H}$, repeat the computation of (1.3): letting $dh$ be the left-invariant measure on $\mathbb{H}$,

$$\int_{\mathbb{H}} |F_{g,f}(h)|^2 dh = \int_{\mathbb{R}^2} |\langle g, \pi_k(1, y, x) f\rangle|^2 dydx = \int_{\mathbb{R}^2} |\langle g, M_{ky} T_x f\rangle|^2 dydx.$$

Now

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left|\langle g, M_{ky} T_x f\rangle\right|^2 dxdy = \int_{\mathbb{R}} \int_{\mathbb{R}} \left|\left([M_{-ky} g] * f^*\right)(x)\right|^2 dxdy.$$

In the last equality above, $*$ stands for the usual convolution and $f^*(x) = \overline{f(-x)}$. A standard computation shows that the function $x \mapsto ([M_{-ky} g] * f^*)(x)$ is the inverse Fourier transform of

$$\widehat{M_{-ky} g}\,\widehat{f^*}: \xi \mapsto \widehat{g}(\xi + ky)\overline{\widehat{f}}(\xi). \tag{4.1}$$

Hence, for each $y \in \mathbb{R}$, $x \mapsto ([M_{-ky} g] * f^*)(x)$ belongs to $L^2(\mathbb{R})$ if the function (4.1) belongs to $L^2(\mathbb{R})$, in which case

$$\int_{\mathbb{R}} |[M_{-ky} g] * f^*(x)|^2 dx = \int_{\mathbb{R}} |\widehat{g}(\xi + ky)\overline{\widehat{f}}(\xi)|^2 d\xi.$$

But

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}} |\widehat{g}(\xi + ky)\overline{\widehat{f}}(\xi)|^2 d\xi dy &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |\widehat{g}(\xi + ky)|^2 dy\Big) |\overline{\widehat{f}}(\xi)|^2 d\xi \\ &= \frac{1}{|k|} \|g\|^2 \|f\|^2.\end{aligned}$$

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We conclude that for a.e. $y$, the function (4.1) does belong to $L^2(\mathbb{R})$, and hence

$$\int_{\mathbb{H}} |F_{g,f}(h)|^2 dh = \int_{\mathbb{R}} \int_{\mathbb{R}} |g(\xi + ky)\overline{\widehat{f}}(\xi)|^2 d\xi dy = \frac{1}{|k|} \|f\|^2 \|g\|^2 < \infty. \qquad \blacksquare$$

It is worth noting that Conjecture 2.1 implies Conjecture 1.1 was also proved in [16, Proposition 1.1].

**Proof of Theorem 4.6** Suppose that the Heisenberg Translate Conjecture (Conjecture 2.1) is true. To prove the HRT Conjecture (Conjecture 1.1), let $\phi \in L^2(\mathbb{R})$, $\phi \neq 0$, and let $\mathcal{F} \subset \mathbb{R}^2$ be finite.

Consider $F(h) = F_{\phi,\phi}(h) = \langle \phi, \pi_1(h)\phi \rangle$ and put $\mathcal{E} = \{(1, y, x) : (y, x) \in \mathcal{F}\}$. Clearly, $\mathcal{E}$ is co-central, and by Lemma 4.7, $F$ defines a non-zero element of $L^2(\mathbb{H})$. Observe also that for $h = (\tau, y, x)$, the definition of $\pi_1$ implies that $F(\tau, y, x) = \tau^{-1}F(1, y, x)$; thus, $F \in L^2(\mathbb{H})_1$. By our assumption that Conjecture 2.1 is true, $\mathcal{L}(F, \mathcal{E})$ is linearly independent.

Now for each $h \in \mathcal{E}$,

$$L_h F(h) = \langle \phi, \pi(h)^{-1}\pi(h)\phi \rangle = \langle \pi(h)\phi, \pi(h)\phi \rangle,$$

so

$$\mathcal{L}(F, \mathcal{E}) = \{F_{\pi_1(h)\phi,\phi} : h \in \mathcal{E}\}.$$

Hence, $\pi_1(\mathcal{E})\phi$ is linearly independent. By Lemma 3.2, $\mathcal{G}(\phi, \mathcal{F})$ is independent.

Conversely, suppose that the HRT Conjecture (Conjecture 1.1) is true. Let $F \in L^2(\mathbb{H})$, $F \neq 0$, and let $\mathcal{E} \subset \mathbb{H}$ finite and co-central. By Conclusion 2.2, we can assume that $F \in L^2(\mathbb{R}^2)^\perp$, meaning that $F = \sum_{k\neq 0} F_k$ as above. Choose $k$ such that $F_k \neq 0$; by Lemma 3.3, the operator $\pi_k(F) = \pi_k(F_k)$ is non-zero, so there is $\phi \in L^2(\mathbb{R})$ such that $\pi_k(F)\phi$ is a non-zero element of $L^2(\mathbb{R})$. Define the finite subset $\mathcal{F}_k$ of $\mathbb{R}^2$, as in Lemma 3.2. By assumption, $\mathcal{G}(\pi_k(F)\phi, \mathcal{F}_k)$ is linearly independent, and hence by Lemma 3.2, $\pi_k(\mathcal{E})\pi_k(F)\phi$ is linearly independent. But then

$$\pi_k(\mathcal{E})\pi_k(F)\phi = \{\pi_k(L_h F)\phi : h \in \mathcal{E}\} = \pi_k\Big(\mathcal{L}(F, \mathcal{E})\Big)\phi$$

is linearly independent, so $\mathcal{L}(F, \mathcal{E})$ is linearly independent. $\blacksquare$

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